APPROXIMATION BY α-BASKAKOV–JAIN TYPE OPERATORS

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Dedicated to professor H. M. Srivastava on the occasion of his 80^{th} birthday

Abstract. In this manuscript, we consider the Baskakov-Jain type operators involving two parameters α and τ. Some approximation results concerning the weighted approximation are discussed. Also, we find a quantitative Voronovskaja type asymptotic theorem and Grüss Voronovskaya type approximation theorem for these operators. Some numerical examples to illustrate the approximation of these operators to certain functions are also given.
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1. Introduction

Aral and Erbay [5] introduced Baskakov operators based on $\alpha \in [0,1]$ as follows:

$$\mathscr{B}_m^{(\alpha)}(u;z) = \sum_{i=0}^{\infty} b_{m,i}^{(\alpha)}(z) u\left(\frac{i}{m}\right), \quad z \in [0,\infty), \tag{1.1}$$

where

$$b_{m,i}^{(\alpha)}(z) = \frac{z^{i-1}}{(1+z)^{m+i-1}}\left[\frac{\alpha z}{(1+z)}\binom{m+i-1}{i} - (1-\alpha)(1+z)\binom{m+i-3}{i-2} + (1-\alpha)z\binom{m+i-1}{i}\right].$$

For the special case $\alpha = 1$, the operators $\mathscr{B}_m^{(\alpha)}(f;z)$ reduces to the Baskakov operators [6].

Gupta [13] presented a general family of Durrmeyer type operators and derived some direct results. Kajla and Agrawal [19] defined Durrmeyer type modification of Szász operator involving Charlier polynomials. Kajla et al. [22] considered a Durrmeyer type generalization of the operators (1.1) and gave the uniform convergence results. In 2020, Mohiuddine et al. [26] Baskakov-Durrmeyer type operators based on the parameters and studied quantitative approximation results. Very recently, Mohiuddine et al. [23] introduced Stancu-Kantorovich variant of the operators (1.1) and studied the direct results. For article related to such type of a study we refer the reader to (cf. [2–4, 7, 8, 10–12, 14–21, 24, 25, 27–30] etc.) and reference therein.

Let $\tau > 0$ and $\alpha \in [0,1]$. For $\gamma > 0$ and $C_\gamma[0,\infty) := \{u \in C[0,\infty) : |u(v)| \leq N_u e^{\gamma v}, \text{ for some } N_u > 0\}$, we construct a α-Baskakov-Jain type operators as follows:

$$\mathscr{G}_{m,\tau}^{(\alpha)}(u;z) = \sum_{i=0}^{\infty} b_{m,i}^{(\alpha)}(z) \int_0^{\infty} b_{m,i}^{\tau}(v) u(v) dv, \tag{1.2}$$

where $b_{m,i}^{\tau}(v) = \dfrac{\tau}{B\left(i+1, \frac{m}{\tau}\right)} \dfrac{(\tau v)^i}{(1+\tau v)^{\frac{m}{\tau}+i+1}}$ and $b_{m,i}^{(\alpha)}(z)$ is defined as above.

In this article, we will study the order of convergence of these operators in a weighted space and asymptotic type formula, quantitative and Grüss Voronovskaya type approximation theorem.

2. Basic Results

Lemma 1. *For $z \in [0, \infty)$, the moments of the operators $\mathscr{G}_{m,\tau}^{(\alpha)}(u; z)$ are given by*

(i) $\mathscr{G}_{m,\tau}^{(\alpha)}(e_0; z) = 1;$

(ii) $\mathscr{G}_{m,\tau}^{(\alpha)}(e_1; z) = \dfrac{z(m + 2\alpha - 2)}{(m - \tau)} + \dfrac{1}{(m - \tau)};$

(iii) $\mathscr{G}_{m,\tau}^{(\alpha)}(e_2; z) = \dfrac{mz^2(-3 + m + 4\alpha)}{(m - 2\tau)(m - \tau)} + \dfrac{z(4m + 10(-1 + \alpha))}{(m - 2\tau)(m - \tau)} + \dfrac{2}{(m - 2\tau)(m - \tau)};$

