

THE TOTAL CO-INDEPENDENT DOMINATION NUMBER OF SOME GRAPH OPERATIONS

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ABSTRACT. A set D of vertices of a graph G is a total dominating set if every vertex of G is adjacent to at least one vertex of D . The total dominating set D is called a total co-independent dominating set if the subgraph induced by $V(G) - D$ is edgeless. The minimum cardinality among all total co-independent dominating sets of G is the total co-independent domination number of G . In this article we study the total co-independent domination number of the join, strong, lexicographic, direct and rooted products of graphs.

1. INTRODUCTION

Throughout this article we consider simple graphs $G = (V(G), E(G))$ of order n and size m . That is, graphs that are finite, undirected, and without loops or multiple edges. Given a vertex v of G , $N(v)$ represents the *open neighborhood* of v , i.e., the set of all neighbors of v in G , and the *degree* of v is $\delta(v) = |N(v)|$. If $\delta(v) = n - 1$, then we say that v is a *universal vertex* of G . The *minimum* and *maximum degrees* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. By $G[D]$ we denote the induced subgraph of G on $D \subseteq V(G)$. By the union of two graphs $G \cup H$ we mean the disjoint union. In particular, $kG = \cup_{i=1}^k G$ is a disjoint union of k copies of a graph G . We use the notation $[k]$ for the set of integer numbers $\{1, \dots, k\}$.

Given a graph G , a set $D \subseteq V(G)$ is a *total dominating set* of G if every vertex in $V(G)$ is adjacent to at least one vertex in D . The *total domination number* of G is the minimum cardinality among all total dominating sets of G and is denoted by $\gamma_t(G)$. A $\gamma_t(G)$ -*set* is a total dominating set of cardinality $\gamma_t(G)$. For more information on total domination we suggest the relatively recent survey [15] and the book [16]. A set of vertices is *independent* if it induces an edgeless graph. The *independence number* of G is the cardinality of a maximum independent set

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of G and is denoted by $\alpha(G)$. An independent set of cardinality $\alpha(G)$ is called an $\alpha(G)$ -set. A set S of vertices of G is a *vertex cover* if every edge of G is incident in at least one vertex of S . The *vertex cover number* of G , denoted by $\beta(G)$, is the smallest cardinality among all vertex covers of G . We refer to a $\beta(G)$ -set in G as a vertex cover of cardinality $\beta(G)$. The following well-known result, due to Gallai [12], states the relationship between the independence number and the vertex cover number of a graph.

Theorem 1.1 (Gallai, 1959 [12]). *For any graph G of order n , $\beta(G) + \alpha(G) = n$.*

The theory of domination and independence in graphs has attracted the attention of many researchers since several years ago. The number of works, results and open problems in this area span a wide range of directions, from highly theoretical aspects and various practical applications, to a large number of connections with other topics in graph theory itself and in some external areas. One can easily notice these facts, by simply making some specialized searches in well-known databases like MathSciNet, Scopus or JCR. The idea of studying a variant of a dominating set whose complement is independent has previously been explored, for instance, in the Ph.D. thesis [19], although it was probably introduced earlier.

In recent years, a reborn interest has arisen in research concerning connections between domination and independence in graphs. One of the ideas behind this interest comes from a Roman domination structure whose “complement” has independent properties. Particular and remarkable cases are observed in [1, 8] for co-independent Roman domination, in [2] for co-independent double Roman domination, and in [7, 18] for co-independent total Roman domination. Other similar parameters not related to Roman domination are [3, 4, 11]. It is then our goal here to continue making some contributions to this topic which concerns dominating sets whose complements form an independent set. We remark that some of the articles referenced above use the term “outer-independence” instead of “co-independence”.

A total dominating set D of a graph G without isolated vertices is called a *total co-independent dominating set* (or TC-ID set for short) if the set of vertices $V(G) - D$ is independent. The *total co-independent domination number* of G is the minimum cardinality among all TC-ID sets of G and is denoted by $\gamma_{t,coi}(G)$. A TC-ID set of cardinality $\gamma_{t,coi}(G)$ is a $\gamma_{t,coi}(G)$ -set.

The total co-independent domination number of a graph G has been introduced in [23], where a few of its combinatorial properties were considered. Among them, a couple of almost trivial bounds in terms of $\alpha(G)$ and the order of G were proved for $\gamma_{t,coi}(G)$. The following result is an example of this.

Theorem 1.2 ([23]). *For any graph G of order n without isolated vertices we have $\gamma_{t,coi}(G) \geq n - \alpha(G)$.*

In addition, exact values for some families of graphs were presented in [23]. We mention $\gamma_{t,coi}(P_n) = n - \lceil \frac{n}{3} \rceil$, $\gamma_{t,coi}(C_n) = n - \lfloor \frac{n}{3} \rfloor$, $\gamma_{t,coi}(K_n) = n - 1$ and $\gamma_{t,coi}(K_{s,t}) = \min\{s, t\} + 1$, which we use later in Section 3.

The parameter of total co-independent domination number was studied also in [5] for the case of trees, and in [6] from a combinatorial and complexity point

of view. Until now, and to the best of our knowledge, this parameter has not been studied for product graphs. In this sense, it is our goal to make several contributions to this topic of total co-independent domination, and so, to begin studying the join, direct, strong and lexicographic product graphs, and the rooted and corona product graphs. We end this section with an easy general result that will come in handy later for the rooted product.

Lemma 1.3. *Let H be a graph of order $n \geq 3$ different from the star graph. If H has a universal vertex $v \in V(H)$, then $\gamma_{t,coi}(H) = n - \alpha(H)$.*

Proof. Let v be a universal vertex of H . If $V(H) - \{v\}$ is a $\gamma_{t,coi}(H)$ -set, then $\alpha(H) = 1$ and $H \cong K_n$ and the result follows. Otherwise, let A be an arbitrary $\alpha(H)$ -set. Clearly, $v \notin A$ and the set $V(H) - A$ induces a connected subgraph of H with at least two vertices, because v is a universal vertex and H is not a star. Moreover, $V(H) - A$ also totally dominates H because v is universal. Hence, $\gamma_{t,coi}(H) \leq n - \alpha(H)$. Theorem 1.2 completes the proof. \square

2. THE JOIN, THE LEXICOGRAPHIC AND THE STRONG PRODUCTS OF GRAPHS

The join graph $G \vee H$ of the graphs G and H is the graph with vertex set $V(G \vee H) = V(G) \cup V(H)$ and edge set $E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The following simple lemma is a part of folklore, and it can be found for instance in [22].

