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MULTIDIMENSIONAL COMMON FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN DISLOCATED METRIC SPACES

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Abstract. Motivated by the F-contraction introduced by Wardowski [Fixed Point Theory Appl. **2012**, 2012:94], we introduce three types of multidimensional Ciric-type rational F-contractions for multivalued mappings in dislocated metric spaces. Using these contractions, we establish fixed points of N-order for multivalued mappings. Our result generalizes the main result obtained by Rasham et al. [J. Fixed Point Theory Appl. **20** (2018), no. 1, Paper no. 45].

1. Introduction and preliminaries

The fundamental theorem in metric fixed point theory is the well-known Banach contraction principle [8], introduced by the Polish mathematician S. Banach in 1922. Several authors extended and generalized this theorem by weakening the contractive condition and by generalizing the metric space in various ways. In 2012 Wardowski [27] introduced a new type of contraction mapping known as F-contraction, and using it he proved a fixed point theorem that generalizes the Banach contraction principle. After that, Secelean [24] replaced the second condition of F-contraction by an equivalent but simpler condition and proved a couple of fixed point theorems. As an application he investigated an iterative function system composed of F-contractions and extended some results in the literature. Then Abbas et al. [1] introduced common fixed point theorems for F-contractions and proved period point results for F-contractions as an application of their results. Afterwards, a generalized multivalued F-contraction mapping was introduced by Acar et al. [2], who established a few fixed point results. The study on multivalued F-contraction was further explored in [3, 25]. More results on F-contraction and related fixed point theorems are found in [4, 12, 14, 17, 28].

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Dislocated metric spaces are a generalization of partial metric spaces [15]. They were introduced by Hitzler and Seda in 2000 [11]. In 2012 Amini-Harandi [7] reintroduced dislocated metric spaces by a different name, as metric-like spaces, and proved fixed point theorems using various contractions in complete metric-like spaces. In 2015, Karapinar et al. [14] introduced conditionally F-contraction mappings and proved fixed point theorems in complete metric-like spaces. In the same year, Alsulami et al. [6] introduced modified F-contractive mappings in complete metric-like spaces and proved existence and uniqueness of fixed points of such mappings. Very recently, Rasham et al. [20] established common fixed point theorems for new Ciric-type rational multivalued F-contractions in dislocated metric spaces and solved Volterra-type integral equations as an application of these results.

The idea of a fixed point for the map $T : X^N \to X$ was introduced by Presic in 1965 [18, 19]. Later, in 2010, Samet and Vetro [22] introduced a nice definition of fixed points of N-order, which are a generalized version of the fixed points introduced by Presic. After that, many authors redefined this concept with several names and established several fixed point results [5, 9, 13, 16, 21]. Generally this type of fixed point theorems are known as multidimensional fixed point theorems. Multidimensional fixed point theorems for multivalued mappings are introduced by Deshpande and Handa [10]. Sawangsup et al. [23] introduced the concept of $F_{\mathcal{R}}$-contraction and proved several fixed point theorems, which are generalizations of the results obtained by Wardowski. They also proved corresponding multidimensional fixed point theorems using $F_{\mathcal{R}^{\mathcal{N}}}$-contractions.

In this paper we introduce three types of multidimensional Ciric-type rational F-contractions. Using these contractions we prove results on mutivalued fixed points of N-order in dislocated metric spaces.

Now we recall some definitions and results that will be necessary for our proofs.

Definition 1.1 ([11])**.** Let X be a non-empty set. A mapping $d_l : X \times X \to [0, \infty)$ is said to be a *dislocated metric* on X if, for every $x, y, z \in X$,

(i) $d_l(x, y) = 0 \Rightarrow x = y$;
(ii) $d_l(x, y) = d_l(y, x)$;
(iii) $d_l(x, y) \leq d_l(x, z) + d_l(z, y)$.

Then the pair (X, d_l) is said to be a *dislocated metric space.*

Example 1.2. If $X = \mathbb{R}^+ \cup \{0\}$ then $d_l(x, y) = x + y$ defines a dislocated metric d_l on X.

Definition 1.3 ([11])**.** Let (X, d_l) be a dislocated metric space.

(i) A sequence $\{x_n\}$ in (X, d_l) *dislocated-converges* to $x \in X$ if $\lim_{n\to\infty} d_l(x_n, x) = 0$.
(ii) A sequence $\{x_n\}$ is said to be a *dislocated Cauchy sequence* if, given any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $d_l(x_m, x_n) < \epsilon$ or $\lim_{n,m\to\infty} d_l(x_n, x_m) = 0$.
(iii) A dislocated metric space (X, d_l) is said to be *complete* if every Cauchy sequence in X converges to a point $x \in X$ and $d_l(x, x) = 0$.

Definition 1.4 ([26])**.** Let K be a non-empty subset of a dislocated metric space X and let $x \in X$. An element $y_0 \in K$ is called a *best approximation* in K if

$$d_l(x,K) = d_l(x,y_0), \quad \text{where } d_l(x,K) = \inf_{y\in K} d_l(x,y).$$

If each $x \in X$ has at least one best approximation in K, then K is called a *proximinal set*. We denote by $P(X)$ the set of all closed proximinal subsets of X.

Definition 1.5 ([26])**.** The function $H_{d_l} : P(X) \times P(X) \to \mathbb{R}^+$, defined by

$$H_{d_l}(A,B) = \max\left\{\sup_{a\in A} d_l(a,B), \sup_{b\in B} d_l(A,b)\right\}$$

is called a *dislocated Hausdorff metric* on $P(X)$.

Definition 1.6 ([27])**.** Let the function $F : \mathbb{R}^+ \to \mathbb{R}$ satisfy the following properties:

- (F_1) F is strictly increasing, i.e., for all $x, y \in \mathbb{R}^+$, if $x < y$ then $F(x) < F(y)$.
- (F_2) For all sequences $\{x_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} x_n = 0$ if and only if $\lim_{n\to\infty} F(x_n) = -\infty$.
- (F_3) There exists $k \in (0,1)$ such that $\lim_{x\to 0^+} x^k F(x) = 0$.

A mapping $T : X \to X$ is said to be an *F-contraction* if there exists $\tau > 0$ such that, for all $x, y \in X$,

$$d(Tx,Ty) > 0 \ \Rightarrow\ \tau + F(d(Tx,Ty)) \leq F(d(x,y)). \tag{1.1}$$

We denote by $\mathcal{F}$ the set of all mappings satisfying the conditions (F_1)–(F_3).