(iv)
$$\mathscr{G}_{m,\tau}^{(\alpha)}(e_3; z) = \frac{m(1 + m)z^3(-4 + m + 6\alpha)}{(m - 3\tau)(m - 2\tau)(m - \tau)} + \frac{3mz^2(-11 + 3m + 14\alpha)}{(m - 3\tau)(m - 2\tau)(m - \tau)} + \frac{18z(-3 + m + 3\alpha)}{(m - 3\tau)(m - 2\tau)(m - \tau)} + \frac{6}{(m - 3\tau)(m - 2\tau)(m - \tau)};$$

(v)
$$\begin{aligned}\mathscr{G}_{m,\tau}^{(\alpha)}(e_4; z) &= \frac{m(1 + m)(2 + m)z^4(-5 + m + 8\alpha)}{(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)} + \frac{4m(1 + m)z^3(-19 + 4m + 27\alpha)}{(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)}\\ &+ \frac{24mz^2(-13 + 3m + 16\alpha)}{(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)} + \frac{z(96m + 336(-1 + \alpha))}{(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)}\\ &+ \frac{24}{(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)},\end{aligned}$$

(vi)
$$\begin{aligned}\mathscr{G}_{m,\tau}^{(\alpha)}(e_5; z) &= \frac{m(1 + m)(2 + m)(3 + m)z^5(-6 + m + 10\alpha)}{(m - 5\tau)(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)}\\ &+ \frac{5m(1 + m)(2 + m)z^4(-29 + 5m + 44\alpha)}{(m - 5\tau)(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)} + \frac{100m(1 + m)z^3(-11 + 2m + 15\alpha)}{(m - 5\tau)(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)}\\ &+ \frac{600mz^2(-5 + m + 6\alpha)}{(m - 5\tau)(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)} + \frac{600z(-4 + m + 4\alpha)}{(m - 5\tau)(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)}\\ &+ \frac{120}{(m - 5\tau)(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)},\end{aligned}$$

(vii)
$$\begin{aligned}\mathscr{G}_{m,\tau}^{(\alpha)}(e_6; z) &= \frac{m(1 + m)(2 + m)(3 + m)(4 + m)z^6(-7 + m + 12\alpha)}{(m - 6\tau)(m - 5\tau)(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)}\\ &+ \frac{6m(1 + m)(2 + m)(3 + m)z^5(-41 + 6m + 65\alpha)}{(m - 6\tau)(m - 5\tau)(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)}\\ &+ \frac{90m(1 + m)(2 + m)z^4(-33 + 5m + 48\alpha)}{(m - 6\tau)(m - 5\tau)(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)}\\ &+ \frac{600m(1 + m)z^3(-25 + 4m + 33\alpha)}{(m - 6\tau)(m - 5\tau)(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)}\\ &+ \frac{1800mz^2(-17 + 3m + 20\alpha)}{(m - 6\tau)(m - 5\tau)(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)}\\ &+ \frac{z(4320m + 19440(-1 + \alpha))}{(m - 6\tau)(m - 5\tau)(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)}\\ &+ \frac{720}{(m - 6\tau)(m - 5\tau)(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)}.\end{aligned}$$

Lemma 2. *From Lemma 1, we have*

$$\mathscr{G}_{m,\tau}^{(\alpha)}((v - z); z) = \frac{z(\tau + 2\alpha - 2)}{(m - \tau)} + \frac{1}{(m - \tau)}; \quad \mathscr{G}_{m,\tau}^{(\alpha)}((v - z)^2; z) = \frac{z^2(m + \tau(-8 + 2\tau + m + 8\alpha))}{(m - 2\tau)(m - \tau)} + \frac{2z(-5 + 2\tau + m + 5\alpha)}{(m - 2\tau)(m - \tau)} + \frac{2}{(m - 2\tau)(m - \tau)};$$

