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TREES WITH A UNIQUE MAXIMUM INDEPENDENT SET AND THEIR LINEAR PROPERTIES

DANIEL A. JAUME, GONZALO MOLINA, AND RODRIGO SOTA

Abstract. Trees with a unique maximum independent set encode the maximum matching structure in every tree. In this work we study some of their linear properties and give two graph operations, stellare and S-coalescence, which allow building all trees with a unique maximum independent set. The null space structure of any tree can be understood in terms of these graph operations.

1. Introduction

Trees with a unique maximum independent set were first characterized in 1985 by Hopkins and Staton [4]. Sander and Sander [8] gave in 2005 another characterization in terms of the FOX algorithm. In [5] a characterization using linear algebra and the null decomposition of trees was given. Furthermore, it was proved in [5] that every tree can be decomposed into a forest of subtrees with a unique (perfect) maximum matching, and a forest of subtrees with a unique maximum independent set. This last forest encodes the variable part of the maximum matching structure of a tree.

In this work we study many properties of trees with a unique maximum independent set. We call them *independent trees*. Furthermore, we describe all these properties in terms of their atom forest (spanning forest of strong unique independence subtrees); see Section 3. We also give two graph operations, stellare and S-coalescence, which allow building every tree from very simple subtrees.

This paper is organized as follows. In Section 2, we work with atom trees. These are independent trees with the property that the complement of their unique maximum independent set is also an independent set of each tree, see [4]. In Section 3, trees with a unique independent set are studied in depth. Finally, in

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Section 4, we give two graph operations which allow building every independent tree.

Let us now introduce some notation required later on. All graphs in this work are labeled (even when we do not write the labels), finite, undirected, and with neither loops nor multiple edges. Let G be a graph. By $V(G)$ we denoted its set of vertices, and $v(G) := |V(G)|$ (following Bapat [1], we use uppercase letters for sets and lowercase letters for their cardinalities). Similarly, we denote by $E(G)$ its set of edges, and by $e(G)$ the size of G. The neighborhood of $v \in V(G)$, denoted by $N(v)$, is the set $\{u \in V(G) : u \sim v\}$, where $u \sim v$ means that $\{u, v\} \in E(G)$. The neighborhood of a subset S of vertices of G is $N(S) := \bigcup_{v \in S} N(v)$. The closed neighborhood of v, denoted by $N[v]$, is the set of vertices $\{v\} \cup N(v)$. The closed neighborhood of S, denoted by $N[S]$, is the set $S \cup N(S)$. For all graph-theoretic notions not defined here, the reader is referred to [3]. Let $u, v \in V(G)$. By $G+\{u, v\}$ we denote the graph obtained by adding the edge $\{u, v\}$ to $E(G)$. Let $e \in E(G)$. By $G - e$ we denote the graph obtained by removing the edge e from G, thus $E(G-e) = E(G)-\{e\}$. By $\deg(v)$ we denote the degree of a vertex v of a graph G, that is, $\deg(v) = |N(v)|$. A vertex v of a graph G is a pendant vertex if $\deg(v) = 1$. We write $[n]$ instead of $\{1, \ldots, n\}$.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. By $\mathbb{R}^G$ we denote the vector space of all functions from $V(G)$ to $\mathbb{R}$. Let $\vec{x} \in \mathbb{R}^G$ and $v \in V(G)$. We write $\vec{x}_v$ instead of $\vec{x}(v)$. By e_v we denote the standard basis vector at v, i.e., $e_v(u) = 1$ if $v = u$ and $e_v(u) = 0$ if $v \neq u$. With θ we denote the null vector of $\mathbb{R}^G$. For all linear algebra-theoretic notions not defined here, the reader is referred to [6].

2. Atom trees and null decomposition of trees

The null space of a graph G, denoted by $\mathcal{N}(G)$, is the null space of its adjacency matrix. Thus, $\mathcal{N}(G) = \mathcal{N}(A(G))$. The nullity of G is the nullity of its adjacency matrix: $\operatorname{null}(G) = \operatorname{null}(A(G))$.

The null space of any graph decomposes the vertices of the graph into three sets: the support, the core, and the invertible part, see [5].

Definition 2.1. Let G be a graph. The *support* of G, denoted by $\operatorname{Supp}(G)$, is the set of vertices of G

$$\{v \in V(G) : \exists \vec{x} \in \mathcal{N}(G) \text{ such that } \vec{x}_v \neq 0\}.$$

The *core* of G, denoted by $\operatorname{Core}(G)$, is the set of vertices $N(\operatorname{Supp}(G)) - \operatorname{Supp}(G)$. The *invertible* part of G, denoted by $\operatorname{Inv}(G)$, is $V(G) - N[\operatorname{Supp}(G)]$.

The vertices in $\operatorname{Supp}(G)$ are called supported vertices of T. Following Bapat [1], we write $\operatorname{supp}(T)$ instead of $|\operatorname{Supp}(T)|$. The vertices in $\operatorname{Core}(G)$ are called core vertices of T; we write $\operatorname{core}(T)$ instead of $|\operatorname{Core}(T)|$. The vertices in $\operatorname{Inv}(G)$ are called invertible vertices of T; we write $\operatorname{inv}(T)$ instead of $|\operatorname{Inv}(T)|$.

The next lemma is a rephrasing of Lemma 2.2 in [5].

Lemma 2.2 ([5])**.** *Let G be a graph. If $S \subset \operatorname{Supp}(G)$, then there exists $\vec{x} \in \mathcal{N}(G)$ such that for all $v \in S$, $\vec{x}_v \neq 0$.*

An independent set I of a graph G is a set of vertices of G pairwise non-neighbors. The independence number of G, denoted by $\alpha(G)$, is the maximum cardinality of the independent sets of G. The set of all maximum independent sets of a graph G is denoted by $\mathcal{I}(G)$, and its cardinality by $a(G)$, i.e., $a(G) = |\mathcal{I}(G)|$.

Lemma 2.3 ([5, Lemma 2.6]). *If T is a tree, then* $\text{Supp}\,(T)$ *is an independent set of T.*

The following lemma is a slightly generalized version of Lemma 3.4 in [5].

Lemma 2.4. *Let T be tree. If $v \in \text{Core}\,(T)$, then $|N(v) \cap \text{Supp}\,(T)| \geqslant 2$.*

Proof. Clearly $|N(v) \cap \text{Supp}\,(S)| > 0$. By Lemma 2.2 there exists $\vec{x} \in \mathcal{N}\,(T)$ such that for all $u \in \text{Supp}\,(T)$, we have $\vec{x}_u \neq 0$. Since $\sum_{w\sim v} \vec{x}_w = 0$, there are at least two vertices of $N(v)$ such that their respective coordinates in $\vec{x}$ are nonzero, and this is precisely the assertion of the lemma. □

Trees with a unique maximum independent set were first characterized in 1985 by Hopkins and Staton [4]. In that work they introduced the notion of *strong maximum independent set*: Let I be a maximum independent set of G; if $I^c := V(G) - I$ is also an independent set of G, then we say that I is a strong maximum independent set of G. Graphs with a strong maximum independent set are bipartite.