Lemma 2.1 ([22]). *For any two graphs G and H ,*

$$\alpha(G \vee H) = \max\{\alpha(G), \alpha(H)\}.$$

Theorem 2.2. *Let G and H be any graphs with $n_X = |V(X)|$ and $m_X = |E(X)|$ for $X \in \{G, H\}$. If $m_G \geq m_H$, then $\gamma_{t,coi}(G \vee H)$ equals*

$$\begin{cases} n_G + n_H - \max\{\alpha(G), \alpha(H)\} + 1, & \text{if } (m_G > m_H = 0 \wedge \alpha(G) < n_H \wedge \delta(G) = 0) \\ & \vee m_G = 0; \\ n_G + n_H - \max\{\alpha(G), \alpha(H)\}, & \text{otherwise.} \end{cases}$$

Proof. By Lemma 2.1 and Theorem 1.2 we obtain the inequality $\gamma_{t,coi}(G \vee H) \geq n_G + n_H - \max\{\alpha(G), \alpha(H)\}$.

Let A_G be an $\alpha(G)$ -set and let A_H be an $\alpha(H)$ -set. Notice that $D_G = V(G \vee H) - A_G$ and $D_H = V(G \vee H) - A_H$ are TC-ID sets of $G \vee H$ whenever both G and H , respectively, contain at least one edge. In this case $\gamma_{t,coi}(G \vee H) \leq \min\{|D_G|, |D_H|\} = n_G + n_H - \max\{\alpha(G), \alpha(H)\}$. Therefore the equality follows when $m_G > 0$ and $m_H > 0$.

Suppose next that $m_G > 0$ and $m_H = 0$, which means that $\alpha(H) = n_H$ and $H \cong \bar{K}_t$ for some positive integer t . Again D_G is a TC-ID set because $m_G > 0$. If $\alpha(G) \geq n_H$, then we have $\gamma_{t,coi}(G \vee H) \leq |D_G| = n_G + n_H - \alpha(G) = n_G + n_H - \max\{\alpha(G), \alpha(H)\}$ and equality holds again. So, let $\alpha(G) < n_H = t$. If $\delta(G) > 0$, then $D_H = V(G)$ is a TC-ID set because $V(G)$ contains no isolated vertex. In this case $\gamma_{t,coi}(G \vee H) \leq |D_H| = n_G = n_G + n_H - t = n_G + n_H - \max\{\alpha(G), \alpha(H)\}$ and equality holds again. If $\delta(G) = 0$, then $D_H = V(G)$ is not a TC-ID set because D_H

is not a total dominating set of $G \vee H$. Notice that $V(H)$ is the unique $\alpha(G \vee H)$ -set because $\alpha(G) < n_H$ and every independent set of $G \vee H$ is contained either in $V(G)$ or in $V(H)$. Therefore, $\gamma_{t,coi}(G \vee H) > n_G + n_H - \max\{\alpha(G), \alpha(H)\}$. On the other hand, the set $V(G) \cup \{h\}$, for some $h \in V(H)$, is a TC-ID set of $G \vee H$ of cardinality $n_G + 1 = n_G + n_H - t + 1 = n_G + n_H - \max\{\alpha(G), \alpha(H)\} + 1$ and we have $\gamma_{t,coi}(G \vee H) = n_G + n_H - \max\{\alpha(G), \alpha(H)\} + 1$.

Finally, let $m_H = m_G = 0$, which means that $G \cong \overline{K}_s$ and $H \cong \overline{K}_t$. Again D_G and D_H are not TC-ID sets of $G \vee H$ because $V(H)$ and $V(G)$, respectively, are not total dominating sets of $G \vee H$. As before, the inequality $\gamma_{t,coi}(G \vee H) > n_G + n_H - \max\{\alpha(G), \alpha(H)\}$ follows. The equality $\gamma_{t,coi}(G \vee H) = n_G + n_H - \max\{\alpha(G), \alpha(H)\} + 1$ now follows from the fact that $V(G) \cup \{h\}$, with $h \in V(H)$, and $V(H) \cup \{g\}$, with $g \in V(G)$, are TC-ID sets of $G \vee H$. \square

Let G and H be two graphs. The *lexicographic product* of G and H is the graph $G \circ H$ whose vertex set is $V(G \circ H) = V(G) \times V(H)$, and $(g, h)(g', h') \in E(G \circ H)$ if and only if $gg' \in E(G)$ or $(g = g' \text{ and } hh' \in E(H))$. The set $G^h = \{(g, h) : g \in V(G)\}$ is called the *G -layer through h* , and similarly, $H^g = \{(g, h) : h \in V(H)\}$ is called the *H -layer through g* . Clearly, subgraphs of $G \circ H$ induced by G^h and H^g are isomorphic to G and H , respectively. The map $p_G : V(G) \times V(H) \rightarrow V(G)$ defined by $p_G((g, h)) = g$ is called a *projection map* to G . The map p_H is defined similarly. Notice that $K_2 \circ H \cong H \vee H$ presents a connection between the lexicographic product and the join of graphs.

In order to obtain the total co-independent domination number of the lexicographic product of two graphs, we need the following result.

Theorem 2.3 ([13]). *For any two graphs G and H , $\alpha(G \circ H) = \alpha(G)\alpha(H)$.*

It is straightforward to observe that $G \circ H$ is connected whenever $G \not\cong K_1$ is connected. Moreover, if G is not connected, then one can treat every component of $G \circ H$ separately with respect to $\gamma_{t,coi}(G \circ H)$. Hence, we can assume that G is connected.

Let G be a graph without isolated vertices. By $\mathcal{I}(G)$ we denote the set of all maximal independent sets of G . By a *maximal independent set* we mean an independent set which is not contained in any other independent set (notice that any maximum independent set is maximal, but the contrary is not always true). The set $D_G = V(G) - A_G$ is a dominating set of G for any $A_G \in \mathcal{I}(G)$ because of the maximality of A_G . Denote by D_G^* the set of all isolated vertices from $G[D_G]$. Finally, let A_G^* be a minimum subset of A_G such that A_G^* dominates D_G^* .

