# GOTZMANN MONOMIALS IN FOUR VARIABLES

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ABSTRACT. It is a widely open problem to determine which monomials in the $n$-variable polynomial ring $K[x_1, \ldots, x_n]$ over a field $K$ have the Gotzmann property, i.e. induce a Borel-stable Gotzmann monomial ideal. Since 2007, only the case $n \leq 3$ was known. Here we solve the problem for the case $n = 4$. The solution involves a surprisingly intricate characterization.

## 1. INTRODUCTION

Let $K$ be a field and let $R_n = K[x_1, \ldots, x_n]$ be the $n$-variable polynomial algebra over $K$ endowed with its usual grading $\deg(x_i) = 1$ for all $i$. We denote by $S_n \subset R_n$ the set of all monomials $u = x_1^{a_1} \cdots x_n^{a_n}$ in $R_n$, and by $S_{n,d} \subset S_n$ the subset of monomials of degree $\deg(u) = \sum_i a_i = d$.

A monomial ideal $J \subseteq R_n$ is said to be *Borel-stable* or *strongly stable* if for any monomial $v \in J$ and any variable $x_j$ dividing $v$, one has $x_i v / x_j \in J$ for all $1 \leq i \leq j$. Given a monomial $u \in S_n$, let $\langle u \rangle$ denote the smallest Borel-stable monomial ideal in $R_n$ containing $u$, and let $B(u)$ denote the unique minimal system of monomial generators of $\langle u \rangle$. Then $B(u)$ may be described as the smallest set of monomials containing $u$ and stable under the operations $v \mapsto v x_i / x_j$ whenever $x_j$ divides $v$ and $i \leq j$.

Recall that a homogeneous ideal $I \subseteq R_n$ is a *Gotzmann ideal* if, from a certain degree on, its Hilbert function attains Macaulay's lower bound. See e.g. [4, 7] for more details. Determining which homogeneous ideals are Gotzmann ideals is notoriously difficult. This will be illustrated in this paper, where our determination of all monomials $u$ in $S_4$ such that the ideal $\langle u \rangle$ is a Gotzmann ideal involves a surprisingly complicated formula. We introduce the following definition.

**Definition 1.1.** We say that a monomial $u \in S_n$ is a *Gotzmann monomial* if its associated Borel-stable monomial ideal $\langle u \rangle$ is a Gotzmann ideal.

Determining all Gotzmann monomials in $S_n$ is a widely open problem. Indeed, the current knowledge about it is limited to the case $n \leq 3$. Specifically, for $n \leq 2$

all monomials in $S_1$ or $S_2$ are Gotzmann, whereas for $n = 3$ *the monomial $x_1^a x_2^b x_3^t$ is Gotzmann in $S_3$ if and only if $t \geq \binom{b}{2}$.* The latter result can be deduced from [13, Proposition 8]. A short proof using the general tools developed in this paper will be given in the last section.

The above result for $n = 3$ illustrates a general property of Gotzmann monomials, proved in [4] using Gotzmann's persistence theorem.

**Theorem 1.2.** *Let $u \in S_n$.*

   (1) *There exists $k \in \mathbb{N}$ such that $u x_n^k$ is Gotzmann in $S_n$.*
   (2) *If $u$ is Gotzmann in $S_n$, then so is $u x_n$.*

This reduces the determination of Gotzmann monomials in $S_n$ to the following question. Given $u_0 \in S_{n-1}$, what is the least exponent $t \geq 0$ such that $u_0 x_n^t$ is a Gotzmann monomial in $S_n$?

Our main result in this paper is the classification of all Gotzmann monomials in $S_4$ (see Theorem 7.7); we state that *a monomial $u = x_1^a x_2^b x_3^c x_4^t$ is a Gotzmann monomial in $S_4$ if and only if*

$$t \geq \binom{\binom{b}{2}}{2} + \frac{b+4}{3}\binom{b}{2} + (b+1)\binom{c+1}{2} + \binom{c+1}{3} - c.$$

Interestingly, before achieving this rather intricate characterization, all the easy-to-perform computer-algebraic experiments we ran in order to get a clue at it were of no help. Only the conceptual tools developed below allowed us to formulate and prove this result. Completing the analogous task in $S_n$ for $n \geq 5$ remains an open problem.

1.1. **Some related results.** In 2000, Aramova, Avramov, and Herzog posed the open problem of determining which monomial ideals are Gotzmann ideals [2]. Some partial answers have since emerged. In 2003, the first author characterized all principal Borel ideals with Borel generator up to degree 4 which are Gotzmann [4]. In 2006, Mermin classified Lexlike ideals, i.e. ideals which are generated by initial segments of "lexlike" sequences [10]. In 2007, Murai classified Gotzmann ideals in the polynomial ring in three variables [13]. In 2008, Murai and Hibi described all Gotzmann ideals in $K[x_1, \ldots, x_n]$ with fewer than $n$ generators [11]. In 2008, L. Sorrenti, A. Olteanu, and O. Olteanu classified Gotzmann ideals which are generated by segments in the lexicographic order [14]. In 2012, Hoefel characterized all Gotzmann edge ideals [8]. In 2012, Hoefel and Mermin described all Gotzmann squarefree monomial ideals [9]. See also [15] for related results.

1.2. **Contents.** In Section 2, we recall or introduce basic notions such as lexsegments and lexintervals, the sets of *gaps* and *cogaps* of a monomial, the *maxgen monomial* of a set of monomials, and finally Gotzmann monomials and criteria in terms of gaps and cogaps to recognize them. In Section 3, we focus on properties of the gaps and cogaps of a monomial and how to compute them. In Section 4, we describe the lexicographic predecessor and successor of a monomial. Section 5

is devoted to the determination of the maxgen monomial of lexintervals. In Section 6, we show some specific behaviors of the first and last variables. Finally, in Section 7 we use all the material developed in the preceding sections to prove our main theorem on the characterization of Gotzmann monomials in four variables.

## 2. Background and basic notions

2.1. **Lexsegments and lexintervals.** Recall the definition of the *lexicographic order* on $S_{n,d}$. Let $u, v \in S_{n,d}$. Write $u = \mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ with $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, and similarly $v = \mathbf{x}^{\mathbf{b}}$ with $\mathbf{b} \in \mathbb{N}^n$. By definition, we have

$$u >_{\text{lex}} v$$

if and only if the leftmost nonzero coordinate of $\mathbf{a} - \mathbf{b}$ is positive. Equivalently, let

$$u = x_{i_1} \cdots x_{i_d}, \ v = x_{j_1} \cdots x_{j_d} \in S_{n,d}$$

with $i_1 \leq \cdots \leq i_d$, $j_1 \leq \cdots \leq j_d$. Then $u >_{\text{lex}} v$ if and only if the leftmost nonzero coordinate of $(i_1 - j_1, \ldots, i_d - j_d)$ is negative. For simplicity, we shall omit the subscript and write $\geq$ instead of $\geq_{\text{lex}}$.

We shall need below the following well-known equivalence.

**Lemma 2.1.** *Let $u, v \in S_{n,d}$. Then for all $1 \leq i \leq n$, we have $u > v$ if and only if $x_i u > x_i v$.*

*Proof.* Write $u = \mathbf{x}^{\mathbf{a}}$, $v = \mathbf{x}^{\mathbf{b}}$ with $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$. Then $x_i u = \mathbf{x}^{\mathbf{a}+\mathbf{e}_i}$, $x_i v = \mathbf{x}^{\mathbf{b}+\mathbf{e}_i}$, where $\mathbf{e}_i \in \mathbb{N}^n$ is the basis vector with a 1 at the $i$th coordinate and 0 elsewhere. The statement follows since

$$(\mathbf{a} + \mathbf{e}_i) - (\mathbf{b} + \mathbf{e}_i) = \mathbf{a} - \mathbf{b}. \qquad \square$$

The following notation will be used throughout.

**Notation 2.2.** For $u \in S_{n,d}$, we denote by $L(u)$ the *lexsegment* determined by $u$, i.e.

$$L(u) = \{v \in S_{n,d} \mid v \geq u\}.$$

More generally, for $u_1, u_2 \in S_{n,d}$ such that $u_1 \geq u_2$, we denote by $L(u_1, u_2)$ the *lexinterval* of intermediate monomials, namely

$$L(u_1, u_2) = \{v \in S_{n,d} \mid u_1 \geq v \geq u_2\}.$$

Thus $L(u) = L(x_1^d, u)$ for $u \in S_{n,d}$. Finally, we use the notation

$$L^*(u_1, u_2) = L(u_1, u_2) \setminus \{u_1\} = \{v \in S_{n,d} \mid u_1 > v \geq u_2\}.$$

2.2. **Gotzmann sets.**

**Definition 2.3.** A subset $B \subseteq S_n$ is said to be *Borel-stable* if $u \in B$ implies $x_i u / x_j \in B$ for all $1 \leq i \leq j \leq n$ such that $x_j$ divides $u$.

**Definition 2.4.** A monomial ideal $I \subseteq R_n$ is said to be *Borel-stable* if its set of monomials $I \cap S_n$ is a Borel-stable set.

Let $B \subseteq S_{n,d}$. We define and denote the *shade*[1] of $B$ by

$$\mathrm{Shad}(B) = \{x_i u \mid u \in B, \, i = 1, \ldots, n\} \subseteq S_{n,d+1}.$$

For $i \geq 2$, the $i$-th shade of $B$ is defined recursively by

$$\mathrm{Shad}^i(B) = \mathrm{Shad}(\mathrm{Shad}^{i-1}(B)).$$

**Notation 2.5.** Let $B \subseteq S_{n,d}$. We denote by $B^{\mathrm{lex}}$ the unique lexsegment in $S_{n,d}$ such that $|B^{\mathrm{lex}}| = |B|$.

Thus, there exists a unique monomial $w_B$ in $S_{n,d}$ such that

$$B^{\mathrm{lex}} = L(w_B).$$

**Example 2.6.** Let $B = \left\{x_1^2, x_1 x_2, x_1 x_3, x_2^2\right\} \subseteq S_{4,2}$. The lexsegment of length $|B| = 4$ in $S_{4,2}$ is

$$L(x_1 x_4) = \left\{x_1^2, x_1 x_2, x_1 x_3, x_1 x_4\right\}.$$

Hence $B^{\mathrm{lex}} = \left\{x_1^2, x_1 x_2, x_1 x_3, x_1 x_4\right\} = L(x_1 x_4)$, and thus $w_B = x_1 x_4$.

The following result can be found in [7, Theorem 2.7].