Theorem 2.5. *Let T be a tree. The following statements are equivalent:*

(1) *T has a unique strong maximum independent set.*
(2) *If u and v are pendant vertices of T, then the distance between them is even.*
(3) *The tree T is* $(\text{Supp}\,(T)\,, \text{Core}\,(T))$*-bipartite.*

A tree that satisfies any of these conditions (and hence all of them) is called an atom tree*.*

Proof. By Theorem 3 of [4], statements (1) and (2) are equivalent. Assume that (3) holds. By Lemma 2.4, if v is a pendant vertex of T, then $v \in \text{Supp}\,(T)$. Hence, (3) implies (2). In order to prove that (2) implies (3), fix u, a pendant vertex of T. Any other vertex of T is in a path between u and another pendant vertex of T, named w. By P we denote the unique path from u to w in T. We define $\vec{x}(u, w)$ as follows:

$$\vec{x}(u,w)_v = \begin{cases} 1 & \text{if } d(u,v) \equiv_4 0 \text{ and } v \in V(P), \\ -1 & \text{if } d(u,v) \equiv_4 2 \text{ and } v \in V(P), \\ 0 & \text{otherwise.} \end{cases}$$

If $\vec{x}(u, w) \notin \mathcal{N}\,(T)$, then there exists $z \in V(T)$ such that $d(u, z) \equiv_2 1$ and

$$\sum_{v \in N(z)} \vec{x}(u,w)_v \neq 0.$$

Therefore, there exists a unique $t_1 \in V(P) \cap N(z)$ such that $\vec{x}(u, w)_{t_1} \neq 0$. Since $d(u, z) \equiv_2 1$, z is not a pendant vertex of T. Therefore there must exist a $t_2 \in N(z) \cap (V(T) - V(P))$. Redefine $\vec{x}(u, w)_{t_2} = -\vec{x}(u, w)_{t_1}$, and P as $P + \{t_1, z\} + \{z, t_1\}$. If this new $\vec{x}(u, w)$ is not in $\mathcal{N}\,(T)$, repeat the former process. After a finite number of rounds, we arrive to a vector in the null space of T. Hence, $\{v \in V(T) : d(u,v) \equiv_2 0\} \subset \mathrm{Supp}\,(T)$.

Since T is a tree, the sets $V_0 = \{v \in V(T) : d(u,v) \equiv_2 0\}$ and $V_1 = \{v \in V(T) : d(u,v) \equiv_2 1\}$ form a bipartition of T. Note that $V_1 = N\,(V_0) \subset \mathrm{Supp}\,(T)$. Thus $V_1 \subset \mathrm{Core}\,(T)$. Therefore,

$$V(T) = V_0 \cup V_1 \subset \mathrm{Supp}\,(T) \cup \mathrm{Core}\,(T) \subset V(T),$$

and hence $\mathrm{Supp}\,(T) = V_1$ and $\mathrm{Core}\,(T) = V_2$. □

We usually write $\mathfrak{A}$ for atom trees. In [4], atom trees are called *strong unique independence trees.* A very similar notion can be traced in the work of Sander and Sander [8], in terms of the FOX algorithm.

The null decomposition of trees breaks any tree in two forests: a forest of trees with a unique maximum matching (a perfect matching) and a forest of trees with a unique maximum independent set. Trees with a unique maximum matching have a non-singular adjacency matrix, see [2, 5, 1]. They are called *invertible* or *matching* trees.

Let G be a graph. Given $U \subset V(G)$, the subgraph of G induced by U is denoted by $G\langle U\rangle$. The set of all connected components of a graph G is denoted by $\mathcal{K}(G)$.

Let T be a tree. We set $\mathcal{F}_{\mathrm{indep}}\,(T) := T\langle N[\mathrm{Supp}\,(T)]\rangle$, and $\mathcal{F}_{\mathrm{match}}\,(T) := T\langle \mathrm{Inv}\,(T)\rangle$. The forest $\mathcal{F}_{\mathrm{indep}}\,(T)$ is called the *independence forest* of T. The forest $\mathcal{F}_{\mathrm{match}}\,(T)$ is called the *matching forest* of T. The empty set is simultaneously an independence forest of T and a matching forest of T. These notions were introduced in [5] under the names *S-set* and *N-set* of T.

Theorem 2.6 ([5]). *Let T be a tree. The independence forest of T is a forest of independent trees and the matching forest of T is a forest of matching trees.*

Definition 2.7 ([5]). The *connection edges* of a tree T, denoted by $CE(T)$, is the set of all the edges between a core vertex and an invertible vertex:

$$\{\{u,v\} \in E(T) : u \in \mathrm{Core}\,(T) \text{ and } v \in \mathrm{Inv}\,(T)\}.$$

Let $\mathcal{F}_{\mathrm{null}}\,(T) := \mathcal{F}_{\mathrm{indep}}\,(T) \cup \mathcal{F}_{\mathrm{match}}\,(T)$. This forest associated with T is called the *null forest* of T. The following result is implicit in [5].

Theorem 2.8 ([5]). *If T is a tree, then $E(T - \mathcal{F}_{\mathrm{null}}\,(T)) = CE(T)$.*

We use the same symbol for a forest and for the set of all its connected components: $H \in \mathcal{F}$ means $H \in \mathcal{K}(\mathcal{F})$, where $\mathcal{F}$ is a forest.

Theorem 2.9 ([5]). *If T is a tree, then*

$$\mathrm{Supp}\,(T) = \bigcup_{S \in \mathcal{F}_{\mathrm{indep}}(T)} \mathrm{Supp}\,(S)\,,$$

$$\mathrm{Core}\,(T) = \bigcup_{S \in \mathcal{F}_{\mathrm{indep}}(T)} \mathrm{Core}\,(S)\,,$$

$$\mathrm{Inv}\,(T) = \bigcup_{N \in \mathcal{F}_{\mathrm{match}}(T)} \mathrm{Inv}\,(N)\,.$$

The row space of a graph G, denoted by $\mathcal{R}(G)$, is the row space of its adjacency matrix. Thus, $\mathcal{R}(G) = \mathcal{R}(A(G))$. The rank of a graph G is the rank of its adjacency matrix: $\operatorname{rk}(G) = \operatorname{rk}(A(G))$. By $\nu(G)$ we denote the matching number of G, i.e., the cardinality of a maximum matching of G. By $\mathcal{M}(G)$ we denote the set of all maximum matchings in G. The number of maximum matchings in G is denoted by $m(G)$. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. The independence number $\alpha(G)$ of a graph G and its domination number $\gamma(G)$ are related by $\gamma(G) \leqslant \alpha(G)$.

