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VISCOSITY APPROXIMATION METHOD FOR MODIFIED SPLIT GENERALIZED EQUILIBRIUM AND FIXED POINT PROBLEMS

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Abstract. We introduce a viscosity iterative algorithm for approximating a common solution of a modified split generalized equilibrium problem and a fixed point problem for a quasi-pseudocontractive mapping which also solves some variational inequality problems in real Hilbert spaces. The proposed iterative algorithm is constructed in such a way that it does not require the prior knowledge of the operator norm. Furthermore, we prove a strong convergence theorem for approximating the common solution of the aforementioned problems. Finally, we give a numerical example of our main theorem. Our result complements and extends some related works in the literature.

1. Introduction

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2 respectively. The Split Feasibility Problem (SFP), first introduced in [7] by Censor and Elfving, requires finding a point in a nonempty closed convex subset in one space such that its image under a bounded linear operator is in another nonempty closed convex subset in the image space. That is, find $x^* \in C$ such that

$$Ax^* \in Q,$$

where $A : H_1 \to H_2$ is a bounded linear operator. The SFP arises in many fields in the real world, such as signal processing, image reconstruction, and intensity-modulated radiation therapy problems. For example, see [8, 9, 23] and the references therein. Many well-known iterative algorithms have been established for the SFP; for instance, Byrne [5] proposed the CQ algorithm to study the SFP; Qu and Xiu [20] considered a modified CQ algorithm to study the SFP; and Xu [26]

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introduced a regularized algorithm for studying the SFP and proved a strong convergence result.

The introduction of the SFP to fixed point theory has also yielded some optimization problems such as the split equilibrium problem, the split variational inequality problem, the split inclusion poblem, among others.

Let $A : C \to H$ be a mapping. The Variational Inequality Problem (VIP) is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \tag{1.1}$$

for all $v \in C$. The solution set of (1.1) is denoted by VIP(C, A). The VIP has emerged as a fascinating branch of mathematical and engineering sciences, with a wide range of applications in industry, finance, economics, ecology, and pure and applied sciences; see, for instance, [11, 17, 25].

Another optimization problem which includes the VIP is the Equilibrium Problem (EP), first introduced and studied by Blum and Oettli [4]; see also [24]. Many problems in physics, optimization, and economics can be reduced to finding the solution of EP, which is defined as follows: find $x \in C$ such that

$$F(x, y) \geq 0, \tag{1.2}$$

for all $y \in C$, where $F : C \times C \to \mathbb{R}$ is a bifunction. We denote by EP (1.2) the solution set of (1.2).

Let $F : C \times C \to \mathbb{R}$ be a bifunction and $f : H \to H$ a mapping. The Generalized Equilibrium Problem (GEP) is to find $x \in C$ such that

$$F(x, y) + \langle f(x), y - x\rangle \geq 0, \tag{1.3}$$

for all $y \in C$. We denote by EP(F, f) the solution set of (1.3).

Remark 1.1. If $F \equiv 0$, the GEP (1.3) reduces to VIP (1.1), and when $f \equiv 0$, the GEP (1.3) reduces to EP (1.2).

In 2013, Kazmi and Rizvi [18] introduced and studied the following Split Equilibrium Problem (SEP), which is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \forall\, x \in C, \tag{1.4}$$

and such that

$$y^* = Ax^* \in Q \quad \text{solves} \quad F_2(y^*, y) \geq 0, \quad \forall\, y \in Q, \tag{1.5}$$

where $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are nonlinear bifunctions.

The SEP (1.4)-(1.5) reduces to EP (1.2), if $H_1 \equiv H_2$, $F_1 \equiv F_2$, $A \equiv I$, and $C = Q$.

The Split Variational Inequality Problem (SVIP) was introduced and studied by Censor et al. [10], who defined the problem as follows: find $x^* \in C$ such that

$$\langle f_1(x^*), x - x^*\rangle \geq 0, \quad \forall\, x \in C, \tag{1.6}$$

and such that

$$y^* = Ax^* \in Q \quad \text{solves} \quad \langle f_2(y^*), y - y^*\rangle \geq 0, \quad \forall\, y \in Q, \tag{1.7}$$

where $f_1 : C \to H_1$ and $f_2 : Q \to H_2$ are nonlinear mappings. The SVIP has already been used in practice as a model in intensity-modulated radiation therapy (IMRT) treatment planning and the modelling of many inverse problems arising from phase retrieval and other real-world problems such as data compression or sensor networks in computerized tomography; see for example [14].

Very recently, Cheawchan and Kangtunyakarn [13] introduced the Modified Split Generalized Equilibrium Problem (MSGEP), which is to find $x^* \in C$ such that

$$F_1(x^*, x) + \langle f_1(x^*), x - x^* \rangle \geq 0, \quad \forall\, x \in C, \tag{1.8}$$

and such that

$$y^* = Ax^* \in Q \quad \text{solves} \quad F_2(y^*, y) + \langle f_2(y^*), y - y^* \rangle \geq 0, \quad \forall\, y \in Q, \tag{1.9}$$

where $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are nonlinear bifunctions and $f_1 : C \to H_1$ and $f_2 : Q \to H_2$ are nonlinear mappings. We denote by $\Omega = \{x^* \in EP(F_1, f_1) : Ax^* \in EP(F_2, f_2)\}$ the solution set of MSGEP (1.8)–(1.9).

The MSGEP generalizes the SEP (1.4)-(1.5) and the SVIP (1.6)-(1.7) in the following ways:

(i) if $f_1 \equiv f_2 \equiv 0$ in MSGEP (1.8)-(1.9), then MSGEP (1.8)-(1.9) reduces to SEP (1.4)-(1.5);
(ii) if $F_1 \equiv F_2 \equiv 0$ in MSGEP (1.8)- (1.9), then MSGEP (1.8)-(1.9) reduces to SVIP (1.6)-(1.7).

For solving EP, we assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x, x) = 0$, for all $x \in C$;
(A2) F is monotone, i.e. $F(x, y) + F(y, x) \leq 0$, $\forall\, x, y \in C$;
(A3) for each $x, y, z \in C$, $\limsup_{t \to 0} F(tz + (1-t)x, y) \leq F(x, y)$;
(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall\, y \in C.$$

Let C be a nonempty closed convex subset of a real Hilbert space H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \forall\, y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\|P_C(x) - P_C(y)\| \leq \langle x - y, P_C(x) - P_C(y) \rangle.$$

Moreover, $P_C(x)$ is characterized by the following properties:

$$\langle x - P_C(x), y - P_C(x) \rangle \leq 0$$

and

$$\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2, \quad \forall\, x \in H,\ y \in C.$$

For all $x, y \in H$, it is well known that every nonexpansive operator $T : H \to H$ satisfies the inequality

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x) \rangle \leq \frac{1}{2} \|(T(x) - x) - (T(y) - y)\|^2,$$

and therefore, we have that for all $x \in H$ and $y \in F(T)$.

$$\langle x - T(x), y - T(x) \rangle \leq \frac{1}{2} \|T(x) - x\|^2.$$

We now give some definitions that will be needed later.

Definition 1.2. Let C be a nonempty closed convex subset of a real Hilbert space H. A point $p \in C$ is called a fixed point of a mapping T if $Tp = p$. We denote by $F(T)$ the set of all fixed points of T.

Definition 1.3. Let C be a nonempty closed convex subset of a real Hilbert space H. We say that a nonlinear mapping $T : C \to C$ is

(i) a contraction, if there exists a constant $\phi \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \phi \|x - y\|, \quad \forall\, x, y \in C;$$

(ii) nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall\, x, y \in C;$$

(iii) firmly nonexpansive, if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \quad \forall\, x, y \in C;$$

(iv) firmly quasi-nonexpansive, if $F(T) \neq \emptyset$ and

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 - \|(I - T)x\|^2, \quad \forall\, x \in C \text{ and } x^* \in F(T);$$

(v) strictly pseudo-contractive if there exists $k \in (0, 1]$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k \|(I - T)x - (I - T)y\|^2, \quad \forall\, x, y \in C;$$

(vi) demicontractive, if $F(T) \neq \emptyset$ and there exists $k \in [0, 1)$ such that

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + k \|Tx - x\|^2, \quad \forall\, x \in C \text{ and } x^* \in F(T).$$

From the definitions stated above, we notice that the class of demicontractive mappings includes many nonlinear mappings, such as quasi-nonexpansive and strictly pseudo-contractive with nonempty fixed points sets, as special cases.

