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ORTHOGONALITY OF THE DICKSON POLYNOMIALS OF THE $(k+1)$-TH KIND

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Dedicated to V. E. P., in gratitude for her continuous unbounded support (without measure).

Abstract. We study the Dickson polynomials of the $(k+1)$-th kind over the field of complex numbers. We show that they are a family of co-recursive orthogonal polynomials with respect to a quasi-definite moment functional L_k. We find an integral representation for L_k and compute explicit expressions for all of its moments.

1. Introduction

Let $n \in \mathbb{N}$, $\mathbb{F}_q$ be a finite field, and $a \in \mathbb{F}_q$. The Dickson polynomials $D_n(x;a)$, defined by (see [29, 9.6.1])

$$D_n(x;a) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n}{n-j} \binom{n-j}{j} (-a)^j \, x^{n-2j}, \quad x \in \mathbb{F}_q,$$

were introduced by Leonard Eugene Dickson (1874–1954) in his 1896 Ph.D. thesis "The analytic representation of substitutions on a power of a prime number of letters with a discussion of the linear group" [19], published in two parts in *The Annals of Mathematics* [20, 21]. The Dickson polynomials are the unique monic polynomials satisfying the functional equation (see [29, 9.6.3])

$$D_n\left(y + \frac{a}{y}; a\right) = y^n + \left(\frac{a}{y}\right)^n, \quad y \in \mathbb{F}_{q^2}.$$

See [39] for further algebraic and number theoretic properties of the Dickson polynomials.

Let $\mathbb{N}_0$ denote the set $\mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$. In [58], Wang and Yucas extended the Dickson polynomials to a family depending on a new parameter $k \in \mathbb{N}_0$, which they called *Dickson polynomials of the* $(k+1)$*-th kind.* They defined them by

$$D_{n,k}(x;a) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n-kj}{n-j} \binom{n-j}{j} (-a)^j \, x^{n-2j}, \tag{1.1}$$

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with initial values

$$D_{0,k}(x;a) = 2-k, \quad D_{1,k}(x;a) = x. \tag{1.2}$$

They also showed that the polynomials $D_{n,k}(x;a)$ satisfy the fundamental functional equation

$$D_{n,k}\left(y+\frac{a}{y};a\right) = y^n + \left(\frac{a}{y}\right)^n + k\frac{ay^n - y^2\left(\frac{a}{y}\right)^n}{y^2-a}, \quad y \neq 0.$$

Note that

$$\lim_{y\to\pm\sqrt{a}} \frac{ay^n - y^2\left(\frac{a}{y}\right)^n}{y^2-a} = (n-1)\left(\pm\sqrt{a}\right)^n,$$

and therefore

$$D_{n,k}\left(y+\frac{a}{y};a\right) = [2+(n-1)k]\left(\pm\sqrt{a}\right)^n, \quad y = \pm\sqrt{a}.$$

We clearly have

$$D_{n,0}(x;a) = D_n(x;a) \qquad \text{(Dickson polynomials)}$$

and (see [29, 9.6.1])

$$D_{n,1}(x;a) = E_n(x;a) \qquad \text{(Dickson polynomials of the second kind)}.$$

In fact, since

$$\frac{n-kj}{n-j} = k-(k-1)\frac{n}{n-j},$$

we have (see [58, 2.1])

$$D_{n,k}(x;a) = kE_n(x;a) - (k-1)D_n(x;a).$$

The polynomials $D_{n,k}(x;a)$ also satisfy the fundamental recurrence (see [58, Remark 2.5])

$$D_{n+2,k} = xD_{n+1,k} - aD_{n,k}, \quad n \in \mathbb{N}_0. \tag{1.3}$$

The first few Dickson polynomials of the $(k+1)$-th kind are

$$\begin{aligned}
D_{2,k}(x;a) &= x^2 + a(k-2),\\
D_{3,k}(x;a) &= x^3 + a(k-3)x,\\
D_{4,k}(x;a) &= x^4 + a(k-4)x^2 + a^2(2-k),\\
D_{5,k}(x;a) &= x^5 + a(k-5)x^3 + a^2(5-2k)x.
\end{aligned}$$

They have zeros at

$$\begin{aligned}
&x = \pm\sqrt{a}\sqrt{2-k}, \quad \text{if } n=2,\\
&x = 0,\ \pm\sqrt{a}\sqrt{3-k}, \quad \text{if } n=3,\\
&x = \pm\frac{\sqrt{a}}{\sqrt{2}}\sqrt{4-k\pm\sqrt{(k-2)^2+4}}, \quad \text{if } n=4,\\
&x = 0,\ \pm\frac{\sqrt{a}}{\sqrt{2}}\sqrt{5-k\pm\sqrt{(k-1)^2+4}}, \quad \text{if } n=5,
\end{aligned} \tag{1.4}$$

as can be verified using a mathematical symbolic computation program such as Mathematica.

Remark 1.1. Note that the polynomials $D_{n,k}(x;a)$ are monic for $n \geq 1$, but $D_{0,k}(x;a) = 1$ only for $k = 1$.

In this article, we study the polynomials $D_{n,k}(x;a)$ over the field of complex numbers, with $a > 0$ and $k \in \mathbb{R}$. Our motivation is the three-term recurrence relation (1.3), which suggests that the Dickson polynomials of the $(k+1)$-th kind form a family of orthogonal polynomials with respect to some linear functional L_k. However, from (1.4) we see that for $k > 2$ the polynomials $D_{n,k}(x;a)$ may have a pair of purely imaginary roots. Also, the polynomials $D_{3,3}(x;a)$ and $D_{5,\frac{5}{2}}(x;a)$ have a triple zero at $x = 0$. This implies that the linear functional L_k is quasi-definite ([16, Theorem 2.4.3], [28, Theorem 1]).

The article is organized as follows: in Section 2, we derive some of the main properties of the Dickson polynomials of the $(k+1)$-th kind, including different expressions, a hypergeometric representation, differential equations, and a generating function.

In Section 3, we present some basic results from the theory of orthogonal polynomials that we will need to find the linear functional L_k. We define the co-recursive polynomials associated with a given family of orthogonal polynomials, and list some of their main properties. We also show that a family of polynomials related to $D_{n,k}(x;a)$ are co-recursive polynomials associated with the Chebyshev polynomials of the second kind.

In Section 4 we apply the results of the previous sections to the Dickson polynomials of the $(k+1)$-th kind and obtain a representation for their moment functional L_k. We also find explicit expressions for the moments of L_k.

Finally, in Section 5 we summarize our results. In the hope that our results will be of interest to researchers outside the field of orthogonal polynomials and special functions, we have made the paper as self-contained as possible.

2. Properties of Dickson polynomials

We begin by checking the initial polynomial $D_{0,k}(x;a)$. Since it is not clear from the definition (1.1) that $D_{0,k}(x;a) = 2 - k$, we consider even and odd degrees and obtain the following result.

Proposition 2.1. *The even and odd Dickson polynomials of the $(k+1)$-th kind are given by*

$$D_{2n,k}(x;a) = (2-k)(-a)^n + \sum_{j=1}^{n} \frac{(2-k)\,n + kj}{j+n} \binom{n+j}{2j} (-a)^{n-j} x^{2j} \tag{2.1}$$

and

$$D_{2n+1,k}(x;a) = x \sum_{j=0}^{n} \frac{(2-k)\,n + kj + 1}{j+n+1} \binom{n+j+1}{2j+1} (-a)^{n-j} x^{2j}. \tag{2.2}$$

Proof. From (1.1), we have

$$D_{2n,k}(x;a) = \sum_{j=0}^{n} \frac{2n-kj}{2n-j} \binom{2n-j}{j} (-a)^j \, x^{2n-2j},$$

and by switching the index to $i = n - j$ we get

$$D_{2n,k}(x;a) = \sum_{i=0}^{n} \frac{(2-k)\, n + ki}{i+n} \binom{n+i}{n-i} (-a)^{n-i} \, x^{2i};$$

thus (2.1) follows after using the symmetry of the binomial coefficients,

$$\binom{n}{k} = \binom{n}{n-k}.$$

A similar calculation gives (2.2). □

Next, we will find a representation for $D_{n,k}(x;a)$ in terms of the generalized hypergeometric function

$${}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; x \right) = \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_p)_j}{(b_1)_j \cdots (b_q)_j} \frac{x^j}{j!},$$

where $(u)_j$ denotes the Pochhammer symbol (also called shifted or rising factorial), defined by (see [46, 5.2.4])

$$\begin{aligned} &(a)_0 = 1 \\ &(a)_j = a(a+1)\cdots(a+j-1), \quad j \in \mathbb{N}. \end{aligned}$$

Proposition 2.2. *The Dickson polynomials of the $(k+1)$-th kind admit the hypergeometric representation*

$$D_{n,k}(x;a) = x^n \; {}_3F_2\left(\begin{matrix} -\frac{n-1}{2}, -\frac{n}{2}, 1-\frac{n}{k} \\ 1-n, -\frac{n}{k} \end{matrix} ; \frac{4a}{x^2} \right), \quad k \neq 0. \tag{2.3}$$

