REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 61, No. 2, 2020, Pages 229–247
Published online: October 15, 2020
https://doi.org/10.33044/revuma.v61n2a03

ON THE ATOMIC AND MOLECULAR DECOMPOSITION OF WEIGHTED HARDY SPACES

PABLO ROCHA

Abstract. The purpose of this article is to give another molecular decomposition for members of weighted Hardy spaces, different from that given by Lee and Lin [*J. Funct. Anal.* **188** (2002), no. 2, 442–460], and to review some overlooked details. As an application of this decomposition, we obtain the boundedness on $H^p_w(\mathbb{R}^n)$ of every bounded linear operator on some $L^{p_0}(\mathbb{R}^n)$ with $1 < p_0 < +\infty$, for all weights $w \in \mathcal{A}_\infty$ and all $0 < p \leq 1$ if $1 < \frac{r_w - 1}{r_w} p_0$, or all $0 < p < \frac{r_w-1}{r_w} p_0$ if $\frac{r_w-1}{r_w} p_0 \leq 1$, where r_w is the critical index of w for the reverse Hölder condition. In particular, the well-known results about boundedness of singular integrals from $H^p_w(\mathbb{R}^n)$ into $L^p_w(\mathbb{R}^n)$ and on $H^p_w(\mathbb{R}^n)$ for all $w \in \mathcal{A}_\infty$ and all $0 < p \leq 1$ are established. We also obtain the $H^p_{w^p}(\mathbb{R}^n)$-$H^q_{w^q}(\mathbb{R}^n)$ boundedness of the Riesz potential I_α for $0 < p \leq 1$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and certain weights w.

1. Introduction

The Hardy spaces on $\mathbb{R}^n$ were defined in [4] by C. Fefferman and E. Stein; since then the subject has received considerable attention. One of the most important applications of Hardy spaces is that they are good substitutes for Lebesgue spaces when $p \leq 1$. For example, when $p \leq 1$, it is well known that Riesz transforms are not bounded on $L^p(\mathbb{R}^n)$; however, they are bounded on Hardy spaces $H^p(\mathbb{R}^n)$.

To obtain the boundedness of operators—like singular integrals or fractional type operators—in the Hardy spaces $H^p(\mathbb{R}^n)$, one can appeal to the atomic or molecular characterization of $H^p(\mathbb{R}^n)$, which means that a distribution in H^p can be represented as a sum of atoms or molecules. The atomic decomposition of elements in $H^p(\mathbb{R}^n)$ was obtained by Coifman in [2] (for $n = 1$), and by Latter in [8] (for $n \geq 1$). In [20], Taibleson and Weiss gave the molecular decomposition of elements in $H^p(\mathbb{R}^n)$. Then the boundedness of linear operators in H^p can be deduced, in principle, from their behavior on atoms or molecules. However, it must be mentioned that M. Bownik in [1], based on an example of Y. Meyer, constructed a linear functional defined on a dense subspace of $H^1(\mathbb{R}^n)$, which maps all $(1, \infty, 0)$ atoms into bounded scalars, and yet cannot be extended to a bounded linear functional on the whole $H^1(\mathbb{R}^n)$. This implies that it does not suffice to

2010 *Mathematics Subject Classification.* 42B15, 42B25, 42B30.
Key words and phrases. Weighted Hardy spaces; Singular integrals; Fractional operators.

check that an operator from a Hardy space H^p, $0 < p \leq 1$, into some quasi Banach space X, maps atoms into bounded elements of X to establish that this operator extends to a bounded operator on H^p. Bownik's example is, in a certain sense, pathological. Fortunately, if T is a classical operator, then the uniform boundedness of T on atoms implies the boundedness from H^p into L^p; this follows from the boundedness on L^s, $1 < s < \infty$, of T, and since one always can take an atomic decomposition which converges in the norm of L^s (see [21] and [14]).

The weighted Lebesgue spaces $L^p_w(\mathbb{R}^n)$ are a generalization of the classical Lebesgue spaces $L^p(\mathbb{R}^n)$, replacing the Lebesgue measure dx by the measure $w(x)\,dx$, where w is a non-negative measurable function. Then one can define the weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ by generalizing the definition of $H^p(\mathbb{R}^n)$ (see [18]). It is well known that the harmonic analysis on these spaces is relevant if the "weights" w belong to the class $\mathcal{A}_\infty$. The atomic characterization of $H^p_w(\mathbb{R}^n)$ has been given in [5] and [18]. The molecular characterization of $H^p_w(\mathbb{R}^n)$ was developed independently by X. Li and L. Peng in [10] and by M.-Y. Lee and C.-C. Lin in [9]. In both works the authors obtained the boundedness of the classical singular integrals on H^p_w for $w \in \mathcal{A}_1$. We extend these results for all $w \in \mathcal{A}_\infty$.

Given $w \in \mathcal{A}_\infty$, a w-(p, p_0, d) atom is a measurable function $a(\cdot)$ with support in a ball B such that

(1) $\|a\|_{L^{p_0}} \leq \dfrac{|B|^{1/p_0}}{w(B)^{1/p}}$, and

(2) $\int x^\alpha a(x)\,dx = 0$, for all multi-indices $|\alpha| \leq d$,

where the parameters p, p_0, and d satisfy certain restrictions. We remark that our definition of atom differs from that given in [5, 18].

One of our main results is Theorem 2.9 of Section 2 below, which states the following:

If $w \in \mathcal{A}_\infty$ and f belongs to a dense subspace of H^p_w, then there exist a sequence of w-(p, p_0, d) atoms $\{a_j\}$ and a sequence of scalars $\{\lambda_j\}$ with $\sum_j |\lambda_j|^p \leq c\|f\|^p_{H^p_w}$ such that $f = \sum_j \lambda_j a_j$, where the series converges to f in $L^s(\mathbb{R}^n)$, for all $s > 1$.

With this result we avoid any problems that could arise with respect to establishing the boundedness of classical operators on H^p_w. The verification of the convergence in L^s for the infinite atomic decomposition was sometimes an overlooked detail. As far as the author knows, the above result has been proved for w-(p, ∞, d) atoms in $\mathbb{R}$ by J. García-Cuerva in [5], and for w-(p, ∞, d) atoms in $\mathbb{R}^n$ by D. Cruz-Uribe et al. in [3].

Given $w \in \mathcal{A}_\infty$, we say that a measurable function $m(\cdot)$ is a w-(p, p_0, d) molecule centered at a ball $B = B(x_0, r)$ if it satisfies the following conditions:

(m1) $\|m\|_{L^{p_0}(B(x_0,2r))} \leq |B|^{\frac{1}{p_0}} w(B)^{-\frac{1}{p}}$.

(m2) $|m(x)| \leq w(B)^{-\frac{1}{p}} \left(1 + \frac{|x-x_0|}{r}\right)^{-2n-2d-3}$ for all $x \in \mathbb{R}^n \setminus B(x_0, 2r)$.

(m3) $\int_{\mathbb{R}^n} x^\alpha m(x)\,dx = 0$ for every multi-index α with $|\alpha| \leq d$.

Our definition of molecule is an adaptation from that given in [13] by E. Nakai and Y. Sawano in the setting of variable Hardy spaces. It is clear that a w-(p, p_0, d) atom is a w-(p, p_0, d) molecule. The pointwise inequality in (m2) seems a good substitute for “the loss of compactness in the support of an atom”.

In Section 3, we obtain the following result (Theorem 3.3 below):

Let $0 < p \leq 1$, $w \in \mathcal{A}_\infty$, and let $f \in \mathcal{S}'(\mathbb{R}^n)$ be such that $f = \sum_j \lambda_j m_j$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\{\lambda_j\}$ is a sequence of positive numbers belonging to $\ell^p(\mathbb{N})$ and the functions m_j are (p, p_0, d)-molecules centered at B_j with respect to the weight w. Then $f \in H^p_w(\mathbb{R}^n)$, with

$$\|f\|^p_{H^p_w} \leq C_{w,p,p_0} \sum_j \lambda_j^p.$$

With these results in Section 4 we re-establish the boundedness on H^p_w and from H^p_w into L^p_w of certain singular integrals, for all $w \in \mathcal{A}_\infty$ and all $0 < p \leq 1$. We also obtain the $H^p_{w^p}$-$H^q_{w^q}$ boundedness of the Riesz potential I_α, for $0 < p \leq 1$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and certain weights w.

Notation. The symbol $A \lesssim B$ stands for the inequality $A \leq cB$ for some constant c. We denote by $B(x_0, r)$ the ball centered at $x_0 \in \mathbb{R}^n$ of radius r. Given a ball $B(x_0, r)$ and a constant $c > 0$, we set $cB = B(x_0, cr)$. For a measurable subset $E \subset \mathbb{R}^n$ we denote by $|E|$ and χ_E the Lebesgue measure of E and the characteristic function of E, respectively. Given a real number $s \geq 0$, we write $\lfloor s \rfloor$ for the integer part of s. As usual we denote by $\mathcal{S}(\mathbb{R}^n)$ the space of smooth and rapidly decreasing functions and with $\mathcal{S}'(\mathbb{R}^n)$ the dual space. If β is the multi-index $\beta = (\beta_1, \ldots, \beta_n)$, then $|\beta| = \beta_1 + \cdots + \beta_n$.

