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GEOMETRY OF POINTWISE CR-SLANT WARPED PRODUCTS IN KAEHLER MANIFOLDS

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Abstract. We call a submanifold M of a Kaehler manifold $\tilde{M}$ a *pointwise CR-slant warped product* if it is a warped product, $B \times_f N_\theta$, of a CR-product $B = N_T \times N_\perp$ and a proper pointwise slant submanifold N_θ with slant function θ, where N_T and $N_\perp$ are complex and totally real submanifolds of $\tilde{M}$. We prove that if a pointwise CR-slant warped product $B \times_f N_\theta$ with $B = N_T \times N_\perp$ in a Kaehler manifold is weakly $\mathfrak{D}^\theta$-totally geodesic, then it satisfies

$$\|\sigma\|^2 \geq 4s \left\{ (\csc^2\theta + \cot^2\theta)\|\nabla^T(\ln f)\|^2 + (\cot^2\theta)\|\nabla^\perp(\ln f)\|^2 \right\},$$

where N_T, $N_\perp$, and N_θ are complex, totally real and proper pointwise slant submanifolds of $\tilde{M}$, respectively, and $s = \frac{1}{2}\dim N_\theta$. In this paper we also investigate the equality case of the inequality. Moreover, we give a non-trivial example and provide some applications of this inequality.

1. Introduction

A warped product $N_1 \times_f N_2$ of two Riemannian manifolds (N_1, g_1) and (N_2, g_2) is the product manifold $N_1 \times N_2$ equipped with the warped product metric

$$g = g_{N_1} + f^2 g_{N_2},$$

where $f : N_1 \to \mathbb{R}^+$ is a positive smooth function on N_1. The function f is called the *warping function* [1, 12]. When the warping function f is constant, $M = N_1 \times_f N_2$ is simply a Riemannian product. It is known that, for a vector field X on N_1 and a vector field Z on N_2, we have

$$\nabla_X Z = \nabla_Z X = X(\ln f)Z, \tag{1.1}$$

where ∇ is the Levi-Civita connection on M. Further, it is well-known that N_1 is totally geodesic and N_2 is totally umbilical in $N_1 \times_f N_2$ (see e.g. [1, 6, 7, 9]).

A submanifold M of a Kaehler manifold $\tilde{M}$ is called a *CR-product* if it is locally the Riemannian product $N_T \times N_\perp$ of a complex submanifold N_T and a totally real submanifold $N_\perp$ of $\tilde{M}$ (see [2]). B.-Y. Chen had determined CR-products in

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complex space forms in [2, 3]. In [6] Chen proved that there do not exist warped product submanifolds of the form $N_\perp \times_f N_T$ in any Kaehler manifolds $\tilde{M}$, where N_T is a complex submanifold and $N_\perp$ is a totally real submanifold of $\tilde{M}$. On the other hand, he introduced the notion of *CR-warped products* in $\tilde{M}$ as warped product submanifolds of the form $N_T \times_f N_\perp$. Since then there are many results on warped product submanifolds studied by many geometers (see, for instance, [7, 8, 11, 12, 15, 17, 18, 20, 21, 22, 23, 24, 25, 26]). In particular, B. Sahin in [20] proved that there are no warped product submanifolds of the form $N_T \times_f N_\lambda$ in Kaehler manifolds, where N_λ is a slant submanifold with slant angle $\lambda \in (0, \frac{\pi}{2})$. He also studied warped product submanifolds of the form $N_T \times_f N_\theta$ in [22], where N_θ is a pointwise slant submanifold, instead of a slant submanifold. Sahin named such submanifolds *pointwise semi-slant* warped product submanifolds. Also, Sahin proved in [21] that there do not exist warped product submanifolds of the form $N_\perp \times_f N_\lambda$ in Kaehler manifolds such that $N_\perp$ is totally real and N_λ is a proper slant. Further, he also studied in [21] warped product *hemi-slant submanifolds* which are of the form $N_\lambda \times_f N_\perp$. Warped product hemi-slant submanifolds were extended to warped product *pointwise hemi-slant submanifolds* and studied in [23, 26].

In this paper, we introduce and investigate a more general family of warped product submanifolds, called *pointwise CR-slant warped products*, in Kaehler manifolds, which are of the form $B \times_f N_\theta$, where $B = N_T \times N_\perp$ is a CR-product and N_θ is a proper pointwise slant submanifold of $\tilde{M}$.

Our main purpose is to establish the following sharp inequality:

$$\|\sigma\|^2 \geq 4s\left\{\left(\csc^2\theta + \cot^2\theta\right)\|\nabla^T(\ln f)\|^2 + (\cot^2\theta)\|\nabla^\perp(\ln f)\|^2\right\}, \tag{1.2}$$

for weakly $\mathfrak{D}^\theta$-totally geodesic, pointwise CR-slant warped products in a Kaehler manifold, where $s = \frac{1}{2}\dim N_\theta$ and $\|\sigma\|$ is the norm of the second fundamental form.

The paper is organized as follows: In Section 2, we give basic definitions and formulas. Definition and a non-trivial example of CR-slant warped products are given in Section 3. In Section 4, we provide useful lemmas and propositions for the proof of our main theorem. The proof of inequality (1.2) and the study of the equality case of (1.2) are given in Section 5. The last section provides several applications of the main theorem.

2. Preliminaries

A $2m$-manifold $\tilde{M}^{2m}$ with an almost complex structure J and a Riemannian metric $\langle\ ,\ \rangle$ is said to be a *Kaehler manifold* if it satisfies (see e.g. [4, 10, 27])

$$J^2 = -I, \quad \langle JU, JV\rangle = \langle U, V\rangle, \quad (\tilde{\nabla}_U J)V = 0 \tag{2.1}$$

for vector fields U, V tangent to $\tilde{M}$, where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$ and I is the identity transformation.