In 2015, Chang et al. [12] introduced a new type of nonlinear mapping called quasi-pseudo-contractive mapping, as follows:

Definition 1.4. An operator $T : C \to C$ is said to be quasi-pseudo-contractive if $F(T) \neq \emptyset$ and

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + \|Tx - x\|^2, \quad \forall\, x \in C \text{ and } x^* \in F(T).$$

It is obvious that this class of mappings contains the class of demicontractive mappings, see [12].

Definition 1.5. A bounded linear operator D on H is called strongly positive if there exists a constant $\beta > 0$ such that

$$\langle Dx, x\rangle \geq \beta \left\|x\right\|^2, \quad \forall\, x \in C.$$

Definition 1.6. Let H be a real Hilbert space and C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is said to be demiclosed at 0 if for any bounded sequence $\{x_n\} \subset C$ such that $\{x_n\}$ converges weakly to x and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, we have that $Tx = x$.

The viscosity iterative algorithm is one of the algorithms that have been used extensively by authors to approximate solutions of fixed point problems and optimization problems. The algorithm is constructed in such a way that it also solves some variational inequality problem (see [6, 21, 28] and the references therein). In 2017, Deepho et al. [16] considered the viscosity iterative algorithm to approximate a common element of the set of solutions of a split variational inclusion problem of a finite family of k-strictly pseudo-contractive nonself mappings. They proved a strong convergence result under suitable conditions, which also solves some variational inequality problem. The following iteration process was used to approximate the aforementioned problems:

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2})Ax_n), \\ y_n = \beta_n u_n + (1-\beta_n)\displaystyle\sum_{i=1}^{N} \eta_{i=1}^n T_i u_n, \\ x_{n+1} = \alpha_n \tau g(x_n) + (I - \alpha_n D)y_n, \quad n \geq 1, \end{cases}$$

where $\alpha_n, \beta_n \in (0,1)$, $\lambda > 0$, g is a contraction mapping with coefficient $\rho \in (0,1)$, $\sum_{i=1}^N \eta_{i=1}^n = 1$, $\{T_i\}_{i=1}^N$ is a finite family of k_i-strictly pseudo-contraction mappings, and $J_\lambda^{B_i}(i = 1,2)$ is the resolvent of the maximal monotone mappings.

In 2018, Abass et al. [1] introduced an iterative algorithm that does not require the prior knowledge of the operator norm to approximate the common solution of SEP and fixed point problem for an infinite family of quasi-nonexpansive multivalued mappings. Using their iterative algorithm, they prove a strong convergence result.

Very recently, Cheawchan and Kangtunyakarn [13] introduced a new iterative algorithm for finding a common element of the set of solutions of variational inequality problems and the set of solutions of MSGEP without assuming the demiclosedness condition. They proved the following theorem.

Theorem 1.7 ([13]). *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D_1, D_2 : C \to H_1$ be α-, β-inverse strongly monotone mappings respectively. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying (A1)–(A4). Let $\{T_i\}_{i=1}^\infty$ be a finite family of quasi-nonexpansive mappings of C into itself with $\cap_{i=1}^N F(T_i) \neq \emptyset$. Let $f_1 : H_1 \to H_1$ be a ρ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be a firmly nonexpansive mapping. Assume that*

$\Pi := VI(C,D_1)\cap VI(C,D_2)\cap\cap_{i=1}^N F(T_i)\cap\Omega\neq\emptyset$. For $x_1,u\in C$, let $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be sequences generated by

$$\begin{cases} u_n = T_r^{F_1}(I-rf_1)(x_n+\gamma A^*(T_s^{F_2}(I-sf_2)-I)Ax_n), \\ y_n = P_C(I-d_1D_1)(au_n+(I-a)P_C(I-d_2D_2)u_n), \\ x_{n+1} = \alpha_n u+\beta_n x_n+\gamma_n P_C\Big(I-\lambda_n\Big(\displaystyle\sum_{i=1}^N k_i(I-T_i)\Big)\Big)y_n, \quad \forall\, n\in\mathbb{N}, \end{cases}$$

where $d_1\in(0,2\alpha)$, $d_2\in(0,2\beta)$, $r\in(0,2\rho)$, $s\in(0,1)$, $a\in[0,1]$, $0<k_i<1$, with $\sum_{i=1}^N k_i=1$, $\gamma\in(0,\frac{1}{L})$, L is the spectral radius of the operator A^*A, and A^* is the adjoint of A. Also, $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n+\beta_n+\gamma_n=1$ for all $n\in\mathbb{N}$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^\infty\alpha_n=\infty$;
(ii) $0<c\le\beta_n,\gamma_n\le d<1$ *for some* $c,d>0$ *for all* $n\ge1$;
(iii) $\sum_{i=1}^\infty\lambda_n<\infty$ *and* $0<\lambda_n<1$;
(iv) $\sum_{n=1}^\infty|\alpha_{n+1}-\alpha_n|<\infty$, $\sum_{n=1}^\infty|\beta_{n+1}-\beta_n|<\infty$.

Then $\{x_n\}$ converges strongly to $z=P_\Pi u$.

Motivated by the works of Abass et al. [1], Deepho et al. [16], Cheawchan and Kangtunyakarn [13], we propose a viscosity iterative algorithm that does not require any knowledge of the spectral radii to approximate a common solution of MSGEP and fixed point problem for a quasi-pseudo-contractive mapping, which is also a solution of some variational inequality problem. We prove a strong convergence of the iterative scheme to a solution of the aforementioned problems in the framework of real Hilbert spaces. Furthermore, we give a numerical example of our main result.

2. Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In what follows, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively.

Lemma 2.1. *Let H be a real Hilbert space. Then for all $x,y\in H$ and $\alpha\in(0,1)$ we have*

$$\|\alpha x+(1-\alpha)y\|^2=\alpha\|x\|^2+(1-\alpha)\|y\|^2-\alpha(1-\alpha)\|x-y\|^2;$$
$$\|x+y\|^2\le\|x\|^2+2\langle y,x+y\rangle;$$
$$2\langle x,y\rangle=\|x\|^2+\|y\|^2-\|x-y\|^2=\|x+y\|^2-\|x\|^2-\|y\|^2.$$

Lemma 2.2 ([13]). *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2 respectively. Let $F_1:C\times C\to\mathbb{R}$ and $F_2:Q\times Q\to\mathbb{R}$ be bifunctions satisfying (A1)–(A4). Let $f_1:H_1\to H_1$ be a ρ-inverse strongly monotone mapping and $f_2:H_2\to H_2$ be a firmly nonexpansive mapping. Then*

$T_r^{F_1}(I - rf_1)$ and $T_s^{F_2}(I - sf_2)$ are nonexpansive mappings, where $r \in (0, 2\rho)$, $s \in (0,1)$, and $T_r^{F_1} : H_1 : C$ is defined by

$$T_r^{F_1}(x) = \{z \in C : F_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall\, y \in C\},$$

for all $x \in H_1$ and $T_s^{F_2} : H_2 \to Q$ is defined by

$$T_s^{F_2}(\overline{x}) = \{\overline{z} \in Q : F_2(\overline{z},y) + \frac{1}{s}\langle y - \overline{z}, \overline{z} - \overline{x} \geq 0,\ \forall\, y \in Q\},$$

for all $\overline{x} \in H_2$.

Recall that a Banach space X is said to satisfy Opial's condition if for any sequence $\{x_n\}$ in X which converges weakly to x^*,

$$\limsup_{n\to\infty} \|x_n - x^*\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall\, y \in X \text{ with } y \neq x^*.$$

Lemma 2.3 ([15])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \to C$ as follows:*

$$T_r^F x = \big\{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall\, y \in C\big\}, \tag{2.1}$$

for all $x \in H$. Then the following hold:

(i) T_r^F is nonempty and single-valued;

(ii) T_r^F is firmly nonexpansive, that is, $\forall\, x, y \in H$,

$$\left\|T_r^F x - T_r^F y\right\|^2 \leq \langle T_r^F x - T_r^F y, x - y\rangle;$$

(iii) $F(T_r^F) = EP(F)$;

(iv) $EP(F)$ is closed and convex.