$$\begin{aligned}\mathscr{G}_{m,\tau}^{(\alpha)}((v - z)^4; z) &= \frac{z^4\big(24\tau^4 - 10m + 3m^2 + 16m\alpha + 2\tau m(-32 + 3m + 48\alpha) + \tau^2 m(-8 + 3m + 80\alpha) + 2\tau^3(-96 + 23m + 96\alpha)\big)}{(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)}\\ &+ \frac{z^3\big(96\tau^3 - 76m + 184\tau^2 m + 12m^2 + 720\tau^2(-1 + \alpha) + 108m\alpha + 12\tau m(-9 + m + 21\alpha)\big)}{(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)}\\ &+ \frac{z^2\big(144\tau^2 - 96m + 204\tau m + 12m^2 + 864\tau(-1 + \alpha) + 168m\alpha\big)}{(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)} + \frac{z(96\tau + 72m + 336(-1 + \alpha))}{(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)} + \frac{24}{(m - 4\tau)(m - 3\tau)(m - 2\tau)(m - \tau)}.\end{aligned}$$

Remark 1. *We have*

$$\begin{aligned}\lim_{m\to\infty} m\ \mathscr{J}^{\alpha,1}_{m,\tau}(z) &= 1+z(\tau+2\alpha-2),\\ \lim_{m\to\infty} m\ \mathscr{J}^{\alpha,2}_{m,\tau}(z) &= z(2+z+\tau z),\\ \lim_{m\to\infty} m^2\ \mathscr{J}^{\alpha,4}_{m,\tau}(z) &= 3z^2(2+z+\tau z)^2,\\ \lim_{m\to\infty} m^3\ \mathscr{J}^{\alpha,6}_{m,\tau}(z) &= 15z^3(2+z+\tau z)^3,\end{aligned}$$

where $\mathscr{J}^{\alpha,s}_{m,\tau} := \mathscr{G}^{(\alpha)}_{m,\tau}((v-z)^s;z), s=1,2,4,6.$

3. Direct Results

Theorem 1. *Let* $u\in C_\gamma[0,\infty)$. *Then* $\lim_{m\to\infty}\mathscr{G}^{(\alpha)}_{m,\tau}(u;z)=u(z)$, *uniformly in each compact subset of* $[0,\infty)$.

Proof. By the application of Bohman-Korovkin Result and Lemma 1, the proof of this theorem is direct. □

3.1. Voronovskaja type theorem.

Theorem 2. *Let* $u\in C_\gamma[0,\infty)$. *If* u'' *exists at a point* $z\in[0,\infty)$, *then*

$$\lim_{m\to\infty} m\left[\mathscr{G}^{(\alpha)}_{m,\tau}(u;z)-u(z)\right]=(1+z(\tau+2\alpha-2))\,u'(z)+\frac{1}{2}z(2+z+\tau z)u''(z).$$

Proof. From Taylor's theorem, we have

$$u(v)=u(z)+u'(z)(v-z)+\frac{1}{2}u''(z)(v-z)^2+\varpi(v,z)(v-z)^2, \tag{3.1}$$

where $\lim_{v\to z}\varpi(v,z)=0$. Applying the linear operator $\mathscr{G}^{(\alpha)}_{m,\tau}$, we may write

$$\begin{aligned}\mathscr{G}^{(\alpha)}_{m,\tau}(u;z)-u(z)&=\mathscr{G}^{(\alpha)}_{m,\tau}((v-z);z)u'(z)+\frac{1}{2}\mathscr{G}^{(\alpha)}_{m,\tau}((v-z)^2;z)u''(z)\\ &\quad+\mathscr{G}^{(\alpha)}_{m,\tau}(\varpi(v,z)(v-z)^2;z).\end{aligned}$$

The Cauchy-Schwarz inequality implies

$$m\mathscr{G}^{(\alpha)}_{m,\tau}(\varpi(v,z)(v-z)^2;z)\le\sqrt{\mathscr{G}^{(\alpha)}_{m,\tau}(\varpi^2(v,z);z)}\sqrt{m^2\mathscr{G}^{(\alpha)}_{m,\tau}((v-z)^4;z)}. \tag{3.2}$$

As $\varpi^2(z,z)=0$ and $\varpi^2(\cdot,z)\in C_\gamma[0,\infty)$, we have

$$\lim_{m\to\infty}\mathscr{G}^{(\alpha)}_{m,\tau}(\varpi^2(v,z);z)=\varpi^2(z,z)=0. \tag{3.3}$$