We also have

$$D_{n,0}(x;a) = x^n \; {}_2F_1\left(\begin{matrix} -\frac{n-1}{2}, -\frac{n}{2} \\ 1-n \end{matrix} ; \frac{4a}{x^2} \right).$$

Proof. Let

$$D_{n,k}(x;a) = x^n \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} c_j,$$

with

$$c_j = \frac{n-kj}{n-j} \binom{n-j}{j} (-a)^j \, x^{-2j}. \tag{2.4}$$

We have $c_0 = 1$ and $c_j = 0$ for $j > \frac{n}{2}$. Using (2.4), we get

$$\frac{c_{j+1}}{c_j} = \frac{(2j-n+1)\,(2j-n)\,(jk+k-n)}{(j-n+1)\,(jk-n)\,(j+1)} \frac{a}{x^2}. \tag{2.5}$$

Let $k \neq 0$. Then,

$$\frac{c_{j+1}}{c_j} = \frac{\left(j - \frac{n}{2} + \frac{1}{2}\right)\left(j - \frac{n}{2}\right)\left(j + 1 - \frac{n}{k}\right)}{(j - n + 1)\left(j - \frac{n}{k}\right)(j+1)} \frac{4a}{x^2},$$

and it follows that

$$c_j = \frac{\left(-\frac{n-1}{2}\right)_j \left(-\frac{n}{2}\right)_j \left(1 - \frac{n}{k}\right)_j}{(1-n)_j \left(-\frac{n}{k}\right)_j} \frac{1}{j!} \left(\frac{4a}{x^2}\right)^j.$$

Thus,

$$D_{n,k}(x;a) = x^n \sum_{j=0}^{\infty} \frac{\left(-\frac{n-1}{2}\right)_j \left(-\frac{n}{2}\right)_j \left(1 - \frac{n}{k}\right)_j}{(1-n)_j \left(-\frac{n}{k}\right)_j} \frac{1}{j!} \left(\frac{4a}{x^2}\right)^j.$$

If $k = 0$, we see from (2.5) that

$$\frac{c_{j+1}}{c_j} = \frac{4\left(j + \frac{1-n}{2}\right)\left(j - \frac{n}{2}\right)}{(j - n + 1)(j+1)} \frac{a}{x^2},$$

and therefore

$$c_j = \frac{\left(-\frac{n-1}{2}\right)_j \left(-\frac{n}{2}\right)_j}{(1-n)_j} \frac{1}{j!} \left(\frac{4a}{x^2}\right)^j.$$

Hence,

$$D_{n,0}(x;a) = x^n \sum_{j=0}^{\infty} \frac{\left(-\frac{n-1}{2}\right)_j \left(-\frac{n}{2}\right)_j}{(1-n)_j} \frac{1}{j!} \left(\frac{4a}{x^2}\right)^j. \qquad \square$$

Remark 2.3. A representation of $D_{n,2}(x;a)$ in terms of associated Legendre functions of the first and second kinds [46, 14.3] was given by N. Fernando and S. Manukure [23].

Proposition 2.4. *For $n \in \mathbb{N}_0$, the Dickson polynomials of the $(k+1)$-th kind satisfy the following relations:*

$$D_{n,k+2} - 2D_{n,k+1} + D_{n,k} = 0,$$

$$\begin{aligned} -(x^2 - 4a)&\left[(k-1)nx^2 + a(k-2)(kn - 2n - k)\right] D''_{n,k} \\ &+ x\left[(k-1)nx^2 + a\left(6k + 4n + 3k^2 n - 4kn - 3k^2\right)\right] D'_{n,k} \\ &+ \left[(k-1)n^3x^2 + an\left(-k - 2n + kn\right)\left(-2k - 2n + kn\right)\right] D_{n,k} = 0, \end{aligned} \tag{2.6}$$

$$\begin{aligned} (x^2 - 4a)^2 D^{(iv)}_{n,k} + 10x(x^2 - 4a)D'''_{n,k} + \left[(23 - 2n^2)x^2 + 8a(n^2 - 4)\right] D''_{n,k} \\ - 3(2n^2 - 3)xD'_{n,k} + n^2(n^2 - 4)D_{n,k} = 0, \end{aligned}$$

and

$$(x^2 - 4a)D''_{n,k} - 4nD_{n+1,k}D'_{n,k} + (2n+3)\,xD'_{n,k} + n\,(n+2)\,D_{n,k} = 0.$$

Proof. All the identities can be automatically found and proved using the hypergeometric representation (2.3) and the Mathematica package HolonomicFunctions [33]. $\square$

Remark 2.5. The differential equation (2.6) already appeared in [58, Lemma 2.7].

We can use the recurrence relation (1.3) to obtain a different representation for the polynomials $D_{n,k}(x;a)$.

Proposition 2.6. *For $x \neq \pm 2\sqrt{a}$, the Dickson polynomials of the $(k+1)$-th kind are given by*

$$D_{n,k}(x;a) = \left(1 + k\frac{x-\Delta}{2\Delta}\right)\left(\frac{x+\Delta}{2}\right)^n + \left(1 - k\frac{x+\Delta}{2\Delta}\right)\left(\frac{x-\Delta}{2}\right)^n, \tag{2.7}$$

where

$$\Delta = \sqrt{x^2 - 4a}.$$

We also have

$$D_{n,k}(\pm 2\sqrt{a};a) = (kn + 2 - k)\left(\pm\sqrt{a}\right)^n. \tag{2.8}$$

Proof. Let us assume that we can write

$$D_{n,k}(x;a) = R^n, \tag{2.9}$$

for some function $R(x,k,a)$. Using (2.9) in the recurrence (1.3), we obtain

$$R^2 - xR + a = 0,$$

and therefore

$$R_{\pm} = \frac{x \pm \Delta}{2}$$

with

$$\Delta = \sqrt{x^2 - 4a}.$$

It follows that the general solution of (1.3) is given by

$$D_{n,k}(x;a) = C_1(x;a,k)\left(\frac{x+\Delta}{2}\right)^n + C_2(x;a,k)\left(\frac{x-\Delta}{2}\right)^n. \tag{2.10}$$

Using the initial conditions (1.2) in (2.10), we get

$$C_1(x;a,k) + C_2(x;a,k) = 2 - k,$$
$$C_1(x;a,k)\left(\frac{x+\Delta}{2}\right) + C_2(x;a,k)\left(\frac{x-\Delta}{2}\right) = x.$$

Thus, assuming that $x \neq \pm 2\sqrt{a}$, we get

$$C_1(x;a,k) = 1 + k\frac{x-\Delta}{2\Delta}, \quad C_2(x;a,k) = 1 - k\frac{x+\Delta}{2\Delta}.$$

To verify (2.8), we replace it in the recurrence (1.3), and obtain

$$\begin{aligned}(kn+2+k)\left(\pm\sqrt{a}\right)^{n+2} - \left(\pm 2\sqrt{a}\right)(kn+2)\left(\pm\sqrt{a}\right)^{n+1} + a(kn+2-k)\left(\pm\sqrt{a}\right)^n \\ = a\left(\pm\sqrt{a}\right)^n\left[(kn+2+k) - 2(kn+2) + (kn+2-k)\right] = 0.\end{aligned}$$

□

Using (2.7), we can obtain a generating function for the polynomials $D_{n,k}(x;a)$.

Proposition 2.7. *The ordinary generating function of the polynomials $D_{n,k}(x;a)$ is given by*

$$G(z;x,k,a)=\sum_{n=0}^{\infty}D_{n,k}(x;a)z^{n}=\frac{2-k+(k-1)\,xz}{az^{2}-xz+1}.$$

Proof. From (2.7), we have (as formal power series)

$$\begin{aligned}G(z;x,k,a)&=\sum_{n=0}^{\infty}D_{n,k}(x;a)z^{n}\\&=\left(1+k\frac{x-\Delta}{2\Delta}\right)\sum_{n=0}^{\infty}\left(z\frac{x+\Delta}{2}\right)^{n}+\left(1-k\frac{x+\Delta}{2\Delta}\right)\sum_{n=0}^{\infty}\left(z\frac{x-\Delta}{2}\right)^{n}\\&=\left(1+k\frac{x-\Delta}{2\Delta}\right)\frac{1}{1-\left(z\frac{x+\Delta}{2}\right)}+\left(1-k\frac{x+\Delta}{2\Delta}\right)\frac{1}{1-\left(z\frac{x-\Delta}{2}\right)}.\end{aligned}$$

Thus,

$$G(z;x,k,a)=4\frac{2-k+(k-1)\,xz}{\left(x^{2}-\Delta^{2}\right)z^{2}-4xz+4},$$

and the result follows since

$$\Delta^{2}=x^{2}-4a.$$ ☐

Remark 2.8. The same generating function was obtained in [58, Lemma 2.6] using the recurrence (1.3).

3. Orthogonal polynomials

Let $\{\mu_n\}$ be a sequence of complex numbers and let $\mathcal{L}:\mathbb{C}[x]\to\mathbb{C}$ be a linear functional defined by

$$\mathcal{L}\left[x^{n}\right]=\mu_{n},\quad n\in\mathbb{N}_{0}.$$

Then $\mathcal{L}$ is called the *moment functional* determined by the moment sequence $\{\mu_n\}$. The number μ_n is called the *moment* of order n.