Throughout this paper, C will denote a positive constant, not necessarily the same at each occurrence.

2. Preliminaries

2.1. **Weighted theory.** A weight is a non-negative locally integrable function on $\mathbb{R}^n$ that takes values in $(0, \infty)$ almost everywhere, i.e., the weights are allowed to be zero or infinity only on a set of Lebesgue measure zero.

Given a weight w and $0 < p < \infty$, we denote by $L^p_w(\mathbb{R}^n)$ the space of all functions f satisfying $\|f\|^p_{L^p_w} := \int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx < \infty$. When $p = \infty$, we have that $L^\infty_w(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$, with $\|f\|_{L^\infty_w} = \|f\|_{L^\infty}$. If E is a measurable set, we use the notation $w(E) = \int_E w(x)\, dx$.

Let f be a locally integrable function on $\mathbb{R}^n$. The function

$$M(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy,$$

where the supremum is taken over all balls B containing x, is called the uncentered Hardy–Littlewood maximal function of f.

We say that a weight w belongs to $\mathcal{A}_1$ if there exists $C > 0$ such that

$$M(w)(x) \leq C w(x), \quad \text{a.e. } x \in \mathbb{R}^n;$$

the best possible constant is denoted by $[w]_{\mathcal{A}_1}$. Equivalently, a weight w belongs to $\mathcal{A}_1$ if there exists $C > 0$ such that for every ball B

$$\frac{1}{|B|} \int_B w(x)\, dx \leq C \operatorname*{ess\,inf}_{x \in B} w(x). \tag{2.1}$$

Remark 2.1. If $w \in \mathcal{A}_1$ and $0 < r < 1$, then by Hölder's inequality we have that $w^r \in \mathcal{A}_1$.

For $1 < p < \infty$, we say that a weight $w \in \mathcal{A}_p$ if there exists $C > 0$ such that for every ball B

$$\left(\frac{1}{|B|} \int_B w(x)\, dx \right) \left(\frac{1}{|B|} \int_B [w(x)]^{-\frac{1}{p-1}}\, dx \right)^{p-1} \leq C.$$

It is well known that $\mathcal{A}_{p_1} \subset \mathcal{A}_{p_2}$ for all $1 \leq p_1 < p_2 < \infty$. Also, if $w \in \mathcal{A}_p$ with $1 < p < \infty$, then there exists $1 < q < p$ such that $w \in \mathcal{A}_q$. We denote by $\widetilde{q}_w := \inf\{q > 1 : w \in \mathcal{A}_q\}$ the *critical index of* w.

Lemma 2.2. *If $w \in \mathcal{A}_p$ for some $1 \leq p < \infty$, then the measure $w(x)\, dx$ is doubling: precisely, for all $\lambda > 1$ and all balls B we have*

$$w(\lambda B) \leq \lambda^{np} [w]_{\mathcal{A}_p} w(B),$$

where λB denotes the ball with the same center as B and radius λ times the radius of B.

Theorem 2.3 ([11, Theorem 9])**.** *Let $1 < p < \infty$. Then*

$$\int_{\mathbb{R}^n} [Mf(x)]^p w(x)\, dx \leq C_{w,p,n} \int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx,$$

for all $f \in L^p_w(\mathbb{R}^n)$ if and only if $w \in \mathcal{A}_p$.

Given $1 < p \leq q < \infty$, we say that a weight $w \in \mathcal{A}_{p,q}$ if there exists $C > 0$ such that for every ball B

$$\left(\frac{1}{|B|} \int_B [w(x)]^q\, dx \right)^{1/q} \left(\frac{1}{|B|} \int_B [w(x)]^{-p'}\, dx \right)^{1/p'} \leq C < \infty.$$

For $p = 1$, we say that a weight $w \in \mathcal{A}_{1,q}$ if there exists $C > 0$ such that for every ball B

$$\left(\frac{1}{|B|} \int_B [w(x)]^q\, dx \right)^{1/q} \leq C \operatorname*{ess\,inf}_{x \in B} w(x).$$

When $p = q$, this definition is equivalent to $w^p \in \mathcal{A}_p$.

Remark 2.4. From the inequality (2.1) it follows that if a weight $w \in \mathcal{A}_1$, then $0 < \operatorname{ess\,inf}_{x \in B} w(x) < \infty$ for each ball B. Thus $w \in \mathcal{A}_1$ implies that $w^{\frac{1}{q}} \in \mathcal{A}_{p,q}$, for each $1 \leq p \leq q < \infty$.

Given $0 < \alpha < n$, we define the fractional maximal operator M_α by

$$M_\alpha f(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |f(y)|\, dy,$$

where f is a locally integrable function and the supremum is taken over all balls B containing x.

The fractional maximal operator satisfies the following weighted inequality.

Theorem 2.5 ([12, Theorem 3])**.** *If* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, *and* $w \in \mathcal{A}_{p,q}$, *then*

$$\left(\int_{\mathbb{R}^n} [M_\alpha f(x)]^q w^q(x)\, dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p w^p(x)\, dx \right)^{1/p},$$

for all $f \in L^p_{w^p}(\mathbb{R}^n)$.

A weight w satisfies the reverse Hölder inequality with exponent $s > 1$, denoted by $w \in RH_s$, if there exists $C > 0$ such that for every ball B,

$$\left(\frac{1}{|B|} \int_B [w(x)]^s\, dx \right)^{\frac{1}{s}} \leq C \frac{1}{|B|} \int_B w(x)\, dx;$$

the best possible constant is denoted by $[w]_{RH_s}$. We observe that if $w \in RH_s$, then by Hölder's inequality, $w \in RH_t$ for all $1 < t < s$, and $[w]_{RH_t} \leq [w]_{RH_s}$. Moreover, if $w \in RH_s$, $s > 1$, then $w \in RH_{s+\epsilon}$ for some $\epsilon > 0$. We denote by $r_w = \sup\{r > 1 : w \in RH_r\}$ *the critical index of* w *for the reverse Hölder condition.*

It is well known that a weight w satisfies the condition $\mathcal{A}_\infty$ if and only if $w \in \mathcal{A}_p$ for some $p \geq 1$ (see [6, Corollary 7.3.4]). So $\mathcal{A}_\infty = \cup_{1 \leq p < \infty} \mathcal{A}_p$. Also, $w \in \mathcal{A}_\infty$ if and only if $w \in RH_s$ for some $s > 1$ (see [6, Theorem 7.3.3]). Thus $1 < r_w \leq +\infty$ for all $w \in \mathcal{A}_\infty$.

Another remarkable result about the reverse Hölder classes was discovered by Stromberg and Wheeden. They proved in [19] that $w \in RH_s$, $1 < s < +\infty$, if and only if $w^s \in \mathcal{A}_\infty$.

Given a weight w, $0 < p < \infty$, and a measurable set E, we set $w^p(E) = \int_E [w(x)]^p\, dx$. The following result is an immediate consequence of the reverse Hölder condition.

Lemma 2.6. *For* $0 < \alpha < n$, *let* $0 < p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *If* $w^p \in RH_{\frac{q}{p}}$ *then*

$$[w^p(B)]^{-\frac{1}{p}} [w^q(B)]^{\frac{1}{q}} \leq [w^p]^{1/p}_{RH_{q/p}} |B|^{-\frac{\alpha}{n}},$$

for each ball B *in* $\mathbb{R}^n$.

2.2. **Weighted Hardy spaces.** Topologize $\mathcal{S}(\mathbb{R}^n)$ by the collection of semi-norms $\|\cdot\|_{\alpha,\beta}$, with α and β multi-indices, given by

$$\|\varphi\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)|.$$

For each $N \in \mathbb{N}$, we set $\mathcal{S}_N = \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \|\varphi\|_{\alpha,\beta} \leq 1, |\alpha|, |\beta| \leq N\}$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. We denote by $\mathcal{M}_N$ the grand maximal operator given by

$$\mathcal{M}_N f(x) = \sup_{t>0} \sup_{\varphi \in \mathcal{S}_N} \left| \left(t^{-n} \varphi(t^{-1} \cdot) * f \right)(x) \right|.$$

Given a weight $w \in \mathcal{A}_\infty$ and $p > 0$, the weighted Hardy space $H^p_w(\mathbb{R}^n)$ consists of all tempered distributions f such that

$$\|f\|_{H^p_w(\mathbb{R}^n)} = \|\mathcal{M}_N f\|_{L^p_w(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} [\mathcal{M}_N f(x)]^p w(x)\, dx \right)^{1/p} < \infty.$$

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a function such that $\int \phi(x)\, dx \neq 0$. For $f \in \mathcal{S}'(\mathbb{R}^n)$, we define the maximal function $M_\phi f$ by

$$M_\phi f(x) = \sup_{t>0} \left| \left(t^{-n} \phi(t^{-1} \cdot) * f \right)(x) \right|.$$

For N sufficiently large, we have $\|M_\phi f\|_{L^p_w} \simeq \|\mathcal{M}_N f\|_{L^p_w}$ (see [18]).

In what follows we consider the set

$$\widehat{\mathcal{D}}_0 = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \hat{\phi} \in C^\infty_c(\mathbb{R}^n) \text{ and } \operatorname{supp}(\hat{\phi}) \subset B(0,\delta) \text{ for some } \delta > 0\}.$$

The following theorem is crucial to get the main results.