**Theorem 2.7.** *For any subset $B \subseteq S_{n,d}$, one has*

$$|\mathrm{Shad}(B)| \geq |\mathrm{Shad}(B^{\mathrm{lex}})|.$$

*Proof.* See [7]. □

**Definition 2.8.** A subset $B \subseteq S_{n,d}$ is said to be a *Gotzmann set* if equality in Theorem 2.7 is achieved, i.e. if

$$|\mathrm{Shad}(B)| = |\mathrm{Shad}(B^{\mathrm{lex}})|.$$

Recall that a homogeneous ideal $I \subseteq R_n$ is a *Gotzmann ideal* if, from a certain degree on, its Hilbert function attains Macaulay's lower bound. Gotzmann sets are linked to Gotzmann ideals by the following result. See e.g. [15] for more details.

**Proposition 2.9.** *Let $B \subseteq S_{n,d}$ with $d \geq 1$. Then the ideal $(B)$ of $R_n$ spanned by $B$ is a Gotzmann ideal if and only if $B$ is a Gotzmann set.*

The next lemma is crucial in the characterization of Borel-stable sets which are Gotzmann sets. We first introduce some notation.

**Notation 2.10.** Let $u \in S_n$ be a monomial distinct from 1. We denote by $\max(u)$ the largest index $i \leq n$ such that $x_i$ divides $u$.

**Notation 2.11.** Let $B \subseteq S_{n,d}$ be a set of monomials of degree $d \geq 1$. For all $1 \leq i \leq n$, we denote by $m_i(B)$ the number of monomials $u \in B$ such that $\max(u) = i$.

Of course, we have $|B| = m_1(B) + \cdots + m_n(B)$.

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[1]'Shad' should stand for *shade* as in combinatorial set theory [1], and not for *shadow* as written in [4, 7]. The *shadow* of $B$ actually corresponds to the set of all monomials $u/x_j$ with $u \in B$ and $x_j$ dividing $u$.

**Lemma 2.12.** *Let $B \subseteq S_{n,d}$ be a Borel-stable set. Then $B$ is a Gotzmann set if and only if*

$$m_i(B) = m_i(B^{\mathrm{lex}})$$

*for all $1 \leq i \leq n$.*

*Proof.* See [7] and Lemma 1.6 in [4]. □

Given $B \subseteq S_{n,d}$, it will be useful to collect the numbers $m_i(B)$ for $1 \leq i \leq n$ as a single monomial. This gives rise to the following definition.

**Definition 2.13.** Let $B \subseteq S_{n,d}$ be a set of monomials of degree $d \geq 1$. Let $m_i = m_i(B)$ for $1 \leq i \leq n$. The *maxgen monomial* of $B$ is defined by

$$\mathrm{maxgen}(B) = x_1^{m_1} \cdots x_n^{m_n}.$$

Note that $\deg(\mathrm{maxgen}(B)) = |B|$. We may now rephrase Lemma 2.12 using the maxgen monomial.

**Lemma 2.14.** *Let $B \subseteq S_{n,d}$ be a Borel-stable set. Then $B$ is a Gotzmann set if and only if*

$$\mathrm{maxgen}(B) = \mathrm{maxgen}(B^{\mathrm{lex}}).$$

*Proof.* Follows from Lemma 2.12 and the definition of $\mathrm{maxgen}(B)$. □

2.3. **The maxgen monomial revisited.** Given $B \subseteq S_{n,d}$ with $d \geq 1$, we now describe $\mathrm{maxgen}(B)$ in a slightly more useful way. First, some preliminaries.

**Notation 2.15.** Let $u \in S_n$ be a monomial distinct from 1. We denote by

- $\min(u)$ the smallest index $i \geq 1$ such that $x_i$ divides $u$;
- $\lambda(u) = x_j$, where $j = \max(u)$.

Thus $\lambda(u)$ divides $u$, and it is the "last", or lexicographically smallest, variable with this property. This yields a function

$$\lambda \colon S_{n,d} \setminus \{1\} \longrightarrow S_{n,1} = \{x_1, \ldots, x_n\}.$$

For instance, if $u = x_2^3 x_3 x_4^2$, then $\min(u) = 2$, $\max(u) = 4$, and $\lambda(u) = x_4$.

**Lemma 2.16.** *Let $B \subseteq S_{n,d}$. Then*

$$\mathrm{maxgen}(B) = \prod_{w \in B} \lambda(w) = \prod_{w \in B} x_{\max(w)}.$$

*Proof.* Directly follows from the definitions. □

Thus, $\mathrm{maxgen}(B)$ may be viewed as the maximal index generating function of all monomials in $B$.

We shall sometimes tacitly use the following easy observation.

**Remark 2.17.** If $B \subseteq B' \subseteq S_{n,d}$, then $\mathrm{maxgen}(B)$ divides $\mathrm{maxgen}(B')$.

2.4. **Gaps and cogaps.**

**Notation 2.18.** Let $u \in S_n$. We denote by $B(u)$ the smallest Borel-stable subset of $S_n$ containing $u$.

Observe that if $u \in S_{n,d}$, then $B(u) \subseteq S_{n,d}$.

**Lemma 2.19.** *Let $u \in S_{n,d}$. Then $B(u) \subseteq L(u)$.*

*Proof.* Let $v \in B(u)$. Then $v$ is obtained from $u$ by repeated operations of the form

$$u' \mapsto x_i u'/x_j,$$

where $u' \in B(u)$, $x_j$ divides $u'$, and $i \leq j$. Since $x_i u'/x_j \geq u'$ at each such step, it follows that $v \geq u$, whence $v \in L(u)$. □

For our present purposes, it is of particular interest to consider the set difference $L(u) \setminus B(u)$. The following concept first arose in [4].

**Definition 2.20.** Let $u \in S_{n,d}$. We set $\mathrm{gaps}(u) = L(u) \setminus B(u)$, and we call *gaps of u* the elements of this set.
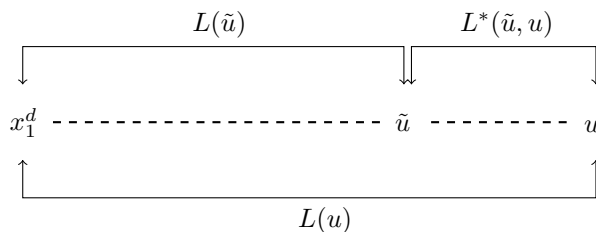
**Notation 2.21.** Let $u \in S_{n,d}$. We denote by $\tilde{u} \in S_{n,d}$ the unique monomial such that

$$L(\tilde{u}) = B(u)^{\mathrm{lex}}.$$

Since $B(u)$ and $B(u)^{\mathrm{lex}}$ have the same cardinality by definition, we have

$$|L(\tilde{u})| = |B(u)|.$$

Moreover, since $B(u) \subseteq L(u)$ by the above lemma, we have $x_1^d \geq \tilde{u} \geq u$ and so $L(\tilde{u}) \subseteq L(u)$. Here is an illustration of the situation:



Since $|L(\tilde{u})| = |B(u)|$, we have

$$|L(u) \setminus L(\tilde{u})| = |L(u) \setminus B(u)| = |\mathrm{gaps}(u)|.$$

This motivates our definition of $\mathrm{cogaps}(u)$, a lexinterval with the same cardinality as $\mathrm{gaps}(u)$.

**Definition 2.22.** Let $u \in S_{n,d}$. We set $\mathrm{cogaps}(u) = L(u) \setminus L(\tilde{u})$. That is, $\mathrm{cogaps}(u)$ is the lexinterval of cardinality $g = |\mathrm{gaps}(u)|$ with smallest element $u$. Equivalently,

$$\mathrm{cogaps}(u) = L^*(\tilde{u}, u).$$

By construction, we have $|\operatorname{gaps}(u)| = |\operatorname{cogaps}(u)|$ and two partitions of $L(u)$, namely:

$$L(u) = B(u) \sqcup \operatorname{gaps}(u),$$
$$L(u) = L(\tilde{u}) \sqcup \operatorname{cogaps}(u).$$

**Example 2.23.** Let $n = 4$, $d = 2$, and $u = x_2 x_3$. Then

$$B(u) = \left\{ x_1^2, x_1 x_2, x_1 x_3, x_2^2, x_2 x_3 \right\},$$
$$L(u) = \left\{ x_1^2, x_1 x_2, x_1 x_3, x_1 x_4, x_2^2, x_2 x_3 \right\},$$
$$\operatorname{gaps}(u) = L(u) \setminus B(u) = \{ x_1 x_4 \}.$$

The unique monomial $\tilde{u} \in L(u)$ such that $|L(\tilde{u})| = |B(u)|$ is $\tilde{u} = x_2^2$. Hence $\operatorname{cogaps}(u) = L(u) \setminus L(\tilde{u}) = \{ x_2 x_3 \}$.

A word of caution regarding $L(u)$ and $B(u)$ is needed here.

**Remark 2.24.** For the lexsegment determined by $u \in S_n$, one should write $L_n(u)$ rather than $L(u)$. Indeed, let $m < n$ be positive integers. Then $S_m \subset S_n$ canonically. Let now $u \in S_m$. Then $L_m(u) \neq L_n(u)$ in general. For instance, with $u = x_2 x_3$ as above, we have

$$L_3(u) = \left\{ x_1^2, x_1 x_2, x_1 x_3, x_2^2, x_2 x_3 \right\} = L_4(u) \setminus \{ x_1 x_4 \}.$$

Consequently, one should also write $\operatorname{gaps}_n(u)$ rather than $\operatorname{gaps}(u)$. However, we shall systematically omit the index $n$ since it will be fixed in any given discussion below. On the other hand, the set $B(u)$ is independent of $n$.

### 2.5. Gotzmann monomials.

**Definition 2.25.** Let $u \in S_{n,d}$. We say that $u$ is a *Gotzmann monomial* if $B(u)$ is a Gotzmann set.

**Remark 2.26.** Note that Gotzmann monomials in $S_n$ may no longer be Gotzmann monomials in $S_{n+1}$. For instance, $x_2 x_3$ is Gotzmann in $S_3$ but not in $S_4$.

Our determination of Gotzmann monomials in $S_3$ and $S_4$ will use the following general characterization.