Lemma 2.4 ([19])**.** *Assume that D is a strongly positive bounded linear operator on a Hilbert space H with a coefficient $\overline{\tau} > 0$ and $0 < \mu < \|D\|^{-1}$. Then $\|I - \mu D\| \leq 1 - \mu\overline{\tau}$.*

Lemma 2.5 ([12])**.** *Let H be a real Hilbert space and $T : H \to H$ be an L-Lipschitzian mapping with $L \geq 1$. Denote $K := (1-\xi)I + \xi T\big((1-\eta)I + \eta T\big)$ if $0 < \xi < \eta < \frac{1}{1+\sqrt{1+L^2}}$. Then the following conclusions hold:*

(1) $F(T) = F(T((1-\eta)I + \eta T)) = F(K)$;

(2) if T is demiclosed at 0, then K is also demiclosed at 0;

(3) in addition, if $T : H \to H$ is quasi-pseudocontractive, then the mapping K is quasi-nonexpansive, that is,

$$\|Kx - u^*\| \leq \|x - u^*\|, \quad \forall\, x \in H \text{ and } u^* \in F(T) = F(K).$$

Lemma 2.6 ([19])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that $f : C \to C$ is a contraction with coefficient $\mu \in (0,1)$ and D is a strongly positive linear bounded operator with a coefficient $\overline{\sigma} > 0$. Then, for $0 < \sigma < \frac{\overline{\sigma}}{\mu}$,*

$$\langle x - y, (D - \sigma f)x - (D - \sigma f)y\rangle \geq (\overline{\sigma} - \sigma\mu)\, \|x - y\|^2, \quad x, y \in H.$$

That is, $D - \sigma f$ is strongly monotone with coefficient $\overline{\sigma} - \sigma\mu$.

Lemma 2.7 ([26]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\sigma_n)a_n + \sigma_n\delta_n, \quad n > 0,$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a real sequence satisfying

(i) $\sum_{n=1}^{\infty} \sigma_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main result

Lemma 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4) and $f : H \to H$ be any nonlinear mapping. Then, for $0 < r_1 \leq r_2$, we have that*

$$\left\|T_{r_1}^F(I - r_1 f)x - T_{r_2}^F(I - r_2 f)x\right\| \leq \left\|x - T_{r_2}^F(I - r_2 f)x\right\|, \quad \forall\, x \in H.$$

Proof. Let $z = T_{r_1}^F(I - r_1 f)x$ and $w = T_{r_2}^F(I - r_2 f)x$. Then, from (2.1) we obtain

$$F(z,w) + \frac{1}{r_1}\langle w - z, z - (I - r_1 f)x\rangle \geq 0.$$

Similarly, we obtain

$$F(w,z) + \frac{1}{r_2}\langle z - w, w - (I - r_2 f)x\rangle \geq 0.$$

Adding the last two inequalities and using condition (A2) we obtain

$$\langle x - z - r_1 f x, z - w\rangle - \frac{r_1}{r_2}\langle x - w - r_2 f x, z - w\rangle \geq 0.$$

That is,

$$\langle (x - r_1 f x - z) - \Big(\frac{r_1}{r_2}x - r_1 f x - \frac{r_1}{r_2}w\Big), z - w\rangle \geq 0,$$

which implies that

$$\langle x - z, z - w\rangle \geq \frac{r_1}{r_2}\langle x - w, z - w\rangle.$$

Thus, from Lemma 2.1 we obtain

$$\|x-w\|^2 - \|x-z\|^2 - \|x-z\|^2 - \|z-w\|^2 \geq \frac{r_1}{r_2}\big(\|x-w\|^2 + \|w-z\|^2 - \|x-z\|^2\big).$$

Since $\frac{r_1}{r_2} \leq 1$, we get

$$\left(1 + \frac{r_1}{r_2}\right)\|z-w\|^2 \leq \left(1 - \frac{r_1}{r_2}\right)\|x-w\|^2,$$

which implies that

$$\|z-w\|^2 \leq \left(\frac{r_2 - r_1}{r_2 + r_1}\right)\|x-w\|^2 \leq \|x-w\|^2. \qquad \square$$

Lemma 3.2. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $T : H_1 \to H_2$ be an L-Lipschitzian and quasi-pseudocontractive mapping with $L \geq 1$. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying (A1)–(A4) and let D be a strongly positive bounded linear operator on H_1 with coefficient $\overline{\tau} > 0$. Let $f_1 : H_1 \to H_1$ be a ρ_1-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be a ρ_2-inverse strongly monotone mapping. Assume that $\Gamma := F(T) \cap \Omega \neq \emptyset$ and g is a contraction mapping with coefficient $\mu \in (0,1)$. Let the sequences $\{u_n\}$, $\{y_n\}$, and $\{x_n\}$ be generated for arbitrary $x_1 \in H$ by*

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - r_n f_1)(x_n + \gamma_n A^*(T_{s_n}^{F_2}(I - s_n f_2) - I)Ax_n), \\ y_n = \alpha_n u_n + (1-\alpha_n)((1-\xi_n)I + \xi_n T(1-\eta_n)I + \eta_n T)u_n, \\ x_{n+1} = \beta_n \tau g(x_n) + (I - \beta_n D)y_n, \quad n \geq 1, \end{cases} \tag{3.1}$$

where $K := ((1-\xi_n)I + \xi_n T(1-\eta_n)I + \eta_n T)$, $0 < r \leq r_n < 2\rho_1$, $0 < s \leq s_n < 2\rho_2$, $\alpha_n \in (0,1)$, and the step size γ_n is chosen in such a way that for some $\epsilon > 0$,

$$\gamma_n \in \left(\epsilon, \frac{\left\|(T_{s_n}^{F_2}(I - s_n f_2) - I)Ax_n\right\|^2}{\left\|A^*(T_{s_n}^{F_2}(I - s_n f_2) - I)Ax_n\right\|^2} - \epsilon \right), \tag{3.2}$$

for all $T_{s_n}^{F_2}(I - s_n f_2)Ax_n \neq Ax_n$, and $\gamma_n = \gamma$ otherwise (γ being any nonnegative real number), with the sequence $\{\beta_n\}$ satisfying the following conditions:

(i) $\beta_n \in (0,1)$, $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
(ii) $0 < \tau < \frac{\overline{\tau}}{\mu}$;
(iii) $0 < a < \xi_n < \eta_n < b < \frac{1}{1+\sqrt{1+L^2}}$, $\forall\, n \geq 1$.

Then, the sequence $\{x_n\}$ generated by (3.1) *is bounded.*

Proof. Let $p \in \Gamma$. We have $p = T_{r_n}^{F_1}(I - r_n f_1)p$ and $Ap = T_{s_n}^{F_2}(I - s_n f_2)Ap$. From Lemma 2.2, we have that $(T_{r_n}^{F_1}(I - r f_1))$ and $(T_{s_n}^{F_2}(I - s_n f_2))$ are nonexpansive mappings. Using (3.1) and (3.2), we obtain

$$\begin{aligned} \|u_n - p\|^2 &= \left\|T_{r_n}^{F_1}(I - r_n f_1)(x_n + \gamma_n A^*(T_{s_n}^{F_2}(I - s_n f_2)Ax_n) - T_{r_n}^{F_1}(I - r_n f_1)p\right\|^2 \\ &\leq \left\|x_n + \gamma_n A^*(T_{s_n}^{F_2}(I - s_n f_2) - I)Ax_n) - p\right\|^2 \\ &\leq \|x_n - p\|^2 + \gamma_n^2 \left\|A^*(T_{s_n}^{F_2}(I - s_n f_2) - I)Ax_n\right\|^2 \\ &\quad + 2\gamma_n \langle x_n - p, A^*(T_{s_n}^{F_2}(I - s_n f_2) - I)Ax_n \rangle, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
2\gamma_n\langle x_n-p, A^*(T^{F_2}_{s_n}(I-s_nf_2)-I)Ax_n\rangle \\
&= 2\gamma_n\langle A(x_n-p), (T^{F_2}_{s_n}(I-s_nf_2)-I)Ax_n\rangle \\
&= 2\gamma_n\langle A(x_n-p)+(T^{F_2}_{s_n}(I-s_nf_2)-I)Ax_n \\
&\quad -(T^{F_2}_{s_n}(I-s_nf_2)-I)Ax_n, (T^{F_2}_{s_n}(I-s_nf_2)-I)Ax_n\rangle \\
&= 2\gamma_n\big\{\langle T^{F_2}_{s_n}(I-s_nf_2)Ax_n-Ap, (T^{F_2}_{s_n}(I-sf_2)-I)Ax_n\rangle \\
&\quad -\left\|(T^{F_2}_{s_n}(I-s_nf_2)-I)Ax_n\right\|^2\big\} \\
&\leq 2\gamma_n\big\{\tfrac{1}{2}\left\|T^{F_2}_{s_n}(I-s_nf_2)-I)Ax_n\right\|^2-\left\|(T^{F_2}_{s_n}(I-s_nf_2)-I)Ax_n\right\|^2\big\} \\
&\leq -\gamma_n\left\|(T^{F_2}_{s_n}(I-s_nf_2)-I)Ax_n\right\|^2.
\end{aligned} \tag{3.4}$$