A moment functional $\mathcal{L}$ is called *positive-definite* if $\mathcal{L}\left[r(x)\right]>0$ for every polynomial $r(x)$ that is not identically zero and is non-negative for all real x. Otherwise, $\mathcal{L}$ is called *quasi-definite*.

A sequence $\{P_n\}\subset\mathbb{C}[x]$, with $\deg\left(P_n\right)=n$, is called an *orthogonal polynomial sequence* with respect to $\mathcal{L}$ provided that

$$\mathcal{L}\left[P_{n}P_{m}\right]=h_{n}\delta_{n,m},\quad n,m\in\mathbb{N}_{0},$$

where $h_n\neq 0$ and $\delta_{n,m}$ is Kronecker's delta [13].

One of the fundamental properties of orthogonal polynomials is that they satisfy a three-term recurrence relation.

Theorem 3.1. *Let $\mathcal{L}$ be a moment functional and let $\{P_n\}$ be the sequence of monic orthogonal polynomials associated with it. Then, there exist $\beta_n\in\mathbb{C}$ and $\gamma_n\in\mathbb{C}\setminus\{0\}$ such that the polynomials $P_n(x)$ are a solution of the three-term recurrence relation*

$$P_{n+1}=\left(x-\beta_{n}\right)P_{n}-\gamma_{n}P_{n-1},\quad n\in\mathbb{N},\tag{3.1}$$

with initial conditions

$$P_0(x) = 1, \quad P_1(x) = x - \beta_0. \tag{3.2}$$

Proof. See [13, Theorem 4.1]. □

A linearly independent solution of (3.1) with initial conditions

$$P_0^*(x) = 0, \quad P_1^*(x) = 1 \tag{3.3}$$

is given by the so-called *associated orthogonal polynomials* $P_n^*(x)$ [13, 4.3]. Note that $\deg P_n^*(x) = n - 1$.

The converse of Theorem 3.1 is known as Favard's theorem.

Theorem 3.2. *Let $\{P_n\}$ be a sequence of polynomials satisfying the three-term recurrence relation* (3.1) *with $\beta_n \in \mathbb{C}$ and $\gamma_n \in \mathbb{C} \setminus \{0\}$. Then, there exists a* unique *linear functional $\mathcal{L}$ such that*

$$\mathcal{L}[P_0] = 1, \quad \mathcal{L}[P_n P_m] = h_n \delta_{n,m},$$

with

$$h_0 = P_0, \quad h_1 = \gamma_1, \quad h_n = \gamma_n h_{n-1},\ n = 2, 3, \dots.$$

Proof. See [13, Theorem 4.4]. □

Remark 3.3. It follows from (1.2) and (1.3) that (at least for $k \neq 2$) $\{D_{n,k}\}$ is a sequence of monic (for $n \geq 1$) orthogonal polynomials with respect to a moment functional L_k* satisfying

$$L_k[D_{n,k} D_{m,k}] = h_n(k) \delta_{n,m}, \tag{3.4}$$

with

$$h_0(k) = 2 - k, \quad h_n(k) = a^n,\ n \in \mathbb{N}. \tag{3.5}$$

In Section 4 we will find a representation for the moment functional L_k.

Proposition 3.4. *Let $\mathcal{L}$ be a moment functional and let $\{P_n\}$ be the sequence of monic orthogonal polynomials associated with it. Then, the following are equivalent:*

(a) All the moments of odd order are zero,

$$\mathcal{L}\left[x^{2n+1}\right] = 0, \quad n \in \mathbb{N}_0.$$

(b) The polynomials $P_n(x)$ satisfy

$$P_n(-x) = (-1)^n P_n(x), \quad n \in \mathbb{N}_0.$$

Proof. See [13, Theorem 4.3]. □

Proposition 3.5. *Let $k \neq 2$ and let $\mu_n(k)$ denote the moments of the linear functional defined by* (3.4)*. Then, we have*

$$\mu_0(k) = \frac{1}{2 - k}, \tag{3.6}$$

$$\mu_{2n+1}(k) = 0, \quad n \in \mathbb{N}_0, \tag{3.7}$$

*In the remainder of the paper, L_k will denote the moment functional associated with the polynomials $D_{n,k}(x; a)$.

and

$$\mu_{2n} = -\sum_{j=0}^{n-1} \frac{(2-k)\,n + kj}{j+n} \binom{n+j}{2j} (-a)^{n-j}\, \mu_{2j}, \quad n \in \mathbb{N}.$$

The first few nonzero moments are

$$\mu_2(k) = a, \quad \mu_4(k) = -a^2\,(k-3)\,, \quad \mu_6(k) = a^3(k^2 - 6k + 10),$$
$$\mu_8(k) = -a^4(k^3 - 9k^2 + 29k - 35).$$

Proof. From (3.5), we see that

$$2 - k = h_0 = L_k\left[D_{0,k}^2\right] = D_{0,k}^2 L_k[1] = (2-k)^2\,\mu_0,$$

from which (3.6) follows.

Using (1.1), it is clear that

$$D_{n,k}(-x;a) = (-1)^n D_{n,k}(x;a), \tag{3.8}$$

and Proposition 3.4 gives

$$\mu_{2n+1}(k) = 0, \quad n = 0, 1, \ldots.$$

From (2.1) we have

$$D_{2n,k}(x;a) = \sum_{j=0}^{n} \frac{(2-k)\,n + kj}{j+n} \binom{n+j}{2j} (-a)^{n-j}\, x^{2j},$$

and therefore

$$\begin{aligned} 0 = L_k\left[D_{2n,k}\right] &= \sum_{j=0}^{n} \frac{(2-k)\,n + kj}{j+n} \binom{n+j}{2j} (-a)^{n-j}\, \mu_{2j}(k) \\ &= \sum_{j=0}^{n-1} \frac{(2-k)\,n + kj}{j+n} \binom{n+j}{2j} (-a)^{n-j}\, \mu_{2j}(k) + \mu_{2n}(k). \end{aligned}$$

□

Remark 3.6. In Section 4 we will find a closed-form expression for $\mu_{2n}(k)$.

The task of finding an explicit integral representation for the functional $\mathcal{L}$ is called a *moment problem* [1], [34], [50]. A moment functional $\mathcal{L}$ is called *determinate* if there exists a unique (up to an additive constant) distribution $\psi(x)$ such that

$$\mathcal{L}\left[x^n\right] = \int_{\Lambda} x^n d\psi(x), \tag{3.9}$$

where the set Λ is called the support of the distribution ψ. Otherwise, $\mathcal{L}$ is called *indeterminate* [7], [55].

A criterion to decide if the moment functional $\mathcal{L}$ is determinate is due to Torsten Carleman [50, P 59]: If $\gamma_n > 0$ and

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{\gamma_n}} = \infty,$$

then $\mathcal{L}$ is determinate. Since for the Dickson polynomials we have $\gamma_n = a$, it follows that the moment problem is determinate.

One method to find a distribution function satisfying (3.9) is given by Markov's theorem.

Theorem 3.7. *Let the moment functional $\mathcal{L}$, supported on the set $\Lambda \subset \mathbb{C}$, be determinate, let $\{P_n\}$ be the monic orthogonal polynomials with respect to $\mathcal{L}$, and let $\{P_n^*\}$ be the associated polynomials. Then,*

$$\lim_{n\to\infty} \mathcal{L}[1] \frac{P_n^*(z)}{P_n(z)} = \mathcal{L}\left[\frac{1}{z-x}\right], \quad z \notin \Lambda,$$

where the convergence is uniform on compact subsets of $\mathbb{C} \setminus \Lambda$.

Proof. See [57]. □

The function

$$\mathcal{S}(z) = \mathcal{L}\left[\frac{1}{z-x}\right]$$

is called the *Stieltjes transform* of $\mathcal{L}$ [56]. Its asymptotic behavior is (see [30, Section 12.9])

$$\mathcal{S}(z) \sim \frac{\mu_0}{z} + \frac{\mu_1}{z^2} + \frac{\mu_2}{z^3} + \cdots, \quad z \to \infty. \tag{3.10}$$

3.1. **Co-recursive polynomials.** Let $\{q_n\}$ be a sequence of polynomials satisfying (3.1) with initial conditions

$$q_0(x) = u, \quad q_1(x) = x - v. \tag{3.11}$$

The polynomials $q_n(x)$ are called *co-recursive* with parameters (u, v). They were introduced by T. Chihara in [12], where he considered the case $u = 1$; see also [5], [38], [48], and [51].

Note that when $u = 1$ we can consider the polynomials $q_n(x)$ to be solutions of the perturbed three-term recurrence relation

$$xq_n = q_{n+1} + \widetilde{\beta}_n q_n + \gamma_n q_{n-1}, \quad n \in \mathbb{N}_0$$

with initial conditions

$$q_{-1}(x) = 0, \quad q_0(x) = 1,$$

where

$$\widetilde{\beta}_n = \beta_n + (v - \beta_0)\,\delta_{n,0}.$$

Orthogonal polynomials that are solutions of three-term recurrence relations with finite perturbations of the recurrence coefficients are called *co-polynomials*; see [11], [37], and [40].