Theorem 2.10 ([5])**.** *If T is a tree, then*

$$\begin{aligned}
\operatorname{null}(T) &= \operatorname{supp}(T) - \operatorname{core}(T), \\
\operatorname{rk}(T) &= 2\operatorname{core}(T) + \operatorname{inv}(T), \\
\nu(T) &= \operatorname{core}(T) + \frac{\operatorname{inv}(T)}{2}, \\
m(T) &= \prod_{S \in \mathcal{F}_{\text{indep}}(T)} m(S), \\
\alpha(T) &= \operatorname{supp}(T) + \frac{\operatorname{inv}(T)}{2}, \\
a(T) &= \prod_{N \in \mathcal{F}_{\text{match}}(T)} a(N).
\end{aligned}$$

Corollary 2.11. *Let T be a tree. Then $\nu(T) = \alpha(T) - \operatorname{null}(T)$.*

3. Trees with a unique maximum independent set

In this section, we study some linear properties of trees with a unique maximum independent set.

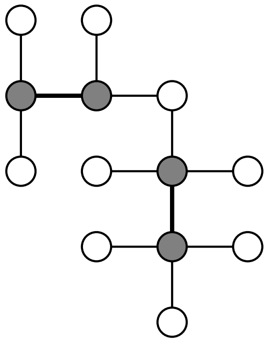
Theorem 3.1 ([4, Theorem 6] and [5, Section 3 and Corollary 4.18])**.** *Let T be a tree. The following statements are equivalent:*

1. $a(T) = 1$.
2. $\operatorname{Supp}(T)$ *is the unique maximum independent set of T.*
3. $N[\operatorname{Supp}(T)] = V(T)$.
4. *T has a spanning forest $\mathcal{F}$ such that each component of $\mathcal{F}$ is an atom tree, and each edge in $E(T) - E(\mathcal{F})$ joins two core vertices of T.*

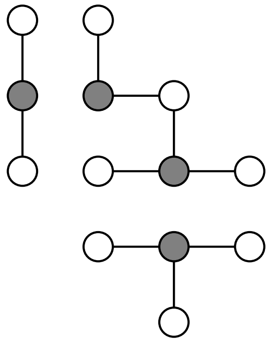
A tree that satisfies any of these conditions (and hence all of them) is called an independent tree.

Therefore, independent trees can be built from a forest of atom trees. In the next section we give two graph-theoretic operations, with linear-algebraic flavor, that allow building every independent tree.

Figure 1 shows an independent tree and its spanning forest of atom trees. In [5] the independent trees are called *S-trees*, because the spotlight was on the linear structure of the null space of the adjacency matrix. The independent trees can also be characterized via the FOX algorithm, see [8].



(A) An independent tree T



(B) Its atom forest $\mathcal{F}_{\text{atom}}(T)$

FIGURE 1

Definition 3.2. The *bond edges* of a tree T, denoted by $BE(T)$, is the set of all the edges between core vertices:

$$\{\{u, v\} \in E(T) : u, v \in \operatorname{Core}(T)\}.$$

We prove that for each independent tree there exists just one spanning forest of atoms: $\mathcal{F}_{\text{atom}}(T) := \mathcal{K}(T - BE(T))$, see Theorem 3.5. This forest is important to understand the maximum matching structure of any tree. See Corollary 3.8.

Let T be an independent tree. Let $\mathcal{F}$ be a spanning forest of atoms of T such that $E(T) - E(\mathcal{F}) \subset BE(T)$, let and $\mathfrak{A} \in \mathcal{F}$. Then

(1) $\deg_T(v) = \deg_{\mathfrak{A}}(v)$, for all $v \in \operatorname{Supp}(T) \cap V(\mathfrak{A})$;
(2) if v is a pendant vertex of $\mathfrak{A}$, then v is a pendant vertex of T.

We need the following notation introduced in [5]. Given a graph G, let $\vec{x}$ be a vector of $\mathbb{R}^G$. Let H be a subgraph of G. The vector obtained when we restrict $\vec{x}$ to the coordinates (vertices) associated with H is denoted by $\vec{x}\downharpoonleft^G_H$. By $\vec{y}\upharpoonleft^G_H$ we denote the lift of vector $\vec{y} \in \mathbb{R}^H$ to a vector of $\mathbb{R}^G$: for any $u \in V(G) - V(H)$, $(\vec{y}\upharpoonleft^G_H)_u := 0$, and for any $u \in V(H)$, $(\vec{y}\upharpoonleft^G_H)_u := \vec{y}_u$.

Theorem 3.3. *Let T be an independent tree and let $\mathcal{F}$ be a spanning forest of atoms of T such that $E(T) - E(\mathcal{F}) \subset BE(T)$. Then the following statements hold:*

(1) *If $\mathfrak{A} \in \mathcal{F}$, then*
  (a) $\operatorname{Supp}(\mathfrak{A}) = \operatorname{Supp}(T) \cap V(\mathfrak{A})$,
  (b) $\operatorname{Core}(\mathfrak{A}) = \operatorname{Core}(T) \cap V(\mathfrak{A})$.
(2) $\operatorname{Supp}(T) = \bigcup_{\mathfrak{A}\in\mathcal{F}} \operatorname{Supp}(\mathfrak{A})$.
(3) $\operatorname{Core}(T) = \bigcup_{\mathfrak{A}\in\mathcal{F}} \operatorname{Core}(\mathfrak{A})$.

Proof. By Lemma 2.2, we can take $\vec{x} \in \mathcal{N}(T)$ such that $\vec{x}_v \neq 0$ if and only if $v \in \operatorname{Supp}(T)$.

Claim 1: $\vec{x}\downharpoonleft^T_{\mathfrak{A}} \in \mathcal{N}(\mathfrak{A})$.

Therefore, $\operatorname{Supp}(T) \cap V(\mathfrak{A}) \subset \operatorname{Supp}(\mathfrak{A})$. Hence, $\operatorname{Core}(\mathfrak{A}) \subset \operatorname{Core}(T) \cap V(\mathfrak{A})$.

Claim 2: $\operatorname{Core}(T) \cap V(\mathfrak{A}) \subset \operatorname{Core}(\mathfrak{A})$.