On substituting (3.4) into (3.3), we have

$$\begin{aligned}
\|u_n-p\|^2 &\leq \|x_n-p\|^2+\gamma_n^2\left\|A^*(T^{F_2}_{s_n}(I-s_nf_2)-I)Ax_n\right\|^2 \\
&\quad -\gamma_n\left\|(T^{F_2}_{s_n}(I-s_nf_2)-I)Ax_n\right\|^2 \\
&= \|x_n-p\|^2-\gamma_n\big[\left\|(T^{F_2}_{s_n}(I-s_nf_2)-I)Ax_n\right\|^2 \\
&\quad -\gamma_n\left\|(T^{F_2}_{s_n}(I-s_nf_2)-I)Ax_n\right\|^2\big].
\end{aligned} \tag{3.5}$$

Since $\gamma_n \in \left(\epsilon, \frac{\left\|(T^{F_2}_{s_n}(I-s_nf_2)-I)Ax_n\right\|^2}{\left\|A^*(T^{F_2}_{s_n}(I-s_nf_2)-I)Ax_n\right\|^2}-\epsilon\right)$, we obtain

$$\|u_n-p\|^2 \leq \|x_n-p\|^2. \tag{3.6}$$

Using (3.1) and (3.6), we have

$$\begin{aligned}
\|y_n-p\|^2 &= \|\alpha_nu_n+(1-\alpha_n)(1-\xi_n)I+\xi_nT(1-\eta_n)I+\eta_nT(u_n-p)\|^2 \\
&= \|\alpha_n(u_n-p)+(1-\alpha_n)(Ku_n-p)\|^2 \\
&\leq \alpha_n\|u_n-p\|^2+(1-\alpha_n)\|Ku_n-p\|^2-\alpha(1-\alpha_n)\|Ku_n-u_n\|^2 \\
&\leq \alpha_n\|u_n-p\|^2+(1-\alpha_n)\|u_n-p\|^2 \\
&\leq \|u_n-p\|^2 \\
&\leq \|x_n-p\|^2.
\end{aligned} \tag{3.7}$$

Furthermore, using Lemma 2.4, (3.1), and (3.7), we obtain

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\beta_n[\tau g(x_n)-Dp]+(1-\beta_nD)(y_n-p)\| \\
&\leq (1-\beta_n\overline{\tau})\|y_n-p\|+\beta_n\|\tau g(x_n)-Dp\| \\
&\leq (1-\beta_n\overline{\tau})\|y_n-p\|+\beta_n[\|\tau g(x_n)-\tau g(p)\|+\|\tau g(p)-Dp\|] \\
&\leq [1-(\overline{\tau}-\tau\mu)\beta_n]\|x_n-p\|+\beta_n\|\tau g(p)-Dp\|.
\end{aligned}$$

It follows from induction that

$$\|x_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|\tau g(p) - Dp\|}{\overline{\tau} - \tau\mu}\right\}, \quad n \geq 1.$$

Hence $\{x_n\}$ is bounded. Consequently, we deduce that $\{u_n\}$ and $\{y_n\}$ are all bounded. ☐

Theorem 3.3. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $T : H_1 \to H_2$ be an L-Lipschitzian and quasi-pseudocontractive mapping with $L \geq 1$. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying (A1)–(A4) and let D be a strongly positive bounded linear operator on H_1 with coefficient $\overline{\tau} > 0$. Let $f_1 : H_1 \to H_1$ be a ρ_1-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be a ρ_2-inverse strongly monotone mapping. Assume that $\Gamma := F(T) \cap \Omega \neq \emptyset$ and that g is a contraction mapping with coefficient $\mu \in (0,1)$. Let $K := ((1-\xi_n)I + \xi_n T((1-\eta_n)I + \eta_n T)$, $0 < r \leq r_n < 2\rho_1$, $0 < s \leq s_n < 2\rho_2$, $\alpha_n \in (0,1)$, and the step size γ_n is chosen in such a way that, for some $\epsilon > 0$,*

$$\gamma_n \in \left(\epsilon, \frac{\left\|(T_{s_n}^{F_2}(I - s_n f_2) - I)Ax_n\right\|^2}{\left\|A^*(T_{s_n}^{F_2}(I - s_n f_2) - I)Ax_n\right\|^2} - \epsilon\right),$$

for all $T_{s_n}^{F_2}(I - s_n f_2)Ax_n \neq Ax_n$ and $\gamma_n = \gamma$ otherwise (γ being any nonnegative real number), with the sequences $\{\beta_n\}$, $\{\xi_n\}$, $\{\eta_n\}$ satisfying the following conditions:

(i) $\beta_n \in (0,1)$, $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
(ii) $0 < a < \xi_n < \eta_n < b < \frac{1}{1+\sqrt{1+L^2}}$, $\forall\, n \geq 1$;
(iii) $0 < \tau < \frac{\overline{\tau}}{\mu}$;
(iv) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.

Then, the sequence $\{x_n\}$ generated by (3.1) converges strongly to $x^* \in \Gamma$ which solves the variational inequality

$$\langle (D - \tau g)x^*, x^* - x\rangle \leq 0, \ \forall\, x \in \Gamma,$$

where $x = P_\Gamma(I + \tau g - D)x$.

Proof. Let $p \in \Gamma$. Then, applying Lemma 2.4 and (3.5), we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\beta_n[\tau g(x_n)-Dp]+(I-\beta_n D)(y_n-p)\|^2 \\
&\le (1-\beta_n\overline{\tau})^2\|y_n-p\|^2+\beta_n^2\|\tau g(x_n)-Dp\|^2 \\
&\quad +2\beta_n(1-\beta_n\overline{\tau})\|\tau g(x_n)-Dp\|\,\|y_n-p\| \\
&\le (1-\beta_n\overline{\tau})^2\|u_n-p\|^2+\beta_n^2\|\tau g(x_n)-Dp\|^2 \\
&\quad +2\beta_n(1-\beta_n\overline{\tau})\|\tau g(x_n)-Dp\|\,\|y_n-p\| \\
&\le (1-\beta_n\overline{\tau})^2\big[\,\|x_n-p\|^2+\gamma_n^2\left\|A^*(T_{s_n}^{F_2}(I-s_nf_2)-I)Ax_n\right\|^2 \\
&\quad -\gamma_n\left\|(T_{s_n}^{F_2}(I-s_nf_2)-I)Ax_n\right\|^2\,\big]+\beta_n^2\|\tau g(x_n)-Dp\| \\
&\quad +2\beta_n(1-\beta_n\overline{\tau})\|\tau g(x_n)-Dp\|\,\|y_n-p\| \\
&\le (1-\beta_n\overline{\tau})^2\|x_n-p\|^2-\gamma_n\big[\,\left\|(T_{s_n}^{F_2}(I-s_nf_2)-I)Ax_n\right\|^2 \\
&\quad -\gamma_n\left\|A^*(T_{s_n}^{F_2}(I-s_nf_2)-I)Ax_n\right\|^2\,\big] \\
&\quad +\beta_n^2\|\tau g(x_n)-Dp\|^2+2\beta_n(1-\beta_n\overline{\tau})\|\tau g(x_n)-Dp\|\,\|y_n-p\|.
\end{aligned} \tag{3.8}$$

It follows from (3.8) and the condition $\gamma_n \in \left(\epsilon, \frac{\left\|(T_{s_n}^{F_2}(I-s_nf_2)-I)Ax_n\right\|^2}{\left\|A^*(T_{s_n}^{F_2}(I-s_nf_2)-I)Ax_n\right\|^2}-\epsilon\right)$ that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\beta_n\overline{\tau})^2\|x_n-p\|^2-\epsilon\left\|A^*(T_{s_n}^{F_2}(I-s_nf_2)-I)Ax_n\right\|^2 \\
&\quad +\beta_n^2\|\tau g(x_n)-Dp\|^2+2\beta_n(1-\beta_n\overline{\tau})\|\tau g(x_n)-Dp\|\,\|y_n-p\|.
\end{aligned} \tag{3.9}$$

We now divide our proof into two cases.