Let M be a submanifold of $\tilde{M}^{2m}$. Denote by $\Gamma(TM)$ the Lie algebra of vector fields on M and let $\Gamma(T^\perp M)$ denote the set of vector fields normal to M. The formulas of Gauss and Weingarten are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{2.2}$$
$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$, where D denotes the normal connection of the normal bundle $T^\perp M$ and A is the shape operator of M. The second fundamental form σ and the shape operator A of M are related by

$$\langle \sigma(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle,$$

where $\langle\ ,\ \rangle$ denotes the inner products on M and $\tilde{M}$ with respect to their metrics.

For any vector field X tangent to M, we put

$$JX = PX + FX, \tag{2.3}$$

where PX is the tangential component and FX is the normal component of JX.

For any vector field ξ normal to the submanifold M, we put

$$J\xi = t\xi + f\xi, \tag{2.4}$$

where $t\xi$ and $f\xi$ are the tangential and normal components of $J\xi$, respectively.

A submanifold M of a Kaehler manifold $\tilde{M}^{2m}$ is called a *complex submanifold* if $J(T_xM) \subset T_xM$, $\forall\, x \in M$, where T_xM denotes the tangent space of M at $x \in M$. It is called *totally real* if $J(T_xM) \subset T_x^\perp M$, $\forall\, x \in M$, where $T_x^\perp M$ is the normal space of M at $x \in M$ (see [14]).

Definition 2.1. A submanifold M of a Kaehler manifold is called *pointwise slant* if for each $x \in M$, the *Wirtinger angle* $\theta(X)$ between JX and T_xM is independent of the choice of $X \in T_xM \setminus \{0\}$. The function $\theta : TM \setminus \{0\} \to \mathbb{R}$ is called the *slant function* of the pointwise slant submanifold M (see [13, 16, 19]).

A pointwise slant submanifold is called a *slant submanifold* if its slant function θ is globally constant; such θ is called the *slant angle* of the slant submanifold (see [4, 5]). Clearly, the same definitions apply to submanifolds in almost Hermitian manifolds. Obviously, complex and totally real submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively,

Definition 2.2. A pointwise slant submanifold M of a Kaehler manifold $\tilde{M}$ is called *proper* if its slant function θ satisfies $0 < \theta < \frac{\pi}{2}$. Thus proper pointwise slant submanifolds are neither complex nor totally real.

From Lemma 2.1 of [13], we know that a submanifold M of an almost Hermitian manifold $\tilde{M}$ is a pointwise slant submanifold if and only if

$$P^2 X = -(\cos^2\theta)X, \tag{2.5}$$

for some real-valued function θ defined on M. The following relations are immediate consequences of (2.5):

$$\langle PX, PY \rangle = \cos^2\theta\, \langle X, Y \rangle,$$
$$\langle FX, FY \rangle = \sin^2\theta\, \langle X, Y \rangle$$

for $X, Y \in \Gamma(TM)$. The next relation for pointwise slant submanifolds of an almost Hermitian manifold follows easily from (2.1) and (2.5):

$$tFX = -\sin^2\theta\, X, \quad fFX = -FPX \tag{2.6}$$

for $X \in \Gamma(TM)$.

Notation 2.3. Let $\mathbb{R}^n$ denote the Cartesian n-space and let

$$\mathbb{E}^{2m} = \{(x_1, \dots, x_m, y_1, \dots, y_m) : x_i, y_i \in \mathbf{R}, i = 1, \dots, n\}$$

be the Euclidean $2m$-space with natural coordinates $(x_1, \dots, x_m, y_1, \dots, y_m)$ and with the standard Euclidean metric $g_0 = \sum_{i=1}^m (dx_i^2 + dy_i^2)$. Consider the almost complex structure J on $\mathbb{E}^{2m}$ given by

$$J(x_1, \dots, x_m, y_1, \dots, y_m) = (-y_1, \dots, -y_m, x_1, \dots, x_m). \tag{2.7}$$

Then the complex Euclidean m-space $\mathbb{C}^m = (\mathbb{E}^{2m}, J)$ is a flat Kaehler manifold.

3. Pointwise CR-slant warped products: definition and example

Now, we provide the following definitions.

Definition 3.1. Let M be a submanifold of a Kaehler manifold $(\tilde{M}, \tilde{g}, J)$. A distribution $\mathfrak{D}$ on M is called a *complex distribution* (resp., a *totally real distribution*) if $J(\mathfrak{D}) \subset \mathfrak{D}$ (resp., $J(\mathfrak{D}) \subset T^\perp M$).

Definition 3.2. A distribution $\mathfrak{D}$ on M is called *pointwise slant* if for each given point $p \in M$ and any unit vector $X \in \mathfrak{D}_p$, the angle $\theta(X)$ between JX and $\mathfrak{D}_p$ is independent of the choice of X. The function θ is called the *slant function.* In particular, if the angle $\theta(X)$ between JX and $\mathfrak{D}_p$ is also independent of the choice of point $p \in M$, then the distribution $\mathfrak{D}$ is called a *slant distribution.*