Let $\{P_n\}$ and $\{P_n^*\}$ be the linearly independent solutions of (3.1) with initial conditions (3.2) and (3.3). Then, we can represent $q_n(x)$ as a linear combination

$$q_n(x) = AP_n(x) + BP_n^*(x).$$

Using (3.11), we see that

$$A = u, \quad B = (1-u)\,x + u\beta_0 - v,$$

and therefore

$$q_n(x) = uP_n(x) + [(1-u)\,x + u\beta_0 - v]\,P_n^*(x), \quad n \in \mathbb{N}_0. \tag{3.12}$$

Linear combinations of orthogonal polynomials have been studied by many authors; see [3], [35], [42], [43], and [44].

Note that the associated polynomials for both sequences are the same, i.e., $q_n^*(x) = P_n^*(x)$. Thus, we have

$$\frac{q_n(z)}{q_n^*(z)} = u\frac{P_n(z)}{P_n^*(z)} + (1-u)\,z + u\beta_0 - v,$$

and assuming that the moment problem is determined, we can use Markov's theorem and obtain

$$\frac{L_q[1]}{S_q(z)} = \frac{uL_P\,[1]}{S_P(z)} + (1-u)\,z + u\beta_0 - v,$$

where $S_q(z)$, $S_p(z)$ and L_q, L_p denote the Stieltjes transforms and linear functionals associated with the sequences $\{q_n\}$ and $\{P_n\}$, respectively. Solving for $S_q(z)$, we obtain

$$S_q(z) = \frac{L_q[1]S_P(z)}{[(1-u)\,z + u\beta_0 - v]\,S_P(z) + uL_P[1]}. \tag{3.13}$$

The rational transformation (3.13) is a particular case of the general *spectral transformations* of the Stieltjes transform (see [59])

$$\widetilde{S}(z) = \frac{A(z)S(z) + B(z)}{C(z)S(z) + D\,(z)},$$

where $A(z)$, $B(z)$, $C(z)$, $D(z)$ are polynomials; see also [8], [10], [24], [36], and [47].

Suppose that $L_P[1] = 1$, and let us denote the polynomial $(1-u)\,z + u\beta_0 - v$ appearing in (3.13) by $Q\,(z)$. If the function $S_P(z)$ is a solution of the quadratic equation

$$AQS_P^2 + (BQ + uA - C)\,S_P + uB = 0, \tag{3.14}$$

for some polynomials $A(z)$, $B(z)$, $C(z)$, then we have

$$\frac{S_q(z)}{L_q[1]} = \frac{S_P\,(z)}{Q(z)S_P(z) + u} = \frac{A\,(z)\,S_P(z) + B(z)}{C(z)}.$$

Stieltjes transforms satisfying quadratic equations with polynomial coefficients are called *second degree forms*; see [2], [6], [45], and [49].

The linear transformation

$$\frac{S_q(z)}{L_q[1]} = \frac{A\,(z)\,S_P(z) + B(z)}{C(z)}$$

can be written as a composition of three basic transformations:

(1) The *Uvarov transformation* [41, 54], defined by

$$\frac{L_U[r]}{\lambda_U} = L_P[r] + Mr\,(\omega)\,, \quad r \in \mathbb{C}[x],$$

or by

$$\frac{S_U(z)}{\lambda_U} = S(z) + \frac{M}{z-\omega},$$

where $M + L_P[1] \neq 0$ and

$$\lambda_U = \frac{L_U[1]}{L_P[1] + M}.$$

(2) The *Christoffel transformation* [9, 14], defined by

$$\frac{L_C[r]}{\lambda_C} = L_P\left[(x-\omega)\, r(x)\right], \quad r \in \mathbb{C}\left[x\right],$$

or by

$$\frac{S_C(z)}{\lambda_C} = (z-\omega)\, S(z) - L_P[1],$$

where $L_P\left[x-\omega\right] \neq 0$ and

$$\lambda_C = \frac{L_C[1]}{L_P\left[x-\omega\right]}.$$

(3) The *Geronimus transformation* [17, 18, 26, 27], defined by

$$\frac{L_G[r]}{\lambda_G} = L_P\left[\frac{r\,(x)}{x-\omega}\right] + Mr\,(\omega)\,, \quad r \in \mathbb{C}[x],$$

or by

$$(z-\omega)\,\frac{S_G(z)}{\lambda_G} = S(z) - S(\omega) + M,$$

where $M - S(\omega) \neq 0$ and

$$\lambda_G = \frac{L_G[1]}{M - S(\omega)}.$$

If the coefficients in the three-term recurrence relation (3.1) are constant, then we see that $P_n^*(x)$ and $P_{n-1}(x)$ satisfy the same recurrence, and have the same initial conditions. Therefore, $P_n^*\,(x) = P_{n-1}(x)$ and we obtain

$$x = \frac{P_{n+1}}{P_n} + \beta + \gamma\frac{P_{n-1}}{P_n} = \frac{P_{n+1}}{P_{n+1}^*} + \beta + \gamma\frac{P_n^*}{P_n}.$$

Using Markov's theorem, we conclude that

$$z = \frac{L_P[1]}{S_P(z)} + \beta + \gamma\frac{S_P(z)}{L_P[1]}.$$

Assuming that $L_P[1] = 1$, we find that $S_P\,(z)$ is the solution of the quadratic equation

$$\gamma S_P^2 - (z-\beta)\, S_P + 1 = 0. \tag{3.15}$$

Multiplying (3.14) by γ, (3.15) by AQ, and subtracting, we get

$$\left[\gamma\,(BQ + uA - C) + (z-\beta)\, AQ\right] S_P + u\gamma B - AQ = 0.$$

Therefore,

$$A = u\gamma, \quad B = Q, \quad C = Q^2 + u\,(z-\beta)\, Q + u^2\gamma,$$

or, replacing $Q(z)$ by $(1-u)\,z + u\beta - v$,

$$B = (1-u)z + u\beta - v, \quad C = (z-v)\left[(1-u)z + u\beta - v\right] + u^2\gamma.$$

We conclude that

$$\frac{S_q(z)}{L_q[1]} = \frac{u\gamma S_P(z) + (1-u)\,z + u\beta - v}{(z-v)\,[(1-u)\,z + u\beta - v] + u^2\gamma}. \tag{3.16}$$

Polynomial solutions of three-term recurrence relations with constant coefficients

$$xq_n = q_{n+1} + \beta q_n + \gamma q_{n-1}, \quad q_0(x) = u, \quad q_1(x) = x - v$$

were analyzed in [15], where the authors concluded that the linear functional L_q was of the form

$$\frac{L_q[r]}{L_q[f]} = \frac{1}{2\pi\gamma} \int_{\beta-2\sqrt{\gamma}}^{\beta+2\sqrt{\gamma}} r(x) \frac{\sqrt{4\gamma - (x-\beta)^2}}{f(x)}\, dx + M_1 r(y_1) + M_2 r(y_2), \quad r \in \mathbb{C}[x],$$

with

$$f(x) = (x-v)\,[(1-u)\,x + u\beta - v] + u^2\gamma = (1-u)\,(x-y_1)\,(x-y_2).$$

They only considered the case where $u > 0$ and all parameters (including the roots of f) are real numbers. For the masses M_1, M_2 they obtained

$$M_i = \frac{2}{\sqrt{\left[(\beta-v)^2 + 4\gamma\,(u-1)\right]_+}} \left[\frac{u\gamma}{|y_i - v|} - \frac{|y_i - v|}{u}\right]_+, \quad u \neq 1,$$

and

$$M_1 = M_2 = \left[1 - \frac{\gamma}{(\beta-v)^2}\right]_+, \quad u = 1,$$

where

$$[x]_+ = \frac{x + |x|}{2}.$$

The solution of (3.15) having the right asymptotic behavior is

$$S_P(z) = \frac{z - \beta - \sqrt{(z-\beta)^2 - 4\gamma}}{2\gamma} \sim \frac{1}{z} + \frac{\beta}{z^2}, \quad z \to \infty,$$

and it follows that an integral representation of the linear functional L_P is given by

$$L_P[r] = \frac{1}{2\pi\gamma} \int_{\beta-2\sqrt{\gamma}}^{\beta+2\sqrt{\gamma}} r(x) \sqrt{4\gamma - (x-\beta)^2}\, dx, \quad r \in \mathbb{C}[x].$$

Using the change of variables

$$x = \beta + 2y\sqrt{\gamma}$$

we obtain

$$L_P[r] = \frac{2}{\pi} \int_{-1}^{1} r\left(\beta + 2y\sqrt{\gamma}\right) \sqrt{1-y^2}\, dy.$$

Therefore, it is enough to study the linear functional

$$L_U[r] = \frac{2}{\pi}\int_{-1}^{1} r(y)\sqrt{1-y^2}\,dy$$

associated to the Chebyshev polynomials of the second kind, which we will define in the next subsection.