Theorem 2.7 ([18, Theorem 1, p. 103]). *Let w be a doubling weight on $\mathbb{R}^n$. Then $\widehat{\mathcal{D}}_0$ is dense in $H^p_w(\mathbb{R}^n)$, $0 < p < \infty$.*

2.2.1. *Weighted atom theory.* Let $w \in \mathcal{A}_\infty$ with critical index $\widetilde{q}_w$ and critical index r_w for the reverse Hölder condition. Let $0 < p \leq 1$, $\max\left\{1, p\left(\frac{r_w}{r_w - 1}\right)\right\} < p_0 \leq +\infty$, and $d \in \mathbb{Z}$ such that $d \geq \left\lfloor n\left(\frac{\widetilde{q}_w}{p} - 1\right)\right\rfloor$. We say that a function $a(\cdot)$ is a w-(p, p_0, d) atom centered at $x_0 \in \mathbb{R}^n$ if

(a1) $a \in L^{p_0}(\mathbb{R}^n)$ with support in the ball $B = B(x_0, r)$.

(a2) $\|a\|_{L^{p_0}(\mathbb{R}^n)} \leq |B|^{\frac{1}{p_0}} w(B)^{-\frac{1}{p}}$.

(a3) $\int x^\alpha a(x)\, dx = 0$ for all multi-indices α such that $|\alpha| \leq d$.

We observe that the condition $\max\left\{1, p\left(\frac{r_w}{r_w-1}\right)\right\} < p_0 < +\infty$ implies that $w \in RH_{\left(\frac{p_0}{p}\right)'}$. If $r_w = +\infty$, then $w \in RH_t$ for each $1 < t < +\infty$. So, if $r_w = +\infty$ and since $\lim_{t \to +\infty} \frac{t}{t-1} = 1$, we put $\frac{r_w}{r_w - 1} = 1$. For example, if $w \equiv 1$, then $\widetilde{q}_w = 1$ and $r_w = +\infty$, and our definition of atom coincides in this case with the definition of atom in the classical Hardy spaces.

Lemma 2.8. *Let $w \in \mathcal{A}_\infty$ with critical index $\widetilde{q}_w$ and critical index r_w. If $a(\cdot)$ is a w-(p, p_0, d) atom, then $a(\cdot) \in H^p_w(\mathbb{R}^n)$. Moreover, there exists a positive constant C independent of the atom a such that $\|a\|_{H^p_w} \leq C$.*

Proof. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \phi(x)\, dx \neq 0$. Since ϕ has a radial majorant that is a non-increasing, bounded, and integrable function, we have that

$$M_\phi a(x) \leq c M a(x), \quad \text{for all } x \in \mathbb{R}^n.$$

In view of the moment condition of a we have

$$(a * \phi_t)(x) = \int [\phi_t(x - y) - q_{x,t}(y)] a(y)\, dy, \quad \text{if } x \in \mathbb{R}^n \setminus B(x_0, 4r),$$

where $q_{x,t}$ is the degree d Taylor polynomial of the function $y \to \phi_t(x-y)$ expanded around x_0. By the standard estimate of the remainder term of the Taylor expansion, the condition (a2), and Hölder's inequality, we obtain that

$$\begin{aligned} M_\phi a(x) &\leq c\|a\|_1 r^{d+1}|x-x_0|^{-n-d-1} \\ &\leq cw(B)^{-1/p} r^{n+d+1}|x-x_0|^{-n-d-1} \\ &\leq cw(B)^{-1/p}[M(\chi_B)(x)]^{\frac{n+d+1}{n}}, \quad \text{if } x \in \mathbb{R}^n \setminus B(x_0,4r). \end{aligned}$$

Therefore,

$$\int [M_\phi a(x)]^p w(x)\,dx \leq c \int \left(\chi_{B(x_0,4r)}[Ma(x)]^p + \frac{[M(\chi_B)(x)]^{\frac{(n+d+1)p}{n}}}{w(B)} \right) w(x)\,dx.$$

On the right side of this inequality, we apply Hölder's inequality with p_0/p and use that $w \in RH_{\left(\frac{p_0}{p}\right)'}$ $(p_0 > p(\frac{r_w}{r_w-1}))$ and Lemma 2.2 for the first term; for the second term we have that $\frac{(n+d+1)p}{n} > \widetilde{q}_w$, so $w \in \mathcal{A}_{\frac{(n+d+1)p}{n}}$. Then by invoking Theorem 2.3 we obtain

$$\|M_\phi a\|^p_{L^p_w} = \int_{\mathbb{R}^n} [M_\phi a(x)]^p w(x)\,dx \leq C,$$

where the constant C is independent of the w-(p,p_0,d) atom a. Thus $a \in H^p_w(\mathbb{R}^n)$. □

Theorem 2.9. *Let $f \in \hat{\mathcal{D}}_0$, and $0 < p \leq 1$. If $w \in \mathcal{A}_\infty$, then there exist a sequence of w-(p,p_0,d) atoms $\{a_j\}$ and a sequence of scalars $\{\lambda_j\}$ with $\sum_j |\lambda_j|^p \leq c\|f\|^p_{H^p_w}$ such that $f = \sum_j \lambda_j a_j$, where the convergence is both in $L^s(\mathbb{R}^n)$ and pointwise, for each $1 < s < \infty$.*

Proof. Given $f \in \hat{\mathcal{D}}_0$, let $\mathcal{O}_j = \{x : \mathcal{M}_N f(x) > 2^j\}$ and let $\mathcal{F}_j = \{Q^j_k\}_k$ be the Whitney decomposition associated to $\mathcal{O}_j$ such that $\bigcup_k Q^{j*}_k = \mathcal{O}_j$. Fixed $j \in \mathbb{Z}$, we define the set

$$E^j = \{(i,k) \in \mathbb{Z} \times \mathbb{Z} : Q^{j+1*}_i \cap Q^{j*}_k \neq \emptyset\}$$

and let $E^j_k = \{i : (i,k) \in E^j\}$ and $E^j_i = \{k : (i,k) \in E^j\}$. Following the proof in [16, Ch. III, §2.3, pp. 107–109], we have a sequence of functions A^j_k such that

- (i) $\operatorname{supp}(A^j_k) \subset Q^{j*}_k \cup \bigcup_{i \in E^j_k} Q^{j+1*}_i$ and $|A^j_k(x)| \leq c2^j$ for all $k,j \in \mathbb{Z}$.
- (ii) $\int x^\alpha A^j_k(x)\,dx = 0$ for all α with $|\alpha| \leq d$ and all $k,j \in \mathbb{Z}$.
- (iii) The sum $\sum_{j,k} A^j_k$ converges to f in the sense of distributions.

From (i) we obtain

$$\sum_k |A^j_k| \leq c2^j \left(\sum_k \chi_{Q^{j*}_k} + \sum_k \chi_{\bigcup_{i \in E^j_k} Q^{j+1*}_i} \right);$$

following the proof of Theorem 5 in [14] we obtain

$$\leq c2^j \left(\chi_{\mathcal{O}_j} + \sum_k \sum_{i \in E_k^j} \chi_{Q_i^{j+1*}} \right) = c2^j \left(\chi_{\mathcal{O}_j} + \sum_i \sum_{k \in E_i^j} \chi_{Q_i^{j+1*}} \right)$$
$$\leq c2^j \left(\chi_{\mathcal{O}_j} + 84^n \sum_i \chi_{Q_i^{j+1*}} \right) \leq c2^j \left(\chi_{\mathcal{O}_j} + \chi_{\mathcal{O}_{j+1}} \right) \leq c2^j \chi_{\mathcal{O}_j}.$$

By [14, Lemma 4] we have that

$$\sum_{j,k} |A_k^j(x)| \leq c \sum_j 2^j \chi_{\mathcal{O}_j \setminus \mathcal{O}_{j+1}}(x), \quad \text{a.e. } x \in \mathbb{R}^n.$$

Thus, for $1 < s < \infty$ fixed,

$$\begin{aligned} \int \left(\sum_{j,k} |A_k^j(x)| \right)^s dx &\leq c \sum_j \int_{\mathcal{O}_j \setminus \mathcal{O}_{j+1}} 2^{js}\, dx \leq c \sum_j \int_{\mathcal{O}_j \setminus \mathcal{O}_{j+1}} (\mathcal{M}f(x))^s\, dx \\ &\leq c \int_{\mathbb{R}^n} (\mathcal{M}f(x))^s\, dx < \infty, \end{aligned} \tag{2.2}$$

since $f \in \hat{\mathcal{D}}_0 \subset L^s(\mathbb{R}^n)$. From (2.2) and (iii) we obtain that the sum $\sum_{j,k} A_k^j$ converges to f in $L^s(\mathbb{R}^n)$, and $\sum_{j,k} A_k^j(x) = f(x)$ a.e. $x \in \mathbb{R}^n$, for each $1 < s < \infty$.

