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ON THE RESTRICTED PARTITION FUNCTION VIA DETERMINANTS WITH BERNOULLI POLYNOMIALS. II

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ABSTRACT. Let $r \geq 1$ be an integer, $\mathbf{a} = (a_1, \dots, a_r)$ a vector of positive integers, and let $D \geq 1$ be a common multiple of $a_1, \dots, a_r$. We prove that if $D = 1$ or D is a prime number then the restricted partition function $p_{\mathbf{a}}(n) :=$ the number of integer solutions $(x_1, \dots, x_r)$ to $\sum_{j=1}^r a_j x_j = n$, with $x_1 \geq 0, \dots, x_r \geq 0$, can be computed by solving a system of linear equations with coefficients that are values of Bernoulli polynomials and Bernoulli–Barnes numbers.

1. Introduction

Let $\mathbf{a} := (a_1, a_2, \dots, a_r)$ be a sequence of positive integers, $r \geq 1$. The *restricted partition function* associated to $\mathbf{a}$ is $p_{\mathbf{a}} : \mathbb{N} \to \mathbb{N}$, $p_{\mathbf{a}}(n) :=$ the number of integer solutions $(x_1, \dots, x_r)$ of $\sum_{i=1}^r a_i x_i = n$ with $x_i \geq 0$. Let D be a common multiple of $a_1, \dots, a_r$. According to [5], $p_{\mathbf{a}}(n)$ is a quasi-polynomial of degree $r-1$, with the period D, i.e.

$$p_{\mathbf{a}}(n) = d_{\mathbf{a},r-1}(n)n^{r-1} + \cdots + d_{\mathbf{a},1}(n)n + d_{\mathbf{a},0}(n), \quad \text{for all } n \geq 0, \tag{1.1}$$

where $d_{\mathbf{a},m}(n + D) = d_{\mathbf{a},m}(n)$, for all $0 \leq m \leq r-1$, $n \geq 0$, and $d_{\mathbf{a},r-1}(n)$ is not identically zero. The restricted partition function $p_{\mathbf{a}}(n)$ was studied extensively in the literature, starting with the works of Sylvester [15] and Bell [5]. Popoviciu [11] gave a precise formula for $r = 2$. Recently, Bayad and Beck [4, Theorem 3.1] proved an explicit expression of $p_{\mathbf{a}}(n)$ in terms of Bernoulli–Barnes polynomials and the Fourier–Dedekind sums, in the case that $a_1, \dots, a_r$ are pairwise coprime. In [6], we proved that the computation of $p_{\mathbf{a}}(n)$ can be reduced to solving the linear congruency $a_1 j_1 + \cdots + a_r j_r \equiv n \pmod{D}$ in the range $0 \leq j_1 \leq \frac{D}{a_1}, \dots, 0 \leq j_r \leq \frac{D}{a_r}$. In [8], we proved that if a determinant $\Delta_{r,D}$, which depends only on r and D, with entries consisting in values of Bernoulli polynomials is nonzero, then $p_{\mathbf{a}}(n)$ can be computed in terms of values of Bernoulli polynomials and Bernoulli–Barnes numbers. The aim of this paper is to tackle the same problem, from another perspective that relays on the arithmetic properties of Bernoulli polynomials.

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First we recall some definitions. The *Barnes zeta function* associated to $\mathbf{a}$ and $w>0$ is

$$\zeta_{\mathbf{a}}(s,w) := \sum_{n=0}^{\infty} \frac{p_{\mathbf{a}}(n)}{(n+w)^s}, \quad \operatorname{Re} s > r;$$

see [3] and [13] for further details. It is well known that $\zeta_{\mathbf{a}}(s,w)$ is meromorphic on $\mathbb{C}$ with poles at most in the set $\{1,\ldots,r\}$. We consider the function

$$\zeta_{\mathbf{a}}(s) := \lim_{w\searrow 0}(\zeta_{\mathbf{a}}(s,w) - w^{-s}). \tag{1.2}$$

In [6, Lemma 2.6], we proved that

$$\zeta_{\mathbf{a}}(s) = \frac{1}{D^s}\sum_{m=0}^{r-1}\sum_{v=1}^{D} d_{\mathbf{a},m}(v) D^m \zeta\left(s-m, \frac{v}{D}\right), \tag{1.3}$$

where

$$\zeta(s,w) := \sum_{n=0}^{\infty}\frac{1}{(n+w)^s}, \quad \operatorname{Re} s > 1,$$

is the *Hurwitz zeta function*; see also [7]. The *Bernoulli numbers* B_j are defined by

$$\frac{z}{e^z-1} = \sum_{j=0}^{\infty} B_j \frac{z^j}{j!},$$

$B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, $B_4=-\frac{1}{30}$, and $B_n=0$ if n is odd and greater than 1. The *Bernoulli polynomials* are defined by

$$\frac{ze^{xz}}{(e^z-1)} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!}.$$