Collecting (3.2)-(3.3) and Remark 1, we obtain

$$\lim_{m\to\infty} m\mathscr{G}^{(\alpha)}_{m,\tau}(\varpi(v,z)(v-z)^2;z)=0. \tag{3.4}$$

Hence

$$\lim_{m\to\infty} m\left[\mathscr{G}^{(\alpha)}_{m,\tau}(u;z)-u(z)\right]=(1+z(\tau+2\alpha-2))\,u'(z)+\frac{1}{2}z(2+z+\tau z)u''(z).$$

□

4. Weighted approximation

Let $H_\varrho[0,\infty)$ denote the space of all real-valued functions on $[0,\infty)$ satisfying the condition $|u(z)| \leq N_u \varrho(z)$, where $N_u > 0$ is a constant depending only on u and $\varrho(z) = 1 + z^2$ is a weight function. Suppose that $C_\varrho[0,\infty)$ be the space of all continuous functions in $H_\varrho[0,\infty)$ endowed with the norm $\|u\|_\varrho := \sup\limits_{z\in[0,\infty)} \dfrac{|u(z)|}{\varrho(z)}$ and $C^0_\varrho[0,\infty) := \left\{u \in C_\varrho[0,\infty) : \lim\limits_{z\to\infty} \dfrac{|u(z)|}{\varrho(z)} < \infty\right\}$.

Theorem 3. *For each $u \in C^0_\varrho[0,\infty)$ and $r > 0$, we have*

$$\lim_{m\to\infty} \sup_{z\in[0,\infty)} \frac{\left|\mathscr{G}^{(\alpha)}_{m,\tau}(u;z) - u(z)\right|}{(1+z^2)^{1+r}} = 0.$$

Proof. For any fixed $z_0 > 0$, there holds the relation

$$\begin{aligned}
\sup_{z\in[0,\infty)} \frac{\left|\mathscr{G}^{(\alpha)}_{m,\tau}(u;z) - u(z)\right|}{(1+z^2)^{1+r}} &\leq \sup_{z\leq z_0} \frac{\left|\mathscr{G}^{(\alpha)}_{m,\tau}(u;z) - u(z)\right|}{(1+z^2)^{1+r}} + \sup_{z>z_0} \frac{\left|\mathscr{G}^{(\alpha)}_{m,\tau}(u;z) - u(z)\right|}{(1+z^2)^{1+r}} \\
&\leq \sup_{z\leq z_0} \left\{\left|\mathscr{G}^{(\alpha)}_{m,\tau}(u;z) - u(z)\right|\right\} + \sup_{z>z_0} \frac{\left|\mathscr{G}^{(\alpha)}_{m,\tau}(u;z)\right|}{(1+z^2)^{1+r}} \\
&\quad + \sup_{z>z_0} \frac{|u(z)|}{(1+z^2)^{1+r}}.
\end{aligned}$$

Since $|u(v)| \leq \|u\|_\varrho(1+v^2), \forall v \geq 0$

$$\begin{aligned}
\sup_{z\in[0,\infty)} \frac{\left|\mathscr{G}^{(\alpha)}_{m,\tau}(u;z) - u(z)\right|}{(1+z^2)^{1+r}} &\leq \left\|\mathscr{G}^{(\alpha)}_{m,\tau}(u;z) - u(z)\right\|_{C[0,z_0]} + \|u\|_\varrho \sup_{z>z_0} \frac{\left|\mathscr{G}^{(\alpha)}_{m,\tau}(1+v^2;z)\right|}{(1+z^2)^{1+r}} \\
&\quad + \sup_{z>z_0} \frac{\|u\|_\varrho}{(1+z^2)^r} \\
&= I_1 + I_2 + I_3, say. \qquad (4.1)
\end{aligned}$$

Now, in view of Theorem 1, for a given $\epsilon > 0$, $\exists\ m_1 \in \mathbb{N}$ such that

$$I_1 = \left\|\mathscr{G}^{(\alpha)}_{m,\tau}(u;z) - u(z)\right\|_{C[0,z_0]} < \frac{\epsilon}{3}, \text{ for all } m \geq m_1. \qquad (4.2)$$