Let $G \cong C_{2k} = v_1 \dots v_{2k}v_1$. There are exactly two maximum independent sets in $\mathcal{I}(G)$, namely $A_{G,1} = \{v_{2i} : i \in [k]\}$ and $A_{G,2} = \{v_{2i-1} : i \in [k]\}$. Without loss of generality we may assume that $A_G = A_{G,2}$ because $A_{G,1}$ and $A_{G,2}$ are symmetric. It is easy to see that one possibility is $A_G^* = \{v_{4i} : i \in [k/2]\}$ if k is even, and $A_G^* = \{v_{4i} : i \in [(k-1)/2]\} \cup \{v_{2k}\}$ if k is odd.

Theorem 2.4. *Let G be a connected graph of order n_G and let H be any graph of order n_H . If $H \not\cong \overline{K}_t$, then*

$$\gamma_{t,coi}(G \circ H) = n_G n_H - \alpha(G)\alpha(H).$$

Moreover, if $H \cong \overline{K}_t$, then

$$\gamma_{t,coi}(G \circ H) = \min_{A_G \in \mathcal{I}(G)} \{|D_G|n_H + |A_G^*|\}.$$

Proof. Let I be an $\alpha(G \circ H)$ -set. Notice that $V(G \circ H) - I$ is a TC-ID set of $G \circ H$ whenever $H \not\cong \overline{K}_t$. By Theorem 2.3 we obtain $\gamma_{t,coi}(G \circ H) \leq |V(G \circ H) - I| = n_G n_H - \alpha(G \circ H) = n_G n_H - \alpha(G)\alpha(H)$. The equality follows by Theorem 1.2 and Theorem 2.3.

Let now $H \cong \overline{K}_t$ and let first $A_G \in \mathcal{I}(G)$, together with D_G and A_G^* . Every vertex outside of $(D_G \times V(H))$ has a neighbor in $(D_G \times V(H))$, because D_G is a dominating set of G . However, if $g \in D_G^*$, then $(g, h) \in (D_G \times V(H))$ has no neighbor in $(D_G \times V(H))$. Hence, to obtain a TC-ID set we need to add some vertices. Since A_G^* dominates vertices from D_G^* , we see that $(D_G \times V(H)) \cup (A_G^* \times \{h\})$ is a TC-ID set of $G \circ H$ for any $h \in V(H)$. Therefore, $\gamma_{t,coi}(G \circ H) \leq \min_{A_G \in \mathcal{I}(G)} \{|D_G|n_H + |A_G^*|\}$. Let now D be a $\gamma_{t,coi}(G \circ H)$ -set. The set $D'_G = p_G(D)$ is a total dominating set of G because $H \cong \overline{K}_t$. Also for $A = V(G \circ H) - D$, $A_G = p_G(A)$ is an independent set of G , since A is an independent set in $G \circ H$. Let $A_G^* = A_G \cap D'_G$ and let $D_G = D'_G - A_G^*$. We will see that every vertex $g \in A_G^*$ is adjacent to some vertex $g' \in D_G$, which is an isolated vertex in the subgraph of G induced by D_G . If this does not hold, then we obtain a contradiction with D being a $\gamma_{t,coi}(G \circ H)$ -set because $D - \{(g, h)\}$ would be a TC-ID set of $G \circ H$, where $(g, h) \in A$. Now, notice that $|D \cap H^g| \geq 1$ for every $g \in A_G^*$, and $|D \cap H^g| = n_H$ for every $g \in D_G$. Therefore, we have at least $|D_G|n_H + |A_G^*|$ vertices in D . As D_G, A_G and A_G^* have the desired properties for every $\gamma_{t,coi}(G \circ H)$ -set D , we have $\gamma_{t,coi}(G \circ H) \geq \min_{A_G \in \mathcal{I}(G)} \{|D_G|n_H + |A_G^*|\}$. Combining both inequalities yields the stated equality. \square

Notice that $\mathcal{I}(G)$ contains all maximal independent sets, which are not necessarily $\alpha(G)$ -sets. We are not aware of any example where a maximal independent set (that is not an $\alpha(G)$ -set) would yield a better result in Theorem 2.4 than an $\alpha(G)$ -set.

The *strong product* of the graphs G and H is the graph $G \boxtimes H$, with vertex set $V(G \boxtimes H) = V(G) \times V(H)$, and two vertices (g, h) and (g', h') are adjacent in $G \boxtimes H$ if and only if, either

- $g = g'$ and $hh' \in E(H)$ or
- $h = h'$ and $gg' \in E(G)$ or
- $gg' \in E(G)$ and $hh' \in E(H)$.

Notice that $G \boxtimes K_t \cong G \circ K_t$. Also, $G \boxtimes H$ has isolated vertices whenever both G and H have isolated vertices. By H^- we denote the graph obtained from H by deleting all isolated vertices. The upper bound $\gamma_t(G \boxtimes H) \leq \gamma_t(G)\gamma_t(H)$ follows from a result in [20]. The independence number $\alpha(G \boxtimes H)$ is the basis for the so-called Shannon capacity; see [14, Chapter 27.1].

Theorem 2.5. *Let G and H be graphs of order n_G and n_H , respectively. If G has no isolated vertices and H has i_H isolated vertices, then*

$$\gamma_{t,coi}(G \boxtimes H) = n_G(n_H - i_H) - \alpha(G \boxtimes H^-) + i_H \gamma_{t,coi}(G).$$

Proof. First notice that $G \boxtimes H \cong G \boxtimes H^- \cup i_H G$. We need $\gamma_{t,coi}(G)$ vertices in a $\gamma_{t,coi}(G \boxtimes H)$ -set D for every component of $G \boxtimes H$ that is isomorphic to G . Hence, D contains $i_H \gamma_{t,coi}(G)$ vertices outside of $G \boxtimes H^-$.