Definition 3.3. Let M be a submanifold of a Kaehler manifold $\tilde{M}$. Assume that there exist three integrable distributions $\mathfrak{D}^T, \mathfrak{D}^\perp$, and $\mathfrak{D}^\theta$ on M such that

$$TM = \mathfrak{D}^T \oplus \mathfrak{D}^\perp \oplus \mathfrak{D}^\theta,$$

where $\mathfrak{D}^T$ is a complex distribution, $\mathfrak{D}^\perp$ is a totally real distribution, and $\mathfrak{D}^\theta$ is a pointwise slant distribution whose slant function θ has values in $(0, \frac{\pi}{2})$. Denote by $N_T, N_\perp$, and N_θ integrable submanifolds of $\mathfrak{D}^T, \mathfrak{D}^\perp$, and $\mathfrak{D}^\theta$, respectively. We call M a *pointwise CR-slant warped product* if the induced metric g on M is a warped product metric of the form

$$g = g_B + f^2 g_{N_\theta}, \tag{3.1}$$

where g_B is the metric of $B = N_T \times N_\perp$, g_{N_θ} is the metric on N_θ, and f is a positive function depending only on B. This warped product is called *proper* if the warping function f is non-constant. In particular, if the slant function θ of $\mathfrak{D}^\theta$ is a constant in $(0, \frac{\pi}{2})$, then M is called a *CR-slant warped product.*

Notation 3.4. We simply denote the pointwise CR-slant warped product above by $(N_T \times N_\perp) \times_f N_\theta$, or by $N_B \times_f N_\theta$ with $B = N_T \times N_\perp$.

Let $M = B \times_f N_\theta$ be a pointwise CR-slant warped product in $\tilde{M}$ with $B = N_T \times N_\perp$. Then we have

$$TM = \mathfrak{D}^T \oplus \mathfrak{D}^\perp \oplus \mathfrak{D}^\theta, \quad T^\perp M = J\mathfrak{D}^\perp \oplus F\mathfrak{D}^\theta \oplus \mu, \tag{3.2}$$

where μ is a J-invariant subbundle of the normal bundle $T^\perp M$.

We make the following definition.

Definition 3.5. A pointwise CR-slant warped product $(N_T \times N_\perp) \times_f N_\theta$ in $\tilde{M}$ is called *weakly $\mathfrak{D}^\theta$-totally geodesic* if its second fundamental form σ satisfies $\langle \sigma(\mathfrak{D}^\theta, \mathfrak{D}^\theta), J\mathfrak{D}^\perp \rangle = \{0\}$, i.e., $\sigma(\mathfrak{D}^\theta, \mathfrak{D}^\theta)$ has no component in $J\mathfrak{D}^\perp$.

We provide the following example of pointwise CR-slant warped product in $\mathbb{C}^9$ which is weakly $\mathfrak{D}^\theta$-totally geodesic.

Example 3.6. Let $\psi : \mathbb{R}^5 \to \mathbb{C}^9$ be an isometric immersion given by

$$\begin{aligned}\psi(u,\, v,\, w,\, r,\, s) = (&u\cos r, u\cos s, w\cos r, w\cos s, u\sin r,\\ &u\sin s, r, w\sin r, w\sin s, v\cos r, v\cos s, w\sin r,\\ &w\sin s, v\sin r, v\sin s, s, w\cos r, w\cos s),\end{aligned}$$

with $u, v, w > 0$. Then the tangent bundle TM is spanned by

$$\begin{aligned}X_1 &= (\cos r, \cos s, 0, 0, \sin r, \sin s, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),\\ X_2 &= (0, 0, 0, 0, 0, 0, 0, 0, 0, \cos r, \cos s, 0, 0, \sin r, \sin s, 0, 0, 0),\\ X_3 &= \big(0, 0, \cos r, \cos s, 0, 0, 0, \sin r, \sin s,\\ &\qquad 0, 0, \sin r, \sin s,\ 0, 0, 0, \cos r, \cos s\big),\\ X_4 &= (-u\sin r, 0, -w\sin r, 0, u\cos r, 0, 1, w\cos r, 0,\\ &\qquad -v\sin r, 0, w\cos r, 0, v\cos r, 0, 0, -w\sin r, 0),\\ X_5 &= (0, -u\sin s, 0, -w\sin s, 0, u\cos s, 0, 0, w\cos s,\\ &\qquad 0, -v\sin s, 0, w\cos s, 0, v\cos s, 1, 0, -w\sin s).\end{aligned} \tag{3.3}$$

Then we find from (2.7) and (3.3) that

$$\begin{aligned}JX_1 &= (0, 0, 0, 0, 0, 0, 0, 0, 0, \cos r, \cos s, 0, 0, \sin r, \sin s, 0, 0, 0),\\ JX_2 &= -(\cos r, \cos s, 0, 0, \sin r, \sin s, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),\\ JX_3 &= (0, 0, -\sin r, -\sin s, 0, 0, 0, -\cos r, -\cos s,\\ &\qquad 0, 0, \cos r, \cos s, 0, 0, 0, \sin r, \sin s),\\ JX_4 &= (v\sin r, 0, -w\cos r, 0, -v\cos r, 0, 0, w\sin r, 0,\\ &\qquad -u\sin r, 0, -w\sin r, 0, u\cos r, 0, 1, w\cos r, 0),\\ JX_5 &= (0, v\sin s, 0, -w\cos s, 0, -v\cos s, -1, 0, w\sin s,\\ &\qquad 0, -u\sin s, 0, -w\sin s, 0, u\cos s, 0, 0, w\cos s).\end{aligned} \tag{3.4}$$

Since JX_3 is perpendicular to TM, $\mathfrak{D}^\perp = \operatorname{Span}\{X_3\}$ is totally real. Moreover, the distributions $\mathfrak{D}^T = \operatorname{Span}\{X_1, X_2\}$ and $\mathfrak{D}^\theta = \operatorname{Span}\{X_4, X_5\}$ are complex and proper pointwise slant with slant function $\theta = \cos^{-1}(1/(1 + u^2 + v^2 + 2w^2))$, respectively.