Now, we set $a_{j,k} = \lambda_{j,k}^{-1} A_k^j$ with $\lambda_{j,k} = c2^j w(B_k^j)^{1/p}$, where B_k^j is the smallest ball containing Q_k^{j*} as well as all the Q_i^{j+1*} that intersect Q_k^{j*}. Then we have a sequence $\{a_{j,k}\}$ of w-(p, p_0, d) atoms and a sequence of scalars $\{\lambda_{j,k}\}$ such that the sum $\sum_{j,k} \lambda_{j,k} a_{j,k}$ converges to f in $L^s(\mathbb{R}^n)$ and a.e. $x \in \mathbb{R}^n$. On the other hand there exists a universal constant c_1 such that $B_k^j \subset c_1 Q_k^j$, so

$$\sum_{j,k} |\lambda_{j,k}|^p \lesssim \sum_{j,k} 2^{jp} w(B_k^j) \lesssim \sum_{j,k} 2^{jp} w(c_1 Q_k^j) \lesssim c_1^{np} \sum_{j,k} 2^{jp} w(Q_k^j) = c \sum_j 2^{jp} w(\mathcal{O}_j).$$

If $x \in \mathbb{R}^n$, there exists a unique $j_0 \in \mathbb{Z}$ such that $2^{j_0 p} < \mathcal{M}_N f(x)^p \leq 2^{(j_0+1)p}$. So

$$\sum_j 2^{jp} \chi_{\mathcal{O}_j}(x) \leq \sum_{j \leq j_0} 2^{jp} = \frac{2^{(j_0+1)p}}{2^p - 1} \leq \frac{2^p}{2^p - 1} \mathcal{M}_N f(x)^p.$$

From this it follows that

$$\sum_{j,k} |\lambda_{j,k}|^p \leq c \sum_j 2^{jp} w(\mathcal{O}_j) \leq c \frac{2^p}{2^p - 1} \|\mathcal{M}_N f\|_{L_w^p}^p = c \frac{2^p}{2^p - 1} \|f\|_{H_w^p}^p,$$

which proves the theorem. □

Theorem 2.10. *Let T be a bounded linear operator from $L^{p_0}(\mathbb{R}^n)$ into $L^{p_0}(\mathbb{R}^n)$ for some $1 < p_0 < +\infty$. If $w \in \mathcal{A}_\infty$ with critical index r_w, $0 < p \leq 1 < \frac{r_w - 1}{r_w} p_0$ or $0 < p < \frac{r_w - 1}{r_w} p_0 \leq 1$, then T can be extended to an $H_w^p(\mathbb{R}^n)$-$L_w^p(\mathbb{R}^n)$ bounded linear operator if and only if T is uniformly bounded in L_w^p norm on all w-(p, p_0, d) atoms a.*

Proof. Since T is a bounded linear operator on $L^{p_0}(\mathbb{R}^n)$, T is well defined on $H^p_w(\mathbb{R}^n) \cap L^{p_0}(\mathbb{R}^n)$. If T can be extended to a bounded operator from $H^p_w(\mathbb{R}^n)$ into $L^p_w(\mathbb{R}^n)$, then $\|Ta\|_{L^p_w} \leq c_p \|a\|_{H^p_w}$ for all w-atoms a. By Lemma 2.8, there exists a universal constant C such that $\|a\|_{H^p_w} \leq C < \infty$ for all w-atoms a; so $\|Ta\|_{L^p_w} \leq C_p$ for all w-atoms a.

Conversely, taking into account the assumptions on p and p_0, given $f \in \widehat{\mathcal{D}}_0$, by Theorem 2.9 there exists a w-(p, p_0, d) atomic decomposition such that $\sum_j |\lambda_j|^p \lesssim \|f\|_{H^p_w}$ and $\sum_j \lambda_j a_j = f$ in $L^{p_0}(\mathbb{R}^n)$. From the boundedness of T on $L^{p_0}(\mathbb{R}^n)$ we have that the sum $\sum_j \lambda_j T a_j$ converges to Tf in $L^{p_0}(\mathbb{R}^n)$, thus there exists a subsequence of natural numbers $\{k_N\}_{N\in\mathbb{N}}$ such that $\lim_{N\to+\infty} \sum_{j=-k_N}^{k_N} \lambda_j T a_j(x) = Tf(x)$ a.e. $x \in \mathbb{R}^n$; this implies that

$$|Tf(x)| \leq \sum_j |\lambda_j T a_j(x)|, \quad \text{a.e. } x \in \mathbb{R}^n.$$

If $\|Ta\|_{L^p_w} \leq C_p < \infty$ for all w-(p, p_0, d) atoms a, and since $0 < p \leq 1$, we get

$$\|Tf\|^p_{L^p_w} \leq \sum_j |\lambda_j|^p \|Ta_j\|^p_{L^p_w} \leq C_p^p \sum_j |\lambda_j|^p \leq C_p^p \|f\|^p_{H^p_w}$$

for all $f \in \widehat{\mathcal{D}}_0$. By Theorem 2.7, we have that $\widehat{\mathcal{D}}_0$ is a dense subspace of $H^p_w(\mathbb{R}^n)$, so the theorem follows by a density argument. □

3. Molecular decomposition

Our definition of molecule is an adaptation from that given in [13] by E. Nakai and Y. Sawano in the setting of variable Hardy spaces.

Definition 3.1. Let $w \in \mathcal{A}_\infty$ with critical index $\widetilde{q}_w$ and critical index r_w for the reverse Hölder condition. Let $0 < p \leq 1$, $\max\left\{1, p\left(\frac{r_w}{r_w - 1}\right)\right\} < p_0 \leq +\infty$, and $d \in \mathbb{Z}$ such that $d \geq \left\lfloor n\left(\frac{\widetilde{q}_w}{p} - 1\right)\right\rfloor$. We say that a function $m(\cdot)$ is a w-(p, p_0, d) molecule centered at a ball $B = B(x_0, r)$ if it satisfies the following conditions:

(m1) $\|m\|_{L^{p_0}(B(x_0, 2r))} \leq |B|^{\frac{1}{p_0}} w(B)^{-\frac{1}{p}}$.

(m2) $|m(x)| \leq w(B)^{-\frac{1}{p}} \left(1 + \frac{|x - x_0|}{r}\right)^{-2n-2d-3}$ for all $x \in \mathbb{R}^n \setminus B(x_0, 2r)$.

(m3) $\int_{\mathbb{R}^n} x^\alpha m(x)\, dx = 0$ for every multi-index α with $|\alpha| \leq d$.

Remark 3.2. The conditions (m1) and (m2) imply that $\|m\|_{L^{p_0}(\mathbb{R}^n)} \leq c \frac{|B|^{\frac{1}{p_0}}}{w(B)^{\frac{1}{p}}}$, where c is a positive constant independent of the molecule m.

From the definition of molecule it is clear that a w-(p, p_0, d) atom is a w-(p, p_0, d) molecule.

In view of Lemma 2.8, the following theorem assures, among other things, that the pointwise inequality in (m2) is a good substitute for "the loss of compactness in the support of an atom".

Theorem 3.3. *Let $0 < p \leq 1$, $w \in \mathcal{A}_\infty$, and let $f \in \mathcal{S}'(\mathbb{R}^n)$ be such that $f = \sum_j \lambda_j m_j$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\{\lambda_j\}$ is a sequence of positive numbers belonging to $\ell^p(\mathbb{N})$ and the functions m_j are (p, p_0, d)-molecules centered at B_j with respect to the weight w. Then $f \in H^p_w(\mathbb{R}^n)$ with*

$$\|f\|^p_{H^p_w} \leq C_{w,p,p_0} \sum_j \lambda_j^p.$$

Proof. Let $\phi \in C^\infty_c(\mathbb{R}^n)$ be such that $\chi_{B(0,1)} \leq \phi \leq \chi_{B(0,2)}$; we set $\phi_{2^k}(x) = 2^{kn}\phi(2^k x)$, where $k \in \mathbb{Z}$. Since $f = \sum_j \lambda_j m_j$ in the sense of the distributions, we have that

$$|(\phi_{2^k} * f)(x)| \leq \sum_{j=1}^{\infty} \lambda_j |(\phi_{2^k} * m_j)(x)|,$$

for all $x \in \mathbb{R}^n$ and all $k \in \mathbb{Z}$. We observe that the argument used in the proof of Theorem 5.2 in [13] to obtain the pointwise inequality (5.2) in that paper works in this setting, but considering now the conditions (m1), (m2), and (m3). Therefore, we get

$$M_\phi(f)(x) \lesssim \sum_j \lambda_j \chi_{2B_j}(x) M(m_j)(x) + \sum_j \lambda_j \frac{\left[M(\chi_{B_j})(x)\right]^{\frac{n+d_w+1}{n}}}{w(B_j)^{\frac{1}{p}}}, \quad x \in \mathbb{R}^n,$$

where M is the Hardy–Littlewood maximal operator.