Let A be an $\alpha(G \boxtimes H^-)$ -set and let $D^- = V(G \boxtimes H^-) - A$. Since A is a maximum independent set, D^- is a dominating set of $G \boxtimes H^-$. Next we show that D^- is also a total dominating set of $G \boxtimes H^-$. Let $(g, h) \in D^-$. Notice that there exists a vertex $g' \in V(G)$ such that $gg' \in E(G)$ and there exists a vertex $h' \in V(H^-)$ such that $hh' \in E(H^-)$. In $G \boxtimes H^-$, the subgraph induced by $S = \{(g, h), (g, h'), (g', h), (g', h')\}$ is a complete graph K_4 , so $|S \cap A| \leq 1$, and consequently $|S \cap D^-| \geq 3$, implying that $|N((g, h)) \cap D^-| \geq 2$. Hence, D^- is a total dominating set of $G \boxtimes H^-$, and therefore a TC-ID set of $G \boxtimes H^-$ as well. Consequently, $\gamma_{t,coi}(G \boxtimes H^-) \leq |D^-| = n_G(n_H - i_H) - \alpha(G \boxtimes H^-)$.

By Theorem 1.2, we get the equality $\gamma_{t,coi}(G \boxtimes H^-) = n_G(n_H - i_H) - \alpha(G \boxtimes H^-)$. Together with all the components of $G \boxtimes H$ isomorphic to G , we obtain the desired result. \square

3. THE DIRECT PRODUCT OF GRAPHS

The *direct product* of two graphs G and H is the graph $G \times H$, with vertex set $V(G \times H) = \{(g, h) : g \in V(G), h \in V(H)\}$, and two vertices (g, h) and (g', h') are adjacent in $G \times H$ if and only if $gg' \in E(G)$ and $hh' \in E(H)$. The direct product can be considered as a subgraph of the strong product. It has the special property that every edge from $G \times H$ projects to edges in both factors G and H , and it is therefore often considered as the most natural graph product. On the other hand, this brings several problems. Even connectivity is not trivial among direct products. Indeed, $G \times H$ is a connected graph if and only if both G and H are connected and at least one of G and H is non bipartite. Moreover, if both G and H are bipartite, then $G \times H$ has exactly two components (see [25] and also [14]). In addition, and important in our case, $G \times H$ has isolated vertices if and only if at least one of G and H has isolated vertices. G - and H -layers are defined as in the lexicographic product. However, one must note that any layer induces a graph without edges. This already implies the lower bound $\alpha(G \times H) \geq \max\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}$. A remarkable upper bound for $\alpha(G \times H)$ can be found in [24]. Some bounds and exact results on $\gamma_t(G \times H)$ can be found in [9, 10, 20, 21].

It is easy to see that $K_{p,q} \times K_{r,s} \cong K_{pr,qs} \cup K_{ps,qr}$. Since $K_{pr,qs} \cong \overline{K}_{pr} \vee \overline{K}_{qs}$ and $K_{ps,qr} \cong \overline{K}_{ps} \vee \overline{K}_{qr}$, we can use Theorem 2.2 to obtain

$$\gamma_{t,coi}(K_{p,q} \times K_{r,s}) = \min\{pr, qs\} + \min\{ps, qr\} + 2. \tag{3.1}$$

We next present an upper bound for $\gamma_{t,coi}(G \times H)$ from the independent set perspective. Let G be a graph without isolated vertices. Recall that $\mathcal{I}(G)$ is the set of all maximal independent sets of G , that $D_G = V(G) - A_G$ is a dominating

set of G for any $A_G \in \mathcal{I}(G)$, and that A_G^* is a minimum subset of A_G such that A_G^* dominates the set (denoted by D_G^*) of all isolated vertices of $G[D_G]$.

Theorem 3.1. *For any two graphs G and H without isolated vertices of order n_G and n_H , respectively,*

$$\gamma_{t,coi}(G \times H) \leq \min_{\substack{A_G \in \mathcal{I}(G), \\ A_H \in \mathcal{I}(H)}} \{|D_G|n_H + |A_G^*|\gamma_t(H), |D_H|n_G + |A_H^*|\gamma_t(G)\}.$$

Proof. Let A_G be any maximal independent set of a graph G and let T_H be a $\gamma_t(H)$ -set. We will show that $S = (D_G \times V(H)) \cup (A_G^* \times T_H)$ is a TC-ID set of $G \times H$. The set $V(G \times H) - S$ is independent because $p_G(V(G \times H) - S) \subseteq A_G$ is independent.

Next we show that S is a total dominating set of $G \times H$. Let (u, v) be an arbitrary vertex of $G \times H$. If $u \in A_G$, then there exists $g \in D_G$ which is adjacent to u in G because D_G dominates G . There also exists $h \in V(H)$ which is adjacent to v in H because H has no isolated vertices and $(g, h) \in S$ is adjacent to (u, v) . If $u \in D_G - D_G^*$, then u is not an isolated vertex from $G[D_G]$; let $g \in D(G)$ be a neighbor of u in D_G . Again there exists a neighbor h of v in H , and $(g, h) \in S$ is adjacent to (u, v) . So, let $u \in D_G^*$ be an isolated vertex in $G[D_G]$. By the definition of A_G^* there exists a neighbor $g \in A_G^*$ of u . Also, since T_H is a total dominating set of H , there exists a neighbor $h \in T_H$ of v . Hence (u, v) has a neighbor (g, h) in S and S is a total dominating set of $G \times H$. Therefore, S is a TC-ID set.

By symmetric arguments we can show that $S' = (V(G) \times D_H) \cup (T_G \times A_H^*)$ is a TC-ID set of $G \times H$ for an arbitrary maximal independent set A_H of H and a $\gamma_t(G)$ -set T_G . Since the sets A_G, A_H, T_G and T_H were arbitrarily chosen, the upper bound follows. □

Again we are not aware of any example where a maximal independent set (that is not an $\alpha(G)$ -set) would yield a better result in Theorem 3.1 than an $\alpha(G)$ -set.