Example 1.7 ([27])**.** Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by the formula $F(x) = \ln x$. It is clear that F satisfies (F_1)–(F_3). Then each self mapping $T : X \to X$ satisfying (1.1) is an F-contraction such that

$$d(Tx,Ty) \leq e^{-\tau} d(x,y) \quad \text{for all } x,y \in X,\ Tx \neq Ty.$$

Definition 1.8 ([10])**.** Let X be a non-empty set, $T : X^N \to 2^X$ (a collection of all non-empty subsets of X). An element $(x_1, x_2, \dots, x_N) \in X^N$ is said to be an *N-tupled fixed point* (or a *fixed point of N-order*) of the mapping T if

$$\begin{aligned}
x_1 &\in T(x_1, x_2, \dots, x_N),\\
x_2 &\in T(x_2, x_3, \dots, x_N, x_1),\\
&\vdots\\
x_N &\in T(x_N, x_1, \dots, x_{N-1}).
\end{aligned}$$

Lemma 1.9 ([26])**.** *Let (X, d_l) be a dislocated metric space. Let $(P(X), H_{d_l})$ be a dislocated Hausdorff metric space on $P(X)$. Then for all $A, B \in P(X)$ and for each $a \in A$ there exists $b_a \in B$ such that $H_{d_l}(A,B) \geq d_l(a, b_a)$ for all $d_l(a,B) = d_l(a,b_a)$.*

2. Main results

In this section we introduce three types of multidimensional Ciric-type rational F-contractions where F is sub-additive and prove multidimensional fixed point theorems in dislocated metric spaces.

Definition 2.1. Let (X, d_l) be a dislocated metric space and let $F \in \mathcal{F}$ satisfy the sub-additive property, i.e., for a finite number $N \in \mathbb{N}$, $F\Big(\sum_{i=1}^{N} x_i\Big) \leq \sum_{i=1}^{N} F(x_i)$. Let $S, T : X^N \to P(X)$ be two multivalued mappings. The pair (S, T) is said to be

(i) a *multidimensional Ciric-type rational F-contraction of type I* if there exists $\tau > 0$ such that

$$\frac{\tau}{N} + F(H_{d_l}(S_{x_i}, T_{y_i})) \leq \frac{1}{N} F\left(\max\left\{ \sum_{i=1}^{N} d_l(x_i, y_i), \sum_{i=1}^{N} d_l(x_i, S_{x_i}), \sum_{i=1}^{N} d_l(y_i, T_{y_i}), \frac{\sum_{i=1}^{N} d_l(x_i, S_{x_i}) \sum_{i=1}^{N} d_l(y_i, T_{y_i})}{1 + \sum_{i=1}^{N} d_l(x_i, y_i)} \right\}\right); \tag{2.1}$$

(ii) a *multidimensional Ciric-type rational F-contraction of type II* if there exists $\tau > 0$ such that

$$\frac{\tau}{N} + F(H_{d_l}(S_{x_i}, T_{y_i})) \leq \frac{1}{N} F\left(\max\left\{ \sum_{i=1}^{N} d_l(x_i, y_i), \sum_{i=1}^{N} d_l(x_i, S_{x_i}), \sum_{i=1}^{N} d_l(y_i, T_{y_i}), \frac{\sum_{i=1}^{N} d_l(y_i, T_{y_i}) \Big[1 + \sum_{i=1}^{N} d_l(x_i, S_{x_i})\Big]}{1 + \sum_{i=1}^{N} d_l(x_i, y_i)} \right\}\right); \tag{2.2}$$

(iii) a *multidimensional Ciric-type rational F-contraction of type III* if there exists $\tau > 0$ such that

$$\frac{\tau}{2N} + F(H_{d_l}(S_{x_i}, T_{y_i})) \leq \frac{1}{2N} F\left(\max\left\{ \sum_{i=1}^{N} d_l(x_i, y_i), \sum_{i=1}^{N} d_l(x_i, T_{y_i}), \sum_{i=1}^{N} d_l(y_i, S_{x_i}), \frac{\sum_{i=1}^{N} d_l(x_i, T_{y_i}) \Big[1 + \sum_{i=1}^{N} d_l(y_i, S_{x_i})\Big]}{1 + 2\sum_{i=1}^{N} d_l(x_i, y_i)} \right\}\right), \tag{2.3}$$

where the inequalities hold for every $(x_1, x_2, \ldots, x_N), (y_1, y_2, \ldots, y_N) \in X^N$, $S_{x_i} = S(x_i, x_{i+1}, \ldots, x_N, x_1, x_2, \ldots, x_{i-1})$, and $T_{y_i} = T(y_i, y_{i+1}, \ldots, y_N, y_1, y_2, \ldots, y_{i-1})$.

Lemma 2.2. *Let (X,d_l) be a dislocated metric space. Define $D_l : X^N \times X^N \to \mathbb{R}$ by $D_l(A,B) = \sum_{i=1}^N d_l(a_i,b_i)$ for all $A = (a_1,a_2,\dots,a_N)$, $B = (b_1,b_2,\dots,b_N) \in X^N$. Then the following properties are satisfied:*

(i) *(X^N, D_l) is also a dislocated metric space.*
(ii) *Let $\{x^n\}$ be a sequence in X and denote a sequence in X^N by $\{U^n\}$, where $U^n = (x_1^n, x_2^n, \dots, x_N^n)$. Then $\{U^n\} \xrightarrow{D_l} U = (x_1,x_2,\dots,x_N)$ if and only if $\{x_i^n\} \xrightarrow{d_l} x_i$, for all $i = 1,2,\dots,N$.*
(iii) *The sequence $\{U^n = (x_1^n,x_2^n,\dots,x_N^n)\}$ is D_l-Cauchy if and only if each $\{x_i^n\}_{i=1}^N$ is Cauchy in (X,d_l).*
(iv) *(X^N, D_l) is complete if and only if (X,d_l) is complete.*

Proof. We can prove the properties easily using definition of D_l and properties of d_l. □

Throughout this paper we denote

$$S_{x_i^j} = S(x_i^j, x_{i+1}^j, \dots, x_N^j, x_1^j, x_2^j, \dots, x_{i-1}^j)$$

and

$$T_{x_i^j} = T(x_i^j, x_{i+1}^j, \dots, x_N^j, x_1^j, x_2^j, \dots, x_{i-1}^j),$$

for all $i = 1,2\dots,N$, $j = 0,1,2,\dots$, and $(x_i^j, x_{i+1}^j, \dots, x_N^j, x_1^j, x_2^j, \dots, x_{i-1}^j) \in X^N$.

Now for proving our results first we construct a sequence in (X^N, D_l) as follows. Consider an arbitrary element $(x_1^0, x_2^0, \dots, x_N^0) \in X^N$. Construct the sequences $\{x_i^n\}_{i=1}^N$ by taking $x_i^1 \in S_{x_i^0}$ such that $d_l(x_i^0, S_{x_i^0}) = d_l(x_i^0, x_i^1)$ for all $i = 1,2,\dots,N$; $x_i^2 \in T_{x_i^1}$ such that $d_l(x_i^1, T_{x_i^1}) = d_l(x_i^1, x_i^2)$ for all $i = 1,2,\dots,N$; $x_i^3 \in S_{x_i^2}$ such that $d_l(x_i^2, S_{x_i^2}) = d_l(x_i^2, x_i^3)$ for all $i = 1,2,\dots,N$. Continuing this process we get the sequences $\{(x_i^n)\}_{i=1}^N \in X$ with points $x_i^{2n+1} \in S_{x_i^{2n}}$ and $x_i^{2n+2} \in T_{x_i^{2n+1}}$ such that $d_l(x_i^{2n}, S_{x_i^{2n}}) = d_l(x_i^{2n}, x_i^{2n+1})$ and $d_l(x_i^{2n+1}, T_{x_i^{2n+1}}) = d_l(x_i^{2n+1}, x_i^{2n+2})$, for all $i = 1,2,\dots,N$. Denote these sequences in X^N by $\{TS(U^n)\}$ generated by $(x_1^0, x_2^0, \dots, x_N^0)$. Denote, for $N = 1$, the sequence constructed above as $\{TS(x_n)\}$ (see [20]).