They are related to the Bernoulli numbers by

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k} B_{n-k}x^k. \tag{1.4}$$

It is well known (see for instance [2, Theorem 12.13]) that

$$\zeta(-n,w) = -\frac{B_{n+1}(w)}{n+1}, \quad \text{for all } n\in\mathbb{N},\ w>0. \tag{1.5}$$

The *Bernoulli–Barnes polynomials* are defined by

$$\frac{z^r e^{xz}}{(e^{a_1 z}-1)\cdots(e^{a_r z}-1)} = \sum_{j=0}^{\infty} B_j(x;\mathbf{a})\frac{z^j}{j!}.$$

The *Bernoulli–Barnes numbers* are defined by

$$B_j(\mathbf{a}) := B_j(0;\mathbf{a}) = \sum_{i_1+\cdots+i_r=j}\binom{j}{i_1,\ldots,i_r} B_{i_1}\cdots B_{i_r} a_1^{i_1-1}\cdots a_r^{i_r-1}.$$

According to [12, Formula (3.10)], the formula

$$\zeta_{\mathbf{a}}(-n,w) = \frac{(-1)^r n!}{(n+r)!} B_{r+n}(w;\mathbf{a}) \tag{1.6}$$

holds for all $n \in \mathbb{N}$. From (1.2) and (1.6), it follows that

$$\zeta_{\mathbf{a}}(-n) = \frac{(-1)^r n!}{(n+r)!} B_{r+n}(\mathbf{a}), \quad \text{for all } n \geq 1. \tag{1.7}$$

From (1.3), (1.5), and (1.7), it follows that

$$\sum_{m=0}^{r-1} \sum_{v=1}^{D} d_{\mathbf{a},m}(v) D^{n+m} \frac{B_{n+m+1}(\frac{v}{D})}{n+m+1} = \frac{(-1)^{r-1} n!}{(n+r)!} B_{r+n}(\mathbf{a}), \quad \text{for all } n \geq 1. \tag{1.8}$$

Let $\underline{\alpha} : \alpha_1 < \alpha_2 < \cdots < \alpha_{rD}$ be a sequence of integers with $\alpha_1 \geq 2$. Substituting n with $\alpha_j - 1$, $1 \leq j \leq rD$, in (1.8) and multiplying by D, we obtain the system of linear equations

$$\sum_{m=0}^{r-1} \sum_{v=1}^{D} d_{\mathbf{a},m}(v) \frac{D^{\alpha_j+m} B_{\alpha_j+m}(\frac{v}{D})}{\alpha_j+m} = \frac{(-1)^{r-1}(\alpha_j-1)! D}{(\alpha_j+r-1)!} B_{\alpha_j+r-1}(\mathbf{a}), \quad 1 \leq j \leq rD,$$

which has the determinant

$$\Delta_{r,D}(\underline{\alpha}) := \begin{vmatrix} \frac{D^{\alpha_1} B_{\alpha_1}(\frac{1}{D})}{\alpha_1} & \cdots & \frac{D^{\alpha_1} B_{\alpha_1}(1)}{\alpha_1} & \cdots & \frac{D^{\alpha_1} B_{\alpha_1+r-1}(\frac{1}{D})}{\alpha_1+r-1} & \cdots & \frac{D^{\alpha_1} B_{\alpha_1+r-1}(1)}{\alpha_1+r-1} \\ \frac{D^{\alpha_2} B_{\alpha_2}(\frac{1}{D})}{\alpha_2} & \cdots & \frac{D^{\alpha_2} B_{\alpha_2}(1)}{\alpha_2} & \cdots & \frac{D^{\alpha_2} B_{\alpha_2+r-1}(\frac{1}{D})}{\alpha_2+r-1} & \cdots & \frac{D^{\alpha_2} B_{\alpha_2+r-1}(1)}{\alpha_2+r-1} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{D^{\alpha_{rD}} B_{\alpha_{rD}}(\frac{1}{D})}{\alpha_{rD}} & \cdots & \frac{D^{\alpha_{rD}} B_{\alpha_{rD}}(1)}{\alpha_{rD}} & \cdots & \frac{D^{\alpha_{rD}} B_{\alpha_{rD}+r-1}(\frac{1}{D})}{\alpha_{rD}+r-1} & \cdots & \frac{D^{\alpha_{rD}} B_{\alpha_{rD}+r-1}(1)}{\alpha_{rD}+r-1} \end{vmatrix}. \tag{1.9}$$

Note that, with the notation given in [8, (2.10)], we have

$$\Delta_{r,D} = \Delta_{r,D}(0, 1, \ldots, rD-1).$$

Here we omit the condition $\alpha_1 \geq 2$.