Since

$$(\operatorname{Supp}(T) \cap V(\mathfrak{A})) \dot{\cup} (\operatorname{Core}(T) \cap V(\mathfrak{A})) = V(\mathfrak{A}) = \operatorname{Supp}(\mathfrak{A}) \dot{\cup} \operatorname{Core}(\mathfrak{A}),$$

we can conclude that $\operatorname{Supp}(T) \cap V(\mathfrak{A}) = \operatorname{Supp}(\mathfrak{A})$.

Proof of Claim 1. We prove the claim in a coordinate-wise fashion. If $v \in \operatorname{Supp}(\mathfrak{A})$, then

$$(A(\mathfrak{A})\,\vec{x}\downharpoonleft^{T}_{\mathfrak{A}})_v = (A(T)\vec{x})_v = 0.$$

If $v \in \operatorname{Core}(\mathfrak{A})$, then

$$0 = (A(T)\vec{x})_v = (A(\mathfrak{A})\,\vec{x}\downharpoonleft^{T}_{\mathfrak{A}})_v + \sum_{\substack{u \sim v \\ u \notin V(\mathfrak{A})}} \vec{x}_u.$$

If $u \sim v$ and $u \notin V(\mathfrak{A})$, then $\{u, v\} \in E(T) - E(\mathcal{F}) \subset BE(T)$, and $\vec{x}_u = 0$. Therefore, $(A(\mathfrak{A})\,\vec{x}\downharpoonleft^{T}_{\mathfrak{A}})_v = 0$. This proves Claim 1.

Proof of Claim 2. If $v \in \operatorname{Core}(T) \cap V(\mathfrak{A})$, then there exists $u \in \operatorname{Supp}(T)$ such that $u \sim v$. Therefore, since $E(T) - E(\mathcal{F}) \subset BE(T)$, the vertex u is a vertex of $\mathfrak{A}$. Thus, $u \in \operatorname{Supp}(T) \cap V(\mathfrak{A}) \subset \operatorname{Supp}(\mathfrak{A})$. Hence, since $v \in V(\mathfrak{A})$, we conclude that $v \in \operatorname{Core}(\mathfrak{A})$.

The other two statements are now obvious. □

Corollary 3.4. *Let T be an independent tree and let $\mathcal{F}$ be a spanning forest of atoms of T such that $E(T) - E(\mathcal{F}) \subset BE(T)$. Let $e \in E(T) - E(\mathcal{F})$ and $\{T_1, T_2\} := \mathcal{K}(T - e)$. Then $\mathcal{F}_1 := \mathcal{F}\langle(T_1)\rangle$ is a spanning forest of atoms of T_1 such that $E(T_1) - E(\mathcal{F}_1) \subset BE(T_1)$, and $\mathcal{F}_2 := \mathcal{F}\langle V(T_2)\rangle$ is a spanning forest of atoms of T_2 such that $E(T_2) - E(\mathcal{F}_2) \subset BE(T_2)$. Furthermore, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$.*

The following result says that the forest decomposition into atoms of independent trees is unique.

Theorem 3.5. *Let T be an independent tree. If $\mathcal{F}$ is a spanning forest of atoms of T such that $E(T) - E(\mathcal{F}) \subset BE(T)$, then $\mathcal{F} = \mathcal{F}_{\text{atom}}(T)$.*

Proof. By induction on $|BE(T)|$ and Corollary 3.4. □

Theorem 3.6 ([5])**.** *If T is an independent tree and $M \in \mathcal{M}(T)$, then each edge of M matches one supported vertex with one core vertex of T.*

The following result is a direct consequence of Theorem 3.6.

Corollary 3.7. *Let T be an independent tree. The following statements hold:*

(1) *If $M \in \mathcal{M}(T)$, then for each $\mathfrak{A} \in \mathcal{F}_{\text{atom}}(T)$ we have $M \cap E(\mathfrak{A}) \in \mathcal{M}(\mathfrak{A})$.*

(2) *For each $\mathfrak{A} \in \mathcal{F}_{\text{atom}}(T)$, let $M_{\mathfrak{A}} \in \mathcal{M}(\mathfrak{A})$. Then*

$$\bigcup_{\mathfrak{A} \in \mathcal{F}_{\text{atom}}(T)} M_{\mathfrak{A}} \in \mathcal{M}(T).$$

Corollary 3.8. *If T is an independent tree, then*

$$m(T) = \prod_{\mathfrak{A} \in \mathcal{F}_{\text{atom}}(T)} m(\mathfrak{A}).$$

The null decomposition of trees says, among other things, that the "variations" in the maximum matching structure of any tree are determined by its independent forest, while the "variations" in the maximum independent structure of any tree are determined by its matching forest. Parts of the following theorem are implicit in [5].

Theorem 3.9. *Let T be a tree. The following statements are true:*

(1) *If $M \in \mathcal{M}(T)$, then for each $S \in \mathcal{F}_{\text{indep}}(T)$ we have $M \cap E(S) \in \mathcal{M}(S)$, and for each $H \in \mathcal{F}_{\text{match}}(T)$ we have that $M \cap E(H)$ is the perfect matching of H.*

(2) *For each $H \in \mathcal{F}_{\text{null}}(T)$, let $M_H \in \mathcal{M}(H)$. Then*

$$\bigcup_{H \in \mathcal{F}_{\text{null}}(T)} M_H \in \mathcal{M}(T).$$

(3) *If $I \in \mathcal{I}(T)$, then for each $N \in \mathcal{F}_{\text{match}}(T)$ we have $I \cap V(N) \in \mathcal{I}(N)$, and for each $S \in \mathcal{F}_{\text{indep}}(T)$ we have that $I \cap V(S) = \operatorname{Supp}(S)$ is the unique maximum independent set of S.*

(4) *For each $H \in \mathcal{F}_{\text{null}}(T)$, let $I_H \in \mathcal{I}(H)$. Then*

$$\bigcup_{H \in \mathcal{F}_{\text{null}}(T)} I_H \in \mathcal{I}(T).$$

Proof. The proofs of statements (1) and (2) are implicit in [5]. By Theorem 3.3 we know that $\operatorname{Supp}(T) = \bigcup_{S \in \mathcal{F}_{\text{indep}}(T)} \operatorname{Supp}(S)$, and that for all $S \in \mathcal{F}_{\text{indep}}(T)$, $\operatorname{Supp}(S) = \operatorname{Supp}(T) \cap V(S)$. The vertices of T satisfy $V(T) = \bigcup_{S \in \mathcal{F}_{\text{indep}}(T)} V(S) \cup \bigcup_{N \in \mathcal{F}_{\text{match}}(T)} V(N)$, where at most one of the "big" unions can be empty. If I is a maximum independent set of a tree T, then

$$\begin{aligned}
\operatorname{supp}(T) + \frac{\operatorname{inv}(T)}{2} &= |I| \\
&= \sum_{S \in \mathcal{F}_{\text{indep}}(T)} |I \cap V(S)| + \sum_{N \in \mathcal{F}_{\text{match}}(T)} |I \cap V(N)| \\
&\leqslant \sum_{S \in \mathcal{F}_{\text{indep}}(T)} \operatorname{supp}(S) + \sum_{N \in \mathcal{F}_{\text{match}}(T)} \frac{v(N)}{2} \\
&= \operatorname{supp}(T) + \frac{\operatorname{inv}(T)}{2}.
\end{aligned}$$