Example 2.3. Let $X = \mathbb{R}^+ \cup \{0\}$ and $D_l(A,B) = \sum_{i=1}^N \ x_i + y_i$, where $A = (x_1,x_2,\dots,x_N)$ and $B = (y_1,y_2,\dots,y_n)$. Define two mappings S and T as

$$S(x_1,x_2,\dots,x_N) = [x_1 + x_2 + \cdots + x_N, 2(x_1 + x_2 + \cdots + x_N)]$$

and

$$T(x_1,x_2,\dots,x_N) = \Big[\frac{x_1 + x_2 + \cdots + x_N}{2N}, \frac{x_1 + x_2 + \cdots + x_N}{N}\Big].$$

Suppose that $N = 2$ and consider $(0,1)$ is an arbitrary element in X^2. Then we can construct the sequence $\{TS(U^n)\} = \{(0, \frac{1}{2}, 1, \dots), (1, 1, \frac{1}{2}, \dots)\}$ generated by $(0,1)$.

Theorem 2.4. *Let (X, d_l) be a complete dislocated metric space and let (S, T) be a pair of multidimensional Ciric-type rational F-contractions of type I. Then $\{TS(U^n)\} \xrightarrow{D_l} U \in X^N$. Moreover, if (2.1) also holds for U, then U is the common fixed point of N-order of the mappings S and T and $D_l(U, U) = 0$.*

Proof. If

$$\max\left\{\sum_{i=1}^{N} d_l(x_i, y_i), \sum_{i=1}^{N} d_l(x_i, S_{x_i}), \sum_{i=1}^{N} d_l(y_i, T_{y_i}), \frac{\sum_{i=1}^{N} d_l(x_i, S_{x_i}) \ \sum_{i=1}^{N} d_l(y_i, T_{y_i})}{1 + \sum_{i=1}^{N} d_l(x_i, y_i)}\right\} = 0,$$

then clearly $(x_1, x_2, \ldots, x_N) = (y_1, y_2, \ldots, y_N)$ is a common fixed point of N-order of S and T.

Now consider the case

$$\max\left\{\sum_{i=1}^{N} d_l(x_i, y_i), \sum_{i=1}^{N} d_l(x_i, S_{x_i}), \sum_{i=1}^{N} d_l(y_i, T_{y_i}), \frac{\sum_{i=1}^{N} d_l(x_i, S_{x_i}) \ \sum_{i=1}^{N} d_l(y_i, T_{y_i})}{1 + \sum_{i=1}^{N} d_l(x_i, y_i)}\right\} > 0$$

for all $(x_1, x_2, \ldots, x_N), (y_1, y_2, \ldots, y_N) \in \{TS(U^n)\}$, with $x_i \neq y_i$, for all $i = 1, 2, \ldots, N$. Using the contraction condition and Lemma 1.9, we get, for all $i = 1, 2, \ldots, N$,

$$\begin{aligned}
&F(d_l(x_i^{2j+1}, x_i^{2j+2})) \\
&\quad \leq F(H_{d_l}(S_{x_i^{2j}}, T_{x_i^{2j+1}})) \\
&\quad \leq \frac{1}{N} F\Bigg(\max\Bigg\{\sum_{i=1}^{N} d_l\big(x_i^{2j}, x_i^{2j+1}\big), \sum_{i=1}^{N} d_l\big(x_i^{2j}, S_{x_i^{2j}}\big), \\
&\qquad \sum_{i=1}^{N} d_l\big(x_i^{2j+1}, T_{x_i^{2j+1}}\big), \frac{\sum_{i=1}^{N} d_l\big(x_i^{2j}, S_{x_i^{2j}}\big) \ \sum_{i=1}^{N} d_l\big(x_i^{2j+1}, T_{x_i^{2j+1}}\big)}{1 + \sum_{i=1}^{N} d_l\big(x_i^{2j}, x_i^{2j+1}\big)}\Bigg\}\Bigg) - \frac{\tau}{N}
\end{aligned}$$

$$= \frac{1}{N} F\Bigg(\max\Bigg\{ \sum_{i=1}^{N} d_l\big(x_i^{2j}, x_i^{2j+1}\big), \sum_{i=1}^{N} d_l\big(x_i^{2j}, x_i^{2j+1}\big), \sum_{i=1}^{N} d_l\big(x_i^{2j+1}, x_i^{2j+2}\big), \frac{\sum_{i=1}^{N} d_l\big(x_i^{2j}, x_i^{2j+1}\big) \ \sum_{i=1}^{N} d_l\big(x_i^{2j+1}, x_i^{2j+2}\big)}{1 + \sum_{i=1}^{N} d_l\big(x_i^{2j}, x_i^{2j+1}\big)} \Bigg\}\Bigg) - \frac{\tau}{N}$$

$$= \frac{1}{N} F\Bigg(\max\Bigg\{ \sum_{i=1}^{N} d_l\big(x_i^{2j}, x_i^{2j+1}\big), \sum_{i=1}^{N} d_l\big(x_i^{2j+1}, x_i^{2j+2}\big) \Bigg\}\Bigg) - \frac{\tau}{N}.$$

If

$$\max\Bigg\{ \sum_{i=1}^{N} d_l\big(x_i^{2j}, x_i^{2j+1}\big), \sum_{i=1}^{N} d_l\big(x_i^{2j+1}, x_i^{2j+2}\big) \Bigg\} = \sum_{i=1}^{N} d_l\big(x_i^{2j+1}, x_i^{2j+2}\big)$$

then

$$F\big(d_l\big(x_i^{2j+1}, x_i^{2j+2}\big)\big) \leq \frac{1}{N} F\Bigg(\sum_{i=1}^{N} d_l\big(x_i^{2j+1}, x_i^{2j+2}\big) \Bigg) - \frac{\tau}{N}, \quad \forall\, i = 1, 2, \ldots, N.$$

Adding the above inequalities we get

$$\sum_{i=1}^{N} F\big(d_l\big(x_i^{2j+1}, x_i^{2j+2}\big)\big) \leq F\Bigg(\sum_{i=1}^{N} d_l\big(x_i^{2j+1}, x_i^{2j+2}\big) \Bigg) - \tau.$$

Since F is sub-additive,

$$F\Bigg(\sum_{i=1}^{N} d_l\big(x_i^{2j+1}, x_i^{2j+2}\big) \Bigg) \leq F\Bigg(\sum_{i=1}^{N} d_l\big(x_i^{2j+1}, x_i^{2j+2}\big) \Bigg) - \tau,$$

which gives a contradiction. Therefore,

$$\max\Bigg\{ \sum_{i=1}^{N} d_l\big(x_i^{2j}, x_i^{2j+1}\big), \sum_{i=1}^{N} d_l\big(x_i^{2j+1}, x_i^{2j+2}\big) \Bigg\} = \sum_{i=1}^{N} d_l\big(x_i^{2j}, x_i^{2j+1}\big).$$