Obviously, all the three distributions are integrable. Let N_T, $N_\perp$, and N_θ be the leaves of $\mathfrak{D}^T$, $\mathfrak{D}^\perp$, and $\mathfrak{D}^\theta$, respectively. Then the induced metric on the product manifold $N_T \times N_\perp \times N_\theta$ is the warped product metric:

$$\begin{aligned} g &= 2(du^2+dv^2)+4dw^2+(1+u^2+v^2+2w^2)(dr^2+ds^2) \\ &= g_B + f^2 g_{N_\theta}. \end{aligned} \tag{3.5}$$

Hence, M is a proper pointwise CR-slant warped product submanifold of $\mathbb{C}^9$ with warping function $f=\sqrt{1+u^2+v^2+2w^2}$. We derive from (3.3) and (3.5) that

$$\begin{aligned}
&\sigma(X_i,X_j)=0, \quad 1\le i,j\le 3, \\
&\sigma(X_1,X_4)=\frac{-1}{1+u^2+v^2+2w^2}\big((1+v^2+2w^2)\sin r, 0, -uw\sin r, \\
&\qquad 0, -(1+v^2+2w^2)\cos r, 0, u, uw\cos r, 0, \\
&\qquad -uv\sin r, 0, uw\cos r, 0, uv\cos r, 0, 0, -uw\sin r, 0\big), \\
&\sigma(X_1,X_5)=\frac{-1}{1+u^2+v^2+2w^2}\big(0, (1+v^2+2w^2)\sin s, 0, -uw\sin s, \\
&\qquad 0, -(1+v^2+2w^2)\cos s, 0, 0, uw\cos s, 0, \\
&\qquad -uv\sin s, 0, uw\cos s, 0, uv\cos s, u, 0, -uw\sin s\big), \\
&\sigma(X_2,X_4)=\frac{1}{1+u^2+v^2+2w^2}\big(uv\sin r, 0, vw\sin r, 0, -uv\cos r, 0, -v, \\
&\qquad -vw\cos r, 0, -(1+u^2+2w^2)\sin r, 0, -vw\cos r, \\
&\qquad 0, (1+u^2+2w^2)\cos r, 0, 0, vw\sin r, 0\big), \\
&\sigma(X_2,X_5)=\frac{1}{1+u^2+v^2+2w^2}\big(0, uv\sin s, 0, vw\sin s, 0, -uv\cos s, 0, 0, \\
&\qquad -vw\cos s, 0, -(1+u^2+2w^2)\sin s, 0, -vw\cos s, \\
&\qquad 0, (1+u^2+2w^2)\cos s, -v, 0, vw\sin s\big), \\
&\sigma(X_3,X_4)=\frac{1}{1+u^2+v^2+2w^2}\big(2uw\sin r, 0, -(1+u^2+v^2)\sin r, 0, \\
&\qquad -2uw\cos r, 0, -2w, (1+u^2+v^2)\cos r, 0, 2vw\sin r, 0, \\
&\qquad (1+u^2+v^2)\cos r, 0, -2vw\cos r, 0, 0, -(1+u^2+v^2)\sin r, 0\big), \\
&\sigma(X_3,X_5)=\frac{1}{1+u^2+v^2+2w^2}\big(0, 2uw\sin s, 0, -(1+u^2+v^2)\sin s, 0, \\
&\qquad -2uw\cos s, 0, 0, (1+u^2+v^2)\cos s, 0, 2vw\sin s, 0, \\
&\qquad (1+u^2+v^2)\cos s, 0, -2vw\cos s, -2w, 0, -(1+u^2+v^2)\sin s\big), \\
&\sigma(X_4,X_4)=-\frac{1}{2}\big(u\cos r, -u\cos s, w\cos r, -w\cos s, u\sin r, -u\sin s, \\
&\qquad 0, w\sin r, -w\sin s, v\cos r, -v\cos s, w\sin r, -w\sin s, \\
&\qquad v\sin r, -v\sin s, 0, w\cos r, -w\cos s\big), \\
&\sigma(X_4,X_5)=0, \qquad \sigma(X_5,X_5)=-\sigma(X_4,X_4).
\end{aligned} \tag{3.6}$$

By applying (3.4) and (3.6), it is easy to verify that the pointwise CR-slant warped product defined by ψ is weakly $\mathfrak{D}^\theta$-totally geodesic. Further, (3.5) and (3.6) imply that N_θ is not totally umbilical in $\mathbb{C}^9$.

4. Lemmas and propositions

In the following, we use the conventions that X_1, Y_1 are vector fields in $\mathfrak{D}^T$, X_2, Y_2 are vector fields in $\mathfrak{D}^\perp$, and X_3, Y_3 are vector fields in $\mathfrak{D}^\theta$.

Lemma 4.1. *Let $M = B \times_f N_\theta$ be a pointwise CR-slant warped product in a Kaehler manifold $\tilde{M}$ with $B = N_T \times N_\perp$. Then we have:*

(i) $\langle \sigma(X_1, Y_1), FX_3 \rangle = 0$,

(ii) $\langle \sigma(X_1, X_3), FY_3 \rangle = -JX_1(\ln f)\, \langle X_3, Y_3 \rangle - X_1(\ln f)\, \langle X_3, PY_3 \rangle$,

for any $X_1, Y_1 \in \Gamma(\mathfrak{D}^T)$ and $X_3, Y_3 \in \Gamma(\mathfrak{D}^\theta)$.