Since $\lim\limits_{m\to\infty} \sup\limits_{z>z_0} \dfrac{\mathscr{G}^{(\alpha)}_{m,\tau}(1+v^2;z)}{1+z^2} = 1$, it follows that there exists $m_2 \in \mathbb{N}$ such that

$$\sup_{z>z_0} \frac{\mathscr{G}^{(\alpha)}_{m,\tau}(1+v^2;z)}{1+z^2} \leq \frac{(1+z_0^2)^r}{\|u\|_\varrho} \cdot \frac{\epsilon}{3} + 1, \text{for all } m \geq m_2.$$

Hence,

$$\begin{aligned}
I_2 = \|u\|_\varrho \sup_{z>z_0} \frac{\left|\mathscr{G}^{(\alpha)}_{m,\tau}(1+v^2;z)\right|}{(1+z^2)^{1+r}} &\leq \frac{\|u\|_\varrho}{(1+z_0^2)^r} \sup_{z>z_0} \frac{\left|\mathscr{G}^{(\alpha)}_{m,\tau}(1+v^2;z)\right|}{1+z^2} \\
&\leq \frac{\|u\|_\varrho}{(1+z_0^2)^r} + \frac{\epsilon}{3}, \text{ for all } m \geq m_2. \qquad (4.3)
\end{aligned}$$

Choose z_0 to be so large that

$$\frac{\|u\|_\varrho}{(1+z_0^2)^r} < \frac{\epsilon}{6},$$

then

$$I_3 = \sup_{z>z_0} \frac{\|u\|_\varrho}{(1+z^2)^r} \leq \frac{\|u\|_\varrho}{(1+z_0^2)^r} < \frac{\epsilon}{6}. \qquad (4.4)$$

Let $m_0 = \max\{m_1, m_2\}$, then by combining (4.2-4.4)

$$\sup_{z\in[0,\infty)} \frac{\left|\mathscr{G}^{(\alpha)}_{m,\tau}(u;z) - u(z)\right|}{(1+z^2)^{1+r}} < \epsilon, \text{ for all } m \geq m_0.$$

□

In the following we study a quantitative Voronoskaja type result for the operators $\mathscr{G}^{(\alpha)}_{m,\tau}$ for functions u in the weighted space $C_\varrho[0,\infty)$. İspir [31], considered the weighted modulus of continuity $\Omega(u;\sigma)$ as follows:

$$\Omega(u;\sigma) = \sup_{0\leq h<\sigma, z\in[0,\infty)} \frac{|u(z+h) - u(z)|}{(1+h^2)(1+z^2)} \tag{4.5}$$

for $u \in C_\varrho[0,\infty)$. From [31], if $u \in C^0_\varrho[0,\infty)$, then $\Omega(.;\sigma)$ has the properties

$$\lim_{\sigma\to 0} \Omega(u;\sigma) = 0$$

and

$$\Omega(u;\lambda\sigma) \leq 2(1+\lambda)(1+\sigma^2)\Omega(u;\sigma), \ \lambda > 0. \tag{4.6}$$

From the equations (4.5)-(4.6) and $u \in C^0_\varrho[0,\infty)$, we can write

$$\begin{aligned} |u(v) - u(z)| &\leq \left(1+(v-z)^2\right)(1+z^2)\Omega(u;|v-z|) \\ &\leq 2\left(1+\frac{|v-z|}{\sigma}\right)(1+\sigma^2)\Omega(u;\sigma)\left(1+(v-z)^2\right)(1+z^2). \end{aligned} \tag{4.7}$$

Theorem 4. *Let $u \in C^0_\varrho[0,\infty)$ such that $u'(z), u''(z) \in C^0_\varrho[0,\infty)$. Then for sufficiently large m and each $z \in [0,\infty)$,*

$$\left| m\left\{\mathscr{G}^{(\alpha)}_{m,\tau}(u;z) - u(z) - u'(z)\mathscr{G}^{(\alpha)}_{m,\tau}\left((v-z);z\right) - \frac{u''(z)}{2!}\mathscr{G}^{(\alpha)}_{m,\tau}\left((v-z)^2;z\right)\right\}\right| = O(1)\Omega\left(u'';\sqrt{1/m}\right).$$