Then

$$F\big(d_l\big(x_i^{2j+1}, x_i^{2j+2}\big)\big) \leq \frac{1}{N} F\Bigg(\sum_{i=1}^{N} d_l\big(x_i^{2j}, x_i^{2j+1}\big) \Bigg) - \frac{\tau}{N}, \quad \forall\, i = 1, 2, \ldots, N.$$

Then by adding the above inequalities and by applying the sub-additive property of F we get

$$F\Bigg(\sum_{i=1}^{N} d_l\big(x_i^{2j+1}, x_i^{2j+2}\big) \Bigg) \leq F\Bigg(\sum_{i=1}^{N} d_l\big(x_i^{2j}, x_i^{2j+1}\big) \Bigg) - \tau. \tag{2.4}$$

Similarly, we get

$$F\Bigg(\sum_{i=1}^{N} d_l\big(x_i^{2j}, x_i^{2j+1}\big) \Bigg) \leq F\Bigg(\sum_{i=1}^{N} d_l\big(x_i^{2j-1}, x_i^{2j}\big) \Bigg) - \tau. \tag{2.5}$$

Using (2.5) in (2.4) we get

$$F\Big(\sum_{i=1}^{N} d_l\big(x_i^{2j+1}, x_i^{2j+2}\big)\Big) \leq F\Big(\sum_{i=1}^{N} d_l\big(x_i^{2j-1}, x_i^{2j}\big)\Big) - 2\tau.$$

Repeating the above steps we get

$$F\Big(\sum_{i=1}^{N} d_l\big(x_i^{2j+1}, x_i^{2j+2}\big)\Big) \leq F\Big(\sum_{i=1}^{N} d_l\big(x_i^{0}, x_i^{1}\big)\Big) - (2j+1)\tau, \tag{2.6}$$

$$F\Big(\sum_{i=1}^{N} d_l\big(x_i^{2j}, x_i^{2j+1}\big)\Big) \leq F\Big(\sum_{i=1}^{N} d_l\big(x_i^{0}, x_i^{1}\big)\Big) - 2j\tau. \tag{2.7}$$

Combining (2.6) and (2.7) we get

$$F\Big(\sum_{i=1}^{N} d_l\big(x_i^{n}, x_i^{n+1}\big)\Big) \leq F\Big(\sum_{i=1}^{N} d_l\big(x_i^{0}, x_i^{1}\big)\Big) - n\tau. \tag{2.8}$$

Taking limit on both sides we get

$$\lim_{n\to\infty} F\Big(\sum_{i=1}^{N} d_l\big(x_i^{n}, x_i^{n+1}\big)\Big) = -\infty.$$

Using (F_2),

$$\lim_{n\to\infty} \sum_{i=1}^{N} d_l\big(x_i^{n}, x_i^{n+1}\big) = 0. \tag{2.9}$$

By (F_3) there exists a $k \in (0,1)$ such that

$$\lim_{n\to\infty} \Big[\sum_{i=1}^{N} d_l\big(x_i^{n}, x_i^{n+1}\big)\Big]^k F\Big(\sum_{i=1}^{N} d_l\big(x_i^{n}, x_i^{n+1}\big)\Big) = 0. \tag{2.10}$$

Using (2.8) we get

$$\begin{aligned}\Big[\sum_{i=1}^{N} d_l\big(x_i^{n}, x_i^{n+1}\big)\Big]^k \Big[F\Big(\sum_{i=1}^{N} d_l\big(x_i^{n}, x_i^{n+1}\big)\Big) - F\Big(\sum_{i=1}^{N} d\big(x_i^{0}, x_i^{1}\big)\Big)\Big] \\ \leq -n\tau \Big[\sum_{i=1}^{N} d_l\big(x_i^{n}, x_i^{n+1}\big)\Big]^k \leq 0.\end{aligned}$$

Taking the limit and using (2.9) and (2.10) we get

$$\lim_{n\to\infty} n \Big[\sum_{i=1}^{N} d_l\big(x_i^{n}, x_i^{n+1}\big)\Big]^k = 0.$$

Using the definition of convergence, there exists $n_1 \in \mathbb{N}$ such that

$$n \Big[\sum_{i=1}^{N} d_l\big(x_i^{n}, x_i^{n+1}\big)\Big]^k \leq 1, \quad \forall\, n \geq n_1,$$

or

$$\sum_{i=1}^{N} d_l\big(x_i^n, x_i^{n+1}\big) \le \frac{1}{n^{\frac{1}{k}}}, \quad \forall\, n \ge n_1.$$

Now, for $m > n$,

$$d_l\big(x_i^n, x_i^m\big) \le d_l\big(x_i^n, x_i^{n+1}\big) + d_l\big(x_i^{n+1}, x_i^{n+2}\big) + \cdots + d_l\big(x_i^{m-1}, x_i^m\big),\ \forall\, i = 1, 2, \ldots, N.$$

Adding the inequalities for $i = 1, 2, \ldots, N$ we get

$$\begin{aligned}\sum_{i=1}^{N} d_l(x_i^n, x_i^m) &\le \sum_{i=1}^{N} d_l(x_i^n, x_i^{n+1}) + \sum_{i=1}^{N} d_l(x_i^{n+1}, x_i^{n+2}) + \cdots + \sum_{i=1}^{N} d_l(x_i^{m-1}, x_i^m) \\ &\le \frac{1}{n^{\frac{1}{k}}} + \frac{1}{(n+1)^{\frac{1}{k}}} + \cdots + \frac{1}{(m-1)^{\frac{1}{k}}} \\ &\le \frac{1}{n^{\frac{1}{k}}} + \frac{1}{(n+1)^{\frac{1}{k}}} + \cdots \\ &= \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.\end{aligned}$$

The convergence of the series $\sum\limits_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ implies that $\lim\limits_{n,m\to\infty} \sum\limits_{i=1}^{N} d_l(x_i^n, x_i^m) = 0$, i.e., $\lim\limits_{n,m\to\infty} d_l(x_i^n, x_i^m) = 0$ for $i = 1, 2, \ldots, N$, i.e., $\{x_i^n\}_{i=1}^N$ are Cauchy sequences in X. Since (X, d_l) is a complete dislocated metric space, there exist $x_1, x_2, \ldots, x_N \in X$ such that

$$\lim_{n\to\infty} d_l(x_i^n, x_i) = 0, \quad \forall\, i = 1, 2, \ldots, N. \tag{2.11}$$

So by Lemma 2.2, $\{TS(U^n)\} \xrightarrow{D_l} U$, where $U = (x_1, x_2, \ldots, x_N)$.