Case 1: Assume that $\{\|x_n-p\|\}$ is a monotonically nonincreasing sequence. Then $\{x_n\}$ is convergent and clearly

$$\lim_{n\to\infty}\|x_n-p\|=\lim_{n\to\infty}\|x_{n+1}-p\|.$$

Thus from (3.9) we have that

$$\begin{aligned}
\epsilon\left\|A^*(T_{s_n}^{F_2}(I-s_nf_2)-I)Ax_n\right\|^2 &\le (1-\beta_n\overline{\tau})^2\|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\quad +\beta_n^2\|\tau g(x_n)-Dp\|^2+2\beta_n(1-\beta_n\overline{\tau})\|\tau g(x_n)-Dp\|\,\|y_n-p\|.
\end{aligned}$$

Hence, from condition (i) of Theorem 3.3, we obtain

$$\lim_{n\to\infty}\left\|A^*(T_{s_n}^{F_2}(I-s_nf_2)Ax_n\right\|=0. \tag{3.10}$$

Furthermore, from (3.8) and (3.10) we have that

$$\begin{aligned}
\gamma_n\left\|(T_{s_n}^{F_2}(I-s_nf_2)-I)Ax_n\right\|^2 &\le (1-\beta_n\overline{\tau})^2\|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\quad +\gamma_n^2\left\|A^*(T_{s_n}^{F-2}(I-s_nf_2)-I)Ax_n\right\|^2+\beta_n^2\|\tau g(x_n)-Dp\|^2 \\
&\quad +2\beta_n(1-\beta_n\overline{\tau})\|\tau g(x_n)-Dp\|\,\|y_n-p\|,
\end{aligned}$$

which implies from condition (i) of Theorem 3.3 and (3.2) that

$$\lim_{n\to\infty} \left\|(T^{F_2}_{s_n}(I - s_n f_2) - I)Ax_n\right\| = 0. \tag{3.11}$$

Let $w_n = x_n + \gamma_n A^*(T^{F_2}_{s_n}(I - s_n f_2) - I)Ax_n$. Applying inequality (3.6), we have

$$\|w_n - p\| \le \|x_n - p\|. \tag{3.12}$$

Using the property of inverse strongly monotone operator and (3.12), we have

$$\begin{aligned}
\|u_n - p\|^2 &= \left\|T^{F_1}_{r_n}(I - r_n f_1)w_n - T^{F_1}_{r_n}(I - r_n f_1)p\right\|^2 \\
&\le \|(I - r_n f_1)w_n - (I - r_n f_1)p\|^2 \\
&= \|w_n - p\|^2 - 2r_n\langle w_n - p, f_1 w_n - f_1 p\rangle + r_n^2 \|f_1 w_n - f_1 p\|^2 \\
&\le \|x_n - p\|^2 - r_n(2\rho_1 - r_n)\|f_1 w_n - f_1 p\|^2.
\end{aligned} \tag{3.13}$$

From Theorem 3.3, we have that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\beta_n[\tau g(x_n) - Dp] + (I - \beta_n D)(y_n - p)\|^2 \\
&\le (1 - \beta_n\overline{\tau})^2 \|y_n - p\|^2 + \beta_n^2 \|\tau g(x_n) - Dp\|^2 \\
&\quad + 2\beta_n(1 - \beta_n\overline{\tau})\|\tau g(x_n) - Dp\|\|y_n - p\| \\
&\le (1 - \beta_n\overline{\tau})^2 \|u_n - p\|^2 + \beta_n^2 \|\tau g(x_n) - Dp\|^2 \\
&\quad + 2\beta_n(1 - \beta_n\overline{\tau})\|\tau g(x_n) - Dp\|\|y_n - p\|.
\end{aligned} \tag{3.14}$$

Substituting (3.13) into (3.14), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1 - \beta_n\overline{\tau})^2\left[\|x_n - p\|^2 - r_n(2\rho_1 - r_n)\|f_1 w_n - f_1 p\|^2\right] \\
&\quad + \beta_n^2 \|\tau g(x_n) - Dp\|^2 \\
&\quad + 2\beta_n(1 - \beta_n\overline{\tau})\|\tau g(x_n) - Dp\|\|y_n - p\| \\
&\le (1 - \beta_n\overline{\tau})^2 \|x_n - p\|^2 - r_n(2\rho_1 - r_n)\|f_1 w_n - f_1 p\|^2 \\
&\quad + \beta_n^2 \|\tau g(x_n) - Dp\|^2 \\
&\quad + 2\beta_n(1 - \beta_n\overline{\tau})\|\tau g(x_n) - Dp\|\|y_n - p\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
r_n(2\rho_1 - r_n)\|f_1 w_n - f_1 p\|^2 &\le (1 - \beta_n\overline{\tau})^2 \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + \beta_n^2 \|\tau g(x_n) - Dp\|^2 \\
&\quad + 2\beta_n(1 - \beta_n\overline{\tau})\|\tau g(x_n) - Dp\|\|y_n - p\|.
\end{aligned}$$

Therefore, from condition (i) of Lemma 3.2, we obtain

$$\lim_{n\to\infty} \|f_1 w_n - f_1 p\| = 0. \tag{3.15}$$

By the firm nonexpansivity of $T_{r_n}^{F_1}$, we have

$$\begin{aligned}\|u_n - p\|^2 &= \left\|T_{r_n}^{F_1}(I - r_n f_1)w_n - T_{r_n}^{F_1}(I - r_n f_1)p\right\|^2\\ &\leq \langle u_n - p, (I - r_n f_1)w_n - (I - r_n f_1)p\rangle\\ &= \frac{1}{2}(\|u_n - p\|^2 + \|(I - r_n f_1)w_n - (I - r_n f_1)p\|^2\\ &\quad - \|(u_n - p) - (I - r_n f_1)w_n + (I - r_n f_1)p\|^2).\end{aligned}$$

That is,

$$\begin{aligned}\|u_n - p\|^2 &\leq \|(I - r_n f_1)w_n - (I - r_n f_1)p\|^2 - \|(u_n - w_n) + r_n(f_1 w_n - f_1 p)\|^2\\ &\leq \|w_n - p\|^2\\ &\quad - \left(\|u_n - w_n\|^2 + 2r_n\langle u_n - w_n, f_1 w_n - f_1 p\rangle + r_n^2\|f_1 w_n - f_1 p\|^2\right)\\ &\leq \|w_n - p\|^2\\ &\quad - \|u_n - w_n\|^2 + 2r_n\|u_n - w_n\|\|F_1 w_n - f_1 p\| - r_n^2\|f_1 w_n - f_1 p\|^2.\end{aligned} \tag{3.16}$$

From (3.8), (3.12) and (3.16), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1 - \beta_n\overline{\tau})^2\|u_n - p\|^2 + \beta_n^2\|\tau g(x_n) - Dp\|^2\\ &\quad + 2\beta_n(1 - \beta_n\overline{\tau})\|\tau g(x_n) - Dp\|\|y_n - p\|\\ &\leq (1 - \beta_n\overline{\tau})^2[\|w_n - p\|^2 - \|u_n - w_n\|^2 + 2r_n\|u_n - w_n\|\|f_1 w_n - f_1 p\|\\ &\quad - r_n^2\|f_1 w_n - f_1 p\|^2] + \beta_n^2\|\tau g(x_n) - Dp\|^2\\ &\quad + 2\beta_n(1 - \beta_n\overline{\tau})\|\tau g(x_n) - Dp\|\|y_n - p\|\\ &\leq (1 - 2\beta_n\overline{\tau} + (\beta_n\overline{\tau}))\|x_n - p\|^2 - (1 - \beta_n\overline{\tau})^2\|u_n - w_n\|^2\\ &\quad + 2r_n(1 - \beta_n\overline{\tau})^2\|u_n - w_n\|\|f_1 w_n - f_1 p\|\\ &\quad - (1 - \beta_n\overline{\tau})^2 r_n^2\|f_1 w_n - f_1 p\|^2\\ &\quad + \beta_n^2\|\tau g(x_n) - Dp\|^2 + 2\beta_n(1 - \beta_n\overline{\tau})\|\tau g(x_n) - Dp\|\|y_n - p\|\\ &\leq \|x_n - p\|^2 + (\beta_n\overline{\tau})^2\|x_n - p\|^2 - (1 - \beta_n\overline{\tau})^2\|u_n - w_n\|^2\\ &\quad + 2r_n(1 - \beta_n\overline{\tau})^2\|u_n - w_n\|\|f_1 w_n - f_1 p\|\\ &\quad - (1 - \beta_n\overline{\tau})^2 r_n^2\|f_1 w_n - f_1 p\|^2 + \beta_n^2\|\tau g(x_n) - Dp\|^2\\ &\quad + 2\beta_n(1 - \beta_n\overline{\tau})\|\tau g(x_n) - Dp\|\|y_n - p\|,\end{aligned}$$