Proof. Applying Taylor's formula, we have

$$\begin{aligned} u(v) &= u(z) + u'(z)(v-z) + \frac{u''(\beta)}{2!}(v-z)^2 \\ &= u(z) + u'(z)(v-z) + \frac{u''(z)}{2!}(v-z)^2 + h_2(v,z), \end{aligned} \tag{4.8}$$

where β is a number between z and v, we have

$$h_2(v,z) = \frac{u''(\beta) - u''(z)}{2!}(v-z)^2. \tag{4.9}$$

Using the property (4.7) of the weighted modulus of continuity, we may write

$$\begin{aligned} |u''(\beta) - u''(z)| &\leq (1+(\beta-z)^2)(1+z^2)\Omega(u'';|\beta-z|) \\ &\leq (1+(v-z)^2)(1+z^2)\Omega(u'';|v-z|) \\ &\leq 2(1+(v-z)^2)(1+z^2)\left(1+\frac{|v-z|}{\sigma}\right)(1+\sigma^2)\Omega(u'';\sigma), \end{aligned} \tag{4.10}$$

but

$$\left(1+\frac{|v-z|}{\sigma}\right)(1+(v-z)^2) \leq \begin{cases} 2(1+\sigma^2), & |v-z| < \sigma, \\ 2\dfrac{(v-z)^4}{\sigma^4}(1+\sigma^2), & |v-z| \geq \sigma, \end{cases},$$

i.e.

$$\left(1+\frac{|v-z|}{\sigma}\right)(1+(v-z)^2) \leq 2\left(1+\frac{(v-z)^4}{\sigma^4}\right)(1+\sigma^2). \tag{4.11}$$

Collecting the equations (4.9)-(4.11) and choosing $0 < \sigma < 1$, we find that

$$|h_2(v,z)| \leq 2(1+\sigma^2)^2(1+z^2)\Omega(u'';\sigma)\left(1+\frac{(v-z)^4}{\sigma^4}\right)(v-z)^2. \tag{4.12}$$

Operating the operator $\mathscr{G}_{m,\tau}^{(\alpha)}$ and Lemma 2 on both sides of (4.8), we obtain

$$\left|\mathscr{G}_{m,\tau}^{(\alpha)}(u;z) - u(z) - u'(z)\mathscr{G}_{m,\tau}^{(\alpha)}(v-z;z) - \frac{u''(z)}{2!}\mathscr{G}_{m,\tau}^{(\alpha)}\left((v-z)^2;z\right)\right| \leq \mathscr{G}_{m,\tau}^{(\alpha)}(|h_2(v,z)|;z). \tag{4.13}$$

Applying Remark 1 and using equation (4.12), we may write

$$\begin{aligned}\mathscr{G}_{m,\tau}^{(\alpha)}(|h_2(v,z)|;z) &\leq 2(1+\sigma^2)^2(1+z^2)\Omega(u'';\sigma)\mathscr{G}_{m,\tau}^{(\alpha)}\left(\left((v-z)^2+\frac{(v-z)^6}{\sigma^4}\right);z\right)\\ &= 2(1+\sigma^2)^2(1+z^2)\Omega(u'';\sigma)\left(\mathscr{G}_{m,\tau}^{(\alpha)}\left((v-z)^2;z\right)+\frac{1}{\sigma^4}\mathscr{G}_{m,\tau}^{(\alpha)}\left((v-z)^6;z\right)\right)\\ &= 2(1+\sigma^2)^2(1+z^2)\Omega(u'';\sigma)\left(O(1/m)+\frac{1}{\sigma^4}O(1/m^3)\right).\end{aligned}$$

By taking $\sigma = \sqrt{1/m}$, we obtain

$$m\mathscr{G}_{m,\tau}^{(\alpha)}(|h_2(v,z)|;z) = O(1)\Omega\left(u'';\sqrt{1/m}\right). \tag{4.14}$$

Using (4.13) and (4.14), we find that

$$\left|m\left\{\mathscr{G}_{m,\tau}^{(\alpha)}(u;z) - u(z)u'(z)\mathscr{G}_{m,\tau}^{(\alpha)}(v-z;z) - \frac{u''(z)}{2!}\mathscr{G}_{m,\tau}^{(\alpha)}\left((v-z)^2;z\right)\right\}\right| = O(1)\Omega\left(u'';\sqrt{1/m}\right), as\ m\to\infty.$$