3.2. **Chebyshev polynomials.** The monic *Chebyshev polynomials of the second kind* $U_n(x)$ are defined by (see [32, 9.8.36])

$$U_n(x) = 2^{-n}(n+1)\ {}_2F_1\left(\begin{matrix} -n, n+2 \\ \frac{3}{2} \end{matrix}; \frac{1-x}{2}\right).$$

They are a solution of the recurrence relation

$$U_{n+1} - xU_n + \frac{1}{4}U_{n-1} = 0, \tag{3.17}$$

with initial conditions (see [32, 9.8.40])

$$U_0(x) = 1, \quad U_1(x) = x.$$

Note that

$$U_{-1}(x) = 0. \tag{3.18}$$

The polynomials $U_n(x)$ satisfy the orthogonality relation (see [32, 9.8.38])

$$L_U[U_nU_m] = \frac{2}{\pi}\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}\,dx = \delta_{n,m}. \tag{3.19}$$

The Stieltjes transform of the linear functional L_U is given by (see [56])

$$S_U(z) = \frac{2}{\pi}\int_{-1}^{1} \frac{\sqrt{1-x^2}}{z-x}\,dx = 2z\left(1-\sqrt{1-z^{-2}}\right), \quad z\in\mathbb{C}\setminus[-1,1]; \tag{3.20}$$

here and in the rest of the paper

$$\sqrt{} : \mathbb{C} \to \left\{ z\in\mathbb{C} \mid -\frac{\pi}{2} < \arg(z) \le \frac{\pi}{2} \right\}$$

denotes the principal branch of the square root. Note that

$$\sqrt{1-z^{-2}} \sim 1, \quad z\to\infty.$$

It is clear from the three-term recurrence relation (1.3) that the polynomials $D_{n,k}(x;a)$ are related to the polynomials $U_n(x)$. Let us introduce the scaled polynomials $d_n(x;k)$ defined by

$$d_n(x;k) = \left(2\sqrt{a}\right)^{-n} D_{n,k}\left(2\sqrt{a}x\right). \tag{3.21}$$

The polynomials $d_n(x;k)$ are a solution of the recurrence (3.17) satisfied by the polynomials $U_n(x)$, with initial conditions

$$d_0(x;k) = 2-k, \quad d_1(x;k) = x.$$

Therefore, we see that the scaled polynomials $d_n(x;k)$ are co-recursive polynomials (with respect to the Chebyshev polynomials of the second kind) with parameters $u = 2-k$, $v = 0$.

Using (3.12), we have

$$d_n(x;k) = (2-k)\, U_n(x) + (k-1)\, xU_{n-1}(x), \tag{3.22}$$

since

$$d_n^*(x;k) = U_n^*(x) = U_{n-1}(x).$$

Remark 3.8. If we use the values of the monic Chebyshev polynomials at $x = 0$ [46, 18.6.1],

$$U_n(0) = 2^{-n} \cos\left(\frac{n\pi}{2}\right),$$

and the representation (3.22), we get

$$D_{n,k}(0;a) = \left(2\sqrt{a}\right)^n d_n(0;k) = \left(\sqrt{a}\right)^n (2-k) \cos\left(\frac{n\pi}{2}\right),$$

in agreement with (2.1)–(2.2).

If $k = 0, 1, 2$, (3.21) and (3.22) give

$$\begin{aligned}
D_{n,k}(x) &= \left(2\sqrt{a}\right)^n \left[(2-k)\, U_n\left(\frac{x}{2\sqrt{a}}\right) + (k-1)\, \frac{x}{2\sqrt{a}} U_{n-1}\left(\frac{x}{2\sqrt{a}}\right)\right], \\
D_{n,0}(x;a) &= \left(2\sqrt{a}\right)^n \left[2U_n\left(\frac{x}{2\sqrt{a}}\right) - \frac{x}{2\sqrt{a}} U_{n-1}\left(\frac{x}{2\sqrt{a}}\right)\right], \\
D_{n,1}(x;a) &= \left(2\sqrt{a}\right)^n U_n\left(\frac{x}{2\sqrt{a}}\right), \\
D_{n,2}(x;a) &= \left(2\sqrt{a}\right)^{n-1} xU_{n-1}\left(\frac{x}{2\sqrt{a}}\right),
\end{aligned} \tag{3.23}$$

and in particular, for $a = 1$, we have

$$\begin{aligned}
D_n(2x) &= D_{n,0}(2x;1) = 2^{n+1} U_n(x) - x2^n U_{n-1}(x), \\
E_n(2x) &= D_{n,1}(2x;1) = 2^n U_n(x),
\end{aligned}$$

as it was observed in [58].

Remark 3.9. Stoll [52] studied second order recurrences with constant coefficients and general initial conditions and found polynomial decompositions in terms of Chebyshev polynomials.

4. Main results

In this section we find a representation for the linear functional L_k defined by (3.4). Although this seems to be something already considered in [15], we have found that the range of parameters of the polynomials $D_{n,k}(x)$ requires a different analysis.

Let $k \neq 2$. Denote by $l_k[r]$ the linear functional satisfying

$$l_k[1] = \frac{1}{2-k}, \quad l_k\left[d_n d_m\right] = 4^{-n} \delta_{n,m}, \ n, m \in \mathbb{N}, \tag{4.1}$$

and let $s(z;k)$ be its Stieltjes transform. From (3.16) and (3.20), we have

$$s(z;k) = \frac{2z}{2-k} \frac{(k-2)\sqrt{1-z^{-2}}+k}{4(k-1)z^2+(k-2)^2}, \tag{4.2}$$

since $v = \beta = 0$.

Our next objective is to represent the function $s(z;k)$ as the Stieltjes transform of a distribution. We begin with a couple of lemmas.

Lemma 4.1. *Let $\mathcal{L}$ be a linear functional with Stieltjes transform $S(z)$ and*

$$f(x) = (x-\omega_1)(x-\omega_2).$$

Then,

$$\mathcal{L}\left[\frac{1}{f(x)(z-x)}\right] = \frac{S(z)}{f(z)} + \frac{[S(\omega_2)-S(\omega_1)]z + \omega_2 S(\omega_1) - \omega_1 S(\omega_2)}{(\omega_1-\omega_2) f(z)},$$

where we always assume that the functional $\mathcal{L}$ acts on the variable x.

Proof. Since

$$\frac{1}{(x-\omega_1)(x-\omega_2)(z-x)} = \frac{1}{(z-\omega_1)(z-\omega_2)(z-x)} + \frac{1}{(\omega_1-\omega_2)(z-\omega_2)(\omega_2-x)} - \frac{1}{(\omega_1-\omega_2)(z-\omega_1)(\omega_1-x)},$$

we get

$$\mathcal{L}\left[\frac{1}{f(x)(z-x)}\right] = \frac{S(z)}{(z-\omega_1)(z-\omega_2)} + \frac{S(\omega_2)}{(\omega_1-\omega_2)(z-\omega_2)} - \frac{S(\omega_1)}{(\omega_1-\omega_2)(z-\omega_1)},$$

and the conclusion follows. □

For the particular case of the linear functional L_U defined by (3.19) with Stieltjes transform $S_U(z)$ given in (3.20) and $f(x) = x^2 - b^2$, Lemma 4.1 gives

$$L_U\left[\frac{1}{(x^2-b^2)(z-x)}\right] = 2z\frac{\sqrt{1-b^{-2}}-\sqrt{1-z^{-2}}}{z^2-b^2}, \quad z, b \in \mathbb{C}\setminus[-1,1], \tag{4.3}$$

because

$$\frac{S_U(z)}{z^2-b^2} - \frac{S_U(b)}{2b(z-b)} + \frac{S_U(-b)}{2b(z+b)} = \frac{2z\left(1-\sqrt{1-z^{-2}}\right)}{z^2-b^2} + \frac{2b\left(1-\sqrt{1-b^{-2}}\right)}{2b(b-z)} + \frac{2(-b)\left(1-\sqrt{1-b^{-2}}\right)}{2b(b+z)}.$$

Note that since

$$\frac{2}{\pi}\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{dx}{z-x} = \frac{2}{z\sqrt{1-z^{-2}}}, \quad z \in \mathbb{C}\setminus[-1,1],$$

(see [56]), we have

$$\lim_{b^2\to 1^-} L_U\left[\frac{1}{(x^2-b^2)(z-x)}\right] = \frac{2}{\pi}\int_{-1}^{1} \frac{\sqrt{1-x^2}}{x^2-1}\frac{1}{z-x}\,dx$$

$$= -\frac{2}{\pi}\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}\,(z-x)}\,dx = \frac{-2}{z\sqrt{1-z^{-2}}}$$

$$= -2z\frac{\sqrt{1-z^{-2}}}{z^2-1} = \lim_{b^2\to 1^-} 2z\frac{\sqrt{1-b^{-2}}-\sqrt{1-z^{-2}}}{z^2-b^2},$$

and, in particular,

$$\frac{1}{2z\sqrt{1-z^{-2}}} = \frac{1}{4}L_U\left[\frac{1}{(1-x^2)(z-x)}\right]. \tag{4.4}$$

Lemma 4.2. *Let* $\omega(k)$ *be defined by*

$$\omega(k) = \frac{1}{2}\frac{k-2}{\sqrt{k-1}}\mathrm{i}, \quad k \neq 1, \tag{4.5}$$

where $\mathrm{i}^2 = -1$. *Then,*

$$\omega(k) \in \mathbb{C}\setminus[-1,1], \quad k \in \mathbb{R}\setminus\{0,1,2\}.$$

Proof. The result follows immediately from the definition (4.5), since

$$\omega(k) \in (-\infty,-1), \quad k \in (-\infty,0)\cup(0,1),$$
$$\mathrm{i}\omega(k) \in \mathbb{R}, \quad k \in (1,\infty),$$
$$\omega(0) = -1, \quad \omega(2) = 0. \qquad \square$$

The function $s(z;k)$ defined in (4.2) has a branch cut on the segment $[-1,1]$ and (perhaps removable) poles at $z = \pm\omega$ if $k \neq 1$. In the next theorem we split $s(z;k)$ in two parts, one analytic in $\mathbb{C}\setminus[-1,1]$ and the other analytic in $\mathbb{C}\setminus\{\pm\omega\}$.