Since $0 < p \leq 1$, it follows that

$$[M_\phi(f)(x)]^p \lesssim \sum_j \lambda_j^p \chi_{2B_j}(x) [M(m_j)(x)]^p + \sum_j \lambda_j^p \frac{\left[M(\chi_{B_j})(x)\right]^{p\frac{n+d_w+1}{n}}}{w(B_j)}, \quad x \in \mathbb{R}^n,$$

and by integrating with respect to w we get

$$\int [M_\phi(f)(x)]^p w(x)\, dx \lesssim \sum_j \lambda_j^p \int \chi_{2B_j}(x) [M(m_j)(x)]^p w(x)\, dx$$
$$+ \sum_j \lambda_j^p \int \frac{\left[M(\chi_{B_j})(x)\right]^{p\frac{n+d_w+1}{n}}}{w(B_j)} w(x)\, dx.$$

On the right side of this inequality, we apply Hölder's inequality with p_0/p, Remark 3.2, Lemma 2.2, and use that $w \in RH_{\left(\frac{p_0}{p}\right)'}$ ($p_0 > p(\frac{r_w}{r_w-1})$) for the first term; for the second term we have that $\frac{(n+d+1)p}{n} > \widetilde{q}_w$, so $w \in \mathcal{A}_{\frac{(n+d+1)p}{n}}$, and by invoking Theorem 2.3 we obtain

$$\|f\|^p_{H^p_w} \leq C_{w,p,p_0} \sum_j \lambda_j^p.$$

This completes the proof. □

Theorem 3.4. *Let T be a bounded linear operator from $L^{p_0}(\mathbb{R}^n)$ into $L^{p_0}(\mathbb{R}^n)$ for some $1 < p_0 < +\infty$. If $w \in \mathcal{A}_\infty$ with critical index r_w, $0 < p \leq 1 < \frac{r_w-1}{r_w} p_0$ or $0 < p < \frac{r_w-1}{r_w} p_0 \leq 1$, and Ta is a w-(p, p_0, d_2) molecule for each w-(p, p_0, d_1) atom a, then T can be extended to an $H^p_w(\mathbb{R}^n)$-$H^p_w(\mathbb{R}^n)$ bounded linear operator.*

Proof. Taking into account the assumptions on p and p_0, given $f \in \widehat{\mathcal{D}}_0$, from Theorem 2.9 it follows that there exists a sequence of w-(p, p_0, d_1) atoms $\{a_j\}$ and a sequence of scalars $\{\lambda_j\}$ with

$$\sum_j |\lambda_j|^p \lesssim \|f\|^p_{H^p_w}, \tag{3.1}$$

such that $f = \sum_j \lambda_j a_j$ in $L^{p_0}(\mathbb{R}^n)$. From the boundedness of T on $L^{p_0}(\mathbb{R}^n)$ we have that $Tf = \sum_j \lambda_j T a_j$ in $L^{p_0}(\mathbb{R}^n)$ and therefore in $\mathcal{S}'(\mathbb{R}^n)$. By hypothesis Ta_j is a w-(p, p_0, d_2) molecule for all j, so Theorem 3.3 and the inequality (3.1) imply that

$$\|Tf\|^p_{H^p_w} \lesssim \sum_j |\lambda_j|^p \lesssim \|f\|^p_{H^p_w}$$

for all $f \in \widehat{\mathcal{D}}_0$, so the theorem follows from the density of $\widehat{\mathcal{D}}_0$ in $H^p_w(\mathbb{R}^n)$. □

4. Applications

4.1. **Singular integrals.** Let $\Omega \in C^\infty(S^{n-1})$ with $\int_{S^{n-1}} \Omega(u)\, d\sigma(u) = 0$. We define the operator T by

$$Tf(x) = \lim_{\epsilon \to 0^+} \int_{|y|>\epsilon} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\, dy, \quad x \in \mathbb{R}^n. \tag{4.1}$$

It is well known that $\widehat{Tf}(\xi) = m(\xi)\widehat{f}(\xi)$, where the multiplier m is homogeneous of degree 0 and is indefinitely differentiable on $\mathbb{R}^n \setminus \{0\}$. Moreover, if $k(y) = \frac{\Omega(y/|y|)}{|y|^n}$ we have

$$|\partial^\alpha_y k(y)| \leq C|y|^{-n-|\alpha|}, \quad \text{for all } y \neq 0 \text{ and all multi-indices } \alpha. \tag{4.2}$$

Then the operator T is bounded on $L^s(\mathbb{R}^n)$ for all $1 < s < +\infty$ and of weak-type $(1, 1)$ (see [15]).

Let $0 < p \leq 1$ and $d = \left\lfloor n\left(\frac{\widetilde{q_w}}{p} - 1\right)\right\rfloor$. Given a w-$(p, p_0, n + 2d + 2)$ atom $a(\cdot)$ with support in the ball $B(x_0, r)$ we have that

$$\|Ta\|_{L^{p_0}(B(x_0,2r))} \leq C\|a\|_{p_0} \leq C|B|^{1/p_0} w(B)^{-1/p}, \tag{4.3}$$

since T is bounded on $L^{p_0}(\mathbb{R}^n)$. In view of the moment condition of $a(\cdot)$ we obtain

$$\begin{aligned} Ta(x) &= \int_B k(x-y)a(y)\, dy \\ &= \int_B [k(x-y) - q_{n+2d+2}(x, y)]a(y)\, dy, \quad x \notin B = B(x_0, 2r), \end{aligned}$$

where q_{n+2d+2} is the degree $n+2d+2$ Taylor polynomial of the function $y \to k(x-y)$ expanded around x_0. From the estimate (4.2) and the standard estimate of the remainder term of the Taylor expansion, there exists ξ between y and x_0 such that

$$|Ta(x)| \leq C\|a\|_1 \frac{|y-x_0|^{n+2d+3}}{|x-\xi|^{2n+2d+3}} \leq C\frac{r^{2n+2d+3}}{w(B)^{1/p}}|x-x_0|^{-2n-2d-3}, \quad x \notin B(x_0, 2r). \tag{4.4}$$

This inequality and a simple computation allow us to obtain

$$|Ta(x)| \leq Cw(B)^{-\frac{1}{p}} \left(1 + \frac{|x - x_0|}{r}\right)^{-2n-2d-3}, \quad \text{for all } x \notin B(x_0, 2r). \tag{4.5}$$

From the estimate (4.4) we obtain that the function $x \to x^\alpha Ta(x)$ belongs to $L^1(\mathbb{R}^n)$ for each $|\alpha| \leq d$, so

$$|((-2\pi ix)^\alpha Ta)\widehat{\ }(\xi)| = |\partial_\xi^\alpha (m(\xi)\widehat{a}(\xi))| = \left| \sum_{\beta \leq \alpha} c_{\alpha,\beta} \, (\partial_\xi^{\alpha-\beta} m)(\xi) \, (\partial_\xi^\beta \widehat{a})(\xi) \right|$$

$$= \left| \sum_{\beta \leq \alpha} c_{\alpha,\beta} \, (\partial_\xi^{\alpha-\beta} m)(\xi) \, ((-2\pi ix)^\beta a)\widehat{\ }(\xi) \right|.$$

From the homogeneity of the function $\partial_\xi^{\alpha-\beta} m$ we obtain that

$$|((-2\pi ix)^\alpha Ta)\widehat{\ }(\xi)| \leq C \sum_{\beta \leq \alpha} |c_{\alpha,\beta}| \frac{\left|((-2\pi ix)^\beta a)\widehat{\ }(\xi)\right|}{|\xi|^{|\alpha|-|\beta|}}, \quad \xi \neq 0. \tag{4.6}$$

Since $\lim_{\xi \to 0} \frac{\left|((-2\pi ix)^\beta a)\widehat{\ }(\xi)\right|}{|\xi|^{|\alpha|-|\beta|}} = 0$ for each $\beta \leq \alpha$ (see [16, Ch. 3, §5.4, p. 128]), taking the limit as $\xi \to 0$ in (4.6) we get

$$\int_{\mathbb{R}^n} (-2\pi ix)^\alpha Ta(x) \, dx = ((-2\pi ix)^\alpha Ta)\widehat{\ }(0) = 0, \quad \text{for all } |\alpha| \leq d. \tag{4.7}$$

From (4.3), (4.5), and (4.7) it follows that there exists a universal constant $C > 0$ such that $CTa(\cdot)$ is a w-(p, p_0, d) molecule if $a(\cdot)$ is a w-$(p, p_0, n + 2d + 2)$ atom. Taking $p_0 \in (1, +\infty)$ such that $1 < \frac{r_w - 1}{r_w} p_0$ and since T is bounded on $L^{p_0}(\mathbb{R}^n)$, by Theorem 3.4 we get the following result.

Theorem 4.1. *Let T be the operator defined in* (4.1)*. If $w \in \mathcal{A}_\infty$ and $0 < p \leq 1$, then T can be extended to an $H^p_w(\mathbb{R}^n)$-$H^p_w(\mathbb{R}^n)$ bounded operator.*

In particular, the Hilbert transform and the Riesz transforms admit a continuous extension on $H^p_w(\mathbb{R})$ and $H^p_w(\mathbb{R}^n)$, for each $w \in \mathcal{A}_\infty$ and $0 < p \leq 1$, respectively.

Remark 4.2. Let $d = \left\lfloor n\left(\frac{\widetilde{q_w}}{p} - 1\right)\right\rfloor$. If $a(\cdot)$ is a w-(p, p_0, d) atom with $1 < \frac{r_w - 1}{r_w} p_0$, then by proceeding as in the estimation of (4.4) we find that

$$|Ta(x)| \leq C \frac{r^{n+d+1}}{w(B)^{1/p}} |x - x_0|^{-n-d-1}, \quad x \notin B(x_0, 2r),$$

so

$$|Ta(x)| \leq C \frac{[M(\chi_B)(x)]^{\frac{n+d+1}{n}}}{w(B)^{1/p}}, \quad x \notin B(x_0, 2r),$$

where M is the Hardy–Littlewood maximal operator.