**Theorem 2.27.** *Let $u \in S_n$. Then $u$ is a Gotzmann monomial if and only if*

$$\operatorname{maxgen}(\operatorname{gaps}(u)) = \operatorname{maxgen}(\operatorname{cogaps}(u)).$$

*Proof.* It follows from Definition 2.25 and Lemma 2.14 that $u$ is a Gotzmann monomial if and only if

$$\operatorname{maxgen}(B(u)) = \operatorname{maxgen}(B(u)^{\operatorname{lex}}).$$

Now by definition of $\tilde{u}$, we have

$$L(\tilde{u}) = B(u)^{\operatorname{lex}}.$$

Hence $u$ is a Gotzmann monomial if and only if

$$\operatorname{maxgen}(B(u)) = \operatorname{maxgen}(L(\tilde{u})).$$

Since

$$B(u) \sqcup \text{gaps}(u) = L(\tilde{u}) \sqcup \text{cogaps}(u),$$

as both sides coincide with $L(u)$, it follows that

$$\text{maxgen}(B(u)) = \text{maxgen}(L(\tilde{u})) \iff \text{maxgen}(\text{gaps}(u)) = \text{maxgen}(\text{cogaps}(u)),$$

and the proof is complete. $\qquad\square$

Thus, from now on, our task will be to develop tools to compute or determine $\text{gaps}(u)$, $\text{cogaps}(u)$, and their respective maxgen monomials, so as to be able to apply the characterization of Gotzmann monomials provided by Theorem 2.27.

## 3. Some results on gaps

Let $u \in S_{n,d}$. Recall that $B(u) \subseteq L(u)$ and that $\text{gaps}(u) = L(u) \setminus B(u)$. We first describe the gaps of $u$ in an equivalent way.

**Lemma 3.1.** *Let $u, v \in S_{n,d}$ be monomials of degree $d$ in $S_n$. Set $u = x_{i_1} \cdots x_{i_d}$ with $i_1 \leq \cdots \leq i_d$ and $v = x_{j_1} \cdots x_{j_d}$ with $j_1 \leq \cdots \leq j_d$. The following are equivalent:*

*(1) $v$ is a gap of $u$;*
*(2) there exist indices $1 \leq s < t \leq d$ such that*

$$(j_1, \ldots, j_{s-1}) = (i_1, \ldots, i_{s-1}), \quad j_s < i_s, \quad j_t > i_t.$$

*Proof.* We have $v \neq u$ since $v \notin B(u)$. The existence of an index $s$ with the given property then follows from the hypothesis $v \in L(u)$. The existence of an index $t > s$ with the desired property then directly follows from the hypothesis $v \notin B(u)$. $\quad\square$

We need yet another notation which will be used to give a structural description of $\text{gaps}(u)$.

**Notation 3.2.** For a monomial $u = x_{i_1} \cdots x_{i_d}$ with $i_1 \leq \cdots \leq i_d$, and for any integer $0 \leq k \leq d$, we denote by $\text{pre}_k(u)$ the *prefix* of $u$ of degree $k$, defined by

$$\text{pre}_k(u) = x_{i_1} \cdots x_{i_k}.$$

Observe that $\text{pre}_k(u)$ may be characterized as follows: it is the unique monomial $v$ of degree $k$ dividing $u$ and satisfying $\max(v) \leq \min(u/v)$.

**Definition 3.3.** For any $v \in S_{n,k}$, we define subsets $A_1(v), A_2(v) \subset S_{n,k}$ as follows:

$$A_1(v) = B(v) \setminus \{v\},$$
$$A_2(v) = \{w \in S_{n,k} \mid \min(v) + 1 \leq \min(w) \leq n\}.$$

**Proposition 3.4.** *Let $u \in S_{n,d}$. For all $1 \leq k \leq d - 1$, let $u_k = \text{pre}_k(u)$. Then*

$$\text{gaps}(u) = \bigsqcup_{k=1}^{d-1} A_1(u_k) A_2(u/u_k).$$

*Proof.* First, any monomial $v = v_1 v_2$ with $v_1 \in A_1(u_k)$, where $u_k$ is the prefix of $u$ of degree $k$ for some $1 \leq k \leq d-1$, and $v_2 \in A_2(u/u_k)$, is a gap of $u$ by construction and Lemma 3.1.

Conversely, let $v$ be a gap of $u$. Set $u = x_{i_1} \cdots x_{i_d}$ with $i_1 \leq \cdots \leq i_d$ and $v = x_{j_1} \cdots x_{j_d}$ with $j_1 \leq \cdots \leq j_d$. In the notation of Lemma 3.1, let $s$ be such that $i_s < j_s$, and let $t > s$ be the *least index* satisfying $j_t > i_t$. Set $k = t - 1$. Then by construction, the factor $x_{j_1} \cdots x_{j_k}$ of degree $k$ of $v$ belongs to $B(u_k)$, since $i_\alpha \leq j_\alpha$ for all $\alpha < t$ by minimality of $t$, and in fact belongs to $A_1(u_k)$ since $i_s < j_s$, whereas the cofactor $x_{j_{k+1}} \cdots x_{j_d}$ belongs to $A_2(u/u_k)$ since $j_d \geq \cdots \geq j_{k+1} = j_t > i_t$. $\square$

**Corollary 3.5.** *Let $u = x_{i_1} \cdots x_{i_d}$ with $i_1 \leq \cdots \leq i_d$. Then*

$$| \operatorname{gaps}(u)| = \sum_{k=1}^{d-1} \left( \left| B(x_{i_1} \cdots x_{i_k}) \right| - 1 \right) \cdot \left| S_{n-i_{k+1}, d-k} \right|.$$

*Proof.* The number of monomials $w_2 \in S_n$ of degree $d - k$ in the variables $x_{i_{k+1}+1}$, $\ldots, x_n$ is equal to the number of monomials of the same degree in $S_{n-i_{k+1}}$, i.e. to $\left| S_{n-i_{k+1}, d-k} \right|$. $\square$

This prompts us to find good formulas for $|B(w)|$ for any monomial $w$. Here is an inductive approach.

**Proposition 3.6.** *Let $w \in S_n$ and $m = \max(w)$. Let $r \geq 1$ be the largest exponent such that $x_m^r$ divides $w$. Let $v = w/x_m^r$, so that $\max(v) < m$. Then*

$$B(w) = \bigsqcup_{i=0}^{r} B\left(v x_{m-1}^{r-i}\right) x_m^i.$$

*Proof.* Indeed, let $w' \in B(w)$. Then $\max w' \leq m$. If $\max w' < m$, then clearly $w' \in B(v x_{m-1}^r)$. Otherwise, if $\max w' = m$, let $i$ be the largest exponent such that $x_m^i$ divides $w'$, so that $1 \leq i \leq r$. Let $v' = w'/x_m^i$. Then clearly $v' \in B(v x_{m-1}^{r-i})$, so that $w' \in B(v x_{m-1}^{r-i}) x_m^i$. $\square$

**Corollary 3.7.** *As above, let $w = v x_m^r \in S_n$ with $\max(w) = m$ and $\max(v) < m$. Then*

$$|B(w)| = \sum_{i=0}^{r} \left| B(v x_{m-1}^{r-i}) \right|.$$

*Proof.* Directly follows from the above partition of $B(w)$. $\square$

This corollary reduces the computation of $|B(w)|$ for monomials in $S_n$ to that for monomials in $S_{n-1}$.

## 4. Predecessors and successors

**Definition 4.1.** Let $u, v \in S_{n,d}$ such that $u > v$. We say that $u$ *covers* $v$ if there are no intermediate monomials between them, i.e. if for any $w \in S_{n,d}$ such that

$u \geq w \geq v$, we have $w = u$ or $w = v$. In that case, we say that $u$ is the *predecessor* of $v$, that $v$ is the *successor* of $u$, and we write

$$u = \operatorname{pred}(v), \quad v = \operatorname{succ}(u).$$

Since $x_1^d$ and $x_n^d$ are the largest and smallest elements in $S_{n,d}$, respectively, the predecessor of $x_1^d$ and the successor of $x_n^d$ are undefined.

Note that, for all $u \in S_{n,d}$ with $u \notin \{x_1^d, x_n^d\}$, we have

$$\operatorname{succ}(\operatorname{pred}(u)) = \operatorname{pred}(\operatorname{succ}(u)) = u.$$

**Proposition 4.2.** *Let $u \in S_n$ and $g = |\operatorname{gaps}(u)|$. Then $\tilde{u}$ is the $g$th predecessor of $u$, i.e.*

$$\tilde{u} = \operatorname{pred}^g(u).$$

*Proof.* Indeed, recall the partition $L(u) = L(\tilde{u}) \sqcup L^*(\tilde{u}, u)$. Thus $\tilde{u}$, the smallest element of $L(\tilde{u})$, is the predecessor of the largest element of $L^*(\tilde{u}, u) = \operatorname{cogaps}(u)$. Since $\operatorname{cogaps}(u)$ has cardinality $g$ and is a lexinterval ending at $u$, its largest element is $\operatorname{pred}^{g-1}(u)$. Hence $\tilde{u} = \operatorname{pred}(\operatorname{pred}^{g-1}(u))$, as desired. □

Lexintervals ending at a monomial $u$ are made up of iterated predecessors of $u$. This motivates the following notation.

**Notation 4.3.** Let $u \in S_{n,d}$ and $r \leq |L(u)|$. Then,

$$\operatorname{pred}_r(u) = \{\operatorname{pred}^i(u) \mid 0 \leq i < r\}.$$

That is, $\operatorname{pred}_r(u)$ is the set of $r$ predecessors of $u$, including $u$ itself. This set is well defined, since $r \leq |L(u)|$ by hypothesis. Of course $\operatorname{pred}_r(u)$ is a lexinterval, since

$$\operatorname{pred}_r(u) = \{w \in S_{n,d} \mid \operatorname{pred}^r(u) > w \geq u\} = L^*(\operatorname{pred}^r(u), u). \qquad (4.1)$$

We may now reinterpret the lexinterval $\operatorname{cogaps}(u)$ in terms of the above concept.

**Proposition 4.4.** *Let $u \in S_{n,d}$ and $g = |\operatorname{gaps}(u)|$. Then*

$$\operatorname{cogaps}(u) = \operatorname{pred}_g(u).$$

*Proof.* By Definition 2.22, we have $|\operatorname{cogaps}(u)| = |\operatorname{gaps}(u)| = g$, and $\operatorname{cogaps}(u) = L^*(\tilde{u}, u)$, where $\tilde{u} = \operatorname{pred}^g(u)$. The stated equality follows from (4.1). □

As we shall need to determine $\operatorname{pred}^i(u)$ for any given $u \in S_{n,d}$, we need an explicit description of $\operatorname{pred}(u)$. We start with the description of the successor of a monomial in $S_{n,d}$ distinct from $x_n^d$.