Theorem 4.3. *Let* $k \neq 2$ *and* $z \in \mathbb{C}\setminus[-1,1]$. *Then, we have*

$$s(z;k) = s_c(z;k) + \chi(k)s_d(z;k),$$

where $\chi(k)$ *is the characteristic function defined by*

$$\chi(k) = \begin{cases} 0, & k \in [0,2]; \\ 1, & k \in \mathbb{R}\setminus[0,2], \end{cases} \tag{4.6}$$

$s_c(z;k)$ *is the continuous part of* $s(z;k)$,

$$s_c(z;k) = \frac{2}{\pi}\int_{-1}^{1} \frac{\sqrt{1-x^2}}{4(k-1)x^2+(k-2)^2}\frac{1}{z-x}\,dx,$$

and $s_d(z;k)$ *is the discrete part of* $s(z;k)$,

$$s_d(z;k) = \frac{4k}{2-k}\frac{z}{4(k-1)z^2+(k-2)^2}.$$

Proof. Let $k \in \mathbb{R} \setminus \{0,1,2\}$. From (4.2), we have

$$s(z;k) = 2z \frac{\frac{k}{2-k} - \sqrt{1-z^{-2}}}{4(k-1)z^2 + (k-2)^2}.$$

Using (4.5), we get

$$\frac{(k-2)^2}{4(k-1)} = -\omega^2, \quad 1 - \omega^{-2}(k) = \left(\frac{k}{2-k}\right)^2. \tag{4.7}$$

Hence,

$$s(z;k) = \frac{2z}{4(k-1)} \frac{\sqrt{1-\omega^{-2}} - \sqrt{1-z^{-2}} + \frac{k}{2-k} - \left|\frac{k}{2-k}\right|}{z^2-\omega^2}.$$

Since we know from Lemma 4.2 that $\omega(k) \in \mathbb{C} \setminus [-1,1]$, we can use (4.3) with $b = \omega$ and obtain

$$s(z;k) = \frac{1}{4(k-1)} L_U \left[\frac{1}{(x^2-\omega^2)(z-x)}\right] + \frac{2z}{4(k-1)} \frac{\frac{k}{2-k} - \left|\frac{k}{2-k}\right|}{z^2-\omega^2}.$$

But

$$\frac{k}{2-k} - \left|\frac{k}{2-k}\right| = \begin{cases} 0, & k \in [0,2); \\ \frac{2k}{2-k}, & k \in \mathbb{R} \setminus [0,2], \end{cases}$$

and therefore

$$s(z;k) = \begin{cases} s_c(z;k), & k \in (0,2) \setminus \{1\}; \\ s_c(z;k) + s_d(z;k), & k \in \mathbb{R} \setminus [0,2], \end{cases}$$

where

$$s_c(z;k) = L_U \left[\frac{1}{4(k-1)x^2 + (k-2)^2} \frac{1}{z-x}\right]$$

and

$$s_d(z;k) = \frac{2z}{4(k-1)} \frac{2k}{2-k} \frac{1}{z^2-\omega^2}.$$

If $k = 0$, then we have from (4.2)

$$s(z;0) = z \frac{\sqrt{1-z^{-2}}}{2(z^2-1)} = \frac{1}{2z\sqrt{1-z^{-2}}},$$

and using (4.4) we get

$$\frac{1}{2z\sqrt{1-z^{-2}}} = \frac{1}{4} L_U \left[\frac{1}{(1-x^2)(z-x)}\right] = s_c(z;0).$$

If $k = 1$, then we have from (4.2)

$$s(z;1) = 2z\left(1 - \sqrt{1-z^{-2}}\right),$$

and using (3.20) we get

$$s(z;1) = \frac{2}{\pi}\int_{-1}^{1} \frac{\sqrt{1-x^2}}{z-x}\, dx = s_c(z;1). \qquad \square$$

Remark 4.4. Orthogonal polynomials with linear functionals of the form

$$\mathcal{L}[r] = \int_{-1}^{1} r(x) \frac{(1-x)^{\alpha}(1+x)^{\beta}}{f(x)}\, dx, \quad \alpha, \beta = \pm\frac{1}{2},$$

where $f(x)$ is a polynomial, are called *Bernstein–Szegö polynomials* [46, 18.31], [53, 2.6]. These polynomials are examples of the Geronimus transformation applied to the Jacobi polynomials, see [4], [25], [31, 2.7.3], and [59].

Corollary 4.5. *Let $k \neq 2$. Let the linear functional l_k be defined by* (4.1)*, the characteristic function $\chi(k)$ by* (4.6)*, and the function $\omega(k)$ by* (4.5)*. Then, for all $r(x) \in \mathbb{C}[x]$ we have*

$$l_k[r] = l_k^{(c)}[r] + \chi(k) l_k^{(d)}[r], \tag{4.8}$$

where

$$l_k^{(c)}[r] = \frac{2}{\pi}\int_{-1}^{1} \frac{r(x)\sqrt{1-x^2}}{4(k-1)x^2 + (k-2)^2}\, dx, \tag{4.9}$$

$$l_k^{(d)}[r] = k \frac{r(\omega) + r(-\omega)}{2(k-1)(2-k)}, \tag{4.10}$$

and we assume that

$$\lim_{k\to 1} \frac{\chi(k)}{k-1} = 0. \tag{4.11}$$

Proof. The result is a direct consequence of Theorem 4.3, since

$$\begin{aligned} \frac{4k}{2-k} \frac{z}{4(k-1)z^2 + (k-2)^2} &= \frac{k}{(2-k)(k-1)} \frac{z}{z^2 - \omega^2} \\ &= \frac{1}{2} \frac{k}{(k-1)(2-k)} \left(\frac{1}{z-\omega} + \frac{1}{z+\omega} \right). \end{aligned}$$

For $k = 1$, we see from (3.22) that

$$d_n(x;1) = U_n(x),$$

and therefore

$$l_1[r] = L_U[r] = \frac{2}{\pi}\int_{-1}^{1} r(x)\sqrt{1-x^2}\, dx,$$

which agrees with (4.8) for $k = 1$ if we use (4.11). $\square$

Remark 4.6. For $k=2$, we see from (3.22) that

$$d_n(x;2) = xU_{n-1}(x),$$

and therefore we can interpret l_2 as the linear functional

$$l_2[r] = L_U\left[\frac{r(x)}{4x^2}\right] = \frac{1}{2\pi}\int_{-1}^{1} r(x)\frac{\sqrt{1-x^2}}{x^2}\,dx,$$

defined for all polynomials $r(x)$ such that $r(x) = x^2 p(x)$, $p(x) \in \mathbb{C}[x]$. It follows that $d_n(x;2)$ will be a family of orthogonal polynomials for $n \geq 1$.

4.1. **The Dickson polynomials.** We can now apply the previous results to the Dickson polynomials of the $(k+1)$-th kind $D_{n,k}(x;a)$, related to the scaled polynomials $d_n(x;k)$ by (3.21).

Lemma 4.7. *Let $\mathfrak{L}_k : \mathbb{C}[x] \to \mathbb{C}$ be the linear functional defined by*

$$\mathfrak{L}_k[r(x)] = l_k\left[r\left(2\sqrt{a}x\right)\right], \tag{4.12}$$

where l_k is the linear functional defined by (4.1)*. Then, $\mathfrak{L}_k$ satisfies*

$$\begin{aligned} \mathfrak{L}_k\left[D_{0,k}^2\right] &= 2-k, \\ \mathfrak{L}_k\left[D_{n,k}D_{m,k}\right] &= a^n \delta_{n,m}, \quad n,m \in \mathbb{N}, \end{aligned} \tag{4.13}$$

and for $k \neq 2$, its Stieltjes transform is given by

$$S(z;k,a) = \frac{z}{2(2-k)}\frac{(k-2)\sqrt{1-4az^{-2}}+k}{(k-1)z^2+(k-2)^2 a}, \quad z \in \mathbb{C}\setminus\left[-2\sqrt{a},2\sqrt{a}\right]. \tag{4.14}$$

Proof. Using (3.21) in (4.12), we have

$$\begin{aligned} \mathfrak{L}_k\left[D_{n,k}(x)D_{m,k}(x)\right] &= l_k\left[D_{n,k}\left(2\sqrt{a}x\right)D_{m,k}\left(2\sqrt{a}x\right)\right] \\ &= \left(2\sqrt{a}\right)^{n+m} l_k\left[d_n(x;k)d_m(x;k)\right]. \end{aligned}$$

Therefore, (4.1) gives

$$\mathfrak{L}_k\left[D_{0,k}^2\right] = l_k\left[\,(2-k)^2\,\right] = (2-k)^2\, l_k[1] = 2-k$$

and

$$\mathfrak{L}_k\left[D_{n,k}(x)D_{m,k}(x)\right] = \left(2\sqrt{a}\right)^{n+m} 4^{-n}\delta_{n,m}, \quad n,m \in \mathbb{N}.$$

But since

$$\left(2\sqrt{a}\right)^{n+m} 4^{-n}\delta_{n,m} = a^n \delta_{n,m},$$

(4.13) follows.