Hence, for each $S \in \mathcal{F}_{\text{indep}}(T)$, we have $|I \cap V(S)| = \operatorname{supp}(S)$, and for each $N \in \mathcal{F}_{\text{match}}(T)$, we have $|I \cap V(N)| = v(N)/2$. Therefore, $I \cap V(S) = \operatorname{Supp}(S)$, because $I \cap V(S)$ is an independent set of S and S has a unique maximum independent set: $\operatorname{Supp}(S)$. Similarly, for all $N \in \mathcal{F}_{\text{match}}(T)$ we have that $I \cap V(N)$ is an independent set of N and $|I \cap V(N)| = \frac{v(N)}{2}$. Therefore, for all $N \in \mathcal{F}_{\text{match}}(T)$ the set $I \cap V(N)$ is a maximum independent set of N.

Statement (4) is a direct consequence of Theorem 2.8. □

Corollary 3.10. *Let T be a tree. If $I \in \mathcal{I}(T)$, then $I \cap \operatorname{Supp}(T) = \operatorname{Supp}(T)$ and $|I \cap \operatorname{Inv}(T)| = \frac{\operatorname{inv}(T)}{2}$.*

The maximum degree of all the core vertices in an atom tree $\mathfrak{A}$ will be denoted by $\Delta_{\text{core}}(\mathfrak{A})$; it provides a lower bound for the nullity of $\mathfrak{A}$.

Theorem 3.11. *Let $\mathfrak{A}$ be an atom tree. Then* $\operatorname{null}(\mathfrak{A}) \geqslant \Delta_{\text{core}}(\mathfrak{A}) - 1$.

Proof. Let $\mathfrak{A}$ be an atom tree, and let $u \in \operatorname{Core}(\mathfrak{A})$ be such that $\deg(u) = \Delta_{\text{core}}(\mathfrak{A})$. It is clear that

$$\begin{aligned}\operatorname{supp}(\mathfrak{A}) &\geqslant \Delta_{\text{core}}(\mathfrak{A}) + \sum_{\substack{v \in \operatorname{Core}(\mathfrak{A}) \\ d(v,u)=2}} 1 + \sum_{\substack{v \in \operatorname{Core}(\mathfrak{A}) \\ d(v,u)=4}} 1 + \cdots + \sum_{\substack{v \in \operatorname{Core}(\mathfrak{A}) \\ d(v,u)=\operatorname{diam}(\mathfrak{A})-2}} 1 \\ &= \Delta_{\text{core}}(\mathfrak{A}) + (\operatorname{core}(\mathfrak{A}) - 1).\end{aligned}$$

Then, by Theorem 2.10, $\operatorname{null}(\mathfrak{A}) = \operatorname{supp}(\mathfrak{A}) - \operatorname{core}(\mathfrak{A}) \geqslant \Delta_{\text{core}}(\mathfrak{A}) - 1$. □

For any atom tree $\mathfrak{A}$, its rank is $2\operatorname{core}(\mathfrak{A})$. But actually, we can give a basis of its row space.

Definition 3.12. Let $v \in \operatorname{Core}(T)$. The *$v$-bouquet* of T, denoted by $R(v)$, is

$$R(v) := \{u \in \operatorname{Supp}(T) : u \sim v\}.$$

The *v-bouquet vector*, denoted by $\vec{R}(v)$, is

$$\vec{R}(v) = \sum_{u \in R(v)} e_u.$$

The “R” stands for the Spanish word “ramillete”, meaning “bouquet”.

Lemma 3.13. *Let $\mathfrak{A}$ be an atom tree. The set*

$$\mathcal{B}_{rk}(\mathfrak{A}) := \{e_v, \vec{R}(v) \in \mathbb{R}^{\mathfrak{A}} : v \in \operatorname{Core}(\mathfrak{A})\}$$

is a basis of $\mathcal{R}(\mathfrak{A})$, the row space of $\mathfrak{A}$.

Proof. Let A_{*u} be the column of $A(\mathfrak{A})$, the adjacency matrix of $\mathfrak{A}$, associated with the vertex u. If $u \in \operatorname{Supp}(\mathfrak{A})$, then

$$A_{*u} = \sum_{\substack{v \in \operatorname{Core}(\mathfrak{A}) \\ v \sim u}} e_v.$$

If $u \in \operatorname{Core}(\mathfrak{A})$, then

$$A_{*u} = \sum_{\substack{v \in \operatorname{Supp}(\mathfrak{A}) \\ v \sim u}} e_v = \vec{R}(u).$$

Hence, $\mathcal{R}(\mathfrak{A}) \subset \langle \mathcal{B}_{rk}(\mathfrak{A}) \rangle$. Clearly $\mathcal{B}_{rk}(\mathfrak{A})$ is a set of linearly independent vectors, and $|B_{rk}(\mathfrak{A})| = 2\operatorname{core}(\mathfrak{A})$. Therefore, $B_{rk}(\mathfrak{A})$ is a basis of $\mathcal{R}(\mathfrak{A})$. □

Let G be a graph and H a subgraph of G. Let $A \subset \mathbb{R}^H$ and $B \subset \mathbb{R}^G$. By $A \upharpoonleft_H^G$ we denote the set of vectors of $\mathbb{R}^G$ $\{\vec{x} \upharpoonleft_H^G : \vec{x} \in A\}$, and by $B \downharpoonleft_H^G$ we denote the set of vectors of $\mathbb{R}^H$ $\{\vec{y} \downharpoonleft_H^G : \vec{y} \in B\}$.

Theorem 3.14. *Let T be an independent tree. Then*

1. $\operatorname{null}(T) = \sum_{\mathfrak{A}\in\mathcal{F}_{\text{atom}}(T)} \operatorname{null}(\mathfrak{A})$;
2. $\operatorname{rk}(T) = \sum_{\mathfrak{A}\in\mathcal{F}_{\text{atom}}(T)} \operatorname{rk}(\mathfrak{A})$;
3. $\mathcal{N}(T) = \bigoplus_{\mathfrak{A}\in\mathcal{F}_{\text{atom}}(T)} \mathcal{N}(\mathfrak{A}) \upharpoonleft^{T}_{\mathfrak{A}}$; *and*
4. $\mathcal{R}(T) = \bigoplus_{\mathfrak{A}\in\mathcal{F}_{\text{atom}}(T)} \mathcal{R}(\mathfrak{A}) \upharpoonleft^{T}_{\mathfrak{A}}$.