Proof. The first part of the lemma is easy and it can be derived by using (2.2), (1.1), and the orthogonality of vector fields. On the other hand, we have

$$\begin{aligned}\langle \sigma(X_1, X_3), FY_3 \rangle &= \big\langle \tilde{\nabla}_{X_3} X_1, JY_3 - PY_3 \big\rangle \\ &= -\big\langle \tilde{\nabla}_{X_3} JX_1, Y_3 \big\rangle - \big\langle \tilde{\nabla}_{X_3} X_1, PY_3 \big\rangle,\end{aligned}$$

for any $X_1 \in \Gamma(\mathfrak{D}^T)$ and $X_3, Y_3 \in \Gamma(\mathfrak{D}^\theta)$. Therefore, we obtain the required result from (1.1). □

Lemma 4.2. *Let $M = B \times_f N_\theta$ be a pointwise CR-slant warped product in a Kaehler manifold $\tilde{M}$ with $B = N_T \times N_\perp$. Then*

(i) $\langle \sigma(X_1, X_3), JX_2 \rangle = 0$,

(ii) $\langle \sigma(X_1, X_2), FX_3 \rangle = 0$,

for any $X_1 \in \Gamma(\mathfrak{D}^T)$, $X_2 \in \Gamma(\mathfrak{D}^\perp)$, and $X_3 \in \Gamma(\mathfrak{D}^\theta)$.

Proof. From (2.2) and (2.1), we have

$$\langle \sigma(X_1, X_3), JX_2 \rangle = \big\langle \tilde{\nabla}_{X_3} X_1, JX_2 \big\rangle = -\big\langle \tilde{\nabla}_{X_3} JX_1, X_2 \big\rangle.$$

Using (1.1) and the orthogonality of vector fields, we have (i). In a similar way, we obtain (ii). □

Lemma 4.3. *Let $M = B \times_f N_\theta$ be a pointwise CR-slant warped product in a Kaehler manifold $\tilde{M}$ with $B = N_T \times N_\perp$. Then we have:*

(i) $\langle \sigma(X_2, Y_2), FX_3 \rangle = \langle \sigma(X_2, X_3), JY_2 \rangle$,

(ii) $\langle \sigma(X_3, Y_3), JX_2 \rangle - \langle \sigma(X_2, X_3), FY_3 \rangle = X_2(\ln f)\, \langle X_3, PY_3 \rangle$,

for any $X_2, Y_2 \in \Gamma(\mathfrak{D}^\perp)$ and $X_3, Y_3 \in \Gamma(\mathfrak{D}^\theta)$.

Proof. The first part follows easily from (2.2), (2.1), (1.1), and (2.3). For the second part, we have

$$\langle \sigma(X_3, Y_3), JX_2 \rangle = \big\langle \tilde{\nabla}_{X_3} Y_3, JX_2 \big\rangle = -\big\langle \tilde{\nabla}_{X_3} PY_3, X_2 \big\rangle - \big\langle \tilde{\nabla}_{X_3} FY_3, X_2 \big\rangle.$$

Using (1.1) and (2.1), we derive

$$\langle \sigma(X_3, Y_3), JX_2 \rangle = X_2(\ln f)\, \langle X_3, PY_3 \rangle - \big\langle J\tilde{\nabla}_{X_3} FY_3, JX_2 \big\rangle. \tag{4.1}$$

On the other hand, applying (2.4) and (2.6), we have

$$\begin{aligned}\left\langle J\tilde{\nabla}_{X_3}FY_3, JX_2\right\rangle &= \left\langle\tilde{\nabla}_{X_3}tFY_3, JX_2\right\rangle + \left\langle\tilde{\nabla}_{X_3}fFY_3, JX_2\right\rangle\\ &= -\sin^2\theta\left\langle\tilde{\nabla}_{X_3}Y_3, JX_2\right\rangle - \sin(2\theta)X_3(\theta)\left\langle Y_3, JX_2\right\rangle\\ &\quad -\left\langle\tilde{\nabla}_{X_3}FPY_3, JX_2\right\rangle.\end{aligned}$$

Again using (2.6), we derive

$$\begin{aligned}\left\langle J\tilde{\nabla}_{X_3}FY_3, JX_2\right\rangle &= -\sin^2\theta\left\langle\sigma(X_3, Y_3), JX_2\right\rangle + \left\langle\tilde{\nabla}_{X_3}tFPY_3, X_2\right\rangle\\ &\quad + \left\langle\tilde{\nabla}_{X_3}fFPY_3, X_2\right\rangle\\ &= -\sin^2\theta\left\langle\sigma(X_3, Y_3), JX_2\right\rangle - \sin^2\theta\left\langle\tilde{\nabla}_{X_3}PY_3, X_2\right\rangle\\ &\quad - \sin(2\theta)X_3(\theta)\left\langle PY_3, X_2\right\rangle - \left\langle\tilde{\nabla}_{X_3}FP^2Y_3, X_2\right\rangle.\end{aligned}$$

Then (1.1) and (2.5) yield with the orthogonality of vector fields that

$$\begin{aligned}\left\langle J\tilde{\nabla}_{X_3}FY_3, JX_2\right\rangle &= -\sin^2\theta\left\langle\sigma(X_3, Y_3), JX_2\right\rangle + X_2(\ln f)\sin^2\theta\left\langle X_3, PY_3\right\rangle\\ &\quad + \cos^2\theta\left\langle\tilde{\nabla}_{X_3}FY_3, X_2\right\rangle - \sin(2\theta)X_3(\theta)\left\langle FY_3, X_2\right\rangle.\end{aligned} \tag{4.2}$$

Hence (ii) follows from (4.1) and (4.2) by using the orthogonality of vector fields. This completes the proof of the lemma. ☐

Definition 4.4. A pointwise CR-slant warped product $(N_T \times N_\perp) \times_f N_\theta$ is called $\mathfrak{D}^1 \oplus \mathfrak{D}^2$*-mixed totally geodesic* if its second fundamental form σ satisfies

$$\sigma(\mathfrak{D}^1, \mathfrak{D}^2) = \{0\},$$

where $\mathfrak{D}^1$ and $\mathfrak{D}^2$ are distributions from $\{\mathfrak{D}^T, \mathfrak{D}^\perp, \mathfrak{D}^\theta\}$.