The bound of Theorem 3.1 performs rather well. We show this by the results that follow until the end of this section. However, we do not obtain equality in all cases. The smallest example is probably $C_5 \times K_2 \cong C_{10}$. We have $\gamma_{t,coi}(C_{10}) = 7$, but the bound from Theorem 3.1 gives 8. Moreover, this is the smallest member of the family $C_{6k-1} \times K_2 \cong C_{12k-2}$, for an arbitrary positive integer k , for which $\gamma_{t,coi}(C_{12k-2}) = 8k - 1$, but the upper bound from Theorem 3.1 gives $8k$. Another example is $C_5 \times C_5$, where it is not hard to check that

$$V(C_5 \times C_5) - \{(g_0, h_0), (g_0, h_1), (g_0, h_2), (g_0, h_3), (g_0, h_4), (g_2, h_0), (g_2, h_4), (g_3, h_2)\}$$

is a TC-ID set of $C_5 \times C_5$, by taking $g_0g_1g_2g_3g_4g_0$ as the first cycle C_5 and $h_0h_1h_2h_3h_4h_0$ as the second C_5 . With this we have $\gamma_{t,coi}(C_5 \times C_5) \leq 17$, but Theorem 3.1 gives 18.

We can use the example $K_{p,q} \times K_{r,s}$ to show that the bound of Theorem 3.1 is sharp. By some straightforward computations, one obtains from Theorem 3.1 that

$$\gamma_{t,coi}(K_{p,q} \times K_{r,s}) \leq \min\{(r + s) \min\{p, q\} + 2, (p + q) \min\{r, s\} + 2\}.$$

With an additional analysis of the relationship \geq between p and q , as well as between r and s , we obtain the same result as in (3.1).

We next study the total co-independent domination number of several examples of direct products. They all show that the bound of Theorem 3.1 is tight.

Proposition 3.2. *For any integer numbers $r \geq t \geq 2$, $\gamma_{t,coi}(K_r \times K_t) = r(t - 1)$.*

Proof. On the one hand, from Theorem 1.2 we have that $\gamma_{t,coi}(K_r \times K_t) \geq rt - \alpha(K_r \times K_t) = rt - \max\{r, t\} = r(t - 1)$. On the other hand, by Theorem 3.1 we obtain $\gamma_{t,coi}(K_r \times K_t) \leq \min\{(r - 1)t, (t - 1)r\} = r(t - 1)$, which gives the equality. \square

Proposition 3.3. *Let $n, r, t \geq 2$ be integers. If $r \geq t$, then $\gamma_{t,coi}(K_{r,t} \times K_n) = nt + 2$.*

Proof. If $n = 2$, then $K_{r,t} \times K_n$ is isomorphic to $K_{r,t} \times K_{1,1}$. Thus, we have $\gamma_{t,coi}(K_{r,t} \times K_{1,1}) = \min\{r, t\} + \min\{r, t\} + 2 = 2t + 2$ by (3.1). Hence, from now on we may assume $n \geq 3$.

Let V_r and V_t be the bipartite sets of $K_{r,t}$ of cardinality r and t , respectively. Let S be a $\gamma_{t,coi}(K_{r,t} \times K_n)$ -set and let $S_r = S \cap (V_r \times V(K_n))$ and $S_t = S \cap (V_t \times V(K_n))$. Suppose first that $V_t \times V(K_n) = S_t$. The set S_t is independent since V_t is independent, and therefore $S_r \neq \emptyset$ because S is a total dominating set. If $S_r = \{(g, h)\}$, then the vertices from $V_t \times \{h\}$ have no neighbor in S , a contradiction. Hence S_r contains at least two vertices. This means that we have $\gamma_{t,coi}(K_{r,t} \times K_n) \geq nt + 2$ in this case. We can ignore the symmetric condition $V_r \times V(K_n) \subseteq S_r$ because we obtain at least $nr + 2 \geq nt + 2$ vertices in $|S|$, a contradiction with S being a $\gamma_{t,coi}(K_{r,t} \times K_n)$ -set when $r > t$.

Now we may assume that $V_t \times V(K_n) \neq S_t$ and $V_r \times V(K_n) \neq S_r$. Since any two vertices $(g, h), (g', h')$ such that $g \in V_r, g' \in V_t$ and $h \neq h'$ are adjacent, it must happen, for the complement of S to be independent, that $V_r \times (V(K_n) - \{h\}) \subseteq S_r$ and $V_t \times (V(K_n) - \{h\}) \subseteq S_t$ for some $h \in V(K_n)$. Consequently, $|S| = |S_r| + |S_t| \geq (r + t)(n - 1) = rn + tn - r - t$. Since $r \geq t \geq 2$ and $n \geq 3$, it happens that $rn - r - t \geq rn - 2r = r(n - 2) \geq r \geq 2$. Thus, again we have $\gamma_{t,coi}(K_{r,t} \times K_n) = |S| \geq tn + 2$.

The other inequality $\gamma_{t,coi}(K_{r,t} \times K_n) \leq \min\{tn + 2, (n - 1)(r + t)\} = tn + 2$ (since $n \geq 3$) follows from Theorem 3.1 and the proof is complete. \square

Proposition 3.4. *For any integers r, t with $r, t \geq 4$,*

$$\gamma_{t,coi}(C_r \times K_t) = \begin{cases} \frac{r}{2}(t + 1), & \text{if } r \equiv 0 \pmod{4}; \\ \frac{(r - 1)}{2}(t + 1) + t, & \text{if } r \equiv 1 \pmod{4}; \\ \frac{r}{2}(t + 1) + 1, & \text{if } r \equiv 2 \pmod{4}; \\ \frac{r + 1}{2}(t + 1) - 2, & \text{if } r \equiv 3 \pmod{4}. \end{cases}$$

Proof. Let $C_r = g_0g_1 \dots g_{r-1}g_0$ (from now on, operations with the subindexes of such vertices are done modulo r) and let $V(K_t) = \{h_1, \dots, h_t\}$. Let S be a $\gamma_{t,coi}(C_r \times K_t)$ -set and let $A = V(C_r \times K_t) - S$. Note that if a vertex (g_i, h_j) does not belong to S for some $h_j \in V(K_t)$, then all the vertices $(g_{i-1}, h_k), (g_{i+1}, h_\ell)$ for every $h_k, h_\ell \in V(K_t) - \{h_j\}$ must belong to S (otherwise A is not independent).