Using (4.2) in (4.12), we get

$$\begin{aligned} \mathfrak{L}_k\left[\frac{1}{z-x}\right] &= l_k\left[\frac{1}{z-2\sqrt{a}x}\right] = \frac{1}{2\sqrt{a}} l_k\left[\frac{1}{\frac{z}{2\sqrt{a}}-x}\right] \\ &= \frac{1}{2\sqrt{a}} s\left(\frac{z}{2\sqrt{a}};k\right) = \frac{1}{2\sqrt{a}}\frac{2\frac{z}{2\sqrt{a}}}{2-k}\frac{(k-2)\sqrt{1-4az^{-2}}+k}{4(k-1)\frac{z^2}{4a}+(k-2)^2}, \end{aligned}$$

and (4.14) follows. ☐

Corollary 4.8. *The linear functional* $\mathfrak{L}_k$ *defined by* (4.12) *is identical to the linear functional* L_k *satisfying* (3.4)*.*

Next, we find a representation for the linear functional L_k.

Theorem 4.9. *Let* $k \neq 2$ *and let* L_k *be the linear functional defined by* (3.4)*. Then,* L_k *admits the representation*

$$L_k[r] = L_k^{(c)}[r] + \chi(k) L_k^{(d)}[r],$$

where $\chi(k)$ *was defined in* (4.6)*,*

$$L_k^{(c)}[r] = \frac{1}{2\pi} \int_{-2\sqrt{a}}^{2\sqrt{a}} \frac{r(t)\sqrt{4a-t^2}}{(k-1)t^2 + (k-2)^2 a} \, dt, \tag{4.15}$$

and

$$L_k^{(d)}[r] = k \frac{r(\Omega) + r(-\Omega)}{2(k-1)(2-k)}, \tag{4.16}$$

with $\Omega(k)$ *defined by*

$$\Omega(k) = 2\sqrt{a}\omega(k) = \sqrt{a}\frac{k-2}{\sqrt{k-1}}\mathrm{i}, \quad k \neq 1. \tag{4.17}$$

Proof. Changing variables to $t = 2\sqrt{a}x$ in the integral

$$I = \int_{-1}^{1} \frac{r(x)\sqrt{1-x^2}}{4(k-1)x^2 + (k-2)^2} \, dx,$$

we obtain

$$I = \frac{1}{4} \int_{-2\sqrt{a}}^{2\sqrt{a}} \frac{r\left(\frac{t}{2\sqrt{a}}\right)\sqrt{4a-t^2}}{(k-1)t^2 + (k-2)^2 a} \, dt.$$

Therefore, from (4.9) we get

$$l_k^{(c)}\left[r\left(2\sqrt{a}t\right)\right] = \frac{1}{2\pi} \int_{-2\sqrt{a}}^{2\sqrt{a}} \frac{r(t)\sqrt{4a-t^2}}{(k-1)t^2 + (k-2)^2 a} \, dt.$$

Also, from (4.10) and (4.17) we have

$$l_k^{(d)}\left[r\left(2\sqrt{a}t\right)\right] = k\frac{r(2\sqrt{a}\omega) + r(-2\sqrt{a}\omega)}{2(k-1)(2-k)} = k\frac{r(\Omega) + r(-\Omega)}{2(k-1)(2-k)}.$$

Therefore, using (4.8) in (4.12) we see that

$$L_k[r] = \frac{1}{2\pi} \int_{-2\sqrt{a}}^{2\sqrt{a}} \frac{r(t)\sqrt{4a-t^2}}{(k-1)t^2 + (k-2)^2 a} \, dt + \chi(k) k \frac{r(\Omega) + r(-\Omega)}{2(k-1)(2-k)}. \qquad \square$$

Although Theorem 4.9 seems to be valid only when $k \neq 2$, we can see that L_2 is well defined.

Lemma 4.10. *Let $k \neq 1$ and let $\Omega(k)$ be defined by* (4.17)*. Then,*

$$D_{n,k}\left(\Omega;a\right)=\left(2-k\right)\left(-\mathrm{i}\sqrt{\frac{a}{k-1}}\right)^{n}. \tag{4.18}$$

Proof. Let us assume that

$$D_{n,k}\left(\Omega;a\right)=b_0B^n$$

for some functions $b_0(k,a)$ and $B(k,a)$. Using (1.3), we have

$$0=b_0B^{n+2}-\Omega b_0B^{n+1}+ab_0B^n=b_0B^n\left(B^2-\Omega B+a\right).$$

Using (1.2), we get

$$b_0=D_{0,k}(\omega;a)=2-k$$

and

$$\Omega=D_{1,k}\left(\Omega;a\right)=\left(2-k\right)B(k,a).$$

Thus,

$$B\left(k,a\right)=\frac{\Omega}{2-k}=-\sqrt{\frac{a}{k-1}}\mathrm{i},$$

and clearly

$$B^2-\Omega B+a=0. \qquad \square$$

It follows from the previous lemma that $L_2^{(d)}$ is well defined.

Proposition 4.11. *Let $\Omega(k)$ be defined by* (4.17) *and $L_k^{(d)}$ by* (4.16)*. Then, for $k \neq 1$,*

$$L_k^{(d)}\left[D_{n,k}D_{m,k}\right]=\frac{\left(2-k\right)k}{k-1}\left[\frac{1+\left(-1\right)^{n+m}}{2}\right]\left(\mathrm{i}\sqrt{\frac{a}{k-1}}\right)^{n+m}.$$

Proof. From (4.18) we have

$$D_{n,k}\left(\Omega;a\right)D_{m,k}\left(\Omega;a\right)=\left(2-k\right)^2\left(-\mathrm{i}\sqrt{\frac{a}{k-1}}\right)^{n+m}.$$

Using (3.8), we get

$$\begin{aligned}D_{n,k}\left(\Omega;a\right)D_{m,k}\left(\Omega;a\right)&+D_{n,k}\left(-\Omega;a\right)D_{m,k}\left(-\Omega;a\right)\\&=\left(2-k\right)^2\left(\mathrm{i}\sqrt{\frac{a}{k-1}}\right)^{n+m}\left[1+\left(-1\right)^{n+m}\right].\end{aligned}$$

Thus,

$$L_k^{(d)}\left[D_{n,k}D_{m,k}\right]=\frac{\left(2-k\right)k}{k-1}\left[\frac{1+\left(-1\right)^{n+m}}{2}\right]\left(\mathrm{i}\sqrt{\frac{a}{k-1}}\right)^{n+m}. \qquad \square$$

We can now extend Theorem 4.9 to all values of k.

Corollary 4.12. *Let $h_n(k)$ be defined by* (3.5) *and $\chi(k)$ by* (4.6)*. Then,*

$$\frac{1}{2\pi}\int_{-2\sqrt{a}}^{2\sqrt{a}} \frac{\sqrt{4a-t^2}D_{n,k}(t)D_{m,k}(t)}{(k-1)t^2+a(k-2)^2}\,dt \\ +\chi(k)\frac{(2-k)\,k}{k-1}\left[\frac{1+(-1)^{n+m}}{2}\right]\left(\mathrm{i}\sqrt{\frac{a}{k-1}}\right)^{n+m} = h_n(k)\delta_{n,m}, \quad n,m\in\mathbb{N}_0. \tag{4.19}$$

Remark 4.13. If we set $k=2$ in (4.19), we obtain

$$\frac{1}{2\pi}\int_{-2\sqrt{a}}^{2\sqrt{a}} \frac{\sqrt{4a-t^2}}{t^2}D_{n,2}(t)D_{m,2}(t)\,dt = h_n(k)\delta_{n,m}, \tag{4.20}$$

which seems to make no sense, since the integrand is singular at $t=0$. However, if we use (3.23) we have

$$D_{n,2}(t;a) = a^{\frac{1}{2}(n-1)}tU_{n-1}\left(\frac{t}{2\sqrt{a}}\right),$$

and we can write (4.20) as

$$a^{\frac{1}{2}(n+m)-1}\frac{1}{2\pi}\int_{-2\sqrt{a}}^{2\sqrt{a}}\sqrt{4a-t^2}U_{n-1}\left(\frac{t}{2\sqrt{a}}\right)U_{m-1}\left(\frac{t}{2\sqrt{a}}\right)dt = h_n(k)\delta_{n,m},$$

or, changing variables to $\tau=\frac{t}{2\sqrt{a}}$,

$$a^{\frac{n+m}{2}}\frac{2}{\pi}\int_{-1}^{1}\sqrt{1-\tau^2}U_{n-1}(\tau)\,U_{m-1}(\tau)\,d\tau = h_n(k)\delta_{n,m}.$$

This agrees with (3.19), since we have $U_{-1}=0$ from (3.18) and $h_0(2)=0$ from (3.5), while for $n,m\geq 1$ we know from (3.5) that

$$a^{-\frac{n+m}{2}}h_n(k)\delta_{n,m} = \delta_{n,m}.$$

Finally, we will use the function $S(z;k,a)$ to find explicit expressions for the moments of L_k.