Now, by Lemma 1.9, we have

$$\frac{\tau}{N} + F(d_l(x_i^{2n+1}, T_{x_i})) \le \frac{\tau}{N} + F(H_{d_l}(S_{x_i^{2n}}, T_{x_i})), \quad \forall\, i = 1, 2, \ldots, N.$$

Since the contraction holds for U, for every $i = 1, 2, \ldots, N$ we have

$$\begin{aligned}&\frac{\tau}{N} + F(d_l(x_i^{2n+1}, T_{x_i})) \\ &\quad \le \frac{1}{N} F\Bigg(\max \Bigg\{ \sum_{i=1}^{N} d_l(x_i^{2n}, x_i), \sum_{i=1}^{N} d_l(x_i^{2n}, S_{x_i^{2n}}), \sum_{i=1}^{N} d_l(x_i, T_{x_i}), \\ &\qquad\qquad \frac{\sum\limits_{i=1}^{N} d_l(x_i^{2n}, S_{x_i^{2n}}) \ \sum\limits_{i=1}^{N} d_l(x_i, T_{x_i})}{1 + \sum\limits_{i=1}^{N} d_l(x_i^{2n}, x_i)} \Bigg\}\Bigg)\end{aligned}$$

$$= \frac{1}{N} F\Bigg(\max\Bigg\{ \sum_{i=1}^{N} d_l(x_i^{2n}, x_i), \sum_{i=1}^{N} d_l(x_i^{2n}, x_i^{2n+1}), \sum_{i=1}^{N} d_l(x_i, T_{x_i}), \frac{\sum_{i=1}^{N} d_l(x_i^{2n}, x_i^{2n+1}) \sum_{i=1}^{N} d_l(x_i, T_{x_i})}{1 + \sum_{i=1}^{N} d_l(x_i^{2n}, x_i)} \Bigg\}\Bigg).$$

Adding the above inequalities for $i = 1, 2, \ldots, N$ and using the sub-additive property of F we get

$$\tau + F\Bigg(\sum_{i=1}^{N} d_l\big(x_i^{2n+1}, T_{x_i}\big) \Bigg) \leq F\Bigg(\max\Bigg\{ \sum_{i=1}^{N} d_l\big(x_i^{2n}, x_i\big), \sum_{i=1}^{N} d_l\big(x_i^{2n}, x_i^{2n+1}\big), \sum_{i=1}^{N} d_l\big(x_i, T_{x_i}\big), \frac{\sum_{i=1}^{N} d_l\big(x_i^{2n}, x_i^{2n+1}\big) \sum_{i=1}^{N} d_l\big(x_i, T_{x_i}\big)}{1 + \sum_{i=1}^{N} d_l\big(x_i^{2n}, x_i\big)} \Bigg\}\Bigg).$$

Since $\tau > 0$,

$$F\Bigg(\sum_{i=1}^{N} d_l\big(x_i^{2n+1}, T_{x_i}\big) \Bigg) \leq F\Bigg(\max\Bigg\{ \sum_{i=1}^{N} d_l\big(x_i^{2n}, x_i\big), \sum_{i=1}^{N} d_l\big(x_i^{2n}, x_i^{2n+1}\big), \sum_{i=1}^{N} d_l\big(x_i, T_{x_i}\big), \frac{\sum_{i=1}^{N} d_l\big(x_i^{2n}, x_i^{2n+1}\big) \sum_{i=1}^{N} d_l\big(x_i, T_{x_i}\big)}{1 + \sum_{i=1}^{N} d_l\big(x_i^{2n}, x_i\big)} \Bigg\}\Bigg).$$

Using (F_1), we get

$$\sum_{i=1}^{N} d_l\big(x_i^{2n+1}, T_{x_i}\big) < \max\Bigg\{ \sum_{i=1}^{N} d_l\big(x_i^{2n}, x_i\big), \sum_{i=1}^{N} d_l\big(x_i^{2n}, x_i^{2n+1}\big), \sum_{i=1}^{N} d_l\big(x_i, T_{x_i}\big), \frac{\sum_{i=1}^{N} d_l\big(x_i^{2n}, x_i^{2n+1}\big) \sum_{i=1}^{N} d_l(x_i, T_{x_i})}{1 + \sum_{i=1}^{N} d_l\big(x_i^{2n}, x_i\big)} \Bigg\}.$$

Taking the limit and using equations (2.9) and (2.11) we get

$$\sum_{i=1}^{N} d_l(x_i, T_{x_i}) < \sum_{i=1}^{N} d_l(x_i, T_{x_i}), \quad \forall\, i = 1, 2, \ldots, N,$$

which is a contradiction; therefore, $d_l(x_i, T_{x_i}) = 0 \;\forall\, i = 1, 2, \ldots, N$, i.e., either $x_i = T_{x_i}$ or $x_i \in T_{x_i} \;\forall\, i = 1, 2, \ldots, N$.

Similarly, using Lemma 1.9 and the inequality

$$\frac{\tau}{N} + F(d_l(x_i^{2n+2}, S_{x_i})) \leq \frac{\tau}{N} + F(H_{d_l}(T_{x_i^{2n+1}}, S_{x_i})), \quad \forall\, i = 1, 2, \ldots, N,$$

we get that $U = (x_1, x_2, \dots, x_N)$ is a fixed point of N-order of the mapping S. Therefore U is a common fixed point of N-order of the mappings S and T.

Finally,

$$\begin{aligned} D_l(U,U) &= \sum_{i=1}^{N} d_l(x_i, x_i) \\ &\leq \sum_{i=1}^{N} d_l(x_i, T_{x_i}) + d_l(T_{x_i}, x_i) \\ &= 0. \end{aligned}$$

□

Theorem 2.4 is a generalization of the results obtained by Rasham et al. [20], which we state as a corollary.

Corollary 2.5 ([20]). *Let (X, d_l) be a complete dislocated metric space and (S,T) two mappings such that for all $x, y \in \{TS(x_n)\}$ we have*

$$\tau + F(H_{d_l}(S_x, T_y)) \leq F\left(\max\left\{d_l(x,y), \frac{d_l(x,S_x).d_l(y,T_y)}{1+d_l(x,y)}, d_l(x,S_x), d_l(y,T_y)\right\}\right), \tag{2.12}$$

where $F \in \mathcal{F}$ and $\tau > 0$. Then $\{TS(x_n)\} \to u \in X$. Moreover, if u satisfies (2.12), *then S and T have a common fixed point u in X and $d_l(u,u) = 0$.*

Proof. If in Theorem 2.4 we take $N = 1$ then we get the result. □

The following example illustrates Corollary 2.5.

Example 2.6 ([20]). Let $X = \mathbb{R}^+ \cup \{0\}$ and $d_l(x,y) = x + y$. Then (X, d_l) is a complete dislocated metric space. Define $S, T : X \to P(X)$ as $S(x) = \left[\frac{1}{3}x, \frac{2}{3}x\right]$ and $T(x) = \left[\frac{1}{5}x, \frac{2}{5}x\right]$ for all $x \in X$. Define $F : \mathbb{R}^+ \to \mathbb{R}$ as $F(x) = \ln(x)$ for all $x \in \mathbb{R}^+$ and $\tau > 0$. As $x, y \in X$ and $\tau = \ln(1.2)$, by taking $x_0 = 7$ we define the sequence $\{TS(x_n)\} = \{7, \frac{7}{3}, \frac{7}{15}, \frac{7}{45}, \dots\}$ in X generated by $x_0 = 7$. Then S and T satisfy (2.12) and so (S,T) have a common fixed point.