**Proposition 4.5.** *Let $u \in S_{n,d}$, distinct from $x_n^d$. Set $u = v x_n^{a_n}$ with $a_n \in \mathbb{N}$ and $\max(v) = m$ with $m \leq n - 1$. Then*

$$\operatorname{succ}(u) = (v/x_m)x_{m+1}^{a_n+1}.$$

*Proof.* This easily follows from the definition of the lexicographic order on $S_{n,d}$. □

**Proposition 4.6.** *Let $u \in S_{n,d}$ be such that $u \neq x_1^d$, and let $m = \max(u)$. Write $u = v x_m^a$ with $a \geq 1$ and $v \in S_{n,d-a}$, so that $\max(v) \leq m - 1$. Then*

$$\operatorname{pred}(u) = v x_{m-1} x_n^{a-1}.$$

*Proof.* Follows from the description above of the successor of a monomial in $S_{n,d}$ distinct from $x_n^d$. □

The next corollary compares $\max(\operatorname{pred}(u))$ with $\max(u)$. There are only two possible outcomes, linked to whether $\lambda(u)^2$ divides $u$ or not; recall that $\lambda(u)$ always divides $u$ by construction.

**Corollary 4.7.** *For all $u \in S_{n,d}$ such that $u \neq x_1^d$, we have*

$$\max(\operatorname{pred}(u)) = \begin{cases} n, & \text{if } \lambda(u)^2 \mid u; \\ \max(u) - 1, & \text{if not.} \end{cases}$$

**Example 4.8.** The lexicographically smallest monomial $u \in S_{4,d}$ such that

$$\max(\operatorname{pred}(u)) \leq 2$$

is $x_2^{d-1}x_3$. That is, for all $v \in S_{4,d}$ such that $v \leq x_2^{d-1}x_3$, we have $\max(v) \geq 3$, i.e. $\lambda(v) \in \{x_3, x_4\}$.

## 5. The maxgen monomial of lexintervals

5.1. **The function $\mu_n$.** We now introduce a function of two monomials $u_1 \geq u_2$ in $S_{n,d}$ which will later be used to give an equivalent description of cogaps. Recall the notation

$$L^*(u_1, u_2) = L(u_1, u_2) \setminus \{u_1\} = \{v \in S_{n,d} \mid u_1 > v \geq u_2\}.$$

**Definition 5.1.** For $u_1, u_2 \in S_{n,d}$ such that $u_1 \geq u_2$, we define $\mu_n(u_2, u_1) \in S_n$ to be the maxgen monomial of the lexinterval $L^*(u_1, u_2)$, i.e.

$$\mu_n(u_2, u_1) = \operatorname{maxgen}(L^*(u_1, u_2)).$$

Equivalently, recalling that maxgen collects the last variables of a set of monomials and takes their product:

$$\mu_n(u_2, u_1) = \prod_{u_1 > v \geq u_2} \lambda(v).$$

Note that by construction, the last variable of $u_1$ is *not* taken into account in $\mu_n(u_2, u_1)$.

**Remark 5.2.** As in Remark 2.24, we have $S_{n,d} \subset S_{n+1,d}$ canonically. Now if $u_1, u_2 \in S_{n,d}$, then $\mu_n(u_2, u_1)$ and $\mu_{n+1}(u_2, u_1)$ differ in general. However, when the number $n$ of variables involved is clear from the context, we shall simply write $\mu(u_2, u_1)$ for $\mu_n(u_2, u_1)$.

The function $\mu$ on $S_{n,d}$ has the following transitive property.

**Lemma 5.3.** *For all $u_1, u_2, u_3 \in S_{n,d}$ such that $u_1 \geq u_2 \geq u_3$, we have*

$$\mu(u_3, u_1) = \mu(u_3, u_2)\mu(u_2, u_1).$$

*Proof.* Directly follows from the definition. □

**Notation 5.4.** For monomials $u_1 \geq u_2$ in $S_{n,d}$, we shall occasionally denote the equality $\mu(u_2, u_1) = w$ as follows:

$$u_2 \xrightarrow{\quad w \quad} u_1.$$

Lemma 5.3 then amounts to *arrow composition*: if $u_1 \geq u_2 \geq u_3$ in $S_{n,d}$, then

$$u_3 \xrightarrow{\quad w_2 \quad} u_2 \xrightarrow{\quad w_1 \quad} u_1$$

is equivalent to

$$u_3 \xrightarrow{\quad w_2 w_1 \quad} u_1.$$

For instance, starting from $x_3^2$ and taking successive predecessors in $S_3$, one has

$$x_3^2 \xrightarrow{\quad x_3 \quad} x_2 x_3 \xrightarrow{\quad x_3 \quad} x_2^2 \xrightarrow{\quad x_2 \quad} x_1 x_3 \xrightarrow{\quad x_3 \quad} x_1 x_2.$$

By arrow composition, this may be summarized as

$$x_3^2 \xrightarrow{\quad x_2 x_3^3 \quad} x_1 x_2,$$

expressing the equality $\mu_3(x_3^2, x_1 x_2) = x_2 x_3^2$.

**Remark 5.5.** If $r \leq |L(u)|$, then $\mathrm{pred}_r(u)$ is defined and we have

$$u \xrightarrow{\quad \mathrm{maxgen}(\mathrm{pred}_r(u)) \quad} \mathrm{pred}^r(u)$$

by construction.

In particular, with $r = |\mathrm{gaps}(u)|$, this yields the following tool in view of effectively applying Theorem 2.27.

**Proposition 5.6.** *Let $u \in S_{n,d}$. Then*

$$\mathrm{maxgen}(\mathrm{cogaps}(u)) = \mu(u, \tilde{u}),$$

*i.e., in arrow notation,* $u \xrightarrow{\quad \mathrm{maxgen}(\mathrm{cogaps}(u)) \quad} \tilde{u}.$

*Proof.* Follows from the above remark and the facts that, if $g = |\mathrm{gaps}(u)|$, then $\tilde{u} = \mathrm{pred}^g(u)$ and $\mathrm{cogaps}(u) = \mathrm{pred}_g(u)$ by Propositions 4.2 and 4.4, respectively. $\square$

**Lemma 5.7.** *Let $v, w \in S_n$. If $\max(v) < \max(w)$, then*

$$\mathrm{pred}(vw) = v\,\mathrm{pred}(w).$$

*Proof.* Let $j = \max(w)$. Then $w = w' x_j^a$ with $\max(w') < j$ and $a \geq 1$. Hence

$$\mathrm{pred}(w) = w' x_{j-1} x_n^{a-1}.$$

Now $vw = vw' x_j^a$, and $\max(vw') < j$ since $\max(v) < j$ by hypothesis. Hence

$$\mathrm{pred}(vw) = vw' x_{j-1} x_n^{a-1} = v\,\mathrm{pred}(w). \qquad \square$$

**Lemma 5.8.** *For all $u_1, u_2 \in S_{n,d}$ such that $u_1 > u_2$ and all $v \in S_n$ such that $\max(v) < \min \mu(u_2, u_1)$, we have*

$$\mu(vu_2, vu_1) = \mu(u_2, u_1).$$

*Proof.* Let $w \in S_{n,d}$ satisfy $u_1 > w \geq u_2$. We have $vu_1 > vw \geq vu_2$, since the product is compatible with the lex order. It follows from the hypothesis that $\max(v) < \max(w)$. Lemma 5.7 then implies $\mathrm{pred}(vw) = v\,\mathrm{pred}(w)$. Therefore $L^*(vu_1, vu_2) = vL^*(u_1, u_2)$, whence the conclusion. $\square$

In order to apply this lemma, we need some control on $\min \mu(u_2, u_1)$. This is provided by the next proposition. First, a lemma.

**Lemma 5.9.** *Let $u, v \in S_{n,d}$. If $u \geq v$ then $\min u \leq \min v$.*

*Proof.* By definition of the lex order, small indices weigh more. Hence if $\min u > \min v$ then $u < v$. $\square$

**Proposition 5.10.** *Let $u_1, u_2 \in S_{n,d}$. If $u_1 > u_2$ then $\min \mu(u_2, u_1) > \min u_1$.*

*Proof.* By Definition 5.1, we have

$$\mu(u_2, u_1) = \prod_{u_1 > v \geq u_2} \lambda(v).$$

For $v < u_1$, the above lemma implies $\min v \geq \min u_1$. Hence $\max v > \min u_1$, for otherwise we would have $\max v = \min v = \min u_1$, implying $v = x_i^d$ for some $i$. But from $v = x_i^d < u_1$, it follows that $\min u_1 < i$, contradicting $\min v = \min u_1$. Having established that $\max v > \min u_1$ for all $v < u_1$, it follows that $\min \mu(u_2, u_1) > \min u_1$, as stated. $\square$

Here are straightforward applications.

**Corollary 5.11.** *For all $u_1, u_2 \in S_{n,d}$ such that $u_1 > u_2$ and all $v \in S_n$ such that $\max v \leq \min u_1$, we have*

$$\mu(vu_2, vu_1) = \mu(u_2, u_1).$$

*Proof.* By the above proposition, we have $\min \mu(u_2, u_1) > \min u_1$. Hence $\max v < \min \mu(u_2, u_1)$ by hypothesis, and the claimed equality then follows from Lemma 5.8. $\square$

**Corollary 5.12.** *Let $m \leq n - 1$ and let $u \in S_{n,d}$ be such that $u < x_m^d$. Then*

$$\min \mu(u, x_m^d) \geq m + 1.$$

*Proof.* Directly follows from the above proposition. $\square$

We now compute $\mu(vx_m^k, vx_{m-1}^k)$ provided $\max v \leq m - 1$. For instance, in $S_{n,d}$ we have by the theorem below:

$$vx_n^k \xrightarrow{\;x_n^k\;} vx_{n-1}^k, \qquad\qquad \text{if } \max v \leq n - 1;$$

$$vx_{n-1}^k \xrightarrow{\;x_{n-1}^k x_n^{\binom{k}{2}}\;} vx_{n-2}^k, \qquad\qquad \text{if } \max v \leq n - 2;$$

$$vx_{n-2}^k \xrightarrow{\;x_{n-2}^k x_{n-1}^{\binom{k}{2}} x_n^{\binom{k+1}{3}}\;} vx_{n-3}^k, \qquad\qquad \text{if } \max v \leq n - 3.$$

In view of a general statement, the following intermediate formula will be useful.