Proposition 4.14. *Let L_k be the linear functional defined by* (3.4)*. Then, the moments of L_k of even order*

$$\mu_{2n}(k) = L_k\left[x^{2n}\right]$$

are given by

$$\mu_{2n}(1) = 2^{2n+1}\binom{\frac{1}{2}}{n+1}(-a)^n, \quad n=0,1,\ldots, \tag{4.21}$$

$$\mu_{2n}(2) = -2^{2n-1}\binom{\frac{1}{2}}{n}(-a)^n, \quad n=1,2,\ldots, \tag{4.22}$$

and if $k \neq 1, 2$,

$$\mu_{2n}(k) = -\frac{1}{2} \frac{(k-2)^{2n}}{(k-1)^{n+1}} (-a)^n \left(\frac{k}{k-2} + \sum_{j=0}^{n} \binom{\frac{1}{2}}{j} \left[\frac{4(k-1)}{(k-2)^2} \right]^j \right). \tag{4.23}$$

Proof. From (3.10) and (4.14), we have

$$\sum_{j=0}^{\infty} \frac{\mu_j(k)}{z^{j+1}} = -\frac{z}{2} \frac{\sqrt{1-4az^{-2}} + \frac{k}{k-2}}{(k-1) z^2 + a(k-2)^2}.$$

Using (3.7), we get

$$\sum_{j=0}^{\infty} \frac{\mu_{2j}(k)}{z^{2j}} = -\frac{1}{2} \frac{\sqrt{1-4az^{-2}} + \frac{k}{k-2}}{k-1 + a(k-2)^2 z^{-2}}.$$

Letting $u = z^{-2}$, we see that

$$\sum_{j=0}^{\infty} \mu_{2j}(k) u^j = -\frac{1}{2} \frac{\sqrt{1-4au} + \frac{k}{k-2}}{k-1 + a(k-2)^2 u},$$

and therefore

$$\begin{aligned} \sqrt{1-4au} + \frac{k}{k-2} &= -2\left[k-1 + a(k-2)^2 u\right] \sum_{j=0}^{\infty} \mu_{2j} u^j \\ &= -\sum_{j=0}^{\infty} 2(k-1)\mu_{2j} u^j - \sum_{j=1}^{\infty} 2a(k-2)^2 \mu_{2(j-1)} u^j. \end{aligned}$$

Since

$$\sqrt{1-4au} = \sum_{j=0}^{\infty} \binom{\frac{1}{2}}{j} (-4au)^j,$$

we obtain

$$1 + \frac{k}{k-2} = -2(k-1)\mu_0,$$

and

$$\binom{\frac{1}{2}}{j} (-4a)^j = -2(k-1)\mu_{2j} - 2a(k-2)^2 \mu_{2(j-1)}, \quad j = 1, 2, \ldots.$$

If $k = 1$, we get

$$\binom{\frac{1}{2}}{j} (-4a)^j = -2a\mu_{2(j-1)}, \quad j = 1, 2, \ldots,$$

or

$$\mu_{2n}(1) = 2^{2n+1} \binom{\frac{1}{2}}{n+1} (-a)^n, \quad n = 0, 1, \ldots.$$

If $k = 2$, we have

$$\mu_{2n}(2) = -2^{2n-1} \binom{\frac{1}{2}}{n} (-a)^n, \quad n = 1, 2, \ldots.$$

If $k \neq 1, 2$, we set $y_j = \mu_{2j}$ and obtain the recurrence

$$y_{j+1} = -\frac{a\,(k-2)^2}{k-1}\,y_j - \frac{(-4a)^{j+1}}{2\,(k-1)}\binom{\frac{1}{2}}{j+1},$$

with

$$y_0 = \frac{1}{2-k}.$$

As it is well known, the general solution of the initial value problem

$$y_{n+1} = c_n y_n + g_n, \quad y_{n_0} = y_0,$$

is (see [22, 1.2.4])

$$y_n = y_0 \prod_{j=n_0}^{n-1} c_j + \sum_{k=n_0}^{n-1}\left(g_k \prod_{j=k+1}^{n-1} c_j\right).$$

Thus,

$$y_n = \frac{1}{2-k}\left[-\frac{a\,(k-2)^2}{k-1}\right]^n - \sum_{j=0}^{n-1}\frac{(-4a)^{j+1}}{2\,(k-1)}\binom{\frac{1}{2}}{j+1}\left[-\frac{a\,(k-2)^2}{k-1}\right]^{n-j-1},$$

or

$$y_n = -\frac{1}{2\,(k-1)}\left[-\frac{a\,(k-2)^2}{k-1}\right]^n\left(\frac{k}{k-2} + \sum_{j=0}^{n}\binom{\frac{1}{2}}{j}\left[\frac{4\,(k-1)}{(k-2)^2}\right]^j\right),$$

and the result follows. □

Remark 4.15. If $k = 0$, we get from (4.23)

$$\mu_{2n}(0) = \frac{(4a)^n}{2}\sum_{j=0}^{n}(-1)^j\binom{\frac{1}{2}}{j},$$

and using the identity (see [46, 26.3.10])

$$\sum_{j=0}^{n}(-1)^j\binom{\alpha}{j} = (-1)^n\binom{\alpha-1}{n},$$

we obtain

$$\mu_{2n}(0) = 2^{2n-1}\binom{-\frac{1}{2}}{n}(-a)^n.$$

This agrees with (4.15), since

$$\mu_{2n}(0) = \frac{1}{2\pi}\int_{-2\sqrt{a}}^{2\sqrt{a}}\frac{t^{2n}}{\sqrt{4a-t^2}}\,dt.$$

When $k = 1$, we have from (4.15)

$$\mu_{2n}(1) = \frac{1}{2\pi a}\int_{-2\sqrt{a}}^{2\sqrt{a}} t^{2n}\sqrt{4a-t^2}\,dt,$$

and therefore (4.21) gives

$$\frac{1}{2\pi a}\int_{-2\sqrt{a}}^{2\sqrt{a}} t^{2n}\sqrt{4a-t^2}\,dt = 2^{2n+1}\binom{\frac{1}{2}}{n+1}(-a)^n,$$

which can be verified directly.

When $k=2$, we can write (see Remark 4.13)

$$\mu_{2n}(2) = \frac{1}{2\pi}\int_{-2\sqrt{a}}^{2\sqrt{a}} t^{2n}\frac{\sqrt{4a-t^2}}{t^2}\,dt = \frac{1}{2\pi}\int_{-2\sqrt{a}}^{2\sqrt{a}} t^{2(n-1)}\sqrt{4a-t^2}\,dt,$$

where $n=1,2,\ldots$. Hence,

$$\mu_{2n}(2) = a\mu_{2(n-1)}(1) = a2^{2(n-1)+1}\binom{\frac{1}{2}}{n}(-a)^{n-1}, \quad n=1,2,\ldots,$$

in agreement with (4.22).

5. Conclusions

We have shown that the Dickson polynomials of the $(k+1)$-th kind defined by

$$D_{n,k}(x;a) = \sum_{j=0}^{\left\lfloor \frac{n}{2}\right\rfloor} \frac{n-kj}{n-j}\binom{n-j}{j}(-a)^j x^{n-2j}$$

satisfy the orthogonality relation

$$\frac{1}{2\pi}\int_{-2\sqrt{a}}^{2\sqrt{a}} \frac{\sqrt{4a-t^2}D_{n,k}(t)D_{m,k}(t)}{(k-1)\,t^2+a\,(k-2)^2}\,dt \\ +\chi(k)\left[\frac{1+(-1)^{n+m}}{2}\right]\frac{(2-k)k}{k-1}\left(\mathrm{i}\sqrt{\frac{a}{k-1}}\right)^{n+m} = h_n(k)\delta_{n,m},$$

where $a>0$, $k\in\mathbb{R}$,

$$\chi(k) = \begin{cases} 0, & k\in[0,2]\,; \\ 1, & k\in\mathbb{R}\setminus[0,2]\,, \end{cases}$$

and

$$h_0(k) = 2-k,$$
$$h_n(k) = a^n, \quad n=1,2,\ldots.$$

We hope that this work will outline some connections between finite fields and orthogonal polynomials, and that it will be of interest to researchers in both areas.

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