which yields

$$\begin{aligned}(1 - \beta_n\overline{\tau})^2\|u_n - w_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (\beta_n\overline{\tau})^2\|x_n - p\|^2\\ &\quad + 2r_n(1 - \beta_n\overline{\tau})^2\|u_n - w_n\|\|f_1 w_n - f_1 p\| - (1 - \beta_n\overline{\tau})^2 r_n^2\|f_1 w_n - f_1 p\|^2\\ &\quad + \beta_n^2\|\tau g(x_n) - Dp\|^2 + 2\beta_n(1 - \beta\overline{\tau})\|\tau g(x_n) - Dp\|\|y_n - p\|.\end{aligned}$$

From condition (i) of Lemma 3.2 and (3.15), we obtain

$$\lim_{n\to\infty} \|u_n - w_n\| = 0. \tag{3.17}$$

Since $w_n = x_n + \gamma_n A^*(T_{s_n}^{F_2}(I - s_n f_2) - I)Ax_n$, we have that

$$\begin{aligned}\|w_n - x_n\| &= \left\|x_n + \gamma_n A^*(T_{s_n}^{F_2}(I - s_n f_2) - I)Ax_n - x_n\right\| \\ &\le \gamma_n \left\|A^*(T_{s_n}^{F_2}(I - s_n f_2) - I)Ax_n\right\|,\end{aligned}$$

which implies from (3.10) that

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{3.18}$$

From (3.1), (3.17), and (3.18), we obtain

$$\|u_n - x_n\| \le \|u_n - w_n\| + \|w_n - x_n\| \to 0, \quad n\to\infty. \tag{3.19}$$

Again,

$$\begin{aligned}\|y_n - p\|^2 &= \|\alpha_n(u_n - p) + (1-\alpha_n)(Ku_n - p)\|^2 \\ &\le \alpha_n \|u_n - p\|^2 + (1-\alpha_n)\|Ku_n - p\|^2 - \alpha(1-\alpha_n)\|Ku_n - u_n\|^2 \\ &\le \alpha_n \|u_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 - \alpha(1-\alpha_n)\|Ku_n - u_n\|^2 \\ &= \|u_n - p\|^2 - \alpha_n(1-\alpha_n)\|Ku_n - u_n\|^2. \end{aligned} \tag{3.20}$$

From (3.1), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 \le{}& (1-\beta_n\overline{\tau})^2\|y_n - p\|^2 + \beta_n\|\tau g(x_n) - Dp\|^2 \\ &+ 2\beta_n(1-\beta_n\overline{\tau})\|\tau g(x_n) - Dp\|\|y_n - p\|.\end{aligned} \tag{3.21}$$

On substituting (3.20) into (3.21), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 \le{}& (1-\beta_n\overline{\tau})^2\big[\,\|u_n - p\|^2 - \alpha_n(1-\alpha_n)\|Ku_n - u_n\|^2\,\big] \\ &+ \beta_n\|\tau g(x_n) - Dp\|^2 + 2\beta_n(1-\beta_n\overline{\tau})\|\tau g(x_n) - Dp\|\|y_n - p\| \\ \le{}& (1-\beta_n\overline{\tau})^2\|x_n - p\|^2 - \alpha_n(1-\alpha_n)(1-\beta_n\overline{\tau})\|Ku_n - u_n\|^2 \\ &+ \beta_n\left\|\tau g(x_n) - Dp\right\|^2 + 2\beta_n(1-\beta_n\overline{\tau})\left\|\tau g(x_n) - Dp\right\|\|y_n - p\|,\end{aligned}$$

which yields

$$\begin{aligned}\alpha_n(1-\alpha_n)(1-\beta_n\overline{\tau})^2\|Ku_n - u_n\|^2 \le{}& (1-\beta_n\overline{\tau})^2\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &+ \beta_n\|\tau g(x_n) - Dp\|^2 + 2\beta_n(1-\beta_n\overline{\tau})\|\tau g(x_n) - Dp\|\|y_n - p\|.\end{aligned} \tag{3.22}$$

Thus, from condition (i) of Lemma 3.2, we obtain

$$\lim_{n\to\infty} \|Ku_n - u_n\| = 0. \tag{3.23}$$

Also, from (3.1) we have

$$\begin{aligned}\|y_n - u_n\| &= \|\alpha_n u_n + (1-\alpha_n)Ku_n - u_n\| \\ &\le (1-\alpha_n)\|Ku_n - u_n\|,\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|y_n - u_n\| = 0. \tag{3.24}$$

Again, from (3.19) and (3.24) we obtain

$$\|y_n - x_n\| \leq \|y_n - u_n\| + \|u_n - x_n\| \to 0, \quad n \to \infty. \tag{3.25}$$

From (3.1) we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \|x_{n+1} - y_n\| + \|y_n - x_n\| \\
&= \|\beta_n \tau g(x_n) + (I - \beta_n D)y_n - y_n\| + \|y_n - x_n\| \\
&\leq \beta_n \|\tau g(x_n) - Dy_n\| + \|y_n - x_n\|.
\end{aligned}$$

From condition (i) of Lemma 3.2 and (3.25) we have that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Since $\{x_n\}$ is bounded, there exists a weakly convergent subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup x^*$. Since every Hilbert space has the Opial property, we have $x_n \rightharpoonup x^*$. On the other hand, from (3.19) we have that $u_n \rightharpoonup x^*$. Using (3.23) and the demiclosedness property of K, we have that $Kx^* = x^*$. Hence $x^* \in F(T)$. Next, we show that $x^* \in \Omega$. Assume that $x^* \notin EP(F_1, f_1)$; since $EP(F_1, f_1) = F(T_r^{F_1}(I - rf_1))$, we obtain $x^* \neq T_r^{F_1}(I - rf_1)x^*$. Using Opial's condition and (3.17), Lemma 2.3, and Lemma 3.1 we obtain

$$\begin{aligned}
\liminf_{j\to\infty} \left\|w_{n_j} - x^*\right\| &< \liminf_{j\to\infty} \left\|w_{n_j} - T_r^{F_1}(I - rf_1)x^*\right\| \\
&\leq \liminf_{j\to\infty} \Big(\left\|w_{n_j} - T_{r_n}^{F_1}(I - r_n f_1)w_{n_j}\right\| \\
&\qquad + \left\|T_{r_n}^{F_1}(I - r_n f_1)w_{n_j} - T_r^{F_1}(I - rf_1)w_{n_j}\right\| \\
&\qquad + \left\|T_r^{F_1}(I - rf_1)w_{n_j} - T_r^{F_1}(I - rf_1)x^*\right\| \Big) \\
&\leq \liminf_{j\to\infty} \Big(\left\|w_{n_j} - u_{n_j}\right\| + \left\|w_{n_j} - x^*\right\| \Big) \\
&\leq \liminf_{j\to\infty} \left\|w_{n_j} - x^*\right\|.
\end{aligned}$$

This is a contradiction, therefore $x^* \in EP(F_1, f_1)$.

Next, we show that $Ax^* \in EP(F_2, f_2)$. Since A is a bounded linear operator, $Ax_{n_j} \rightharpoonup Ax^*$ as $j \to \infty$. Assume that $Ax^* \notin EP(F_2, f_2)$ and since $EP(F_2, f_2) = F(T_{s_n}^{F_2}(I - s_n f_2))$, we obtain $Ax^* \neq T_{s_n}^{F_2}(I - s_n f_2)Ax^*$. Using Opial's condition and (3.11), we obtain, by a similar argument to that given above, $Ax^* \in EP(F_2, f_2)$. Hence, we conclude that $x^* \in \Omega$. Therefore, $x^* \in \Gamma$.