Proof. Statements (1) and (2) follow by applying Theorem 2.10 to independent trees and atom trees in particular, taking into account Theorem 3.3.

Let us prove statement (3). It is clear that all the subspaces $\mathcal{N}(\mathfrak{A})$ are orthogonal. On the one hand, an argument similar to the proof of Claim 1 in Theorem 3.3 proves that if $\vec{x} \in \mathcal{N}(T)$, then $\vec{x}\downharpoonleft^{T}_{\mathfrak{A}} \in \mathcal{N}(\mathfrak{A})$ for $\mathfrak{A} \in \mathcal{F}_{\text{atom}}(T)$. On the other hand, for each $\mathfrak{A} \in \mathcal{F}_{\text{atom}}(T)$, let $\vec{x}(\mathfrak{A}) \in \mathcal{N}(\mathfrak{A})$. Since

$$A(T)\left(\sum_{\mathfrak{A}\in\mathcal{F}_{\text{atom}}(T)} \vec{x}(\mathfrak{A}) \upharpoonleft^{T}_{\mathfrak{A}}\right) = \sum_{\mathfrak{A}\in\mathcal{F}_{\text{atom}}(T)} (A(\mathfrak{A})\vec{x}(\mathfrak{A})) \upharpoonleft^{T}_{\mathfrak{A}} = \theta,$$

we have $\bigoplus_{\mathfrak{A}\in\mathcal{F}_{\text{atom}}(T)} \subset \mathcal{N}(T)$. Statement (3) follows from statement (1). Now we prove (4). Clearly, $\mathcal{R}(\mathfrak{A}) \upharpoonleft^{T}_{\mathfrak{A}}$ are all orthogonal. Let u be a vertex of T. Let $\mathfrak{A} \in \mathcal{F}_{\text{atom}}(T)$ be such that $u \in V(\mathfrak{A})$. Let $A(T)_{*\,u}$ be the column of $A(T)$ associated with the vertex u, and let $A(\mathfrak{A})_{*\,u}$ be the column of $A(\mathfrak{A})$ also associated with the vertex u. Then

$$A(T)_{*\,u} = A(\mathfrak{A})_{*\,u} \upharpoonleft^{T}_{\mathfrak{A}} + \sum_{\{u,v\}\in BE(T)} e_v.$$

Hence, by Lemma 3.13, $\mathcal{R}(T) \subset \bigoplus_{\mathfrak{A}\in\mathcal{F}_{\text{atom}}(T)} \mathcal{R}(\mathfrak{A}) \upharpoonleft^{T}_{\mathfrak{A}}$. Statement (4) follows from statement (2). □

Corollary 3.15. *Let T be an independent tree. Then*

$$\bigcup_{\mathfrak{A}\in\mathcal{F}_{\text{atom}}(T)} \mathcal{B}_{rk}(\mathfrak{A}) \upharpoonleft^{S}_{\mathfrak{A}}$$

is a basis of $\mathcal{R}(T)$.

By $B_{\mathcal{C}}(\mathbb{R}^{G})$ we denote the standard basis of $\mathbb{R}^{G}$. Let T be a tree. The *atom forest* of T, denoted by $\mathcal{F}_{\text{atom}}(T)$, is the forest

$$\bigcup_{S\in\mathcal{F}_{\text{indep}}(T)} \mathcal{F}_{\text{atom}}(S).$$

Corollary 3.16. *Let T be a tree. Then*

$$\bigcup_{N\in\mathcal{F}_{\text{match}}(T)} B_{\mathcal{C}}(\mathbb{R}^{N}) \upharpoonleft^{T}_{N} \cup \bigcup_{S\in\mathcal{F}_{\text{indep}}(T)} \left(\bigcup_{\mathfrak{A}\in\mathcal{F}_{\text{atom}}(T)} B_{\mathcal{R}}(\mathfrak{A}) \upharpoonleft^{S}_{\mathfrak{A}}\right) \upharpoonleft^{T}_{S}$$

is a basis of $\mathcal{R}(T)$.

Similar arguments allow us to prove the following theorem.

Theorem 3.17. *If T is a tree, then*

$$\begin{aligned}
\mathcal{R}(T) &= \bigoplus_{N\in\mathcal{F}_{\mathrm{match}}(T)} \mathcal{R}(N)\upharpoonleft_N^T \oplus \bigoplus_{\mathfrak{A}\in\mathcal{F}_{\mathrm{atom}}(T)} \mathcal{R}(\mathfrak{A})\upharpoonleft_{\mathfrak{A}}^T,\\
\mathrm{rk}(T) &= \sum_{N\in\mathcal{F}_{\mathrm{match}}(T)} \mathrm{rk}(N) + \sum_{\mathfrak{A}\in\mathcal{F}_{\mathrm{atom}}(T)} \mathrm{rk}(\mathfrak{A}),\\
\mathcal{N}(T) &= \bigoplus_{\mathfrak{A}\in\mathcal{F}_{\mathrm{atom}}(T)} \mathcal{N}(\mathfrak{A})\upharpoonleft_{\mathfrak{A}}^T,\\
\mathrm{null}(T) &= \sum_{\mathfrak{A}\in\mathcal{F}_{\mathrm{atom}}(T)} \mathrm{null}(\mathfrak{A}).
\end{aligned}$$

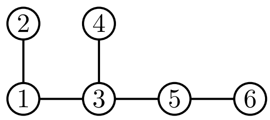
4. Two new graph operations

In this section we define two graph operations under which the independent trees are closed. These operations are important because they allow thinking about independent trees without finding null spaces or a forest of atoms.

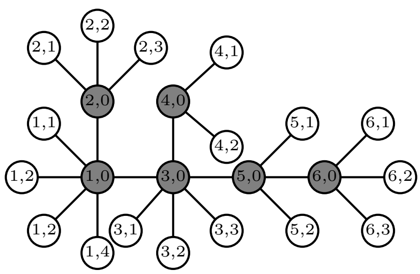
4.1. **Stellare.**

Definition 4.1. Let G be a labeled graph of order n, with labels $[n]$. The $*(k_1,\dots,k_n)$-**stellare** of G is the graph obtained by adding $k_i \geqslant 2$ pendant vertices to vertex i of G.