□

5. Grüss Voronovskaya Type Theorem

Theorem 5. *Let u, w and $uw \in C_\varrho^0[0,\infty)$ such that $u', w', (uw)', u'', w''$ and $(uw)'' \in C_\varrho^0[0,\infty)$. Then, for each $z \in [0,\infty)$,*

$$\lim_{m\to\infty} m\left\{\mathscr{G}_{m,\tau}^{(\alpha)}((uw);z) - \mathscr{G}_{m,\tau}^{(\alpha)}(u;z)\mathscr{G}_{m,\tau}^{(\alpha)}(w;z)\right\} = u'(z)w'(z)z(2+z+\tau z).$$

Proof. Since $(uw)(z) = u(z)w(z), (uw)'(z) = u'(z)w(z)+u(z)w'(z)$ and $(uw)''(z) = u''(z)w(z)+2u'(z)w'(z)+u(z)w''(z)$, we get

$$\begin{aligned}&\mathscr{G}_{m,\tau}^{(\alpha)}((uw);z) - \mathscr{G}_{m,\tau}^{(\alpha)}(u;z)\mathscr{G}_{m,\tau}^{(\alpha)}(w;z)\\ &= \left\{\mathscr{G}_{m,\tau}^{(\alpha)}((uw);z) - u(z).w(z) - (uw)'(z)\mathscr{G}_{m,\tau}^{(\alpha)}(v-z;z) - \frac{(uw)''(z)}{2!}\mathscr{G}_{m,\tau}^{(\alpha)}((v-z)^2;z)\right\}\\ &\quad -w(z)\left\{\mathscr{G}_{m,\tau}^{(\alpha)}(u;z) - u(z) - u'(z)\mathscr{G}_{m,\tau}^{(\alpha)}(v-z;z) - \frac{u''(z)}{2!}\mathscr{G}_{m,\tau}^{(\alpha)}((v-z)^2;z)\right\}\\ &\quad -\mathscr{G}_{m,\tau}^{(\alpha)}(u;z)\left\{\mathscr{G}_{m,\tau}^{(\alpha)}(w;z) - w(z) - w'(z)\mathscr{G}_{m,\tau}^{(\alpha)}(v-z;z) - \frac{w''(z)}{2!}\mathscr{G}_{m,\tau}^{(\alpha)}((v-z)^2;z)\right\}\\ &\quad +\frac{1}{2!}\mathscr{G}_{m,\tau}^{(\alpha)}((v-z)^2;z)\left\{u(z)w''(z) + 2u'(z)w'(z) - w''(z)\mathscr{G}_{m,\tau}^{(\alpha)}(u;z)\right\}\\ &\quad +\mathscr{G}_{m,\tau}^{(\alpha)}(v-z;z)\left\{u(z)w'(z) - w'(z)\mathscr{G}_{m,\tau}^{(\alpha)}(u;z)\right\}.\end{aligned}$$

From Theorems 1 and 4 and Lemma 2, we find that

$$\lim_{m\to\infty} m\left\{\mathscr{G}_{m,\tau}^{(\alpha)}((uw);z) - \mathscr{G}_{m,\tau}^{(\alpha)}(u;z)\mathscr{G}_{m,\tau}^{(\alpha)}(w;z)\right\} = u'(z)w'(z)z(2+z+\tau z).$$

This completes the proof. □

6. Numerical Examples

Example 1. *The convergence of* $\mathscr{G}_{m,\tau}^{(\alpha)}(u;z)$ *operators is illustrated in Figure 1, where* $u(z) = z^3(1+2z)^3$, $z \in [0,1]$, $\alpha = 0.9$ $\tau = 0.5$ *and* $m = 15, 16, 17, 18, 19$. *We observed that when the values of* m *are increasing, the graph of operators* $\mathscr{G}_{m,\tau}^{(\alpha)}(u;z)$ *are going to the graph of the function* u.