**Proposition 5.13.** *For all $2 \leq m \leq n$ and all $k \geq 1$, we have*

$$\mu(x_m^k, x_{m-1}^k) = x_m^k \prod_{1 \leq i \leq k-1} \mu(x_n^i, x_m^i). \tag{5.1}$$

*Proof.* By induction on $k$. For $k = 1$, it is clear that $\mu(x_m, x_{m-1}) = x_m$, since the predecessor of $x_m$ is $x_{m-1}$. Assume now $k \geq 2$ and that formula (5.1) holds for $k - 1$, i.e.

$$\mu(x_m^{k-1}, x_{m-1}^{k-1}) = x_m^{k-1} \prod_{1 \leq i \leq k-2} \mu(x_n^i, x_m^i).$$

Thus, in order to establish (5.1), we only need to show

$$\mu(x_m^k, x_{m-1}^k) = x_m \mu(x_n^{k-1}, x_m^{k-1}) \mu(x_m^{k-1}, x_{m-1}^{k-1}),$$

that is, by Lemma 5.3,

$$\mu(x_m^k, x_{m-1}^k) = x_m \mu(x_n^{k-1}, x_{m-1}^{k-1}). \tag{5.2}$$

In $S_n$, the predecessor of $x_m^k$ is $x_{m-1} x_n^{k-1}$. Hence we have

$$x_m^k \xrightarrow{\quad x_m \quad} x_{m-1} x_n^{k-1} \xrightarrow{\quad \mu(x_{m-1} x_n^{k-1}, x_{m-1} x_m^{k-1}) \quad} x_{m-1} x_m^{k-1}.$$

This means

$$\mu(x_m^k, x_{m-1} x_m^{k-1}) = x_m \mu(x_{m-1} x_n^{k-1}, x_{m-1} x_m^{k-1}). \tag{5.3}$$

But it follows from Corollary 5.11 that

$$\mu(x_{m-1} x_n^{k-1}, x_{m-1} x_m^{k-1}) = \mu(x_n^{k-1}, x_m^{k-1}). \tag{5.4}$$

By Lemma 5.3, we have

$$\mu(x_m^k, x_{m-1}^k) = \mu(x_m^k, x_{m-1} x_m^{k-1}) \mu(x_{m-1} x_m^{k-1}, x_{m-1}^k).$$

Moreover, by (5.3) and (5.4) again, we have

$$\mu(x_m^k, x_{m-1} x_m^{k-1}) = x_m \mu(x_n^{k-1}, x_m^{k-1}).$$

Hence

$$\mu(x_m^k, x_{m-1}^k) = x_m \mu(x_n^{k-1}, x_m^{k-1}) \mu(x_{m-1} x_m^{k-1}, x_{m-1}^k).$$

By Corollary 5.11, we have

$$\mu(x_{m-1} x_m^{k-1}, x_{m-1}^k) = \mu(x_m^{k-1}, x_{m-1}^{k-1}).$$

This proves (5.2) and hence the claimed formula (5.1). $\qquad\square$

Here is the promised general statement. As usual, by convention, an empty product equals 1, as occurs below for $m = n$.

**Theorem 5.14.** *For all $2 \leq m \leq n$, for all $k \geq 1$, and for all $v \in S_n$ such that $\max v \leq m - 1$, we have*

$$\mu(v x_m^k, v x_{m-1}^k) = x_m^k \prod_{i=1}^{n-m} x_{m+i}^{\binom{k-1+i}{i+1}}. \tag{5.5}$$

*Proof.* By Corollary 5.11, we have

$$\mu(vx_m^k, vx_{m-1}^k) = \mu(x_m^k, x_{m-1}^k).$$

Hence, it suffices to establish (5.5) for $v = 1$. We proceed by induction on $k$ and descending induction on $m$. For $k = 1$ and any $m \geq 1$, we have $\mu(x_m, x_{m-1}) = x_m$, and this plainly coincides with the right-hand side of (5.5) since $\binom{1-1+i}{i+1} = 0$ for all $i \geq 1$. Similarly, for $m = n$ and any $k \geq 1$, we have

$$x_n^k \xrightarrow{\quad x_n^k \quad} x_{n-1}^k,$$

since $L^*(x_{n-1}^k, x_n^k) = \{x_{n-1}^{k-i} x_n^i \mid 1 \leq i \leq k\}$ and

$$\mu(x_n^k, x_{n-1}^k) = \text{maxgen}(L^*(x_{n-1}^k, x_n^k)) = \prod_{x_{n-1}^k > v \geq x_n^k} \lambda(v).$$

Equivalently, in formula:

$$\mu(x_n^k, x_{n-1}^k) = x_n^k.$$

Assume now that (5.5) holds for some $m$ such that $n \geq m \geq 2$ and some $k \geq 2$. We now show that (5.5) also holds for $m - 1$. By arrow composition, we have

$$\mu(x_n^i, x_{m-1}^i) = \mu(x_n^i, x_m^i)\mu(x_m^i, x_{m-1}^i).$$

Therefore,

$$
\begin{aligned}
\prod_{i=1}^{k-1} \mu(x_n^i, x_{m-1}^i) &= \prod_{i=1}^{k-1} \mu(x_n^i, x_m^i) \prod_{i=1}^{k-1} \mu(x_m^i, x_{m-1}^i) \\
&= \prod_{i=1}^{n-m} x_{m+i}^{\binom{k-1+i}{i+1}} \prod_{i=1}^{k-1} \mu(x_m^i, x_{m-1}^i) && \text{(by induction on } m) \\
&= \prod_{i=1}^{n-m} x_{m+i}^{\binom{k-1+i}{i+1}} \prod_{i=1}^{k-1} x_m^i \prod_{j=1}^{n-m} x_{m+j}^{\binom{i-1+j}{j+1}} && \text{(by induction on } k) \\
&= x_m^{\binom{k}{2}} \prod_{i=1}^{n-m} x_{m+i}^{\binom{k-1+i}{i+1}} \prod_{i=1}^{k-1} \prod_{j=1}^{n-m} x_{m+j}^{\binom{i-1+j}{j+1}} \\
&= x_m^{\binom{k}{2}} \prod_{i=1}^{n-m} x_{m+i}^{\binom{k-1+i}{i+1}} \prod_{j=1}^{n-m} \prod_{i=1}^{k-1} x_{m+j}^{\binom{i-1+j}{j+1}} \\
&= x_m^{\binom{k}{2}} \prod_{i=1}^{n-m} x_{m+i}^{\binom{k-1+i}{i+1}} \prod_{j=1}^{n-m} x_{m+j}^{\sum_{i=1}^{k-1} \binom{i-1+j}{j+1}}.
\end{aligned}
$$

Exchanging the names of the indices $i, j$ in the last product, we get

$$\prod_{i=1}^{k-1} \mu(x_n^i, x_{m-1}^i) = x_m^{\binom{k}{2}} \prod_{i=1}^{n-m} x_{m+i}^{\binom{k-1+i}{i+1}} \prod_{i=1}^{n-m} x_{m+i}^{\sum_{j=1}^{k-1} \binom{i-1+j}{i+1}}$$

$$= x_m^{\binom{k}{2}} \prod_{i=1}^{n-m} x_{m+i}^{\binom{k-1+i}{i+1} + \sum_{j=1}^{k-1} \binom{i-1+j}{i+1}}$$

$$= x_m^{\binom{k}{2}} \prod_{i=1}^{n-m} x_{m+i}^{\binom{k-1+i}{i+1} + \binom{k-1+i}{i+2}}$$

$$= x_m^{\binom{k}{2}} \prod_{i=1}^{n-m} x_{m+i}^{\binom{k+i}{i+2}}$$

$$= \prod_{i=0}^{n-m} x_{m+i}^{\binom{k+i}{i+2}}.$$

Finally, substituting $i$ with $i-1$ in the last product yields

$$\prod_{i=1}^{k-1} \mu(x_n^i, x_{m-1}^i) = \prod_{i=1}^{n-(m-1)} x_{(m-1)+i}^{\binom{k-1+i}{i+1}}.$$

Hence (5.5) also holds for $m-1$, as claimed. This concludes the proof of the theorem. $\square$

5.2. **Some maxgen computations.** The following result uses the sets $A_1(v), A_2(v)$ introduced in Definition 3.3; it will be needed in view of applying Theorem 2.27.

**Proposition 5.15.** *Let* $u = x_{i_1} \cdots x_{i_d}$ *with* $i_1 \leq \cdots \leq i_d$. *For all* $1 \leq k \leq d-1$, *let* $u_k = \mathrm{pre}_k(u)$. *Then*

$$\mathrm{maxgen}(\mathrm{gaps}(u)) = \prod_{k=1}^{d-1} (\mathrm{maxgen}(A_2(u/u_k)))^{|A_1(u_k)|}.$$

*Proof.* Consider the description of gaps given in Proposition 3.4. For any monomial $w = w_1 w_2 \in A_1(u_k) A_2(u/u_k)$ with $w_1 \in A_1(u_k)$ and $w_2 \in A_2(u/u_k)$, we have $\max(w) = \max(w_2)$, since $\max(w_1) < \min(w_2)$ by construction. Therefore, for all $k$ we have

$$\mathrm{maxgen}(A_1(u_k) A_2(u/u_k)) = \mathrm{maxgen}(A_2(u/u_k))^{|A_1(u_k)|}. \qquad \square$$

Since $A_2(u/u_k)$ is the set of all monomials of degree $\deg(u/u_k)$ in the variables $x_i$ with $\min(u/u_k) + 1 \leq i \leq n$, we will be able to determine $\mathrm{maxgen}(A_2(u/u_k))$ if we can determine $\mathrm{maxgen}(S_{n,d})$ for any $n, d$. Let us proceed to do just that. We start with a recurrence formula.

**Proposition 5.16.** *For all integers* $n, d \geq 1$, *we have*

$$\mathrm{maxgen}(S_{n,d}) = \mathrm{maxgen}(S_{n-1,d}) x_n^{\binom{n+d-2}{d-1}}.$$

*Proof.* Obviously, we have

$$S_{n,d} = \bigsqcup_{i=0}^{d} S_{n-1,d-i} \cdot x_n^i.$$

This follows from writing any $u \in S_{n,d}$ as $u = vx_n^i$ with $v \in S_{n-1,d-i}$. Hence

$$\operatorname{maxgen}(S_{n,d}) = \prod_{i=0}^{d} \operatorname{maxgen}(S_{n-1,d-i} \cdot x_n^i)$$

$$= \operatorname{maxgen}(S_{n-1,d}) \prod_{i=1}^{d} x_n^{|S_{n-1,d-i}|}$$

$$= \operatorname{maxgen}(S_{n-1,d}) x_n^{\sum_{i=1}^{d} |S_{n-1,d-i}|}$$

$$= \operatorname{maxgen}(S_{n-1,d}) x_n^{\sum_{i=1}^{d} \binom{n-2+d-i}{d-i}}$$

$$= \operatorname{maxgen}(S_{n-1,d}) x_n^{\sum_{j=0}^{d-1} \binom{n-2+j}{j}}.$$

We conclude the proof by applying the well-known formula

$$\sum_{j=0}^{d-1} \binom{n-2+j}{j} = \binom{n+d-2}{d-1}. \qquad \square$$

**Corollary 5.17.** *For all $n, d$, we have*

$$\operatorname{maxgen}(S_{n,d}) = \prod_{i=1}^{n} x_i^{\binom{d-2+i}{d-1}}.$$

*Proof.* Use the above induction formula. $\qquad \square$

**Corollary 5.18.** *For all $1 \leq l \leq n$ and all $d$, let $S_{l,n,d}$ be the set of all monomials of degree $d$ in the variables $x_l, \ldots, x_n$. Then we have*

$$\operatorname{maxgen}(S_{l,n,d}) = \prod_{j=l}^{n} x_j^{\binom{d-1+j-l}{d-1}}.$$

*Proof.* Directly follows from the preceding corollary by properly translating indices. $\qquad \square$

We may now inject this information into Proposition 5.15. This yields the following result.