Proposition 1.1. *With the above notation, if $\Delta_{r,D}(\underline{\alpha}) \neq 0$ then*

$$d_{\mathbf{a},m}(v) = \frac{\Delta_{r,D}^{m,v}(\underline{\alpha})}{\Delta_{r,D}(\underline{\alpha})}, \quad \textit{for all } 1 \leq v \leq D,\ 0 \leq m \leq r-1,$$

where $\Delta_{r,D}^{m,v}(\underline{\alpha})$ is the determinant obtained from $\Delta_{r,D}(\underline{\alpha})$, as defined in (1.9), by replacing the $(mD+v)$-th column with the column

$$\left(\frac{(-1)^{r-1}(\alpha_j-1)! D}{(\alpha_j+r-1)!} B_{\alpha_j+r-1}(\mathbf{a}) \right)_{1 \leq j \leq rD-1}.$$

Moreover,

$$p_{\mathbf{a}}(n) = \frac{1}{\Delta_{r,D}(\underline{\alpha})} \sum_{m=0}^{r-1} \Delta_{r,D}^{m,v}(\underline{\alpha}) n^m, \quad \textit{for all } n \in \mathbb{N}.$$

Proof. It follows from (1.8) and (1.9) by Cramer's rule. The last assertion follows from (1.1). □

Our main theorem is the following.

Theorem 1.2. *Let $r \geq 1$ and let $D = 1$ or $D \geq 2$ be a prime number. There exists a sequence of integers $\underline{\alpha} : \alpha_1 < \alpha_2 < \cdots < \alpha_{rD}$, $\alpha_1 \geq 2$, such that $\Delta_{r,D}(\underline{\alpha}) \neq 0$. In particular, we can compute $p_{\mathbf{a}}(n)$ in terms of values of Bernoulli polynomials and Bernoulli–Barnes numbers.*

We believe that the result holds for any integer $D \geq 1$. Unfortunately, our method based on p-adic valuations and congruences for Bernoulli numbers and for the values of Bernoulli polynomials, is not refined enough to prove it.

2. Properties of Bernoulli polynomials

We recall several properties of the Bernoulli polynomials. We have that

$$B_n(1-x) = (-1)^n B_n(x), \quad \text{for all } n \in \mathbb{N}. \tag{2.1}$$

For any integers $n \geq 1$ and $1 \leq v \leq D$, using (1.4), we let

$$\tilde{B}_n(x) := D^n(B_n(x) - B_n) = \sum_{j=1}^{n-1} \binom{n}{j} D^j (xD)^{n-j}. \tag{2.2}$$

According to [1, Theorem 1], we have that

$$\tilde{B}_n\left(\frac{v}{D}\right) \in \mathbb{Z}, \quad \text{for all } 1 \leq v \leq D. \tag{2.3}$$

According to a result of T. Clausen and C. von Staudt (see [9, 14]), we have that

$$B_{2n} = A_{2n} - \sum_{p-1|2n} \frac{1}{p}, \quad \text{for all } n \geq 1, \tag{2.4}$$

where $A_{2n} \in \mathbb{Z}$ and the sum is over all the primes p such that $p - 1 \mid 2n$.

Let p be a prime. For any integer a, the p-adic order of a is $v_p(a) := \max\{k : p^k \mid a\}$, if $a \neq 0$, and $v_p(0) = \infty$. For $q = \frac{a}{b} \in \mathbb{Q}$, the p-adic order of q is $v_p(q) := v_p(a) - v_p(b)$. Note that (2.4) implies

$$v_p(B_{2n}) = \begin{cases} -1, & p-1 \mid 2n; \\ \geq 0, & p-1 \nmid 2n. \end{cases} \tag{2.5}$$

Lemma 2.1. *For any integer $n \geq 1$, we have that:*

1. *$\tilde{B}_n(\frac{1}{2}) = 0$ if n is odd, and $\tilde{B}_n(\frac{1}{2}) \equiv 1 \pmod 2$ if n is even.*
2. *If p is a prime, then $\tilde{B}_n(\frac{v}{p}) \equiv v^n \pmod p$, for all $1 \leq v \leq p-1$.*

Proof. (1) From (2.1) it follows that $B_n(\frac{1}{2}) = 0$ if n is odd. Hence, as $B_n = 0$, we get

$$\tilde{B}_n\left(\frac{1}{2}\right) = D^n\left(B_n\left(\frac{1}{2}\right) - B_n\right) = 0.$$

Assume that n is even. According to (2.2), we have

$$\tilde{B}_n\left(\frac{1}{2}\right) = \sum_{j=0}^{n} \binom{n}{j} B_j 2^j.$$

Since $2 \mid 2nB_1 = -n$ and $v_2(2^j B_j) \geq 1$ for any $j \geq 2$, the conclusion follows immediately.