Suppose first that there exists an edge between two vertices of $p_{C_r}(A)$, that is, for instance, $(g_i, h_j), (g_{i-1}, h_j) \in A$. By the reasons above, $(K_t^{g_{i+1}} - \{(g_{i+1}, h_j)\}) \subseteq S$, $(K_t^{g_i} - \{(g_i, h_j)\}) \subseteq S$, $(K_t^{g_{i-1}} - \{(g_{i-1}, h_j)\}) \subseteq S$ and also $(K_t^{g_{i-2}} - \{(g_{i-2}, h_j)\}) \subseteq S$ for A to be independent. In such a case, there exist at most four vertices outside of S in four consecutive layers $K_t^{g_{i-2}}, K_t^{g_{i-1}}, K_t^{g_i}$ and $K_t^{g_{i+1}}$. Since $t \geq 4$, we will see that this does not give a minimum cardinality to S and that it is better to have a whole layer $K_t^{g_{i+1}}$ outside of S .

If there are no edges between vertices of $p_{C_r}(A)$, then the whole layer $K_t^{g_i}$ can be in A . In such a case, it must happen that $K_t^{g_{i-1}}, K_t^{g_{i+1}} \subseteq S$. Moreover, $K_t^{g_{i-2}} \cap S \neq \emptyset$ and $K_t^{g_{i+2}} \cap S \neq \emptyset$, because some vertices from $K_t^{g_{i-2}}$ and $K_t^{g_{i+2}}$ must dominate vertices from $K_t^{g_{i-1}}$ and $K_t^{g_{i+1}}$, respectively. Furthermore, to dominate all vertices from $K_t^{g_{i-1}}$ and $K_t^{g_{i+1}}$ we need at least $\gamma_t(K_t) = 2$ vertices from $K_t^{g_{i-2}}$ and $K_t^{g_{i+2}}$, respectively. Also, if $K_t^{g_{i-2}}$ or $K_t^{g_{i+2}}$ are not subsets of S , then $K_t^{g_{i-3}}$ and $K_t^{g_{i+3}}$ must be in S because $\gamma_t(K_t) = 2$. We conclude that among any four consecutive K_t -layers $K_t^{g_{i-3}}, K_t^{g_{i-2}}, K_t^{g_{i-1}}$ and $K_t^{g_i}$ at least two of them are completely contained in S and in addition one of the other two contains at least $\gamma_t(K_t) = 2$ vertices in S . Since this is better than in the previous paragraph, we can ignore that option.

This means that for every $i \in \{0, \dots, r-1\}$, $|S \cap (\{g_{i-3}, g_{i-2}, g_{i-1}, g_i\} \times V(K_t))| \geq 2t + 2$. Therefore,

$$\gamma_{t,coi}(C_r \times K_t) = |S| = \frac{1}{4} \sum_{i=0}^{r-1} |S \cap (\{g_{i-3}, g_{i-2}, g_{i-1}, g_i\} \times V(K_t))| \geq \frac{r(t+1)}{2}. \quad (3.2)$$

Notice that in the above argument we can have also $K_t^{g_{i-2}}, K_t^{g_{i+2}} \subseteq S$ and $K_t^{g_{i-3}} \cap S = \emptyset$ and $K_t^{g_{i+3}} \cap S = \emptyset$. This also yields $K_t^{g_{i-4}}, K_t^{g_{i+4}} \subseteq S$ for S to be co-independent. In this case we have only $2t$ in $S \cap \{g_{i-3}, g_{i-2}, g_{i-1}, g_i\}$, but on the other hand at least $3t$ vertices in $S \cap \{g_{i-4}, g_{i-3}, g_{i-2}, g_{i-1}\}$. Because $t \geq 4$ this approach gives more vertices in S and we can ignore it. Next we proceed by analyzing four different situations.

Case 1: $r \equiv 0 \pmod{4}$. Clearly, $A_{C_r} = \{g_{2i-1} : i \in [r/2]\}$ is in $\mathcal{I}(C_r)$ and, in this case, $A_{C_r}^* = \{g_{4i-1} : i \in [r/4]\}$ is one option. By Theorem 3.1, we obtain $\gamma_{t,coi}(C_r \times K_t) \leq \min\{\frac{rt}{2} + \frac{2r}{4}, (t-1)r + 0\} = \frac{r}{2}(t+1)$, where the last equality occurs since $t \geq 3$. The other inequality comes from (3.2), and we have the desired equality in this case.

Case 2: $r \equiv 1 \pmod{4}$. Now, $A_{C_r} = \{g_{2i-1} : i \in [(r-1)/2]\}$ is in $\mathcal{I}(C_r)$ and, in this case, $A_{C_r}^* = \{g_{4i-1} : i \in [(r-1)/4]\}$. Clearly, $|D_{C_r}| = \frac{r+1}{2}$ and $|A_{C_r}^*| = \frac{r-1}{4}$. For these sets, Theorem 3.1 leads to $\gamma_{t,coi}(C_r \times K_t) \leq \min\{\frac{(r+1)t}{2} + \frac{2(r-1)}{4}, (t-1)r + 0\} = \frac{(r-1)(t+1)}{2} + t$, where the last equality occurs since $t \geq 4$ and $r \geq 3$.

For the lower bound, notice that the comments before (3.2) allow us to claim that there exist at least four consecutive K_t -layers, all four of them having nonempty intersection with S , because $r \equiv 1 \pmod{4}$. Furthermore, among these four layers, at least three of them are completely contained in S . One of them is treated separately in the sum of (3.2), and we get

$$\begin{aligned} \gamma_{t,coi}(C_r \times K_t) &= |S| = \frac{1}{4} \sum_{i=0}^{r-1} |S \cap (\{g_{i-3}, g_{i-2}, g_{i-1}, g_i\} \times V(K_t))| \\ &\geq \frac{(r-1)(2t+2)}{4} + t = \frac{(r-1)(t+1)}{2} + t. \end{aligned}$$

Case 3: $r \equiv 2 \pmod{4}$. The set $A_{C_r} = \{g_{2i-1} : i \in [r/2]\}$ is in $\mathcal{I}(C_r)$ and we can choose $A_{C_r}^* = \{g_{4i-3} : i \in [(r+2)/4]\}$. For the set A_{C_r} we have $|D_{C_r}| = \frac{r}{2}$ and $|A_{C_r}^*| = \frac{r+2}{4}$. Now, by Theorem 3.1, we obtain $\gamma_{t,coi}(C_r \times K_t) \leq \min\{\frac{rt}{2} + \frac{2(r+2)}{4}, (t-1)r+0\} = \frac{r}{2}(t+1) + 1$, where the last equality is due to the fact that $t \geq 4$ and $r \geq 3$.