We now show that $\limsup_{j\to\infty} \langle (D - \tau g)x, x - x_n \rangle \leq 0$, where $x = P_\Gamma(I + \tau g - D)x$. Indeed, the subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges weakly to x^*. We obtain, by the property of metric projection P_Γ,

$$\begin{aligned}\limsup_{n\to\infty}\langle (D-\tau g)x, x-x_n\rangle &= \lim_{j\to\infty}\langle (D-\tau g)x, x-x_{n_j}\rangle \\ &= \langle (D-\tau g)x, x-x^*\rangle \\ &= \langle (I+\tau g-D)x-x, x^*-x\rangle \\ &\leq 0.\end{aligned}$$

Also, we show the uniqueness of a solution of the variational inequality

$$\langle (D-\tau g)x, x-x^*\rangle \leq 0, \quad x^*\in\Gamma. \tag{3.26}$$

Suppose that $x\in\Gamma$ and $x^*\in\Gamma$ are both solutions of (3.26); then

$$\langle (D-\tau g)x, x-x^*\rangle \leq 0$$

and

$$\langle (D-\tau g)x^*, x^*-x\rangle \leq 0.$$

Adding the last two inequalities we have

$$\langle (D-\tau g)x-(D-\tau g)x^*, x-x^*\rangle \leq 0.$$

Since $D-\tau g$ is strongly monotone by Lemma 2.6, we have that $x=x^*$. Hence the uniqueness is proved.

Lastly, we prove that $x_n\to x^*$ as $n\to\infty$. From (3.1) and (3.7) we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \langle \beta_n\tau g(x_n)+(I-\beta_n D)y_n-x^*, x_{n+1}-x^*\rangle \\ &= \beta_n\langle \tau g(x_n)-Dx^*, x_{n+1}-x^*\rangle + \langle (I-\beta_n D)(y_n-x^*), x_{n+1}-x^*\rangle \\ &\leq \beta_n\tau\langle g(x_n)-g(x^*), x_{n+1}-x^*\rangle + \beta_n\langle \tau g(x^*)-Dx^*, x_{n+1}-x^*\rangle \\ &\quad + (1-\beta_n\overline{\tau})\,\|y_n-x^*\|\,\|x_{n+1}-x^*\| \\ &\leq \beta_n\tau\mu\,\|x_n-x^*\|\,\|x_{n+1}-x^*\| + \beta_n\langle \tau g(x^*)-Dx^*, x_{n+1}-x^*\rangle \\ &\quad + (1-\beta_n\overline{\tau})\,\|x_n-x^*\|\,\|x_{n+1}-x^*\| \\ &= [1-(\overline{\tau}-\tau\mu)\beta_n]\,\|x_n-x^*\|\,\|x_{n+1}-x^*\| \\ &\quad + \beta_n\langle \tau g(x^*)-Dx^*, x_{n+1}-x^*\rangle \\ &\leq \frac{1-(\overline{\tau}-\tau\mu)\beta_n}{2}\big(\,\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2\,\big) \\ &\quad + \beta_n\langle \tau g(x^*)-Dx^*, x_{n+1}-x^*\rangle \\ &\leq \frac{1-(\overline{\tau}-\tau\mu)\beta_n}{2}\,\|x_n-x^*\|^2+\frac{1}{2}\,\|x_{n+1}-x^*\|^2 \\ &\quad + \beta_n\langle \tau g(x^*)-Dx^*, x_{n+1}-x^*\rangle.\end{aligned}$$

Then, it follows that

$$\|x_{n+1}-x^*\|^2 \leq [1-(\overline{\tau}-\tau\mu)\beta_n]\,\|x_n-x^*\|^2+\beta_n(\overline{\tau}-\tau\mu)\frac{2\langle \tau g(x^*)-Dx^*, x_{n+1}-x^*\rangle}{(\overline{\tau}-\tau\mu)}.$$

By conditions (i) and (ii) of Lemma 3.2, we obtain $\lim_{n\to\infty}\|x_n-x^*\|=0$ using Lemma 2.7.

Case 2: Assume that $\{\|x_n - p\|\}$ is not a monotonically increasing sequence. Set $\Theta_n = \|x_n - p\|^2$ and let $\sigma : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) defined by

$$\sigma(n) := \max\{k \in \mathbb{N} : k \leq n,\ \Theta_k \leq n,\ \Theta_k \leq \Theta_{k+1}\}.$$

Clearly, σ is a non-decreasing sequence such that $\sigma(n) \to 0$ as $n \to \infty$ and $\Theta_{\sigma(n)} \leq \Theta_{\sigma(n)+1}$, for $n \geq n_0$. It follows from (3.22) that

$$\begin{aligned}\alpha_{\sigma(n)}(1-\alpha_{\sigma(n)})(1-\beta_{\sigma(n)}\overline{\tau})^2 &\left\|Ku_{\sigma(n)} - u_{\sigma(n)}\right\|^2 \\ &\leq (1-\beta_{\tau(n)}\overline{\tau})^2 \left\|x_{\sigma(n)} - p\right\|^2 - \left\|x_{\sigma(n)+1} - p\right\|^2 \\ &\quad + \beta_{\sigma(n)} \left\|\tau g(x_{\sigma(n)}) - Dp\right\|^2 \\ &\quad + 2\beta_{\sigma(n)}(1-\beta_{\sigma(n)}\overline{\tau}) \left\|\tau g(x_{\sigma(n)}) - Dp\right\| \left\|y_{\sigma(n)} - p\right\|.\end{aligned}$$

Therefore, since $\lim_{n\to\infty} \beta_{\sigma(n)} = 0$, we obtain

$$\lim_{n\to\infty} \left\|Ku_{\sigma(n)} - u_{\sigma}\right\| = 0.$$

Following the same argument as in Case 1, we conclude that $\{x_\sigma\}$, $\{y_\sigma\}$, and $\{u_\sigma\}$ converge weakly to $p \in F(K) \cap \Omega$. Now, for all $n \geq n_0$, we have

$$\begin{aligned}0 &\leq \left\|x_{\sigma(n)+1} - x^*\right\|^2 - \left\|x_{\sigma(n)} - x^*\right\|^2 \\ &\leq (1-\beta_{\sigma(n)}\overline{\tau}) \left\|x_{\sigma(n)} - x^*\right\|^2 + \beta_{\sigma(n)}^2 \left\|\tau g(x_\sigma) - Dx^*\right\|^2 \\ &\quad + 2\beta_{\sigma(n)}(1-\beta_{\sigma(n)})\overline{\sigma} \left\|\tau g(x_{\sigma(n)}) - Dx^*\right\| \left\|x_{\sigma(n)} - x^* \right\| - \left\| x_{\sigma(n)} - x^*\right\|^2 \\ &= -\beta_{\sigma(n)}\overline{\sigma} \left\|x_{\tau(n)} - x^*\right\|^2 + \beta_{\sigma(n)}^2 \left\|\tau g(x_{\sigma(n)}) - Dx^*\right\|^2 \\ &\quad + 2\beta_{\sigma(n)}(1-\beta_{\sigma(n)}\overline{\sigma}) \left\langle \tau g(x_{\tau(n)}) - Dx^*, x_{\sigma(n)+1} - x^*\right\rangle.\end{aligned}$$

Thus,

$$\begin{aligned}\left\|x_{\sigma(n)} - x^*\right\|^2 &\leq \frac{\beta_{\sigma(n)}}{\overline{\sigma}} \left\|\tau g(x_{\sigma(n)}) - Dx^*\right\|^2 \\ &\quad + \frac{2(1-\beta_{\sigma(n)}\overline{\sigma})}{\overline{\sigma}} \left\langle \tau g(x_{\sigma(n)}) - Dx^*, x_{\sigma(n)+1} - x^*\right\rangle.\end{aligned}$$

Since $\beta_{\sigma(n)} \to 0$ as $n \to \infty$ and $\limsup_{n\to\infty} \langle \tau g(x_{\sigma(n)}) - Dx^*, x_{\sigma(n)+1} - x^*\rangle \leq 0$, we conclude that $\{x_\sigma\}$ converges to x^*. □