(2) According to (2.2), we have that

$$\tilde{B}_n\left(\frac{a}{p}\right) = \sum_{j=0}^{n} \binom{n}{j} B_j v^{n-j} p^j.$$

From (2.5), we have that $v_p(p^j B_j) \geq 1$ for $j \geq 1$, hence the conclusion follows immediately. □

Lemma 2.2. *If p is a prime such that $p \nmid D$ then*

$$\tilde{B}_p\left(\frac{v}{D}\right) \equiv 0 \pmod{p}, \quad \textit{for all } 1 \leq v \leq D-1.$$

Proof. We have that

$$\tilde{B}_p\left(\frac{v}{D}\right) = \sum_{j=0}^{p} \binom{p}{j} B_j v^{p-j} D^j.$$

Since $v_p(B_j) \geq 0$ for $j \leq p-2$, it follows that

$$v_p\left(\binom{p}{j} B_j\right) \geq 1, \quad \text{for all } 1 \leq j \leq p-2.$$

On the other hand, it follows from (2.4) that

$$v^p + \binom{p}{p-1} B_{p-1} v D^{p-1} \equiv v^p - vD^{p-1} \equiv v^p - v \equiv 0 \pmod{p}.$$

Hence, we get the required result. □

3. Preliminary results

Proposition 3.1 (Case $D = 1$)**.** *Let $p_1 < p_2 < \cdots < p_r$ be some primes such that $p_1 > 2$ and $p_{j+1} - p_j > r$, for all $1 \leq j \leq r-1$. Let $\alpha_j := p_j - j$, $1 \leq j \leq r$. We have that $\Delta_{r,1}(\underline{\alpha}) \neq 0$.*

Proof. Note that (2.1) implies $B_n(1) = B_n$ for any $n \geq 2$. It follows that

$$\Delta_{r,1}(\underline{\alpha}) = \begin{vmatrix} \frac{B_{\alpha_1}}{\alpha_1} & \frac{B_{\alpha_1+1}}{\alpha_1+1} & \cdots & \frac{B_{\alpha_1+r-1}}{\alpha_1+r-1} \\ \frac{B_{\alpha_2}}{\alpha_2} & \frac{B_{\alpha_1+1}}{\alpha_2+1} & \cdots & \frac{B_{\alpha_2+r-1}}{\alpha_2+r-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{B_{\alpha_r}}{\alpha_r} & \frac{B_{\alpha_1+1}}{\alpha_r+1} & \cdots & \frac{B_{\alpha_r+r-1}}{\alpha_r+r-1} \end{vmatrix}. \tag{3.1}$$

From (2.4) it follows that $v_{p_j}(B_{\alpha_j+j-1}) = -1$ and $v_{p_j}(B_{\alpha_j+k-1}) \geq 0$, for all $1 \leq k \leq r$ with $k \neq j$. Moreover, if $1 \leq \ell < j \leq r$, then, by hypothesis, $v_{p_j}(B_{\alpha_\ell+k-1}) \geq 0$ for any $1 \leq k \leq r$ (we implicitly used the fact that $B_n = 0$ if $n \geq 3$ is odd). It follows that, in the expansion of $\Delta_{r,1}(\underline{\alpha})$ written in (3.1), the term