For the lower bound, we use a similar argument as in Case 2. Now, there exist at least five consecutive K_t -layers, all of them with nonempty intersection with S , because $r \equiv 2 \pmod{4}$. Clearly, at least three of them are completely contained in S . (Notice that one layer with empty intersection with S is possible among them, however this yields that the remaining four are completely contained in S , which is not optimal as $t \geq 4$.) We deal with two of them separately in the sum of (3.2), and obtain

$$\begin{aligned} \gamma_{t,coi}(C_r \times K_t) &= |S| = \frac{1}{4} \sum_{i=0}^{r-1} |S \cap (\{g_{i-3}, g_{i-2}, g_{i-1}, g_i\} \times V(K_t))| \\ &\geq \frac{(r-2)(2t+2)}{4} + t + 2 = \frac{r(t+1)}{2} + 1. \end{aligned}$$

Case 4: $r \equiv 3 \pmod{4}$. The set $A_{C_r} = \{g_{2i-1} : i \in [(r-1)/2]\}$ is independent and $A_{C_r}^* = \{g_{4i-1} : i \in [(r-3)/4]\}$. With this, $|D_{C_r}| = \frac{r+1}{2}$ and also $|A_{C_r}^*| = \frac{r-3}{4}$. Again by Theorem 3.1, we get $\gamma_{t,coi}(C_r \times K_t) \leq \min\{\frac{(r+1)t}{2} + \frac{2(r-3)}{4}, (t-1)r+0\} = \frac{r+1}{2}(t+1) - 2$, where the last equality holds because $t \geq 4$ and $r \geq 3$.

For the lower bound, there exist at least two consecutive K_t -layers completely contained in S , because $r \equiv 3 \pmod{4}$. We consider them and one neighboring layer separately in the sum of (3.2) and get

$$\begin{aligned} \gamma_{t,coi}(C_r \times K_t) &= |S| = \frac{1}{4} \sum_{i=0}^{r-1} |S \cap (\{g_{i-3}, g_{i-2}, g_{i-1}, g_i\} \times V(K_t))| \\ &\geq \frac{(r-3)(2t+2)}{4} + 2t = \frac{(r+1)(t+1)}{2} - 2. \quad \square \end{aligned}$$

The next proposition is stated without a proof because one can use a similar approach as in the proof of Proposition 3.4.

Proposition 3.5. *For any integer numbers r, t with $r \geq 7$ and $t \geq 3$,*

$$\gamma_{t,coi}(P_r \times K_t) = \begin{cases} \frac{r}{2}(t+1), & \text{if } r \equiv 0 \pmod{4}; \\ \frac{r-1}{2}(t+1), & \text{if } r \equiv 1 \pmod{4}; \\ \frac{r-2}{2}(t+1) + t, & \text{if } r \equiv 2 \pmod{4}; \\ \frac{r-3}{2}(t+1) + t + 2, & \text{if } r \equiv 3 \pmod{4}. \end{cases}$$

4. THE ROOTED AND CORONA PRODUCTS OF GRAPHS

Given a graph G of order n and a graph H with root vertex v , the *rooted product* $G \circ_v H$ is defined as the graph obtained from G and H by taking one copy of G and n copies of H , and identifying the i th vertex of G with the vertex v in the i th copy of H for every $i \in [n]$. If H or G is isomorphic to K_1 , then $G \circ_v H$ is equal to G or H , respectively. In this sense, to obtain the rooted product $G \circ_v H$, hereafter we only consider graphs G and H of order at least two. For every $x \in V(G)$, H_x will denote the copy of H in $G \circ_v H$ containing x . A formula for the independence number of rooted product graphs can be found in [17].

We need to introduce the following definitions. A *near total co-independent dominating set*, abbreviated near-TC-ID set, of a graph G , relative to a vertex v , is a set $D \subseteq V(G)$ satisfying the following:

- (i) $v \in D$;
- (ii) $V(G) - D$ is an independent set;
- (iii) every vertex $u \in D - \{v\}$ is adjacent to at least one vertex in D .

The minimum cardinality among all near-TC-ID sets of G relative to v is called the *near total co-independent domination number of G relative to v* , which we denote as $\gamma_{nt,coi}(G; v)$. A near-TC-ID set of G relative to v with cardinality $\gamma_{nt,coi}(G; v)$ is called a $\gamma_{nt,coi}(G; v)$ -set. Notice that every TC-ID set of G that contains v is a near-TC-ID set of G relative to v .

Next, we present a useful result.

Theorem 4.1. *Let G be a graph of order at least three without isolated vertices. If $v \in V(G)$, then*

$$\gamma_{t,coi}(G) - 1 \leq \gamma_{nt,coi}(G; v) \leq \gamma_{t,coi}(G) + 1.$$

Furthermore,

- (a) *if there is a $\gamma_{t,coi}(G)$ -set that contains v , then $\gamma_{t,coi}(G) - 1 \leq \gamma_{nt,coi}(G; v) \leq \gamma_{t,coi}(G)$;*
- (b) *if v does not belong to any $\gamma_{t,coi}(G)$ -set, then $\gamma_{t,coi}(G) \leq \gamma_{nt,coi}(G; v) \leq \gamma_{t,coi}(G) + 1$.*

Proof. Let D be a $\gamma_{nt,coi}(G; v)$ -set. Suppose there exists a $\gamma_{t,coi}(G)$ -set S that contains v . We have $\gamma_{nt,coi}(G; v) \leq \gamma_{t,coi}(G)$ because $v \in S$. If D is also a TC-ID set of G , then $\gamma_{t,coi}(G) \leq |D| = \gamma_{nt,coi}(G; v)$, which implies that $\gamma_{nt,coi}(G; v) = \gamma_{t,coi}(G)$. If D is not a TC-ID set of G , then $N(v) \cap D = \emptyset$. Let $u \in N(v)$ and

notice that $D' = D \cup \{u\}$ is a TC-ID set of G . So, $\gamma_{t,coi}(G) \leq |D'| = |D| + 1 = \gamma_{nt,coi}(G; v) + 1$, which completes the proof of (a).