In the following, let $*G$ denote an arbitrary, but otherwise fixed, stellare of G. An example is shown in Figure 2. Instead of saying "a stellare of a tree" we just say "a stellare tree".



(A) A tree T



(B) $*(4,3,3,2,2,3)T$

Figure 2. $*(4,3,3,2,2,3)T$ is a stellare tree of T

Theorem 4.2. *If T is a tree, then $*T$ is an independent tree, with* $\mathrm{Core}\,(*T) = V(T)$ *and* $\mathrm{Supp}\,(*T) = V(*T) - V(T)$.

Proof. Take $v \in V(*T) - V(T)$. By the stellare definition, there exist $u \in V(T)$ and $w \in V(*T)$ such that $u \sim v$, the vertices v, w are neighbors of u in $*T$, and $w \notin V(T)$. Let $\vec{x}$ be a vector of $\mathbb{R}^{*T}$ such that

$$\vec{x}_i = \begin{cases} 1 & \text{if } i = v, \\ -1 & \text{if } i = w, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $A(*T)\,\vec{x} = \theta$. Hence, $V(*T)\backslash V(T) = \mathrm{Supp}\,(*T)$. By the stellare definition we have that $N[V(*T) - V(T)] = V(*T)$. Hence, we conclude that $*T$ is an independent tree, $\mathrm{Core}\,(*T) = V(T)$, and $\mathrm{Supp}\,(*T) = V(*T)\backslash V(T)$. ☐

Corollary 4.3. *Let T be a tree with labels $[n]$, and let $k_1, \ldots, k_n$ be a list of n integers, each greater than or equal to 2. Then*

(1) $\mathrm{null}\,(*(k_1, \ldots, k_n)T) = \sum_{i=1}^{n} k_i - n \geqslant n \geqslant \mathrm{null}\,(T)$, *where equality holds if and only if $n = 1$ and $k_1 = 2$;*
(2) $\mathrm{rk}\,(*T) = 2n > \mathrm{rk}\,(T)$*;*
(3) $\alpha(*(k_1, \ldots, k_n)T) = \sum_{i=1}^{n} k_i \geqslant 2n > \alpha(T)$*;*
(4) $\nu(*T) = n > \nu(T)$*;*
(5) $m(*(k_1, \ldots, k_n)T) = \prod_{i=1}^{n} k_i$*;*
(6) $\gamma(*T) = n > \gamma(T)$*, and $V(T)$ is the only minimum (and total, if $n \geqslant 2$) dominating set of any $*T$.*

Proof. Statements (1) and (2) follow from Theorem 2.10 and Theorem 4.2. Statement (3) follows from Theorem 2.10, Theorem 4.2, and Theorem 2.10. Statements (4) and (5) follow from Theorem 2.10, Theorem 4.2, and Theorem 2.10. Statement (6) is clear. ☐

Theorem 4.4. *Let T be a tree and $*T$ a stellare of T. Then the set of vectors of $\mathbb{R}^{*T}$,*

$$\mathcal{B}_{rk}(T) := \{e_v, \vec{R}(v) \in \mathbb{R}^{(*T)} : v \in V(T)\}$$

is a basis of $\mathcal{R}\,(*T)$.

Proof. By Corollary 4.3, $\mathrm{rk}\,(*T) = 2v(T)$. Therefore, we only need to prove that the columns of the adjacency matrix of $*T$ are linear combinations of the vectors of $\mathcal{B}_{rk}$. For $v \in V(*T)$, let A_v denote the column of $A(*T)$ associated with the vertex v. Thus, if $v \in \mathrm{Supp}\,(*T)$, then $A_v = e_w$, for some $w \in \mathrm{Core}\,(*T) = V(T)$ and $w \sim v$. If $v \in \mathrm{Core}\,(*T) = V(T)$, then

$$A_v = e_{R(v)} + \sum_{\substack{w \in V(T) \\ w \sim v}} e_w.$$ ☐

Given a tree T with labels $[n]$, the **stellare labeling** of $*(k_1, \ldots, k_n)T$ is the set

$$\{(u, w) : u \in [n] \text{ and } w \in \{0, 1, \ldots, k_u\}\},$$

where the vertices labeled with $(u, 0)$ are the core vertices of $*T$, and the vertices labeled with (u, w), with $w \in \{1, \ldots, k_u\}$, are the supported vertices of $*T$ which are neighbors of u. See Figure 2.

Lemma 4.5. *Let T be a tree of order n and $*(k_1, \ldots, k_n)T$ a stellare of T. Then the following set of vectors is a basis of the null space of $*(k_1, \ldots, k_n)T$:*

$$\mathcal{B}_{\text{null}}(T) := \{\vec{b}(i, j) \in \mathbb{R}^{*T} : i \in [n],\ j \in \{2, \ldots, k_i\}\},$$

where

$$\vec{b}(i, j)_{(u,w)} = \begin{cases} 1 & \text{if } u = i \text{ and } w = 1, \\ -1 & \text{if } u = i \text{ and } w = j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The set $\mathcal{B}_{\text{null}}(T)$ is a set of linear independent vectors. A direct computation shows that $A(*T)\vec{b} = \theta$ for all $\vec{b} \in \mathcal{B}_{\text{null}}(T)$. As $|\mathcal{B}_{\text{null}}(T)| = \operatorname{supp}(*T) - \operatorname{core}(*T)$, by Theorem 2.10 the set $\mathcal{B}_{\text{null}}$ is a basis of the null space of $*(k_1, \ldots, k_n)T$. □

4.2. **S-coalescence.**

Definition 4.6. Let $T_1, \ldots, T_k$ be k disjoint independent trees. Let $v_i \in \operatorname{Supp}(T_i)$, for each $i \in [k]$. The **S-coalescence** of $(T_1, v_1), \ldots, (T_k, v_k)$, denoted by

$$\circledast_{i=1}^{k}(T_i, v_i),$$

is the tree obtained by identifying all the vertices v_i, denoting this single vertex by v^*. Let $N_{T_i}(v_i)$ be the neighborhood of v_i in T_i. Then $\circledast_{i=1}^{k}(T_i, v_i)$ is the tree with the set of vertices

$$V\left(\circledast_{i=1}^{k}(T_i, v_i)\right) = \left(\bigcup_{1 \leqslant i \leqslant k} (V(T_i) - \{v_i\})\right) \cup \{v^*\},$$

and the set of edges

$$E\left(\circledast_{i=1}^{k}(T_i, v_i)\right) = \{\{u, v^*\} : u \in N_{T_i}(v_i)\} \cup \bigcup_{i=1}^{k} E(T_i) - \{\{u, v_i\} : u \in N_{T_i}(v_i)\}.$$

The following theorem states that independent trees are closed under the operation of S-coalescence.