Theorem 2.7. *Let (X, d_l) be a complete dislocated metric space and (S,T) a pair of multidimensional Ciric-type rational F-contractions of type II. Then $\{TS(U^n)\} \xrightarrow{D_l} U \in X^N$. Moreover, if* (2.2) *also holds for U, then S and T have a common fixed point of N-order U in X^N and $D_l(U,U) = 0$.*

Proof. If

$$\max\left\{\sum_{i=1}^{N} d_l(x_i, y_i), \sum_{i=1}^{N} d_l(x_i, S_{x_i}), \sum_{i=1}^{N} d_l(y_i, T_{y_i}), \frac{\sum_{i=1}^{N} d_l(y_i, T_{y_i})\left[1 + \sum_{i=1}^{N} d_l(x_i, S_{x_i})\right]}{1 + \sum_{i=1}^{N} d_l(x_i, y_i)}\right\} = 0 \tag{2.13}$$

then clearly $(x_1, x_2, \ldots, x_N) = (y_1, y_2, \ldots, y_N)$ is a common fixed point of N-order of S and T.

If the left hand side of (2.13) is greater than zero, then using Lemma 1.9 and the contraction condition (2.2) we get, for all $i = 1, 2, \ldots, N$,

$$\begin{aligned}
&F\big(d_l(x_i^{2j+1}, x_i^{2j+2})\big)\\
&\quad\le F\big(H_{d_l}(S_{x_i^{2j}}, T_{x_i^{2j+1}})\big)\\
&\quad\le \frac{1}{N}F\Bigg(\max\Bigg\{\sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1}), \sum_{i=1}^{N} d_l(x_i^{2j}, S_{x_i^{2j}}), \sum_{i=1}^{N} d_l(x_i^{2j+1}, T_{x_i^{2j+1}}),\\
&\qquad\qquad \frac{\sum_{i=1}^{N} d_l(x_i^{2j+1}, T_{x_i^{2j+1}})\Big[1+\sum_{i=1}^{N} d_l(x_i^{2j}, S_{x_i^{2j}})\Big]}{1+\sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1})}\Bigg\}\Bigg) - \frac{\tau}{N}\\
&\quad= \frac{1}{N}F\Bigg(\max\Bigg\{\sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1}), \sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1}), \sum_{i=1}^{N} d_l(x_i^{2j+1}, x_i^{2j+2}),\\
&\qquad\qquad \frac{\sum_{i=1}^{N} d_l(x_i^{2j+1}, x_i^{2j+2})\Big[1+\sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1})\Big]}{1+\sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1})}\Bigg\}\Bigg) - \frac{\tau}{N}\\
&\quad= \frac{1}{N}F\Bigg(\max\Bigg\{\sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1}), \sum_{i=1}^{N} d_l(x_i^{2j+1}, x_i^{2j+2})\Bigg\}\Bigg) - \frac{\tau}{N}.
\end{aligned}$$

The remaining part of the proof is similar to that of Theorem 2.4. □

Corollary 2.8. *Let (X, d_l) be a complete dislocated metric space and let $S, T : X \to P(X)$ be such that for every $x, y \in \{TS(x_n)\}$ we have*

$$\begin{aligned}
&\tau + F(H_{d_l}(S_x, T_y))\\
&\qquad \le F\Big(\max\Big\{d_l(x, y), d_l(x, S_x), d_l(y, T_y), \frac{d_l(y, T_y)[1 + d_l(x, S_x)]}{1 + d_l(x, y)}\Big\}\Big), \qquad (2.14)
\end{aligned}$$

where $F \in \mathcal{F}$ and $\tau > 0$. Then $\{TS(x_n)\} \to u \in X$. Moreover, if u satisfies (2.14)*, then S and T have a common fixed point u in X and $d_l(u, u) = 0$.*

Theorem 2.9. *Let (X, d_l) be a complete dislocated metric space and (S, T) a pair of multidimensional Ciric-type rational F-contractions of type III. Then $\{TS(U^n)\} \xrightarrow{D_l} U \in X^N$. Moreover, if* (2.3) *also holds for U, then S and T have a common fixed point of N-order U in X^N and $D_l(U, U) = 0$.*

Proof. If

$$\max\left\{\sum_{i=1}^{N} d_l(x_i,y_i), \sum_{i=1}^{N} d_l(x_i,T_{y_i}), \sum_{i=1}^{N} d_l(y_i,S_{x_i}), \frac{\sum_{i=1}^{N} d_l(x_i,T_{y_i})\left[1+\sum_{i=1}^{N} d_l(y_i,S_{x_i})\right]}{1+2\sum_{i=1}^{N} d_l(x_i,y_i)}\right\} = 0 \quad (2.15)$$

then we get $x_i = y_i$ for all $i = 1, 2, \ldots, N$ and $(x_1, x_2, \ldots, x_N)$ is a common fixed point of N-order of S and T.

If the left hand side of (2.15) is greater than zero, then using Lemma 1.9 and the contraction condition (2.3) we get, for all $i = 1, 2, \ldots, N$,

$$\begin{aligned} F\Big(d_l(x_i^{2j+1}, x_i^{2j+2})\Big) &\le F\Big(H_{d_l}(S_{x_i^{2j}}, T_{x_i^{2j+1}})\Big) \\ &\le \frac{1}{2N} F\Bigg(\max\Bigg\{\sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1}), \sum_{i=1}^{N} d_l(x_i^{2j}, T_{x_i^{2j+1}}), \sum_{i=1}^{N} d_l(x_i^{2j+1}, S_{x_i^{2j}}), \\ &\qquad \frac{\sum_{i=1}^{N} d_l(x_i^{2j}, T_{x_i^{2j+1}})\left[1+\sum_{i=1}^{N} d_l(x_i^{2j+1}, S_{x_i^{2j}})\right]}{1+2\sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1})}\Bigg\}\Bigg) - \frac{\tau}{2N}. \end{aligned}$$

Now by applying the triangle inequality and by the definition of the sequence $TS(U^n)$, we get

$$\begin{aligned} F\Big(d_l(x_i^{2j+1}, x_i^{2j+2})\Big) &\le \frac{1}{2N} F\Bigg(\max\Bigg\{\sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1}) + \sum_{i=1}^{N} d_l(x_i^{2j+1}, x_i^{2j+2}), 2\sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1})\Bigg\}\Bigg) \\ &\quad - \frac{\tau}{2N}. \end{aligned}$$