Proposition 4.5. *Let $M = B \times_f N_\theta$ be a pointwise CR-slant warped product in a Kaehler manifold $\tilde{M}$ with $B = N_T \times N_\perp$. If M is $\mathfrak{D}^T \oplus \mathfrak{D}^\theta$-mixed totally geodesic, then the warping function f depends only on $N_\perp$.*

Proof. From Lemma 4.1 (ii), we have

$$\left\langle\sigma(JX_1, X_3), FY_3\right\rangle = X_1(\ln f)\left\langle X_3, Y_3\right\rangle + JX_1(\ln f)\left\langle PX_3, Y_3\right\rangle.$$

Replacing X_3 by PX_3 and using (2.5), we get

$$\left\langle\sigma(JX_1, PX_3), FY_3\right\rangle = X_1(\ln f)\left\langle PX_3, Y_3\right\rangle - JX_1(\ln f)\cos^2\theta\left\langle X_3, Y_3\right\rangle.$$

Hence, from Lemma 4.1 (ii) and (3.6), we derive

$$\left\langle\sigma(X_1, X_3), FY_3\right\rangle - \left\langle\sigma(JX_1, PX_3), FY_3\right\rangle = -JX_1(\ln f)\sin^2\theta\left\langle X_3, Y_3\right\rangle. \tag{4.3}$$

Hence, if M is $\mathfrak{D}^T \oplus \mathfrak{D}^\theta$-mixed totally geodesic, the result follows from (4.3). ☐

Proposition 4.6. *Let $M = B \times_f N_\theta$ be a pointwise CR-slant warped product in a Kaehler manifold $\tilde{M}$ with $B = N_T \times N_\perp$. If M is $\mathfrak{D}^\perp \oplus \mathfrak{D}^\theta$-mixed totally geodesic, then the warping function f depends only on N_T.*

Proof. Using the polarization identity in Lemma 4.3 (ii) for vector fields $X_3, Y_3 \in \Gamma(\mathfrak{D}^\theta)$, we find

$$\langle \sigma(X_3, Y_3), JX_2 \rangle = -X_2(\ln f)\, \langle X_3, PY_3 \rangle + \langle \sigma(Y_3, X_2), FX_3 \rangle. \tag{4.4}$$

Thus we find from Lemma 4.3 (ii) and (4.4) that

$$\langle \sigma(X_2, Y_3), FX_3 \rangle - \langle \sigma(X_2, X_3), FY_3 \rangle = X_2(\ln f)\, \langle X_3, PY_3 \rangle.$$

Now, replacing X_3 by PX_3, we obtain

$$\langle \sigma(X_2, Y_3), FPX_3 \rangle - \langle \sigma(X_2, PX_3), FY_3 \rangle = X_2(\ln f) \cos^2\theta\, \langle X_3, Y_3 \rangle. \tag{4.5}$$

Thus, if M is $\mathfrak{D}^\perp \oplus \mathfrak{D}^\theta$-mixed totally geodesic, then (4.5) yields the result. $\square$

Remark 4.7. Propositions 4.5 and 4.6 imply that a pointwise CR-slant warped product $(N_T \times N_\perp) \times_f N_\theta$ is a Riemannian product if it is $\mathfrak{D}^T \oplus \mathfrak{D}^\theta$-mixed totally geodesic and $\mathfrak{D}^\perp \oplus \mathfrak{D}^\theta$-mixed totally geodesic.

5. The main theorem: a sharp inequality for $\|\sigma\|^2$

Consider a pointwise CR-slant warped product $M = B \times_f N_\theta$ in a Kaehler manifold $\tilde{M}$ with $\dim M = n$, $\dim_{\mathbb{R}} \tilde{M} = 2m$, and $B = N_T \times N_\perp$. We choose a local orthonormal frame $\{e_1, \ldots, e_n\}$ of TM such that

$$\begin{aligned}
\mathfrak{D}^\perp &= \mathrm{Span}\{e_1 = e_1, \ldots, e_q\},\\
\mathfrak{D}^T &= \mathrm{Span}\{e_{q+1} = \hat{e}_1, \ldots, e_{q+p} = \hat{e}_p, \ldots, e_{q+p+1} = \hat{e}_{p+1} = J\hat{e}_1,\\
&\qquad \ldots, e_{q+2p} = \hat{e}_{2p} = J\hat{e}_p\}\\
\mathfrak{D}^\theta &= \mathrm{Span}\{e_{2p+q+1} = e_1^*, \ldots, e_{2p+q+s} = e_s^*,\\
&\qquad e_{2p+q+s+1} = \sec\theta P e_1^*, \ldots, e_n = \sec\theta P e_s^*\},
\end{aligned}$$

with $q = \dim N_\perp$, $p = \frac{1}{2}\dim N_T$, and $s = \frac{1}{2}\dim N_\theta$. Also, we choose a local orthonormal frame $\{E_1, \ldots, E_{2m-n}\}$ of the normal bundles of $T^\perp M$ such that

$$\begin{aligned}
J\mathfrak{D}^\perp &= \mathrm{Span}\{E_1 = Je_1, \ldots, E_q = Je_q\},\\
F(\mathfrak{D}^\theta) &= \mathrm{Span}\{E_{q+1} = \csc\theta F e_1^*, \ldots, E_{q+s} = \csc\theta F e_s^*,\\
&\qquad E_{q+s+1} = \csc\theta \sec\theta FP e_1^*, \ldots, E_{q+2s} = \csc\theta\sec\theta FP e_s^*\}\\
\mu &= \mathrm{Span}\{E_{q+2s+1}, \ldots, E_{2m-n}\},
\end{aligned}$$

where μ is a J-invariant normal subbundle of $T^\perp M$.