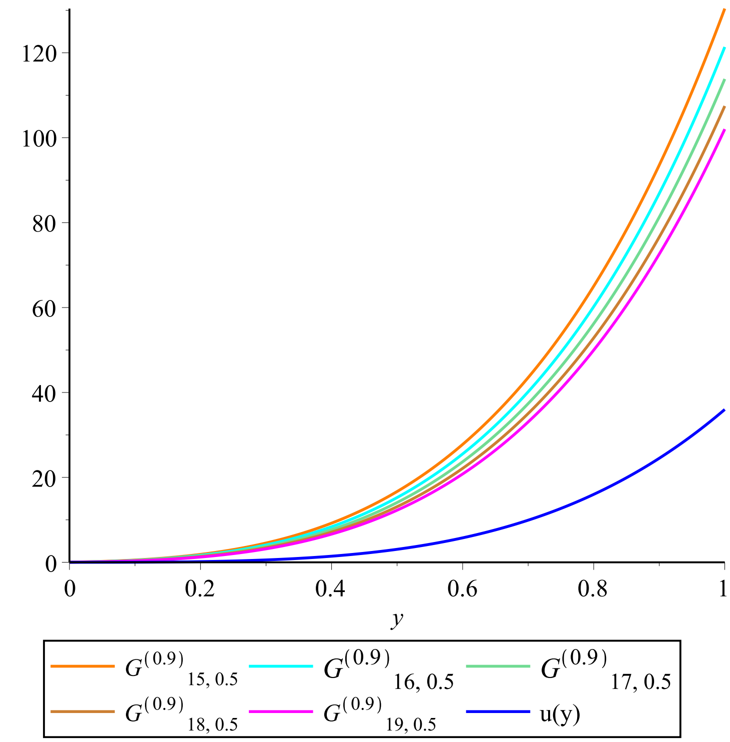

Figure 1. Approximation Process

Example 2. *The convergence of* $\mathscr{G}_{m,\tau}^{(\alpha)}(u;z)$ *operators is illustrated in Figure 2, where* $u(z) = z^2(1+5z)^2$, $z \in [0,1]$, $\alpha = 0.9$ $\tau = 0.5$ *and* $m = 15, 16, 17, 18, 19$. *It is seen that when the values of* m *are increasing, the graph of operators* $\mathscr{G}_{m,\tau}^{(\alpha)}(u;z)$ *are going to the graph of the function* u.

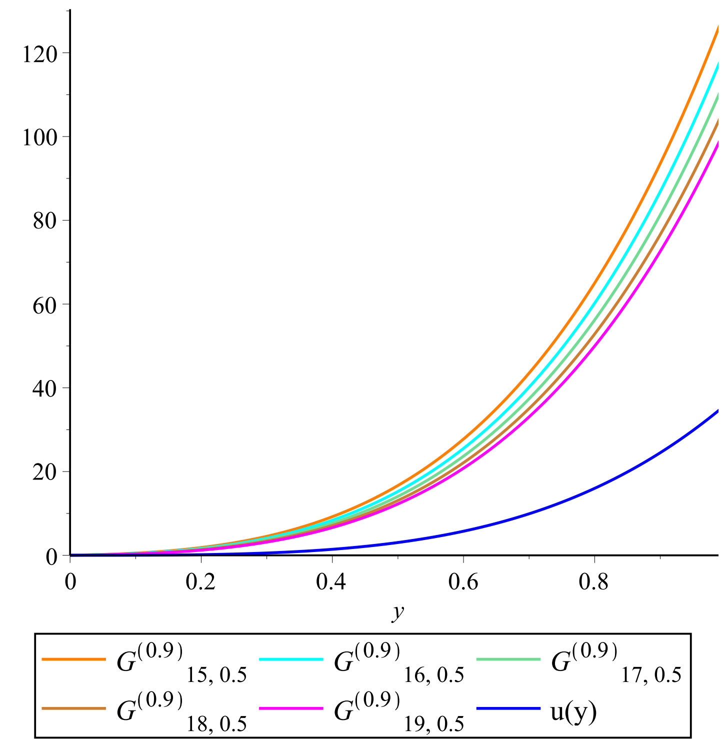

FIGURE 2. Approximation Process

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