Case 1: If

$$\begin{aligned} \max\Bigg\{\sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1}) + \sum_{i=1}^{N} d_l(x_i^{2j+1}, x_i^{2j+2}), 2\sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1})\Bigg\} \\ = \sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1}) + \sum_{i=1}^{N} d_l(x_i^{2j+1}, x_i^{2j+2}), \end{aligned}$$

then

$$F(d_l(x_i^{2j+1}, x_i^{2j+2})) \leq \frac{1}{2N} F\Big(\sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1}) + \sum_{i=1}^{N} d_l(x_i^{2j+1}, x_i^{2j+2})\Big) - \frac{\tau}{2N}.$$

Adding the above inequalities for $i = 1, 2, \ldots, N$ we get

$$\sum_{i=1}^{N} F(d_l(x_i^{2j+1}, x_i^{2j+2})) \leq \frac{1}{2} F\Big(\sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1}) + \sum_{i=1}^{N} d_l(x_i^{2j+1}, x_i^{2j+2})\Big) - \frac{\tau}{2}.$$

Applying the sub-additive property of F on both sides,

$$F\Big(\sum_{i=1}^{N} d_l(x_i^{2j+1}, x_i^{2j+2})\Big) \leq \frac{1}{2} F\Big(\sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1})\Big) + \frac{1}{2} F\Big(\sum_{i=1}^{N} d_l(x_i^{2j+1}, x_i^{2j+2})\Big) - \frac{\tau}{2},$$

i.e.,

$$F\Big(\sum_{i=1}^{N} d_l(x_i^{2j+1}, x_i^{2j+2})\Big) \leq F\Big(\sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1})\Big) - \tau.$$

Following arguments similar to those in Theorem 2.4, we get that the sequences $\{x_i^n\}_{i=1}^N$ are Cauchy sequences in X.

Case 2: If

$$\max\Bigg\{\sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1}) + \sum_{i=1}^{N} d_l(x_i^{2j+1}, x_i^{2j+2}), 2\sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1})\Bigg\} = 2\sum_{i=1}^{N} d_l\big(x_i^{2j}, x_i^{2j+1}\big)$$

then

$$F(d_l(x_i^{2j+1}, x_i^{2j+2})) \leq \frac{1}{2N} F\Big(2\sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1})\Big) - \frac{\tau}{2N}, \quad \forall\, i = 1, 2, \ldots, N.$$

Adding the above sequence for $i = 1, 2, \ldots, N$,

$$\sum_{i=1}^{N} F(d_l(x_i^{2j+1}, x_i^{2j+2})) \leq \frac{1}{2} F\Big(2\sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1})\Big) - \frac{\tau}{2}.$$

Since F is sub-additive,

$$F\Big(\sum_{i=1}^{N} d_l(x_i^{2j+1}, x_i^{2j+2})\Big) \leq F\Big(\sum_{i=1}^{N} d_l(x_i^{2j}, x_i^{2j+1})\Big) - \frac{\tau}{2}.$$

Following steps similar to those in Theorem 2.4, we get that the sequences $\{x_i^n\}_{i=1}^N$ are Cauchy sequences in X.

Since (X, d_l) is a complete dislocated metric space, there exist $(x_1, x_2, \ldots, x_N)$ such that

$$\lim_{n\to\infty} d_l(x_i^n, x_i) = 0, \quad \forall\, i = 1, 2, \ldots, N.$$

Therefore by Lemma 2.2, $\{TS(U^n)\} \xrightarrow{D_l} U$, where $U = (x_1, x_2, \ldots, x_N)$.

Now by Lemma 1.9 for every $i = 1, 2, \ldots, N$ we have

$$
\begin{aligned}
&\frac{\tau}{2N} + F\Big(d_l(x_i^{2n+2}, S_{x_i})\Big) \\
&\quad \le \frac{\tau}{2N} + F\Big(H_{d_l}(T_{x_i^{2n+1}}, S_{x_i}\Big) \\
&\quad = \frac{\tau}{2N} + F\Big(H_{d_l}(S_{x_i}, T_{x_i^{2n+1}})\Big) \\
&\quad \le \frac{1}{2N} F\Bigg(\max\Bigg\{ \sum_{i=1}^{N} d_l(x_i, x_i^{2n+1}), \sum_{i=1}^{N} d_l(x_i, T_{x_i^{2n+1}}), \sum_{i=1}^{N} d_l(x_i^{2n+1}, S_{x_i}), \\
&\qquad\qquad \frac{\sum_{i=1}^{N} d_l(x_i, T_{x_i^{2n+1}}) \Big[1 + \sum_{i=1}^{N} d_l(x_i^{2n+1}, S_{x_i})\Big]}{1 + 2\sum_{i=1}^{N} d_l(x_i, x_i^{2n+1})} \Bigg\}\Bigg) \\
&\quad \le \frac{1}{2N} F\Bigg(\max\Bigg\{ \sum_{i=1}^{N} d_l x_i, x_i^{2n+1}), \sum_{i=1}^{N} d_l(x_i, x_i^{2n+1}) + \sum_{i=1}^{N} d_l(x_i^{2n+1}, T_{x_i^{2n+1}}), \\
&\qquad\qquad \sum_{i=1}^{N} d_l(x_i^{2n+1}, S_{x_i}), \\
&\qquad\qquad \frac{\Big[\sum_{i=1}^{N} d_l(x_i, x_i^{2n+1}) + \sum_{i=1}^{N} d_l(x_i^{2n+1}, T_{x_i^{2n+1}})\Big]\Big[1 + \sum_{i=1}^{N} d_l(x_i^{2n+1}, S_{x_i})\Big]}{1 + 2\sum_{i=1}^{N} d_l(x_i, x_i^{2n+1})} \Bigg\}\Bigg) \\
&\quad \le \frac{1}{2N} F\Bigg(\max\Bigg\{ \sum_{i=1}^{N} d_l(x_i, x_i^{2n+1}), \sum_{i=1}^{N} d_l(x_i, x_i^{2n+1}) + \sum_{i=1}^{N} d_l(x_i^{2n+1}, x_i^{2n+2}), \\
&\qquad\qquad \sum_{i=1}^{N} d_l(x_i^{2n+1}, S_{x_i}), \\
&\qquad\qquad \frac{\Big[\sum_{i=1}^{N} d_l(x_i, x_i^{2n+1}) + \sum_{i=1}^{N} d_l(x_i^{2n+1}, x_i^{2n+2})\Big]\Big[1 + \sum_{i=1}^{N} d_l(x_i^{2n+1}, S_{x_i})\Big]}{1 + 2\sum_{i=1}^{N} d_l(x_i, x_i^{2n+1})} \Bigg\}\Bigg).
\end{aligned}
$$

Adding the above inequalities for $i = 1, 2, \ldots, N$ we get

$$
\begin{aligned}
&\frac{\tau}{2} + \sum_{i=1}^{N} F(d_l(x_i^{2n+2}, S_{x_i})) \\
&\quad \le \frac{1}{2} F\Bigg(\max\Bigg\{ \sum_{i=1}^{N} d_l(x_i, x_i^{2n+1}), \sum_{i=1}^{N} d_l(x_i, x_i^{2n+1}) + \sum_{i=1}^{N} d_l(x_i^{2n+1}, x_i^{2n+2}),
\end{aligned}
$$

$$\sum_{i=1}^{N} d_l(x_i^{2n+1}, S_{x_i}),$$
$$\frac{\Big[\sum\limits_{i=1}^{N} d_l(x_i, x_i^{2n+1}) + \sum\limits_{i=1}^{N} d_l(x_i^{2n+1}, x_i^{2n+2})\Big]\Big[1 + \sum\limits_{i=1}^{N} d_l(x_i^{2n+1}, S_{x_i})\Big]}{1 + 2\sum\limits_{i=1}^{N} d_l(x_i, x_i^{2n+1})}\Bigg\}\Bigg).$$