**Theorem 5.19.** *Let $u = x_{i_1} \cdots x_{i_d}$ with $i_1 \leq \cdots \leq i_d$. Then*

$$\operatorname{maxgen}(\operatorname{gaps}(u)) = \prod_{k=1}^{d-1} \left( \prod_{j=i_{k+1}+1}^{n} x_j^{\binom{d-k-2+j-i_{k+1}}{d-k-1}} \right)^{|B(x_{i_1} \cdots x_{i_k})|-1},$$

*where the internal product is set to 1 if $i_{k+1} = n$.*

*Proof.* The proof follows from Proposition 5.15 together with the above corollary. Using Definition 3.3 for $A_1(v), A_2(v)$, and since $u_k = x_{i_1} \cdots x_{i_k}$, we have

$$|A_1(u_k)| = |B(x_{i_1} \cdots x_{i_k})| - 1.$$

Moreover, $A_2(u/u_k)$ is the set of all monomials of degree $\deg(u/u_k)$ in the variables $x_i$ with $\min(u/u_k)+1 \leq i \leq n$. Therefore, in order to determine $\mathrm{maxgen}(A_2(u/u_k))$, it remains to apply Corollary 5.18, using $l = i_{k+1} + 1$ since $u/u_k = x_{i_{k+1}} \cdots x_{i_d}$ and so $\min(u/u_k) = i_{k+1}$. $\qquad\square$

## 6. On the first and last variables

For the determination of Gotzmann monomials in $S_n$, both variables $x_1$ and $x_n$ behave in some specific ways. This section describes how.

6.1. **Neutrality of $x_1$.** Our purpose here is to show that *a monomial $u \in S_n$ is Gotzmann if and only if $x_1u$ is.* We start with some intermediate results.

**Lemma 6.1.** *Let $u, v \in S_{n,d}$ be such that $u \geq v$. If $x_1$ divides $v$ then $x_1$ divides $u$.*

*Proof.* Write $u = x_{i_1} \cdots x_{i_d}$, $v = x_{j_1} \cdots x_{j_d}$, with $1 \leq i_1 \leq \cdots \leq i_d \leq n$ and $1 \leq j_1 \leq \cdots \leq j_d \leq n$. Without loss of generality, we may assume $u > v$. Hence there exists an index $1 \leq k \leq d$ such that

$$i_1 = j_1, \ldots, i_{k-1} = j_{k-1}, i_k < j_k.$$

Therefore $i_1 \leq j_1$. Assume $x_1$ divides $v$. This is equivalent to $j_1 = 1$. Since $1 \leq i_1 \leq j_1$, we have that $i_1 = 1$, whence $x_1$ divides $u$ and we are done. $\qquad\square$

**Lemma 6.2.** *Let $u \in S_{n,d}$. Then $x_1 L(u) = L(x_1 u)$.*

*Proof.* Let $v \in L(u)$. Then $v \geq u$, whence $x_1 v \geq x_1 u$ by Lemma 2.1, i.e. $x_1 v \in L(x_1 u)$. Therefore $x_1 L(u) \subseteq L(x_1 u)$. Conversely, let $v' \in L(x_1 u) \subseteq S_{n,d+1}$. Then $v' \geq x_1 u$. Hence, mutatis mutandis, $x_1$ divides $v'$ by Lemma 6.1. Thus there exists $v \in S_{n,d}$ such that $v' = x_1 v$. Now $x_1 v \geq x_1 u$ by hypothesis, whence $v \geq u$ by Lemma 2.1 again, i.e. $v \in L(u)$ and so $v' \in x_1 L(u)$. Therefore $L(x_1 u) \subseteq x_1 L(u)$. $\qquad\square$

In particular, the lemma implies that *multiplying any lexsegment by $x_1$ again yields a lexsegment.*

**Lemma 6.3.** *Let $B \subseteq S_{n,d}$. Then $(x_1 B)^{\mathrm{lex}} = x_1 B^{\mathrm{lex}}$.*

*Proof.* We have $|B^{\mathrm{lex}}| = |B| = |x_1 B|$. Applying this to the set $x_1 B$ yields

$$|(x_1 B)^{\mathrm{lex}}| = |x_1 B| = |B| = |B^{\mathrm{lex}}| = |x_1 B^{\mathrm{lex}}|. \tag{6.1}$$

Now $B^{\mathrm{lex}}$ is a lexsegment, whence $x_1 B^{\mathrm{lex}}$ also is by Lemma 6.2 and the comment following it. Moreover, $x_1 B^{\mathrm{lex}}$ has the same cardinality as the lexsegment $(x_1 B)^{\mathrm{lex}}$ by (6.1), whence these two lexsegments coincide. $\qquad\square$

**Proposition 6.4.** *Let $u \in S_{n,d}$. Then $\mathrm{gaps}(x_1 u) = x_1 \mathrm{gaps}(u)$.*

*Proof.* Let $v' \in \mathrm{gaps}(x_1 u)$. Then $v' > x_1 u$, whence $x_1$ divides $v'$ by Lemma 6.1. Let $v \in S_{n,d}$ such that $v' = x_1 v$. Since $x_1 v \in \mathrm{gaps}(x_1 u)$, it follows that $v \in \mathrm{gaps}(u)$, since $v \geq u$ and $v$ cannot belong to $B(u)$, for otherwise $x_1 v$ would belong to $B(x_1 u)$. Hence $v' \in x_1 \mathrm{gaps}(u)$.

Conversely, let $v \in \mathrm{gaps}(u)$. Then $v > u$, whence $x_1 v > x_1 u$ and so $x_1 v \in L(x_1 u)$. Since $v \notin B(u)$, it follows that $x_1 v \notin B(x_1 u)$, whence $x_1 v \in \mathrm{gaps}(x_1 u)$. $\square$

**Theorem 6.5.** *Let $u \in S_n$. Then $u$ is Gotzmann if and only if $x_1 u$ is Gotzmann.*

*Proof.* First, some preliminary steps.

**Step 1.** *We have $B(x_1 u) = x_1 B(u)$.*

Indeed, by applying transformations of the form $v \mapsto v' = x_i v / x_j$ for $v \in B(x_1 u)$ or $v \in x_1 B(u)$, with $x_j$ dividing $v$ and $1 \leq i < j$, the variable $x_1$ is not affected since $j \geq 2$, whence the claimed equality.

**Step 2.** *We have $\widetilde{x_1 u} = x_1 \tilde{u}$.*

Indeed, it suffices to prove that $L(\widetilde{x_1 u}) = L(x_1 \tilde{u})$. On the one hand, we have $L(\widetilde{x_1 u}) = B(x_1 u)^{\mathrm{lex}}$ by definition. Now $B(x_1 u) = x_1 B(u)$ by Step 1. Thus $L(\widetilde{x_1 u}) = (x_1 B(u))^{\mathrm{lex}}$, and $(x_1 B(u))^{\mathrm{lex}} = x_1 B(u)^{\mathrm{lex}}$ by Lemma 6.3, and $x_1 B(u)^{\mathrm{lex}} = x_1 L(\tilde{u})$ by definition. Finally, $x_1 L(\tilde{u}) = L(x_1 \tilde{u})$ by Lemma 6.2 applied to $\tilde{u}$. This concludes the proof of Step 2.

**Step 3.** *For all $B \subseteq S_{n,d}$, we have $\mathrm{maxgen}(x_1 B) = \mathrm{maxgen}(B)$.*

Indeed, this follows from Lemma 2.16 and the obvious equality $\lambda(x_1 v) = \lambda(v)$ for all $v \in S_{n,d}$.

We may now compare the maxgen monomials of $\mathrm{gaps}(u), \mathrm{cogaps}(u)$ with those of $\mathrm{gaps}(x_1 u), \mathrm{cogaps}(x_1 u)$, respectively. First, by Proposition 6.4 and Step 3, we have

$$\mathrm{maxgen}(\mathrm{gaps}(x_1 u)) = \mathrm{maxgen}(\mathrm{gaps}(u)). \qquad (6.2)$$

Symmetrically, we also have

$$\mathrm{maxgen}(\mathrm{cogaps}(x_1 u)) = \mathrm{maxgen}(\mathrm{cogaps}(u)), \qquad (6.3)$$

as we now show:

$$
\begin{aligned}
\mathrm{maxgen}(\mathrm{cogaps}(x_1 u)) &= \mathrm{maxgen}(L(x_1 u) \setminus L(\widetilde{x_1 u})) \\
&= \mathrm{maxgen}(L(x_1 u) \setminus L(x_1 \tilde{u})) \quad \text{(by Step 2)} \\
&= \mathrm{maxgen}(x_1 (L(u) \setminus L(\tilde{u}))) \quad \text{(by Lemma 6.2)} \\
&= \mathrm{maxgen}(L(u) \setminus L(\tilde{u})) \quad \text{(by Step 3)} \\
&= \mathrm{maxgen}(\mathrm{cogaps}(u)).
\end{aligned}
$$

The desired equivalence is now easy to establish. Indeed, it follows from (6.2) and (6.3) that

$$\mathrm{maxgen}(\mathrm{gaps}(x_1 u)) = \mathrm{maxgen}(\mathrm{cogaps}(x_1 u))$$

if and only if

$$\mathrm{maxgen}(\mathrm{gaps}(u)) = \mathrm{maxgen}(\mathrm{cogaps}(u)).$$

Therefore, $x_1 u$ is Gotzmann if and only if $u$ is Gotzmann. $\square$

6.2. **On** $\mathrm{gaps}(ux_n)$**.** For use in the next section, we shall need to control $\mathrm{maxgen}(\mathrm{gaps}(ux_n))$.