Corollary 3.4. Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $T : H_1 \to H_2$ be a nonexpansive mapping. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying (A1)–(A4), and let D be a strongly positive bounded linear operator on H_1 with coefficient $\overline{\tau} > 0$. Let $f_1 : H_1 \to H_1$ be a ρ_1-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be a ρ_2-inverse strongly monotone mapping. Assume that $\Gamma := F(T) \cap \Omega \neq \emptyset$ and that g is a contraction mapping with coefficient $\mu \in (0,1)$. Let the sequences $\{u_n\}$, $\{y_n\}$, and $\{x_n\}$ be generated for arbitrary $x_1 \in H$ by

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - r_n f_1)(x_n + \gamma_n A^*(T_{s_n}^{F_2}(I - s_n f_2) - I)Ax_n), \\ y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ x_{n+1} = \beta_n \tau g(x_n) + (I - \beta_n D)y_n, \ n \geq 1, \end{cases}$$

where $0 < r \leq r_n < 2\rho_1$, $0 < s \leq s_n < 2\rho_2$, $\alpha_n \in (0,1)$, and the step size γ_n is chosen in such a way that for some $\epsilon > 0$,

$$\gamma_n \in \left(\epsilon, \frac{\left\|(T_{s_n}^{F_2}(I - s_n f_2) - I)Ax_n\right\|^2}{\left\|A^*(T_{s_n}^{F_2}(I - s_n f_2) - I)Ax_n\right\|^2} - \epsilon \right),$$

for all $T_{s_n}^{F_2}(I - s_n f_2)Ax_n \neq Ax_n$, and $\gamma_n = \gamma$ otherwise (γ being any nonnegative real number), with the sequence $\{\beta_n\}, \{\xi_n\}, \{\eta_n\}$ satisfying the following conditions:

(i) $\beta_n \in (0,1)$, $\lim_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \beta_n = \infty$;
(ii) $0 < \tau < \frac{\overline{\tau}}{\mu}$;
(iii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.

Then, the sequence $\{x_n\}$ generated by (3.1) converges strongly to $x^* \in \Gamma$ which solves the variational inequality

$$\langle (D - \tau g)x^*, x^* - x\rangle \leq 0, \quad \forall\, x \in \Gamma.$$

4. A numerical example

We consider a numerical example of our algorithm in $\mathbb{R}^2$.

Let $H_1 = H_2 = \mathbb{R}^2$. Then for $z = (z_1, z_2)$, $y = (y_1, y_2)$, (u_1, u_2), (v_1, v_2), and (x_1, x_2), define $F_i(z, y) = -3iz^2 + 2iyz + 4iy^2$, $i = 1, 2$. Then, Lemma 2.3 ensures that we can find $x \in \mathbb{R}^2$ such that

$$\begin{aligned} & F_i(z,y) + \frac{1}{r_n}\langle y - z, z - x\rangle \geq 0, \quad \forall\, y \in \mathbb{R}^2,\ i = 1,2 \\ \iff & -3iz^2 + 2iyz + 4iy^2 + \frac{1}{r_n}(yz - xy - z^2 + xz) \geq 0 \\ \iff & -3ir_n z^2 + 2ir_n yz + 4ir_n y^2 + yz - xy - z^2 + zx. \end{aligned}$$

Let $P(y) = 4ir_n y^2 + (2ir_n z + z - x)y - 3ir_n z^2 - z^2 + zx$. Then P is a quadratic equation in y. Thus, we obtain the determinant as follows:

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (2ir_n z + z - x)^2 - 4ir_n(-3ir_n z^2 - z^2 + zx) \\ &= 16i^2 r_n^2 z^2 + 8ir_n z^2 - 8ir_n xz - 2xz + z^2 + x^2 \\ &= x^2 - 2(4ir_n z + z)x + (4ir_n z + z)^2 \\ &= (x - (4ir_n + 1)z)^2 \\ &\geq 0. \end{aligned}$$

Since $P(y) \geq 0$ for all $y \in \mathbb{R}$, it has at most one solution in $\mathbb{R}$, and then $\Delta \leq 0$. So we have that $z = \frac{x}{4ir_n+1}$. Hence,

$$T_{r_n}^{F_1}(x) = \left(\frac{x_1}{4ir_n+1}, \frac{x_2}{4ir_n+1}\right), \quad i = 1, 2. \tag{4.1}$$

Let $f_i = \frac{x}{2i}$, $i = 1, 2$. Then f_i is ρ_i-inverse strongly monotone with $\rho_i = 2i$, $i = 1, 2$. Let $A(x) = (4x_1 + 3x_2, 3x_1 + 2x_2)$ and $D(x) = 2x$ and $g(x) = \frac{x}{3}$. Then, $\overline{\tau} = 2$ and $\mu = \frac{1}{3}$. Thus, we can take $\tau = 2$ and condition (iii) of Theorem 3.3 is satisfied. From (4.1) we get that $T_{r_n}^{F_1}(x) = \left(\frac{x_1}{4r_n+1}, \frac{x_2}{4r_n+1}\right)$ and $T_{s_n}^{F_2}(x) = \left(\frac{x_1}{8r_n+1}, \frac{x_2}{8r_n+1}\right)$. Thus, we take $r_n = \frac{3n}{2n+1}$ and $s_n = \frac{7n-1}{3n+4}$ for all $n \geq 1$.

Now we define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(x_1, x_2) = -\left(\frac{2\alpha+1}{2}\right)(x_1, x_2), \quad \forall\, \alpha > \frac{1}{2}.$$

Then T is an L-Lipschitzian and quasi-pseudocontractive mapping, with $L = \left(\frac{2\alpha+1}{2}\right)^2 > 1$, $\forall\, \alpha > \frac{1}{2}$. We now take $\beta_n = \frac{1}{n+2}$, $\alpha_n = \frac{n}{2n+3}$, $\eta_n = \frac{1}{2+(\frac{2\alpha+1}{2})^4}$, and $\xi_n = \frac{8}{(2\alpha+1)^4}$ $\forall\, n \geq 1$, $\alpha > \frac{1}{2}$. Then conditions (i) and (ii) of Lemma 3.2 are satisfied. Hence, for $x_1 \in \mathbb{R}^2$, Algorithm 3.1 becomes

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - r_n f_1)(x_n + \gamma_n A^*(T_{s_n}^{F_2}(I - s_n f_2) - I)Ax_n), \\ y_n = \dfrac{n}{2n+3} u_n + \dfrac{n+3}{2n+3}((1-\xi_n)I + \xi_n T((1-\eta_n))I + \eta_n T))u_n, \\ x_{n+1} = \dfrac{2}{3(n+2)} x_n + \left(1 - \dfrac{2}{n+1}\right) y_n. \end{cases}$$

These three cases are displayed in Figure 1 on the next page:

Case 1: $x_0 = (0.1, 0.5)^T$ and $\alpha = 2$.
Case 2: $x_0 = (0.1, 0.5)^T$ and $\alpha = 1$.
Case 3: $x_0 = (1, -2)^T$ and $\alpha = 2$.

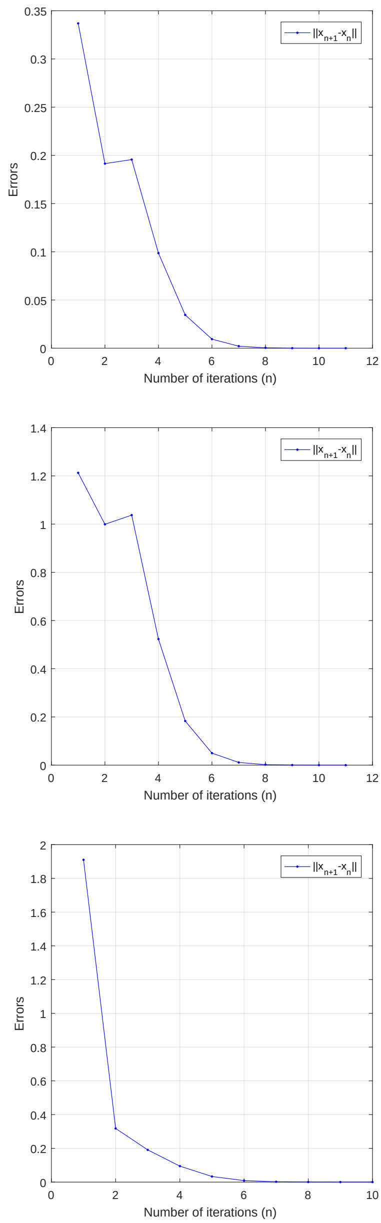

FIGURE 1. Errors vs Number of iterations (n): Case 1 (top); Case 2 (middle); Case 3 (bottom).

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