$$\prod_{j=1}^{r} \frac{D^{\alpha_j+j-1} B_{\alpha_j+j-1}}{\alpha_j + j - 1}$$

cannot be simplified, hence $\Delta_{r,1}(\underline{\alpha}) \neq 0$. □

In the following, we assume that $D \geq 2$ and we consider the determinant

$$\tilde{\Delta}_{r,D}(\underline{\alpha}) := \begin{vmatrix} \frac{\tilde{B}_{\alpha_1}(\frac{1}{D})}{\alpha_1} & \cdots & \frac{\tilde{B}_{\alpha_1}(\frac{D-1}{D})}{\alpha_1} & \cdots & \frac{\tilde{B}_{\alpha_1+r-1}(\frac{1}{D})}{\alpha_1+r-1} & \cdots & \frac{\tilde{B}_{\alpha_1+r-1}(\frac{D-1}{D})}{\alpha_1+r-1} \\ \frac{\tilde{B}_{\alpha_2}(\frac{1}{D})}{\alpha_2} & \cdots & \frac{\tilde{B}_{\alpha_2}(\frac{D-1}{D})}{\alpha_2} & \cdots & \frac{\tilde{B}_{\alpha_2+r-1}(\frac{1}{D})}{\alpha_2+r-1} & \cdots & \frac{\tilde{B}_{\alpha_2+r-1}(\frac{D-1}{D})}{\alpha_2+r-1} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\tilde{B}_{\alpha_{rD-r}}(\frac{1}{D})}{\alpha_{rD-r}} & \cdots & \frac{\tilde{B}_{\alpha_{rD-r}}(\frac{D-1}{D})}{\alpha_{rD-r}} & \cdots & \frac{\tilde{B}_{\alpha_{rD-r}+r-1}(\frac{1}{D})}{\alpha_{rD-r}+r-1} & \cdots & \frac{\tilde{B}_{\alpha_{rD-r}+r-1}(\frac{D-1}{D})}{\alpha_{rD-r}+r-1} \end{vmatrix}. \quad (3.2)$$

Let $p_1 < p_2 < \cdots < p_r$ be some primes such that

$$p_1 \geq \alpha_{r(D-1)} + r \quad \text{and} \quad p_{j+1} - p_j > r, \qquad \text{for all } 1 \leq j \leq r-1.$$

We let

$$\alpha_{rD-r+j} := p_j - j, \quad \text{for all } 1 \leq j \leq r. \quad (3.3)$$

According to Lemma 2.2 and (3.3), we have that

$$v_{p_\ell}\left(\frac{\tilde{B}_{\alpha_{rD-r+j}+j}(\frac{v}{D})}{\alpha_{rD-r+j}+j}\right) \geq 0, \quad \text{for all } 1 \leq j,\ \ell \leq r,\ 1 \leq v \leq D-1. \quad (3.4)$$

On the other hand, since $p_j \geq \alpha_{r(D-1)} + r$, from Lemma 2.2 it follows that

$$v_{p_\ell}\left(\frac{D^{\alpha_t+j}B_{\alpha_t+j}}{\alpha_t+j}\right) \geq 0, \quad v_{p_\ell}\left(\frac{\tilde{B}_{\alpha_t+j}(\frac{v}{D})}{\alpha_t+j}\right) \geq 0, \quad (3.5)$$

for all $1 \leq j, \ell \leq r,\ 1 \leq t \leq r(D-1),\ 1 \leq v \leq D-1$. Also, from (2.5) and (3.3), it follows that

$$v_{p_\ell}\left(\frac{B_{\alpha_{rD-r+j}+j}(\frac{v}{D})}{\alpha_{rD-r+j}+j}\right) \geq 0, \quad v_{p_j}\left(\frac{B_{\alpha_{rD-r+j}+j}(\frac{v}{D})}{\alpha_{rD-r+j}+j}\right) = -1, \quad (3.6)$$

for $1 \leq j,\ \ell \leq r,\ j \neq \ell,\ 1 \leq v \leq D-1$. From (1.9), using the basic properties of determinants and (2.2), it follows that

$$\Delta_{r,D}(\underline{\alpha}) = \begin{vmatrix} \frac{\tilde{B}_{\alpha_1}(\frac{1}{D})}{\alpha_1} & \cdots & \frac{\tilde{B}_{\alpha_1}(\frac{D-1}{D})}{\alpha_1} & \frac{D^{\alpha_1}B_{\alpha_1}}{\alpha_1} & \cdots & \frac{\tilde{B}_{\alpha_1+r-1}(\frac{1}{D})}{\alpha_1+r-1} & \cdots & \frac{D^{\alpha_1}B_{\alpha_1+r-1}}{\alpha_1+r-1} \\ \frac{\tilde{B}_{\alpha_2}(\frac{1}{D})}{\alpha_2} & \cdots & \frac{\tilde{B}_{\alpha_2}(\frac{D-1}{D})}{\alpha_2} & \frac{D^{\alpha_2}B_{\alpha_2}}{\alpha_2} & \cdots & \frac{\tilde{B}_{\alpha_2+r-1}(\frac{1}{D})}{\alpha_2+r-1} & \cdots & \frac{D^{\alpha_2}B_{\alpha_2+r-1}}{\alpha_2+r-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\tilde{B}_{\alpha_{rD}}(\frac{1}{D})}{\alpha_{rD}} & \cdots & \frac{\tilde{B}_{\alpha_{rD}}(\frac{D-1}{D})}{\alpha_{rD}} & \frac{D^{\alpha_{rD}}B_{\alpha_{rD}}}{\alpha_{rD}} & \cdots & \frac{\tilde{B}_{\alpha_{rD}+r-1}(\frac{1}{D})}{\alpha_{rD}+r-1} & \cdots & \frac{D^{\alpha_{rD}}B_{\alpha_{rD}+r-1}}{\alpha_{rD}+r-1} \end{vmatrix}. \quad (3.7)$$