Lemma 4.3. *Let $p_0 \in (1, +\infty)$ be such that $1 < \frac{r_w - 1}{r_w} p_0$. If T is the operator defined in* (4.1) *and $0 < p \leq 1$, then there exists a universal constant $C > 0$ such that $\|Ta\|_{L^p_w} \leq C$ for all w-(p, p_0, d) atoms $a(\cdot)$.*

Proof. Given a w-(p, p_0, d) atom $a(\cdot)$, let $2B = B(x_0, 2r)$, where $B = B(x_0, r)$ is the ball containing the support of $a(\cdot)$. We write

$$\int_{\mathbb{R}^n} |Ta(x)|^p w(x)\,dx = \int_{2B} |Ta(x)|^p w(x)\,dx + \int_{\mathbb{R}^n\setminus 2B} |Ta(x)|^p w(x)\,dx = I + II.$$

Since T is bounded on $L^{p_0}(\mathbb{R}^n)$ and $w \in RH_{(\frac{p_0}{p})'}$ $\big(p \le 1 < \frac{r_w-1}{r_w} p_0\big)$, Hölder's inequality applied with $\frac{p_0}{p}$ and the condition (a2) give

$$I \le C\|a\|_{p_0}^p |B|^{-p/p_0} w(B) = C.$$

From Remark 4.2 and since $w \in \mathcal{A}_{p\frac{n+d+1}{n}}$ $\big(p\frac{n+d+1}{n} > \widetilde{q}_w\big)$, we get

$$II \le w(B)^{-1} \int_{\mathbb{R}^n} [M(\chi_B)(x)]^{\frac{n+d+1}{n}} w(x)\,dx \le C w^{-1}(B) \int_B w(x)\,dx = C,$$

where the second inequality follows from Theorem 2.3. This completes the proof. $\square$

Theorem 4.4. *Let T be the operator defined in* (4.1). *If $w \in \mathcal{A}_\infty$ and $0 < p \le 1$, then T can be extended to an $H^p_w(\mathbb{R}^n)$-$L^p_w(\mathbb{R}^n)$ bounded operator.*

Proof. The theorem follows from Lemma 4.3 and Theorem 2.10. $\square$

4.2. **The Riesz potential.** For $0 < \alpha < n$, let I_α be the Riesz potential defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} f(y)\,dy, \tag{4.8}$$

$f \in L^s(\mathbb{R}^n)$, $1 \le s < \frac{n}{\alpha}$. A well-known result of Sobolev gives the boundedness of I_α from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ for $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. In [17], E. Stein and G. Weiss used the theory of harmonic functions of several variables to prove that these operators are bounded from $H^1(\mathbb{R}^n)$ into $L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$. In [20], M. Taibleson and G. Weiss obtained, using the molecular decomposition, the boundedness of the Riesz potential I_α from $H^p(\mathbb{R}^n)$ into $H^q(\mathbb{R}^n)$, for $0 < p \le 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$; S. Krantz independently obtained the same result in [7]. We extend these results to the context of weighted Hardy spaces using the weighted molecular theory developed in Section 3.

First we recall the definition of the critical indices for a weight w.

Definition 4.5. Given a weight w, we denote by $\widetilde{q}_w = \inf\{q > 1 : w \in \mathcal{A}_q\}$ *the critical index of* w, and we denote by $r_w = \sup\{r > 1 : w \in RH_r\}$ *the critical index of w for the reverse Hölder condition.*

Lemma 4.6. *Let $0 < p < 1$. If $w^{1/p} \in \mathcal{A}_1$, then $p \cdot r_{w^p} \le r_w \le r_{w^p}$.*

Proof. The condition $w^{1/p} \in \mathcal{A}_1$, with $0 < p < 1$, implies that $w^p \in RH_{1/p}$. It is well known that if $w \in RH_r$, then $w \in RH_{r+\epsilon}$ for some $\epsilon > 0$, and thus $1/p < r_{w^p}$. Taking $1/p < t < r_{w^p}$ we have that $1 < pt < t$ and $w^p \in RH_t$, so

$$\left(\frac{1}{|B|}\int_B [w(x)]^{pt}\,dx\right)^{1/pt} = \left(\frac{1}{|B|}\int_B [w^p(x)]^{t}\,dx\right)^{1/pt} \le C\left(\frac{1}{|B|}\int_B w^p(x)\,dx\right)^{1/p}$$
$$\le C\frac{1}{|B|}\int_B w(x)\,dx,$$

where the last inequality follows from Jensen's inequality. This implies that $p\,t < r_w$ for all $t < r_{w^p}$, and thus $p \cdot r_{w^p} \le r_w$.

On the other hand, since $0 < p < 1$ and $w^{1/p} \in \mathcal{A}_1$ we have that $w \in RH_{1/p}$. So $1/p < r_w$; taking $1/p < t < r_w$ it follows that $1 < pt < t$, and therefore $w \in RH_{pt}$. Then

$$\left(\frac{1}{|B|}\int_B [w^p(x)]^{t}\,dx\right)^{1/t} = \left(\frac{1}{|B|}\int_B [w(x)]^{tp}\,dx\right)^{p/pt} \le C\left(\frac{1}{|B|}\int_B w(x)\,dx\right)^{p}$$
$$= C\left(\frac{1}{|B|}\int_B [w^p(x)]^{1/p}\,dx\right)^{p} \le C\frac{1}{|B|}\int_B [w^p(x)]\,dx,$$

where the last inequality follows from the fact that $w^p \in RH_{1/p}$. So $t < r_{w^p}$ for all $t < r_w$, and this gives $r_w \le r_{w^p}$. □

Lemma 4.7. *Let $0 < p < q$. If $w^q \in \mathcal{A}_1$, then $p \cdot r_{w^p} \le q \cdot r_{w^q}$.*

Proof. Since $w^q \in \mathcal{A}_1$ and $0 < p < q$ we have that $w^p \in \mathcal{A}_1$ and $w^p \in RH_{q/p}$. Thus $q/p < r_{w^p}$. Taking $q/p < s < r_{w^p}$ we have that $w^p \in RH_s$ and $1 < ps/q < s$, so

$$\left(\frac{1}{|B|}\int_B [w^q(x)]^{ps/q}\,dx\right)^{q/ps} = \left(\frac{1}{|B|}\int_B [w^p(x)]^{s}\,dx\right)^{q/ps} \le C\left(\frac{1}{|B|}\int_B w^p(x)\,dx\right)^{q/p}$$
$$\le C\frac{1}{|B|}\int_B w^q(x)\,dx,$$

where the last inequality follows from Jensen's inequality. This implies that $\frac{p}{q}\,s < r_{w^q}$ for all $s < r_{w^p}$, and thus $p \cdot r_{w^p} \le q \cdot r_{w^q}$. □

Proposition 4.8. *For $0 < \alpha < n$, let I_α be the Riesz potential defined in* (4.8) *and let $w^{1/s} \in \mathcal{A}_1$, $0 < s < \frac{n}{n+\alpha}$, with $\frac{r_w}{r_w-1} < \frac{n}{\alpha}$. If $s \le p \le \frac{n}{n+\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, then $I_\alpha a(\cdot)$ is a w^q-$\big(q, q_0, \lfloor n(\frac{1}{q}-1)\rfloor\big)$ molecule for each w^p-$\big(p, p_0, 2\lfloor n(\frac{1}{q}-1)\rfloor + 3 + \lfloor \alpha \rfloor + n\big)$ atom $a(\cdot)$, where $\frac{r_w}{r_w-1} < p_0 < \frac{n}{\alpha}$ and $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$.*

Proof. The condition $w^{1/s} \in \mathcal{A}_1$ implies that w^p and w^q belong to $\mathcal{A}_1$, so $\widetilde{q}_{w^p} = \widetilde{q}_{w^q} = 1$. We observe that $2\lfloor n(\frac{1}{q}-1)\rfloor + 3 + \lfloor\alpha\rfloor + n > \lfloor n(\frac{1}{p}-1)\rfloor$, and thus $a(\cdot)$ is an atom with additional vanishing moments.

Now we shall see that $p\frac{r_{w^p}}{r_{w^p}-1} < p_0$ and $q\frac{r_{w^q}}{r_{w^q}-1} < q_0$. The condition $p\frac{r_{w^p}}{r_{w^p}-1} < p_0$ is required in the definition of atom and $q\frac{r_{w^q}}{r_{w^q}-1} < q_0$ in the definition of molecule.

By Lemma 4.6 and by hypothesis we have that

$$p\frac{r_{w^p}}{r_{w^p}-1} \le \frac{r_w}{r_w-1} < p_0. \tag{4.9}$$

Lemma 4.7 and the fact that the function $t \to \frac{t}{t-1}$ is decreasing on the region $(1,+\infty)$ imply that

$$\frac{r_{w^q}}{r_{w^q}-\frac{p}{q}} \leq \frac{r_{w^p}}{r_{w^p}-1}. \tag{4.10}$$

If $\frac{1}{q_0}=\frac{1}{p_0}-\frac{\alpha}{n}$, from (4.9) we have

$$\frac{1}{q_0} < \frac{r_{w^p}-1}{p\,r_{w^p}} - \frac{\alpha}{n}.$$

From (4.10) we obtain

$$\frac{1}{q_0} < \frac{r_{w^q}-\frac{p}{q}}{p\,r_{w^q}} - \frac{\alpha}{n} = \frac{1}{p}\left(1-\frac{p}{q\,r_{w^q}}\right) - \frac{\alpha}{n} = \frac{r_{w^q}-1}{q\,r_{w^q}}.$$

So $q\frac{r_{w^q}}{r_{w^q}-1} < q_0$.