Since $\tau > 0$ and F is sub-additive we get

$$\begin{aligned}
&F\Big(\sum_{i=1}^{N} d_l(x_i^{2n+2}, S_{x_i})\Big)\\
&\leq \frac{1}{2}F\Bigg(\max\Bigg\{\sum_{i=1}^{N} d_l(x_i, x_i^{2n+1}), \sum_{i=1}^{N} d_l(x_i, x_i^{2n+1}) + \sum_{i=1}^{N} d_l(x_i^{2n+1}, x_i^{2n+2}),\\
&\qquad \sum_{i=1}^{N} d_l(x_i^{2n+1}, S_{x_i}),\\
&\qquad \frac{\Big[\sum\limits_{i=1}^{N} d_l(x_i, x_i^{2n+1}) + \sum\limits_{i=1}^{N} d_l(x_i^{2n+1}, x_i^{2n+2})\Big]\Big[1 + \sum\limits_{i=1}^{N} d_l(x_i^{2n+1}, S_{x_i})\Big]}{1 + 2\sum\limits_{i=1}^{N} d_l(x_i, x_i^{2n+1})}\Bigg\}\Bigg).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&2\,F\Big(\sum_{i=1}^{N} d_l(x_i^{2n+2}, S_{x_i})\Big)\\
&\leq F\Bigg(\max\Bigg\{\sum_{i=1}^{N} d_l(x_i, x_i^{2n+1}), \sum_{i=1}^{N} d_l(x_i, x_i^{2n+1}) + \sum_{i=1}^{N} d_l(x_i^{2n+1}, x_i^{2n+2}),\\
&\qquad \sum_{i=1}^{N} d_l(x_i^{2n+1}, S_{x_i}),\\
&\qquad \frac{\Big[\sum\limits_{i=1}^{N} d_l(x_i, x_i^{2n+1}) + \sum\limits_{i=1}^{N} d_l(x_i^{2n+1}, x_i^{2n+2})\Big]\Big[1 + \sum\limits_{i=1}^{N} d_l(x_i^{2n+1}, S_{x_i})\Big]}{1 + 2\sum\limits_{i=1}^{N} d_l(x_i, x_i^{2n+1})}\Bigg\}\Bigg).
\end{aligned}$$

Using the sub-additive property of F we get

$$\begin{aligned}
&F\Big(2\sum_{i=1}^{N} d_l(x_i^{2n+2}, S_{x_i})\Big)\\
&\leq F\Bigg(\max\Bigg\{\sum_{i=1}^{N} d_l(x_i, x_i^{2n+1}), \sum_{i=1}^{N} d_l(x_i, x_i^{2n+1}) + \sum_{i=1}^{N} d_l(x_i^{2n+1}, x_i^{2n+2}),
\end{aligned}$$

$$\sum_{i=1}^{N} d_l(x_i^{2n+1}, S_{x_i}),$$
$$\frac{\Big[\sum_{i=1}^{N} d_l(x_i, x_i^{2n+1}) + \sum_{i=1}^{N} d_l(x_i^{2n+1}, x_i^{2n+2})\Big]\Big[1 + \sum_{i=1}^{N} d_l(x_i^{2n+1}, S_{x_i})\Big]}{1 + 2\sum_{i=1}^{N} d_l(x_i, x_i^{2n+1})}\Bigg\}\Bigg).$$

Since F is strictly increasing,

$$\begin{aligned}
&2\sum_{i=1}^{N} d_l(x_i^{2n+2}, S_{x_i})\\
&< \max\Bigg\{\sum_{i=1}^{N} d_l(x_i, x_i^{2n+1}), \sum_{i=1}^{N} d_l(x_i, x_i^{2n+1}) + \sum_{i=1}^{N} d_l(x_i^{2n+1}, x_i^{2n+2}),\\
&\quad \sum_{i=1}^{N} d_l(x_i^{2n+1}, S_{x_i}),\\
&\quad \frac{\Big[\sum_{i=1}^{N} d_l(x_i, x_i^{2n+1}) + \sum_{i=1}^{N} d_l(x_i^{2n+1}, x_i^{2n+2})\Big]\Big[1 + \sum_{i=1}^{N} d_l(x_i^{2n+1}, S_{x_i})\Big]}{1 + 2\sum_{i=1}^{N} d_l(x_i, x_i^{2n+1})}\Bigg\}.
\end{aligned}$$

Taking the limit we get

$$2\sum_{i=1}^{N} d_l(x_i, S_{x_i}) < \sum_{i=1}^{N} d_l(x_i, S_{x_i}), \quad \forall\, i = 1, 2, \ldots, N,$$

which is a contradiction. Therefore $d_l(x_i, S_{x_i}) = 0$, $\forall\, i = 1, 2, \ldots, N$, i.e., $x_i = S_{x_i}$ or $x_i \in S_{x_i}$. Therefore $U = (x_1, x_2, \ldots, x_N)$ is a fixed point of N-order of the mapping S. Similarly, using Lemma 1.9 and the inequality

$$\frac{\tau}{2N} + F(d_l(x_i^{2n+1}, T_{x_i})) \le \frac{\tau}{2N} + F(H_{d_l}(S_{x_i^{2n}}, T_{x_i})), \quad \forall\, i = 1, 2, \ldots, N,$$

we prove that U is a fixed point of N-order of the mapping T, i.e., U is a common fixed point of N-order of the mappings S and T.

Finally,

$$\begin{aligned}
D_l(U, U) &= \sum_{i=1}^{N} d_l(x_i, x_i)\\
&\le \sum_{i=1}^{N} d_l(x_i, S_{x_i}) + \sum_{i=1}^{N} d_l(S_{x_i}, x_i)\\
&= 0.
\end{aligned}$$

□

Corollary 2.10. *Let (X, d_l) be a complete dislocated metric space and let $S, T : X \to P(X)$ be such that for every $x, y \in \{TS(x_n)\}$ we have*

$$\begin{aligned}&\frac{\tau}{2}+F(H_{d_l}(S_x,T_y))\\&\quad\leq\frac{1}{2}F\Big(\max\Big\{d_l(x,y),d_l(x,T_y),d_l(y,S_x),\frac{d_l(x,T_y)[1+d_l(y,S_x)]}{1+2\ d_l(x,y)}\Big\}\Big),\qquad(2.16)\end{aligned}$$

where $F\in\mathcal{F}$ and $\tau>0$. Then $\{TS(x_n)\}\to u\in X$. Moreover, if u satisfies (2.16)*, then S and T have a common fixed point u in X and $d_l(u,u)=0$.*

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