The following theorem gives a sharp inequality involving the norm $\|\sigma\|$ of the second fundamental form for CR-slant warped products in any Kaehler manifold.

Theorem 5.1. *Let $M = B \times_f N_\theta$ be a pointwise CR-slant warped product in a Kaehler manifold $\tilde{M}$ with $B = N_T \times N_\perp$. If M is weakly $\mathfrak{D}^\theta$-totally geodesic, then*

(i) *The squared norm $\|\sigma\|^2$ of the second fundamental form of M satisfies*

$$\|\sigma\|^2 \geq 4s\big[(\csc^2\theta + \cot^2\theta)\|\nabla^T(\ln f)\|^2 + \cot^2\theta\, \|\nabla^\perp(\ln f)\|^2\big], \tag{5.1}$$

where $\nabla^T(\ln f)$ and $\nabla^\perp(\ln f)$ denote the gradient components of $\ln f$ along N_T and $N_\perp$, respectively, and $s = \frac{1}{2}\dim N_\theta$.

(ii) *The equality sign in* (5.1) *holds identically if and only if B is totally geodesic and N_θ is totally umbilical in $\tilde{M}$.*

(iii) *If the warping function f in $B \times_f N_\theta$ is non-constant, then at least one of $\mathfrak{D}^T \oplus \mathfrak{D}^\theta$ and $\mathfrak{D}^\perp \oplus \mathfrak{D}^\theta$ is non-mixed totally geodesic in $\tilde{M}$.*

Proof. From the definition of σ, we have

$$\|\sigma\|^2 = \sum_{i,j=1}^{n} \langle \sigma(e_i, e_j), \sigma(e_i, e_j)\rangle = \sum_{r=n+1}^{2m-n} \sum_{i,j=1}^{n} \left(\langle \sigma(e_i, e_j), E_r\rangle\right)^2,$$

where $\{e_{n+1}, \dots, e_{2m-n}\}$ is a local orthonormal frame of the normal bundle. From (3.2), the above relation takes the form

$$\begin{aligned}\|\sigma\|^2 =& \sum_{r=1}^{q} \sum_{i,j=1}^{n} \left(\langle \sigma(e_i, e_j), Je_r\rangle\right)^2 + \sum_{r=q+1}^{q+2s} \sum_{i,j=1}^{n} \left(\langle \sigma(e_i, e_j), E_r\rangle\right)^2 \\ &+ \sum_{r=q+2s+1}^{2m-n} \sum_{i,j=1}^{n} \left(\langle \sigma(e_i, e_j), E_r\rangle\right)^2 .\end{aligned} \tag{5.2}$$

Leaving the last μ-components term in (5.2) and using the frame fields given above, we find

$$\begin{aligned}\|\sigma\|^2 \geq& \sum_{r=1}^{q} \sum_{i,j=1}^{2p} \left(\langle \sigma(\hat{e}_i, \hat{e}_j), Je_r\rangle\right)^2 + 2\sum_{r=1}^{q} \sum_{i,j=1}^{2p} \sum_{j=1}^{q} \left(\langle \sigma(\hat{e}_i, e_j), Je_r\rangle\right)^2 \\ &+ 2\sum_{r=1}^{q} \sum_{i,j=1}^{2p} \sum_{j=1}^{2s} \left(\langle \sigma(\hat{e}_i, e_j^*), Je_r\rangle\right)^2 + \sum_{r=1}^{q} \sum_{i,j=1}^{q} \left(\langle \sigma(e_i, e_j), Je_r\rangle\right)^2 \\ &+ 2\sum_{r=1}^{q} \sum_{i=1}^{q} \sum_{j=1}^{2s} \left(\langle \sigma(e_i, e_j^*), Je_r\rangle\right)^2 + \sum_{r=1}^{q} \sum_{i,j=1}^{2s} \left(\langle \sigma(e_i^*, e_j^*), Je_r\rangle\right)^2 \\ &+ \csc^2\theta \sum_{r=1}^{s} \sum_{i,j=1}^{2p} \left[\left(\langle \sigma(\hat{e}_i, \hat{e}_j), Fe_r^*\rangle\right)^2 + \sec^2\theta \left(\langle \sigma(\hat{e}_i, \hat{e}_j), FPe_r^*\rangle\right)^2 \right] \\ &+ 2\csc^2\theta \sum_{r=1}^{s} \sum_{i=1}^{2p} \sum_{j=1}^{q} \left[\left(\langle \sigma(\hat{e}_i, e_j), Fe_r^*\rangle\right)^2 + \sec^2\theta \left(\langle \sigma(\hat{e}_i, e_j), FPe_r^*\rangle\right)^2 \right] \\ &+ 2\csc^2\theta \sum_{r=1}^{s} \sum_{i=1}^{2p} \sum_{j=1}^{2s} \left[\left(\langle \sigma(\hat{e}_i, e_j^*), Fe_r^*\rangle\right)^2 + \sec^2\theta \left(\langle \sigma(\hat{e}_i, e_j^*), FPe_r^*\rangle\right)^2 \right] \\ &+ \csc^2\theta \sum_{r=1}^{s} \sum_{i,j=1}^{q} \left[\left(\langle \sigma(e_i, e_j), Fe_r^*\rangle\right)^2 + \sec^2\theta \left(\langle \sigma(e_i, e_j), FPe_r^*\rangle\right)^2 \right] \\ &+ \csc^2\theta \sum_{r=1}^{s} \sum_{i,j=1}^{2s} \left[\left(\langle \sigma(e_i^*, e_j^*), Fe_r^*\rangle\right)^2 + \sec^2\theta \left(\langle \sigma(e_i^*, e_j^*), FPe_r^*\rangle\right)^2 \right]\end{aligned} \tag{5.3}$$