Proposition 3.2. *With the above assumptions, we have that* $\Delta_{r,D}(\underline{\alpha}) \neq 0$ *if and only if* $\tilde{\Delta}_{r,D}(\underline{\alpha}) \neq 0$.

Proof. The conclusion follows from (3.2), (3.4), (3.5), (3.6), and (3.7), using an argument similar to that in the proof of Proposition 3.1. □

Proposition 3.3 (Case $D = 2$)**.** *With the above assumptions,* $\Delta_{r,2}(\underline{\alpha}) \neq 0$.

Proof. By Proposition 3.2, it is enough to prove that $\tilde{\Delta}_{r,2}(\underline{\alpha}) \neq 0$. We have that

$$\tilde{\Delta}_{r,2}(\underline{\alpha}) = \begin{vmatrix} \frac{\tilde{B}_{\alpha_1}(\frac{1}{2})}{\alpha_1} & \frac{\tilde{B}_{\alpha_1+1}(\frac{1}{2})}{\alpha_1+1} & \cdots & \frac{\tilde{B}_{\alpha_1+r-1}(\frac{1}{2})}{\alpha_1+r-1} \\ \frac{\tilde{B}_{\alpha_2}(\frac{1}{2})}{\alpha_2} & \frac{\tilde{B}_{\alpha_2+1}(\frac{1}{2})}{\alpha_2+1} & \cdots & \frac{\tilde{B}_{\alpha_2+r-1}(\frac{1}{2})}{\alpha_2+r-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\tilde{B}_{\alpha_r}(\frac{1}{2})}{\alpha_r} & \frac{\tilde{B}_{\alpha_r+1}(\frac{1}{2})}{\alpha_1+1} & \cdots & \frac{\tilde{B}_{\alpha_r+r-1}(\frac{1}{2})}{\alpha_r+r-1} \end{vmatrix}. \tag{3.8}$$

We choose $\alpha_j := 2^{j+t} - j + 1$, where $2^t \geq r$. From (2.3) and Lemma 2.1 (1) it follows that

$$v_2\left(\tilde{B}_{\alpha_j+j-1}\left(\frac{1}{2}\right)\right) = 0, \quad v_2\left(\tilde{B}_{\alpha_j+\ell-1}\left(\frac{1}{2}\right)\right) \geq 0, \qquad \text{for all } 1 \leq j, \ell \leq r,\ j \neq \ell. \tag{3.9}$$

On the other hand,

$$j + t = v_2(\alpha_j + j - 1) > v_2(\alpha_j + \ell - 1), \quad \text{for all } 1 \leq j, \ell \leq r,\ j \neq \ell. \tag{3.10}$$

From (3.8), (3.9), and (3.10) it follows that

$$v_2\left(\tilde{\Delta}_{r,2}(\underline{\alpha})\right) = v_2\left(\prod_{j=1}^{r} \frac{\tilde{B}_{\alpha_j+j-1}(\frac{1}{2})}{\alpha_j + j - 1}\right) = -rt - \binom{r}{2}.$$

Hence, $\tilde{\Delta}_{r,2}(\underline{\alpha}) \neq 0$, as required. □

In the following, we assume that $D \geq 3$. Let $N := \left\lfloor \frac{(D-1)r}{2} \right\rfloor$. We also assume that α_t is odd for all $1 \leq t \leq N$, and α_t is even for all $N+1 \leq t \leq r(D-1)$. Let $k := \left\lfloor \frac{D-1}{2} \right\rfloor$ and $\bar{k} = \left\lceil \frac{D-1}{2} \right\rceil$. From (2.1) and (2.2) it follows that

$$\tilde{B}_{\alpha_t+j-1}\left(\frac{D-v}{D}\right) + \tilde{B}_{\alpha_t+j-1}\left(\frac{v}{D}\right) = \begin{cases} 0, & \text{if } \alpha_t + j - 1 \text{ is odd;} \\ 2\tilde{B}_{\alpha_t+j-1}\left(\frac{v}{D}\right), & \text{if } \alpha_t + j - 1 \text{ is even,} \end{cases} \tag{3.11}$$