**Definition 6.6.** Let $u \in S_n$. For all $i \leq n$, we shall use the notation

$$\mathrm{gaps}(u, i) = \{v \in \mathrm{gaps}(u) \mid \max v = i\}.$$

Note that $\mathrm{gaps}(u, 1)$ is empty, for $x_1^d$ cannot be a gap since it obviously belongs to $B(u)$ for all $u \in S_{n,d}$.

**Theorem 6.7.** *Let* $u \in S_n$. *Then for all* $1 \leq j \leq n$, *we have*

$$\mathrm{gaps}(ux_n, j) = \bigsqcup_{i=1}^{j} \mathrm{gaps}(u, i)x_j.$$

Here is an equivalent formulation.

**Theorem 6.8.** *Let* $u \in S_n$. *Then, for any* $w \in S_n$, *we have*

$$w \in \mathrm{gaps}(ux_n) \iff \frac{w}{\lambda(w)} \in \mathrm{gaps}(u).$$

*Proof.* Let $d = \deg(u)$, and write $u = x_{i_1} \cdots x_{i_d}$, with $1 \leq i_1 \leq \cdots \leq i_d \leq n$. We may assume $\deg(w) = d+1$, for otherwise $w$ cannot be a gap of $ux_n$. Set $\max(w) = m$. Let $v = w/\lambda(w)$, and write $v = x_{j_1} \cdots x_{j_d}$, with $1 \leq j_1 \leq \cdots \leq j_d \leq m$.

By Lemma 3.1, $v$ is a gap of $u$ if and only if there exist indices $1 \leq s < t \leq d$ such that $j_s < i_s$ and $j_t > i_t$. If these conditions are met, then since $w = vx_m$ with $m \geq \max(v)$, automatically $w$ is a gap of $ux_n$, still by Lemma 3.1. Conversely, if $w$ is a gap of $ux_n$, and since $\max(w) \leq \max(ux_n) = n$, then the index $t \leq d+1$ given by Lemma 3.4 necessarily satisfies $t \leq d$. Hence $v$ is a gap of $u$. $\square$

**Corollary 6.9.** *Let* $u \in S_n$. *If* $\mathrm{maxgen}(\mathrm{gaps}(u)) = \prod_{i=1}^{n} x_i^{k_i}$ *then*

$$\mathrm{maxgen}(\mathrm{gaps}(ux_n)) = \prod_{j=1}^{n} x_j^{k_1 + \cdots + k_j}.$$

*Proof.* By Theorem 6.7, we have

$$|\mathrm{gaps}(ux_n, j)| = \sum_{i=1}^{j} |\mathrm{gaps}(u, i)x_j| = \sum_{i=1}^{j} |\mathrm{gaps}(u, i)|.$$

The statement now follows from the definition of the maxgen monomial. $\square$

## 7. GOTZMANN MONOMIALS IN $S_2$, $S_3$, $S_4$

This section contains the main result of this paper, namely the characterization of Gotzmann monomials in $S_n$ for $n = 4$. This is achieved in Theorem 7.7. The strategy is as follows. Let $u = x_1^{a_1} \cdots x_{n-1}^{a_{n-1}} x_n^t \in S_n$. We may assume $a_1 = 0$ by Theorem 6.5, according to which $u$ is Gotzmann in $S_n$ if and only if $u/x_1^{a_1}$ is. We first compute $w_1 = \mathrm{maxgen}(\mathrm{gaps}(u))$ using Theorem 5.19. The degree $g$ of $w_1$ gives the numbers of gaps of $u$. We then focus on $\mathrm{cogaps}(u) = \mathrm{pred}_g(u)$ and,

more precisely, compute its maxgen monomial $w_2 = \text{maxgen}(\text{pred}_g(u))$. Finally, requiring $w_1 = w_2$ gives necessary and sufficient conditions on the exponent $t$ of $x_n$ for $u$ to be a Gotzmann monomial.

Before turning to the case $n = 4$, we start by reviewing the known cases $n = 2$ and $3$.

### 7.1. The case $n = 2$.
This is easy. Indeed, *every monomial $u = x_1^a x_2^t$ is Gotzmann in $S_2$*. For in this case, the sets $B(u)$ and $L(u)$ coincide, whence $B(u)^{\text{lex}} = B(u)$ and so $B(u)$ is a Gotzmann set by Lemma 2.12.

### 7.2. The case $n = 3$.
The result below for $n = 3$ may be deduced from [13, Proposition 8]. As an illustration of the strategy briefly described above, we give here an independent short proof using the tools developed in this paper.

**Proposition 7.1.** *Let $u = x_1^a x_2^b x_3^t \in S_3$. Then $u$ is a Gotzmann monomial in $S_3$ if and only if $t \geq \binom{b}{2}$.*

*Proof.* Let $g = |\text{gaps}(u)|$, $w_1 = \text{maxgen}(\text{gaps}(u))$, and $w_2 = \text{maxgen}(\text{cogaps}(u))$. Then $g = \deg(w_1) = \deg(w_2)$. A straightforward computation with Theorem 5.19 yields the monomial

$$w_1 = x_3^{\binom{b}{2}}$$

independent of $t$. Therefore, $g = \binom{b}{2}$. Thus $\text{cogaps}(u) = \text{pred}_g(u)$. Consider now $w_2 = \text{maxgen}(\text{pred}_g(u))$. For all $i \leq t$, we have $\text{pred}^i(u) = x_1^a x_2^{b+i} x_3^{t-i}$. Thus $\lambda(\text{pred}^i(u)) = x_3$ if $i < t$ and $\lambda(\text{pred}^t(u)) = x_2$. Hence $\text{maxgen}(\text{pred}_t(u)) = x_3^t$ and $\text{maxgen}(\text{pred}_{t+1}(u)) = x_2 x_3^t$. Consequently, if $t < g$ then $x_2$ divides $w_2$, by Remark 2.17, and so $w_2 \neq w_1$, whereas if $t \geq g$ then $w_2 = x_3^g = w_1$. Thus $u$ is Gotzmann if and only $t \geq g$, as claimed. $\qquad\square$

### 7.3. The case $n = 4$.
Our purpose in this section is to determine all Gotzmann monomials in four variables. This is achieved in Theorem 7.7. As recalled above, it suffices to consider monomials of the form $x_2^b x_3^c x_4^t$. Implementing our proof strategy requires several preliminary results.

We start by determining $\text{maxgen}(\text{gaps}(x_2^b x_3^c x_4^t))$.

**Proposition 7.2.** *Let $u_0 = x_2^b x_3^c \in S_4$. Then for all $t \geq 0$, we have*

$$\text{maxgen}(\text{gaps}(u_0 x_4^t)) = x_3^{\binom{b}{2}} x_4^{f(t)}, \tag{7.1}$$

*where*

$$f(t) = f(0) + t\binom{b}{2},$$

$$f(0) = \left(\frac{b+1}{3} + c\right)\binom{b}{2} + (b+1)\binom{c+1}{2} + \binom{c+1}{3} - c.$$

*Proof.*

**Case** $t = 0$. This is the longest part of the proof, yet it follows almost mechanically from Theorem 5.19 and a few formulas. In the notation of that result, let us write $u_0 = x_{i_1} \cdots x_{i_d}$, with $i_1 \leq \cdots \leq i_d$, where $d = \deg(u_0) = b + c$. Thus

$$u_0 = \underbrace{x_2 \cdots x_2}_{b \text{ times}} \underbrace{x_3 \cdots x_3}_{c \text{ times}}.$$

Hence for $1 \leq k \leq d$, we have $i_k = 2$ if $k \leq b$, and $i_k = 3$ otherwise. By Theorem 5.19, we have

$$\text{maxgen}(\text{gaps}(u_0)) = \prod_{k=1}^{d-1} \left( \prod_{j=i_{k+1}+1}^{n} x_j^{\binom{d-k-2+j-i_{k+1}}{d-k-1}} \right)^{|B(x_{i_1} \cdots x_{i_k})|-1}$$

$$= \prod_{k=1}^{b-1} \left( \prod_{j=3}^{4} x_j^{\binom{d-k-2+j-2}{d-k-1}} \right)^{|B(x_2^k)|-1} \cdot \prod_{k=b}^{d-1} \left( x_4^1 \right)^{|B(x_2^b x_3^{k-b})|-1}$$

$$= \prod_{k=1}^{b-1} \left( x_3^1 x_4^{d-k} \right)^{|B(x_2^k)|-1} \cdot \prod_{k=b}^{d-1} \left( x_4^1 \right)^{|B(x_2^b x_3^{k-b})|-1}.$$

We now compute the involved exponents. We have $|B(x_2^k)| = k + 1$, as follows from the set equality $B(x_2^k) = \{ x_1^{k-i} x_2^i \mid 0 \leq i \leq k \}$. On the other hand, we have

$$|B(x_2^r x_3^s)| = \binom{s+1}{2} + (r+1)(s+1),$$

as follows from the formula

$$|B(x_2^r x_3^s)| = \sum_{i=0}^{s} |B(x_2^{r+s-i})|$$

of Corollary 3.7, the above formula for $|B(x_2^k)|$, and some straightforward computations.

Inserting these exponent values into the above formula for $\text{maxgen}(\text{gaps}(u_0))$, we get

$$\text{maxgen}(\text{gaps}(u_0)) = \prod_{k=1}^{b-1} \left( x_3 x_4^{d-k} \right)^k \cdot \prod_{k=b}^{d-1} x_4^{\binom{k-b+1}{2} + (b+1)(k-b+1)-1}$$

$$= x_3^{\binom{b}{2}} x_4^{A+B},$$

where

$$A = \sum_{k=1}^{b-1} k(d-k),$$

$$B = \sum_{k=b}^{d-1} \left( \binom{k-b+1}{2} + (b+1)(k-b+1) - 1 \right).$$

By the formula

$$\sum_{k=1}^{b-1} k^2 = \frac{2b-1}{3}\binom{b}{2}$$

and some straightforward computations, we get

$$A = \left(\frac{b+1}{3} + c\right)\binom{b}{2}.$$

Similarly, the formula

$$\sum_{l=1}^{c}\binom{l}{2} = \binom{c+1}{3}$$

and some further straightforward computations yield

$$B = (b+1)\binom{c+1}{2} + \binom{c+1}{3} - c.$$

As $f(0) = A + B$, the proof of formula (7.1) in the case $t = 0$ is complete.