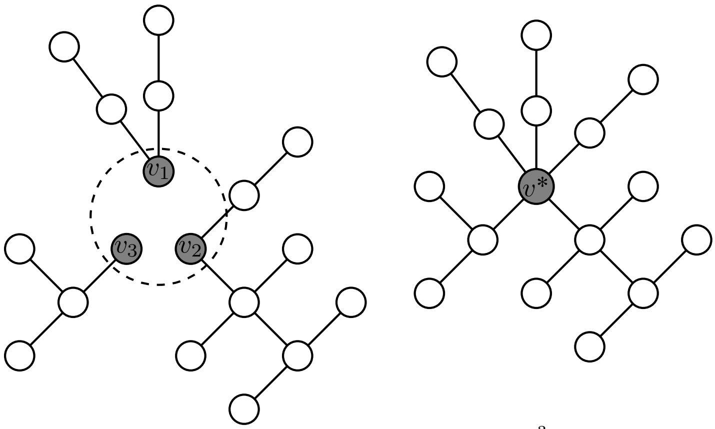
Theorem 4.7. *Let $T_1, \ldots, T_k$ be k disjoint independent trees, and let $v_i \in \operatorname{Supp}(T_i)$. Then $\circledast_{i=1}^{k}(T_i, v_i)$ is an independent tree.*

Proof. It is left to the reader. □

Corollary 4.8. *Let $T_1, \ldots, T_k$ be k disjoint independent trees, and let $v_i \in \operatorname{Supp}(T_i)$ for $i \in [k]$. Then*

(1) $\operatorname{Core}\left(\circledast_{i=1}^{k}(T_i, v_i)\right) = \bigcup_{i=1}^{k} \operatorname{Core}(T_i)$;

(2) $\operatorname{core}\left(\circledast_{i=1}^{k}(T_i, v_i)\right) = \sum_{i=1}^{k} \operatorname{core}(T_i)$;



(A) Independent trees T_1, T_2, and T_3

(B) $\circledast_{i=1}^{3}(T_i, v_i)$

FIGURE 3. S-coalescence of three independent trees

(3) $\operatorname{Supp}\left(\circledast_{i=1}^{k}(T_i, v_i)\right) = \{v^*\} \cup \bigcup_{i=1}^{k} (\operatorname{Supp}(T_i) - \{v_i\})$;

(4) $\operatorname{supp}\left(\circledast_{i=1}^{k}(T_i, v_i)\right) = 1 - k + \sum_{i=1}^{k} \operatorname{supp}(T_i)$;

(5) $\operatorname{rk}\left(\circledast_{i=1}^{k}(T_i, v_i)\right) = \sum_{i=1}^{k} \operatorname{rk}(T_i)$;

(6) $\operatorname{null}\left(\circledast_{i=1}^{k}(T_i, v_i)\right) = 1 - k + \sum_{i=1}^{k} \operatorname{null}(T_i)$;

(7) $\nu\left(\circledast_{i=1}^{k}(T_i, v_i)\right) = \sum_{i=1}^{k} \nu(T_i)$;

(8) $m\left(\circledast_{i=1}^{k}(T_i, v_i)\right) < \prod_{i=1}^{k} m(T_i)$;

(9) $\alpha\left(\circledast_{i=1}^{k}(T_i, v_i)\right) = 1 - k + \sum_{i=1}^{k} \alpha(T_i)$.

Proof. Clearly (1) and (4) follow from (3), and (2) follows from (1). Further, (5) follows from (2) and Theorem 2.10. Statement (6) follows from (2), (4), and (5). Statement (7) follows from Theorem 3.6 and (2). Finally, (9) follows from Theorem 2.10 and (4).

In order to prove (3), let $\vec{y} \in \operatorname{Supp}\left(\circledast_{i=1}^{k}(T_i, v_i)\right)$. Without loss of generality, we assume that $\vec{y}_{v*} = 1$. As

$$A(T_i)\left((y\downarrow^{\circledast_{i=1}^{k}(T_i,v_i)}_{T_i - v_i})\uparrow^{T_i}_{T_i - v_i} + e_{v_i}\right) = \theta,$$

where e_{v_i} is a canonical vector of $\mathbb{R}^{T_i}$, we have

$$\operatorname{Supp}\left(\circledast_{i=1}^{k}(T_i, v_i)\right) \subset \{v^*\} \cup \left(\bigcup_{i=1}^{k} \operatorname{Supp}(T_i) - \{v_i\}\right).$$

Hence, by the proof of Theorem 4.7, (3) follows.

To prove (8), just note that there is an injection between maximum matchings of $\circledast_{i=1}^{k}(T_i, v_i)$ and $\prod_{i=1}^{k} M(T_i)$. Let $M \in \mathcal{M}\big(\circledast_{i=1}^{k}(T_i, v_i)\big)$, and let $u_i \in V(T_i)$ be such that $\{u_i, v^*\} \in M$. Then

$$M - \{u_i, v^*\} + \{u_i, v_i\} \in \prod_{i=1}^{k} M(T_i).$$

But this injection is not onto. Let $M(i) \in \mathcal{M}(T_i)$ be such that $v_i \in V(M(i))$. Clearly, the cardinality of every matching $M \in \mathcal{M}\big(\circledast_{i=1}^{k}(T_i, v_i)\big)$ is less than $\prod_{i=1}^{k} M(i)$. □

It is clear that any S-coalescence of an atom tree is an atom tree.

Proposition 4.9. *Atom trees are closed under S-coalescence.*

The set of all supported vertices with degree greater than one carries structural information about trees. They mark in the tree where the S-coalescences were made.

Theorem 4.10. *Let T be an independent tree. If $v \in \operatorname{Supp}(T)$ and $\deg(v) > 1$, then T is an S-coalescence of independent trees.*

Proof. It is left to the reader. □

We can apply the former decomposition a finite number of times to get the set of stellare trees that form the given independent tree T.

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D. A. Jaume ✉
Departamento de Matemáticas, Facultad de Ciencias Físico-Matemáticas y Naturales, Universidad Nacional de San Luis, and Instituto de Matemáticas Aplicadas de San Luis, IMASL-CONICET, San Luis, Argentina
djaume@unsl.edu.ar

G. Molina
Departamento de Matemáticas, Facultad de Ciencias Físico-Matemáticas y Naturales, Universidad Nacional de San Luis, San Luis, Argentina
lgmolina@unsl.edu.ar

R. Sota
Departamento de Matemáticas, Facultad de Ciencias Físico-Matemáticas y Naturales, Universidad Nacional de San Luis, San Luis, Argentina
rsota@unsl.edu.ar

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