for all $1 \leq t \leq r(D-1)$, $1 \leq v \leq \bar{k}$, and $1 \leq j \leq r$. We consider the determinants

$$\tilde{\Delta}'_{r,D}(\underline{\alpha}) := \begin{vmatrix} \frac{\tilde{B}_{\alpha_1}(\frac{1}{D})}{\alpha_1} & \cdots & \frac{\tilde{B}_{\alpha_1}(\frac{k}{D})}{\alpha_1} & \frac{\tilde{B}_{\alpha_1+1}(\frac{1}{D})}{\alpha_1+1} & \cdots & \frac{\tilde{B}_{\alpha_1+1}(\frac{\bar{k}}{D})}{\alpha_1+1} & \cdots \\ \frac{\tilde{B}_{\alpha_2}(\frac{1}{D})}{\alpha_2} & \cdots & \frac{\tilde{B}_{\alpha_2}(\frac{k}{D})}{\alpha_2} & \frac{\tilde{B}_{\alpha_2+1}(\frac{1}{D})}{\alpha_2+1} & \cdots & \frac{\tilde{B}_{\alpha_2+1}(\frac{\bar{k}}{D})}{\alpha_2+1} & \cdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \frac{\tilde{B}_{\alpha_N}(\frac{1}{D})}{\alpha_N} & \cdots & \frac{\tilde{B}_{\alpha_N}(\frac{k}{D})}{\alpha_N} & \frac{\tilde{B}_{\alpha_{N+1}}(\frac{1}{D})}{\alpha_{N+1}} & \cdots & \frac{\tilde{B}_{\alpha_{N+1}}(\frac{\bar{k}}{D})}{\alpha_{N+1}} & \cdots \end{vmatrix} \tag{3.12}$$

and

$$\tilde{\Delta''}_{r,D}(\underline{\alpha}) := \begin{vmatrix} \frac{\tilde{B}_{\alpha_{N+1}}(\frac{1}{D})}{\alpha_{N+1}} & \cdots & \frac{\tilde{B}_{\alpha_{N+1}}(\frac{\bar{k}}{D})}{\alpha_{N+1}} & \frac{\tilde{B}_{\alpha_{N+1}+1}(\frac{1}{D})}{\alpha_{N+1}+1} & \cdots & \frac{\tilde{B}_{\alpha_{N+1}+1}(\frac{k}{D})}{\alpha_{N+1}+1} & \cdots \\ \frac{\tilde{B}_{\alpha_{N+2}}(\frac{1}{D})}{\alpha_{N+2}} & \cdots & \frac{\tilde{B}_{\alpha_{N+2}}(\frac{\bar{k}}{D})}{\alpha_{N+2}} & \frac{\tilde{B}_{\alpha_{N+2}+1}(\frac{1}{D})}{\alpha_{N+2}+1} & \cdots & \frac{\tilde{B}_{\alpha_{N+2}+1}(\frac{k}{D})}{\alpha_{N+2}+1} & \cdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \frac{\tilde{B}_{\alpha_{rD-r}}(\frac{1}{D})}{\alpha_{rD-r}} & \cdots & \frac{\tilde{B}_{\alpha_{rD-r}}(\frac{\bar{k}}{D})}{\alpha_{rD-r}} & \frac{\tilde{B}_{\alpha_{rD-r}+1}(\frac{1}{D})}{\alpha_{rD-r}+1} & \cdots & \frac{\tilde{B}_{\alpha_{rD-r}+1}(\frac{k}{D})}{\alpha_{rD-r}+1} & \cdots \end{vmatrix}. \tag{3.13}$$

Proposition 3.4. *With the above assumptions, we have that*

$$\tilde{\Delta}_{r,D}(\underline{\alpha}) = C\tilde{\Delta'}_{r,D}(\underline{\alpha})\tilde{\Delta''}_{r,D}(\underline{\alpha}),$$

where $C \neq 0$. *In particular, if* $\tilde{\Delta'}_{r,D}(\underline{\alpha}) \neq 0$ *and* $\tilde{\Delta''}_{r,D}(\underline{\alpha}) \neq 0$ *then* $\tilde{\Delta}_{r,D}(\underline{\alpha}) \neq 0$.