**Case $t \geq 1$.** For all $s \geq 1$, Corollary 6.9 and the above case $t = 0$ imply

$$\mathrm{maxgen}(\mathrm{gaps}(u_0 x_4^s)) = \mathrm{maxgen}(\mathrm{gaps}(u_0 x_4^{s-1}))x_4^{\binom{b}{2}}$$

by induction on $s$. The claimed formula

$$\mathrm{maxgen}(\mathrm{gaps}(u_0 x_4^t)) = x_3^{\binom{b}{2}} x_4^{f(t)}$$

follows by induction on $t$. $\qquad\square$

We now proceed to determine $\mathrm{maxgen}(\mathrm{cogaps}(x_2^b x_3^c x_4^t))$. We first need two lemmas.

**Lemma 7.3.** *For all $r \geq 0$ and $s \geq 1$, we have*

$$\mu(x_2^r x_4^s, x_2^{r+1} x_4^{s-1}) = x_3 x_4^s.$$

*Proof.* Starting from $x_2^r x_4^s$ and taking $s+1$ successive predecessors, Proposition 4.6 yields

$$x_2^r x_4^s \xrightarrow{\;x_4^s\;} x_2^b x_3^s \xrightarrow{\;x_3\;} x_2^{r+1} x_4^{s-1}$$

in arrow notation, i.e. $\mu(x_2^r x_4^s, x_2^b x_3^s) = x_4^s$ and $\mu(x_2^b x_3^s, x_2^{r+1} x_4^{s-1}) = x_3$. The desired formula follows by arrow composition. $\qquad\square$

**Lemma 7.4.** *For all $r \geq 0$ and $1 \leq i \leq s$, we have*

$$\mu(x_2^r x_4^s, x_2^{r+i} x_4^{s-i}) = x_3^i x_4^{\binom{i+1}{2} + i(s-i)}.$$

*Proof.* By induction on $i$. The case $i = 1$ is just Lemma 7.3. By arrow composition, we have

$$\mu(x_2^r x_4^s, x_2^{r+i} x_4^{s-i}) = \prod_{j=0}^{i-1} \mu(x_2^{r+j} x_4^{s-j}, x_2^{r+j+1} x_4^{s-j-1}).$$

Applying Lemma 7.3 again to each factor, we get

$$\mu(x_2^r x_4^s, x_2^{r+i} x_4^{s-i}) = x_3^i x_4^{\sum_{j=0}^{i-1}(s-j)}.$$

Finally, $\sum_{j=0}^{i-1}(s - j) = \binom{i+1}{2} + i(s - i)$ and the proof is complete. $\qquad\square$

**Proposition 7.5.** *We have*

$$x_2^b x_3^c x_4^t \xrightarrow{\phantom{aa} x_3^{\binom{b}{2}} x_4^{h(t)} \phantom{aa}} x_2^{b+\binom{b}{2}} x_4^{c+t-\binom{b}{2}},$$

*where*

$$h(t) = (c + t)\binom{b}{2} - \binom{\binom{b}{2}}{2} - c.$$

*Proof.* Starting from $x_2^b x_3^c x_4^t$ and taking $t + 1$ successive predecessors, Proposition 4.6 yields

$$x_2^b x_3^c x_4^t \xrightarrow{\phantom{aa} x_4^t \phantom{aa}} x_2^b x_3^{c+t} \xrightarrow{\phantom{aa} x_3 \phantom{aa}} x_2^{b+1} x_4^{c+t-1}.$$

Hence

$$\mu(x_2^b x_3^c x_4^t, x_2^{b+1} x_4^{c+t-1}) = x_3 x_4^t. \tag{7.2}$$

From $x_2^{b+1} x_4^{c+t-1}$, we must still reach $x_2^{b+\binom{b}{2}} x_4^{c+t-\binom{b}{2}}$. This can be done using Lemma 7.4 with $(r, s, i) = (b + 1, c + t - 1, \binom{b}{2} - 1)$. We obtain

$$\mu\left(x_2^{b+1} x_4^{c+t-1}, x_2^{b+\binom{b}{2}} x_4^{c+t-\binom{b}{2}}\right) = x_3^{\binom{b}{2}-1} x_4^{\binom{\binom{b}{2}}{2}+(\binom{b}{2}-1)(c+t-\binom{b}{2})}. \tag{7.3}$$

Hence, combining (7.2) and (7.3) using arrow composition, we get

$$\mu\left(x_2^b x_3^c x_4^t, x_2^{b+\binom{b}{2}} x_4^{c+t-\binom{b}{2}}\right) = x_3^{\binom{b}{2}} x_4^{\binom{\binom{b}{2}}{2}+(\binom{b}{2}-1)(c+t-\binom{b}{2})+t}.$$

It remains to show that the exponent of $x_4$ in the monomial of the above right-hand side is equal to $h(t)$. Indeed, we have

$$\binom{\binom{b}{2}}{2} + \left(\binom{b}{2} - 1\right)\left(c + t - \binom{b}{2}\right) + t$$

$$= \binom{\binom{b}{2}}{2} + (c + t)\binom{b}{2} - \left(\binom{b}{2} - 1\right)\binom{b}{2} - c.$$

Since

$$-\left(\binom{b}{2} - 1\right)\binom{b}{2} = -2\binom{\binom{b}{2}}{2},$$

the desired equality with $h(t)$ follows. $\qquad\square$

**Remark 7.6.** By Propositions 7.2 and 7.5, for $t \geq 0$ we have

$$f(t) - h(t) = f(0) - c\binom{b}{2} + \binom{\binom{b}{2}}{2} + c$$

$$= \frac{b+1}{3}\binom{b}{2} + (b+1)\binom{c+1}{2} + \binom{c+1}{3} + \binom{\binom{b}{2}}{2}.$$

In particular, $f(t) - h(t)$ is a positive constant. This will be used below.

Here is our main result.

**Theorem 7.7.** *Let $u = x_1^a x_2^b x_3^c x_4^t \in S_4$. Then $u$ is a Gotzmann monomial in $S_4$ if and only if*

$$t \geq \binom{\binom{b}{2}}{2} + \frac{b+4}{3}\binom{b}{2} + (b+1)\binom{c+1}{2} + \binom{c+1}{3} - c.$$

As expected, the absence of exponent $a$ in this bound on $t$ is consistent with Theorem 6.5.

*Proof.* By Theorem 6.5, we may assume $a = 0$. Let $u_0 = x_2^b x_3^c$, so that $u = u_0 x_4^t$. There are two steps.

**Step 1.** *The monomial $u_0 x_4^t$ is Gotzmann if and only if*

$$t \geq f(t) - h(t) + \binom{b}{2} - c. \tag{7.4}$$

Indeed, by Proposition 7.2, we have

$$\mathrm{maxgen}(\mathrm{gaps}(u_0 x_4^t)) = x_3^{\binom{b}{2}} x_4^{f(t)}. \tag{7.5}$$

Thus $|\mathrm{gaps}(u_0 x_4^t)| = \binom{b}{2} + f(t)$. For $u_0 x_4^t$ to be a Gotzmann monomial, we apply the criterion given by Theorem 2.27. Thus, by (7.5), we need to determine those $t \geq 0$ for which

$$\mathrm{maxgen}(\mathrm{cogaps}(u_0 x_4^t)) = x_3^{\binom{b}{2}} x_4^{f(t)}. \tag{7.6}$$

Now $\mathrm{cogaps}(u_0 x_4^t) = \mathrm{pred}_{\binom{b}{2}+f(t)}(u_0 x_4^t)$ by Proposition 4.4. In order to compute the maxgen monomial of the set of $\binom{b}{2} + f(t)$ predecessors of $u = u_0 x_4^t$, we first compute it for its $\binom{b}{2} + h(t)$ predecessors. Let

$$LI(t) = \mathrm{pred}_{\binom{b}{2}+h(t)}(u_0 x_4^t),$$

$$v(t) = \mathrm{pred}^{\binom{b}{2}+h(t)}(u_0 x_4^t).$$

Then $LI(t) = L^*(v(t), u_0 x_4^t)$, and we seek the maxgen monomial of this lexinterval. By Proposition 7.5, we have

$$u_0 x_4^t = x_2^b x_3^c x_4^t \xrightarrow{\quad x_3^{\binom{b}{2}} x_4^{h(t)} \quad} x_2^{b+\binom{b}{2}} x_4^{c+t-\binom{b}{2}}.$$

Hence

$$v(t) = x_2^{b+\binom{b}{2}} x_4^{c+t-\binom{b}{2}},$$

$$\text{maxgen}(LI(t)) = x_3^{\binom{b}{2}} x_4^{h(t)}.$$

Now, restarting from $v(t)$, it remains to compute $f(t) - h(t)$ more predecessors in order to reach $\text{pred}_{\binom{b}{2}+f(t)}(u_0 x_4^t)$. We will then have

$$\text{maxgen}(\text{cogaps}(u_0 x_4^t)) = \text{maxgen}(LI(t)) \, \text{maxgen}(\text{pred}_{f(t)-h(t)}(v(t)))$$

$$= x_3^{\binom{b}{2}} x_4^{h(t)} \, \text{maxgen}(\text{pred}_{f(t)-h(t)}(v(t))).$$

Therefore, in order to satisfy equality (7.6) for $u_0 x_4^t$ to be a Gotzmann monomial, it is necessary and sufficient to satisfy

$$\text{maxgen}(\text{pred}_{f(t)-h(t)}(v(t))) = x_4^{f(t)-h(t)}.$$

Since $v(t) = x_2^{b+\binom{b}{2}} x_4^{c+t-\binom{b}{2}}$, the above condition is realizable if and only if the exponent of $x_4$ in $v(t)$ is large enough, namely

$$c + t - \binom{b}{2} \geq f(t) - h(t).$$

This condition being equivalent to (7.4), the proof of the claim in Step 1 is complete.

**Step 2.** A straightforward computation on the right-hand side of (7.4) yields

$$f(t) - h(t) + \binom{b}{2} - c = f_0 + t\binom{b}{2} - \left( (c+t)\binom{b}{2} - \binom{\binom{b}{2}}{2} - c \right) + \binom{b}{2} - c$$

$$= f_0 - c\binom{b}{2} + \binom{\binom{b}{2}}{2} + c + \binom{b}{2} - c$$

$$= f_0 - c\binom{b}{2} + \binom{\binom{b}{2}}{2} + \binom{b}{2}$$

$$= \binom{\binom{b}{2}}{2} + \frac{b+4}{3}\binom{b}{2} + (b+1)\binom{c+1}{2} + \binom{c+1}{3} - c.$$

The conjunction of Steps 1 and 2 completes the proof of the theorem. $\qquad\square$

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