$$+ 2\csc^2\theta \sum_{r=1}^{s}\sum_{i=1}^{q}\sum_{j=1}^{2s}\big[\left(\langle\sigma(e_i,e_j^*),Fe_r^*\rangle\right)^2 + \sec^2\theta\left(\langle\sigma(e_i,e_j^*),FPe_r^*\rangle\right)^2\big].$$

There is no relation for warped products for the first, second, fourth, and eleventh terms in (5.3). So, we leave these positive terms. Further, using Lemmas 4.1–4.3 with the hypothesis of the theorem and the orthogonality of vector fields, we obtain

$$\begin{aligned}\|\sigma\|^2 \geq\ & 4s\csc^2\theta\sum_{r=1}^{p}\big[\left(-Je_i(\ln f)\right)^2 + \left(e_i(\ln f)\right)^2\big] \\ & + 4s\cot^2\theta\sum_{r=1}^{p}\big[\left(-Je_i(\ln f)\right)^2 + \left(e_i(\ln f)\right)^2\big] + 4s\cot^2\theta\sum_{r=1}^{q}\left(e_i(\ln f)\right)^2 \\ =\ & 4s(\csc^2\theta+\cot^2\theta)\sum_{r=1}^{2p}\big[\left(e_i(\ln f)\right)^2 + 4s\cot^2\theta\sum_{r=1}^{q}\left(e_i(\ln f)\right)^2\big],\end{aligned}$$

which gives the inequality (5.1). For the equality case, from the omitted terms and the vanishing terms in (5.2) and (5.3), we find

$$\sigma(\mathfrak{D}^\theta,\mathfrak{D}^\theta)=0,\quad \sigma(\mathfrak{D}^T,\mathfrak{D}^T)=0,\quad \sigma(\mathfrak{D}^\perp,\mathfrak{D}^\perp)=0,\quad \sigma(\mathfrak{D}^T,\mathfrak{D}^\perp)=0. \tag{5.4}$$

We also have

$$\sigma(\mathfrak{D}^T,\mathfrak{D}^\theta)\subset F\mathfrak{D}^\theta,\quad \sigma(\mathfrak{D}^\perp,\mathfrak{D}^\theta)\subset F\mathfrak{D}^\theta. \tag{5.5}$$

Since B is totally geodesic in M (cf. e.g. [1, 6]), using this fact together with (5.4), we know that B is totally geodesic in $\tilde{M}$. Also, since N_θ is totally umbilical in M, using this fact together with (5.5), we conclude that N_θ is totally umbilical in $\tilde{M}$. The converse is direct to verify. This gives statement (ii).

Statement (iii) follows from Remark 4.7. □

6. Some applications

The following results follow easily from Theorem 5.1.

Theorem 6.1 ([6]). *If $N_T\times_f N_\perp$ is a CR-warped product in a Kaehler manifold $\tilde{M}$, then we have:*

$$\|\sigma\|^2 \geq 2p\,\left\|\nabla^T(\ln f)\right\|^2, \tag{6.1}$$

where $p=\dim N_\perp$ and $\nabla^T(\ln f)$ denotes the gradient of $\ln f$ on N_T. Moreover, if the equality sign in (6.1) holds identically, then N_T is totally geodesic and $N_\perp$ is totally umbilical in $\tilde{M}$.

Theorem 6.2 ([22]). *If $N_T\times_f N_\theta$ is a warped product pointwise semi-slant submanifold of a Kaehler manifold $\tilde{M}$, then we have*

$$\|\sigma\|^2 \geq 4s(\csc^2\theta+\cot^2\theta)\left\|\nabla^T(\ln f)\right\|^2,\quad \dim N_\theta = 2s. \tag{6.2}$$

Furthermore, if the equality sign in (6.2) holds identically, then N_T is totally geodesic and N_θ is totally umbilical in $\tilde{M}$.

If $\dim N_T=0$, then Theorem 5.1 implies the following.

Theorem 6.3 ([23]). *If $N_\perp \times_f N_\theta$ is a $\mathfrak{D}^\perp \oplus \mathfrak{D}^\theta$-mixed totally geodesic warped product pointwise hemi-slant (pseudo-slant) submanifold of a Kaehler manifold $\tilde{M}$, then its second fundamental form σ satisfies*

$$\|\sigma\|^2 \geq 2s\cos^2\theta\,\|\nabla^\perp(\ln f)\|^2, \quad \dim N_\theta = 2s, \tag{6.3}$$

where $\nabla^\perp(\ln f)$ denotes the gradient of $\ln f$ on $N_\perp$.

Remark 6.4. Theorem 5.1 improves Theorem 6.3, since if M is $\mathfrak{D}^\perp \oplus \mathfrak{D}^\theta$-mixed totally geodesic, then M is Riemannian product of $N_\perp$ and N_θ by Remark 4.7, which is the original statement of Corollary 4.5 of [26]. Thus, Theorem 5.1 implies that the inequality (6.3), obtained in [23], is not sharp.

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