Proof. In (3.2), we add the $(j+tr)$-th column over the $(D-j+tr)$-th column, where $1 \leq j \leq k$ and $0 \leq t \leq r-1$. The conclusion follows from (3.11), (3.12), and (3.13) using the basic properties of determinants. □

4. Proof of Theorem 1.2

The case $D = 1$ was proved in Proposition 3.1. Also, the case $D = 2$ was proved in Proposition 3.3. Assume that $D := p > 2$ is a prime number. Let $k := \lfloor \frac{p-1}{2} \rfloor$. According to Proposition 3.4, it is enough to prove that $\tilde{\Delta'}_{r,p}(\underline{\alpha}) \neq 0$ and $\tilde{\Delta''}_{r,p}(\underline{\alpha}) \neq 0$. Let

$$t_1 < t_2 < \cdots < t_r, \tag{4.1}$$

be a sequence of positive integers, such that $t_1 > \log_p(r-1) :=$ the logarithm of $r-1$ to base p. We define

$$\alpha_{j+(s-1)k} := \begin{cases} 2jp^{t_s} - s + 1, & \text{if } s \text{ is even;} \\ (2j-1)p^{t_s} - s + 1, & \text{if } s \text{ is odd,} \end{cases} \tag{4.2}$$

for all $1 \leq s \leq r$, $1 \leq j \leq k$. From (4.1) and (4.2) it follows that

$$v_p(\alpha_{j+(s-1)k} + s - 1) = t_s, \quad \text{for all } 1 \leq s \leq r,\ 1 \leq j \leq k; \tag{4.3}$$

$$v_p(\alpha_{j+(s-1)k} + \ell) < t_1, \quad \text{for all } 1 \leq s \leq r,\ 1 \leq j \leq k, \text{ and } 0 \leq \ell \leq r-1 \text{ with } \ell \neq s-1. \tag{4.4}$$

On the other hand, from Lemma 2.1 (2) it follows that

$$B_{\alpha_j}\left(\frac{v}{p}\right) \equiv v^{\alpha_j} \pmod{p}, \quad \text{for all } 1 \leq j \leq rp. \tag{4.5}$$

From (4.3), (4.4), and (4.5) it follows that

$$v_p\left(\frac{\widetilde{B}_{\alpha_{j+(s-1)k}+s-1}(\frac{v}{p})}{\alpha_{j+(s-1)k}+s-1}\right) = -t_s, \quad \text{for all } 1 \le s \le r,\ 1 \le j, v \le k, \tag{4.6}$$

$$v_p\left(\frac{\widetilde{B}_{\alpha_{j+(s-1)k}+\ell}(\frac{v}{p})}{\alpha_{j+(s-1)k}+\ell}\right) > -t_1, \quad \text{for all } 1 \le s \le r,\ 1 \le j, v \le k, \quad 0 \le \ell \le r-1 \text{ with } \ell \ne s-1. \tag{4.7}$$

We consider the determinants

$$M_s := \det\left(\widetilde{B}_{\alpha_{j+(s-1)k}+s-1}\left(\frac{v}{p}\right)\right)_{1\le j,v\le k}, \quad 1 \le s \le r. \tag{4.8}$$

From (4.5) it follows that

$$M_s \equiv \det\left(v^{2jp^{t_s}}\right)_{1\le j,v\le k} \equiv \det\left(v^{2j}\right)_{1\le j,v\le k} \pmod p \quad \text{for } s \text{ even}, \tag{4.9}$$

$$M_s \equiv \det\left(v^{2(j-1)p^{t_s}}\right)_{1\le j,v\le k} \equiv \det\left(v^{2j-1}\right)_{1\le j,v\le k} \pmod p \quad \text{for } s \text{ odd}. \tag{4.10}$$

On the other hand, using the Vandermonde formula, we have

$$\det\left(v^{2j}\right)_{1\le j,v\le k} = v^2 \prod_{1\le i<j\le k} (j-i)(j+i) \not\equiv 0 \pmod p, \tag{4.11}$$

$$\det\left(v^{2j-1}\right)_{1\le j,v\le k} = v \prod_{1\le i<j\le k} (j-i)(j+i) \not\equiv 0 \pmod p. \tag{4.12}$$

From (4.9), (4.10), (4.11), and (4.12) it follows that

$$v_p(M_s) = 0, \quad \text{for all } 1 \le s \le r. \tag{4.13}$$

Hence, it follows that $M_s \ne 0$. From (3.12), (4.6), (4.7), (4.8), and (4.13) it follows that

$$v_p\left(\tilde{\Delta'}_{r,p}(\underline{\alpha})\right) = -(t_1+t_2+\cdots+t_r)k.$$

Therefore, $\tilde{\Delta'}_{r,p}(\underline{\alpha}) \ne 0$. Similarly, one can prove that $\tilde{\Delta''}_{r,p}(\underline{\alpha}) \ne 0$.

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