Now we will show that $I_\alpha a(\cdot)$ satisfies the conditions (m1), (m2), and (m3) in the definition of molecule, if $a(\cdot)$ is a w^p-$\big(p,p_0,2\lfloor n(\frac{1}{q}-1)\rfloor+3+\lfloor\alpha\rfloor+n\big)$ atom.

Since I_α is bounded from $L^{p_0}(\mathbb{R}^n)$ into $L^{q_0}(\mathbb{R}^n)$ and $w^p \in RH_{q/p}$, by Lemma 2.6 we get

$$\|I_\alpha a\|_{L^{q_0}(B(x_0,2r))} \leq C\|a\|_{L^{p_0}(\mathbb{R}^n)} \leq C|B|^{1/p_0}(w^p(B))^{-1/p} \leq C|B|^{1/q_0}(w^q(B))^{-1/q},$$

so $I_\alpha a(\cdot)$ satisfies (m1).

Let $d=2\lfloor n(\frac{1}{q}-1)\rfloor+3+\lfloor\alpha\rfloor+n$, and let $a(\cdot)$ be a w^p-(p,p_0,d) atom supported on the ball $B(x_0,r)$. In view of the moment condition of $a(\cdot)$ we obtain

$$I_\alpha a(x) = \int_{B(x_0,r)} \big(|x-y|^{\alpha-n} - q_d(x,y)\big)\, a(y)\, dy, \quad \text{for all } x \notin B(x_0,2r),$$

where q_d is the degree d Taylor polynomial of the function $y \to |x-y|^{\alpha-n}$ expanded around x_0. By the standard estimate of the remainder term of the Taylor expansion, there exists ξ between y and x_0 such that

$$\big||x-y|^{\alpha-n} - q_d(x,y)\big| \leq C|y-x_0|^{d+1}|x-\xi|^{-n+\alpha-d-1},$$

for any $y \in B(x_0,r)$ and any $x \notin B(x_0,2r)$. Since $|x-\xi| \geq \frac{|x-x_0|}{2}$, we get

$$\big||x-y|^{\alpha-n} - q_d(x,y)\big| \leq Cr^{d+1}|x-x_0|^{-n+\alpha-d-1}.$$

This inequality and the condition (a2) allow us to conclude that

$$|I_\alpha a(x)| \leq C\frac{r^{n+d+1}}{(w^p(B))^{1/p}}|x-x_0|^{-n+\alpha-d-1}, \quad \text{for all } x \notin B(x_0,2r). \tag{4.11}$$

Lemma 2.6 and a simple computation give

$$|I_\alpha a(x)| \leq C(w^q(B))^{-1/q}\left(1+\frac{|x-x_0|}{r}\right)^{-2n-2d_q-3}, \quad \text{for all } x \notin B(x_0,2r),$$

where $d_q = \lfloor n(\frac{1}{q}-1)\rfloor$. So $I_\alpha a(\cdot)$ satisfies (m2).

Finally, in [20] Taibleson and Weiss proved that

$$\int_{\mathbb{R}^n} x^\beta I_\alpha a(x)\, dx = 0,$$

for all $0 \leq |\beta| \leq \lfloor n(\frac{1}{q}-1)\rfloor$. This shows that $I_\alpha a(\cdot)$ is a w^q-molecule. The proof of the proposition is therefore concluded. □

Theorem 4.9. *For $0<\alpha<n$, let I_α be the Riesz potential defined in* (4.8). *If $w^{1/s} \in \mathcal{A}_1$ with $0<s<\frac{n}{n+\alpha}$ and $\frac{r_w}{r_w-1}<\frac{n}{\alpha}$, then I_α can be extended to an $H^p_{w^p}(\mathbb{R}^n)$-$H^q_{w^q}(\mathbb{R}^n)$ bounded operator for each $s \leq p \leq \frac{n}{n+\alpha}$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$.*

Proof. Let $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. For the range $p \leq \frac{n}{n+\alpha}$ we have that $p<q\leq 1$. If $p \in [s, \frac{n}{n+\alpha}]$, the condition $w^{1/s} \in \mathcal{A}_1$, $0<s<\frac{n}{n+\alpha}$, implies that w, $w^{1/p}$, and w^p belong to $\mathcal{A}_1$, so $w^q \in \mathcal{A}_1$. Then $\widetilde{q}_{w^p}=\widetilde{q}_{w^q}=1$. We put $d_p=\lfloor n(\frac{1}{p}-1)\rfloor$ and $d_q=\lfloor n(\frac{1}{q}-1)\rfloor$. We recall that in the atomic decomposition, we can always choose atoms with additional vanishing moments (see the corollary in [16, Ch. 3, §2.1.5, p. 105]). That is, if l is any fixed integer with $l>d_p$, then we have an atomic decomposition such that all moments up to order l of our atoms are zero.

For $\frac{r_w}{r_w-1}<p_0<\frac{n}{\alpha}$ we consider $\frac{1}{q_0}=\frac{1}{p_0}-\frac{\alpha}{n}$. We observe that $2\lfloor n(\frac{1}{q}-1)\rfloor+3+\lfloor\alpha\rfloor+n>\lfloor n(\frac{1}{p}-1)\rfloor$. Since $w^{1/p}\in\mathcal{A}_1$, from Lemma 4.6 we have $p\frac{r_{w^p}}{r_{w^p}-1} \leq \frac{r_w}{r_w-1}<p_0$. Thus, given $f\in\widehat{\mathcal{D}}_0$ we can write $f=\sum_j \lambda_j a_j$, where a_j are w^p-$(p,p_0,2\lfloor n(\frac{1}{q}-1)\rfloor+3+\lfloor\alpha\rfloor+n)$ atoms, $\sum_j|\lambda_j|^p \lesssim \|f\|^p_{H^p_{w^p}}$, and the series converges in $L^{p_0}(\mathbb{R}^n)$. Since I_α is a $L^{p_0}(\mathbb{R}^n)$-$L^{q_0}(\mathbb{R}^n)$ bounded operator it follows that $I_\alpha f=\sum_j \lambda_j I_\alpha a_j$ in $L^{q_0}(\mathbb{R}^n)$ and therefore in $\mathcal{S}'(\mathbb{R}^n)$. By Proposition 4.8, we have that the operator I_α maps w^p-$(p,p_0,2\lfloor n(\frac{1}{q}-1)\rfloor+3+\lfloor\alpha\rfloor+n)$ atoms $a(\cdot)$ to w^q-(q,q_0,d_q) molecules $I_\alpha a(\cdot)$, and applying Theorem 3.3 we get

$$\|I_\alpha f\|^q_{H^q_{w^q}} \lesssim \sum_j |\lambda_j|^q \lesssim \left(\sum_j |\lambda_j|^p\right)^{q/p} \lesssim \|f\|^q_{H^p_{w^p}},$$

for all $f\in\widehat{\mathcal{D}}_0$, so the theorem follows from the density of $\widehat{\mathcal{D}}_0$ in $H^p_{w^p}(\mathbb{R}^n)$. □

For $\frac{n}{n+\alpha}<p\leq 1$, we have that $1<q\leq\frac{n}{n-\alpha}$. For this range of q's the space H^q_w can be identified with the space L^q_w. The following theorem contains this range of p's.

Theorem 4.10. *For $0<\alpha<n$, let I_α be the Riesz potential defined in* (4.8). *If $w^{\frac{n}{(n-\alpha)s}} \in \mathcal{A}_1$ with $0<s<1$ and $\frac{r_w}{r_w-1}<\frac{n}{\alpha}$, then I_α can be extended to an $H^p_{w^p}(\mathbb{R}^n)$-$L^q_{w^q}(\mathbb{R}^n)$ bounded operator for each $s\leq p\leq 1$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$.*

Proof. The condition $w^{n/(n-\alpha)s}\in\mathcal{A}_1$, $0<s<1<\frac{n}{n-\alpha}$, implies that w, $w^{1/p}$, w^p, and w^q belong to $\mathcal{A}_1$, for all $s\leq p\leq 1$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$.

We take p_0 such that $\frac{r_w}{r_w-1}<p_0<\frac{n}{\alpha}$; from Lemma 4.6 we have that $p\frac{r_{w^p}}{r_{w^p}-1}\leq\frac{r_w}{r_w-1}<p_0$. Given $f\in\widehat{\mathcal{D}}_0$ we can write $f=\sum\lambda_j a_j$, where the a_j's are w^p-(p,p_0,d) atoms, the scalars λ_j satisfy $\sum_j|\lambda_j|^p\lesssim\|f\|^p_{H^p_{w^p}}$, and the series converges in $L^{p_0}(\mathbb{R}^n)$. For $\frac{1}{q_0}=\frac{1}{p_0}-\frac{\alpha}{n}$, I_α is a bounded operator from $L^{p_0}(\mathbb{R}^n)$ into $L^{q_0}(\mathbb{R}^n)$.

Since $f = \sum_j \lambda_j a_j$ in $L^{p_0}(\mathbb{R}^n)$, we have that

$$|I_\alpha f(x)| \leq \sum_j |\lambda_j||I_\alpha a_j(x)|, \quad \text{a.e. } x \in \mathbb{R}^n. \tag{4.12}$$

If $\|I_\alpha a_j\|_{L^q_{w^q}} \leq C$, with C independent of the w^p-(p, p_0, d) atom $a_j(\cdot)$, then (4.12) allows us to obtain

$$\|I_\alpha f\|_{L^q_{w^q}} \leq C\left(\sum_j |\lambda_j|^{\min\{1,q\}}\right)^{\frac{1}{\min\{1,q\}}} \leq C\left(\sum_j |\lambda_j|^p\right)^{1/p} \lesssim \|f\|_{H^p_{w^p}},$$

for all $f \in \widehat{\mathcal{D}}_0$, so the theorem follows from the density of $\widehat{\mathcal{D}}_0$ in $H^p_{w^p}(\mathbb{R}^n)$.

To conclude the proof we will prove that there exists $C > 0$ such that

$$\|I_\alpha a\|_{L^q_{w^q}} \leq C, \quad \text{for all } w^p\text{-}(p, p_0, d) \text{ atoms } a(\cdot). \tag{4.13}$$

To prove (4.13), let $2B = B(x_0, 2r)$, where $B = B(x_0, r)$ is the ball containing the support of the atom $a(\cdot)$. So

$$\int_{\mathbb{R}^n} |I_\alpha a(x)|^q w^q(x)\, dx = \int_{2B} |I_\alpha a(x)|^q w^q(x)\, dx + \int_{\mathbb{R}^n \setminus 2B} |I_\alpha a(x)|^q w^q(x)\, dx.$$

To estimate the first term in the right side of this equality, we apply Hölder's inequality with $\frac{q_0}{q}$ and use that $w^q \in RH_{\left(\frac{q_0}{q}\right)'}$ ($q_0 > q\frac{r_{w^q}}{r_{w^q}-1}$); thus,

$$\begin{aligned}
\int_{2B} |I_\alpha a(x)|^q w^q(x)\, dx &\leq \|I_\alpha a\|^q_{L^{q_0}} \left(\int_{2B} [w^q(x)]^{\left(\frac{q_0}{q}\right)'} dx\right)^{1/\left(\frac{q_0}{q}\right)'} \\
&\leq C|B|^{q/p_0}(w^p(B))^{-q/p}|2B|^{1/\left(\frac{q_0}{q}\right)'} \left(\frac{1}{|2B|}\int_{2B} w^q(x)\, dx\right) \\
&\leq C|B|^{q\alpha/n}(w^p(B))^{-q/p} w^q(B).
\end{aligned}$$

Lemma 2.6 gives

$$\int_{2B} |I_\alpha a(x)|^q w^q(x)\, dx \leq C. \tag{4.14}$$

From (4.11), taking there $d = \lfloor n(\frac{1}{p} - 1)\rfloor$, we obtain

$$|I_\alpha a(x)| \leq C(w^p(B))^{-1/p}\left[M_{\frac{\alpha n}{n+d+1}}(\chi_B)(x)\right]^{\frac{n+d+1}{n}}, \quad \text{for all } x \notin B(x_0, 2r).$$

So

$$\int_{\mathbb{R}^n \setminus 2B} |I_\alpha a(x)|^q w^q(x)\, dx \leq C(w^p(B))^{-q/p} \int_{\mathbb{R}^n} \left[M_{\frac{\alpha n}{n+d+1}}(\chi_B)(x)\right]^{q\frac{n+d+1}{n}} w^q(x)\, dx, \tag{4.15}$$

Since $d = \lfloor n(\frac{1}{p} - 1)\rfloor$, we have $q\frac{n+d+1}{n} > 1$. We write $\tilde{q} = q\frac{n+d+1}{n}$ and let $\frac{1}{\tilde{p}} = \frac{1}{\tilde{q}} + \frac{\alpha}{n+d+1}$, so $\frac{\tilde{p}}{\tilde{q}} = \frac{p}{q}$ and $w^{q/\tilde{q}} \in \mathcal{A}_{\tilde{p},\tilde{q}}$ (see Remark 2.4). From Theorem 2.5 we obtain

$$\int_{\mathbb{R}^n} \left[M_{\frac{\alpha n}{n+d+1}}(\chi_B)(x)\right]^{q\frac{n+d+1}{n}} w^q(x)\, dx \leq C\left(\int_{\mathbb{R}^n} \chi_B(x) w^p(x)\, dx\right)^{q/p} = C(w^p(B))^{q/p}.$$

This inequality and (4.15) give

$$\int_{\mathbb{R}^n \setminus 2B} |I_\alpha a(x)|^q w^q(x)\, dx \le C. \tag{4.16}$$

Finally, (4.14) and (4.16) allow us to obtain (4.13). This completes the proof. □

To finish, we recover the classical result obtained by Taibleson and Weiss in [20].

Corollary 4.11. *For $0 < \alpha < n$, let I_α be the Riesz potential defined in* (4.8)*. If $0 < p \le 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, then I_α can be extended to an $H^p(\mathbb{R}^n)$-$H^q(\mathbb{R}^n)$ bounded operator.*

Proof. If $w(x) \equiv 1$, then $r_w = +\infty$ and therefore $\frac{r_w}{r_w - 1} = 1$. Applying Theorems 4.9 and 4.10, with $w \equiv 1$, the corollary follows. □

Acknowledgment

I express my thanks to the referee for the useful suggestions and corrections which made the manuscript more readable.

References

[1] M. Bownik, Boundedness of operators on Hardy spaces via atomic decompositions, *Proc. Amer. Math. Soc.* **133** (2005), no. 12, 3535–3542. MR 2163588.

[2] R. R. Coifman, A real variable characterization of H^p, *Studia Math.* **51** (1974), 269–274. MR 0358318.

[3] D. Cruz-Uribe, K. Moen and H. V. Nguyen, The boundedness of multilinear Calderón–Zygmund operators on weighted and variable Hardy spaces, *Publ. Mat.* **63** (2019), no. 2, 679–713. MR 3980937.

[4] C. Fefferman and E. M. Stein, H^p spaces of several variables, *Acta Math.* **129** (1972), no. 3-4, 137–193. MR 0447953.

[5] J. García-Cuerva, Weighted H^p spaces, *Dissertationes Math. (Rozprawy Mat.)* **162** (1979), 63 pp. MR 0549091.

[6] L. Grafakos, *Classical Fourier analysis*, third edition, Graduate Texts in Mathematics, 249, Springer, New York, 2014. MR 3243734.

[7] S. G. Krantz, Fractional integration on Hardy spaces, *Studia Math.* **73** (1982), no. 2, 87–94. MR 0667967.

[8] R. H. Latter, A characterization of $H^p(\mathbf{R}^n)$ in terms of atoms, *Studia Math.* **62** (1978), no. 1, 93–101. MR 0482111.

[9] M.-Y. Lee and C.-C. Lin, The molecular characterization of weighted Hardy spaces, *J. Funct. Anal.* **188** (2002), no. 2, 442–460. MR 1883413.

[10] X. Li and L. Peng, The molecular characterization of weighted Hardy spaces, *Sci. China Ser. A* **44** (2001), no. 2, 201–211. MR 1824320.

[11] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.* **165** (1972), 207–226. MR 0293384.

[12] B. Muckenhoupt and R. Wheeden, Weighted norm inequalities for fractional integrals, *Trans. Amer. Math. Soc.* **192** (1974), 261–274. MR 0340523.

[13] E. Nakai and Y. Sawano, Hardy spaces with variable exponents and generalized Campanato spaces, *J. Funct. Anal.* **262** (2012), no. 9, 3665–3748. MR 2899976.

[14] P. Rocha, A note on Hardy spaces and bounded linear operators, *Georgian Math. J.* **25** (2018), no. 1, 73–76. MR 3767396.

[15] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, 30, Princeton University Press, Princeton, NJ, 1970. MR 0290095.

[16] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, 43, Princeton University Press, Princeton, NJ, 1993. MR 1232192.

[17] E. M. Stein and G. Weiss, On the theory of harmonic functions of several variables. I. The theory of H^p-spaces, *Acta Math.* **103** (1960), 25–62. MR 0121579.

[18] J.-O. Strömberg and A. Torchinsky, *Weighted Hardy spaces*, Lecture Notes in Mathematics, 1381, Springer-Verlag, Berlin, 1989. MR 1011673.

[19] J.-O. Strömberg and R. L. Wheeden, Fractional integrals on weighted H^p and L^p spaces, *Trans. Amer. Math. Soc.* **287** (1985), no. 1, 293–321. MR 0766221.

[20] M. H. Taibleson and G. Weiss, The molecular characterization of certain Hardy spaces, in *Representation theorems for Hardy spaces*, 67–149, Astérisque, 77, Soc. Math. France, Paris, 1980. MR 0604370.

[21] D. Yang and Y. Zhou, A boundedness criterion via atoms for linear operators in Hardy spaces, *Constr. Approx.* **29** (2009), no. 2, 207–218. MR 2481589.

Pablo Rocha
Instituto de Matemática (INMABB), Departamento de Matemática, Universidad Nacional del Sur (UNS)-CONICET, Bahía Blanca, Argentina
pablo.rocha@uns.edu.ar

Received: October 17, 2018
Accepted: August 6, 2019