To prove (b), we assume that v does not belong to any $\gamma_{t,coi}(G)$ -set. Let S be a $\gamma_{t,coi}(G)$ -set. As $v \notin S$, the set $S' = S \cup \{v\}$ is a near-TC-ID set of G relative to v . So $\gamma_{nt,coi}(G; v) \leq |S'| = |S| + 1 = \gamma_{t,coi}(G) + 1$.

Now, suppose that $\gamma_{nt,coi}(G; v) < \gamma_{t,coi}(G)$. Recall that D is a $\gamma_{nt,coi}(G; v)$ -set and let $w \in N(v)$. Notice that $D'' = D \cup \{w\}$ is a TC-ID set of G , so that $\gamma_{t,coi}(G) \leq |D''| = |D| + 1 = \gamma_{nt,coi}(G; v) + 1$. Thus, we obtain $\gamma_{nt,coi}(G; v) = \gamma_{t,coi}(G) - 1$, and hence D'' is a $\gamma_{t,coi}(G)$ -set containing v , which is a contradiction. Therefore, $\gamma_{nt,coi}(G; v) \geq \gamma_{t,coi}(G)$ and the proof of (b) is complete. By (a) and (b), the inequality chain follows. \square

Lemma 4.2. *If D is a $\gamma_{t,coi}(G \circ_v H)$ -set, then*

$$\gamma_{t,coi}(G \circ_v H) \geq \sum_{x \in V(G) \cap D} \gamma_{nt,coi}(H; x) + \sum_{x \in V(G) - D} \gamma_{t,coi}(H).$$

Proof. Let $x \in V(G)$. If $x \in D$, then $V(H_x) \cap D$ is a near-TC-ID set of H_x relative to x , which implies that $|V(H_x) \cap D| \geq \gamma_{nt,coi}(H; x)$. If $x \notin D$, then $V(H_x) \cap D$ is a TC-ID set of H_x because D is a TC-ID set of $G \circ_v H$. So $|V(H_x) \cap D| \geq \gamma_{t,coi}(H)$. Therefore,

$$\begin{aligned} \gamma_{t,coi}(G \circ_v H) &= \sum_{x \in V(G) \cap D} |V(H_x) \cap D| + \sum_{x \in V(G) - D} |V(H_x) \cap D| \\ &\geq \sum_{x \in V(G) \cap D} \gamma_{nt,coi}(H; x) + \sum_{x \in V(G) - D} \gamma_{t,coi}(H). \end{aligned} \quad \square$$

Theorem 4.3. *Let G be a graph of order $n \geq 2$ without isolated vertices. For any graph H of order at least three with root v and without isolated vertices,*

$$\gamma_{t,coi}(G \circ_v H) \in \{n(\gamma_{t,coi}(H) - 1), n\gamma_{t,coi}(H), n(\gamma_{t,coi}(H) + 1) - \alpha(G)\}.$$

Furthermore,

- (i) if $\gamma_{nt,coi}(H; v) = \gamma_{t,coi}(H) - 1$, then $\gamma_{t,coi}(G \circ_v H) = n(\gamma_{t,coi}(H) - 1)$;
- (ii) if $\gamma_{nt,coi}(H; v) = \gamma_{t,coi}(H)$, then $\gamma_{t,coi}(G \circ_v H) = n\gamma_{t,coi}(H)$;
- (iii) if $\gamma_{nt,coi}(H; v) = \gamma_{t,coi}(H) + 1$, then $\gamma_{t,coi}(G \circ_v H) = n(\gamma_{t,coi}(H) + 1) - \alpha(G)$.

Proof. Let D be a $\gamma_{t,coi}(G \circ_v H)$ -set. We consider three cases.

Case 1: $\gamma_{nt,coi}(H; v) = \gamma_{t,coi}(H) - 1$. Let S_H be a $\gamma_{nt,coi}(H; v)$ -set (which contains v). We obtain S by taking a copy of S_H in every copy of H . Such a set is a TC-ID set of $G \circ_v H$ of cardinality $n\gamma_{nt,coi}(H; v) = n(\gamma_{t,coi}(H) - 1)$. Hence, $\gamma_{t,coi}(G \circ_v H) \leq n(\gamma_{t,coi}(H) - 1)$. By Lemma 4.2 we obtain

$$\begin{aligned} \gamma_{t,coi}(G \circ_v H) &\geq \sum_{x \in V(G) \cap D} \gamma_{nt,coi}(H; x) + \sum_{x \in V(G) - D} \gamma_{t,coi}(H) \\ &\geq \sum_{x \in V(G) \cap D} (\gamma_{t,coi}(H) - 1) + \sum_{x \in V(G) - D} \gamma_{t,coi}(H) \\ &= |V(G) \cap D|(\gamma_{t,coi}(H) - 1) + |V(G) - D|\gamma_{t,coi}(H). \end{aligned}$$

Thus, we have

$$n(\gamma_{t,coi}(H) - 1) \geq |V(G) \cap D|(\gamma_{t,coi}(H) - 1) + |V(G) - D|\gamma_{t,coi}(H)$$

and $-n \geq -|V(G) \cap D|$ follows. This holds only when $V(G) \cap D = V(G)$. Hence, $\gamma_{t,coi}(G \circ_v H) = n(\gamma_{t,coi}(H) - 1)$, and (i) follows.

Case 2: $\gamma_{nt,coi}(H; v) = \gamma_{t,coi}(H)$. We construct, as in Case 1, a TC-ID set S of $G \circ_v H$ of cardinality $n\gamma_{nt,coi}(H; v) = n\gamma_{t,coi}(H)$. So $\gamma_{t,coi}(G \circ_v H) \leq n\gamma_{t,coi}(H)$. Again by Lemma 4.2 we get

$$\begin{aligned} \gamma_{t,coi}(G \circ_v H) &\geq \sum_{x \in V(G) \cap D} \gamma_{nt,coi}(H; x) + \sum_{x \in V(G) - D} \gamma_{t,coi}(H) \\ &= \sum_{x \in V(G)} \gamma_{t,coi}(H) + \sum_{x \in V(G) - D} \gamma_{t,coi}(H) \\ &= n\gamma_{t,coi}(H). \end{aligned}$$

Hence, $\gamma_{t,coi}(G \circ_v H) = n\gamma_{t,coi}(H)$, and (ii) follows.

Case 3: $\gamma_{nt,coi}(H; v) = \gamma_{t,coi}(H) + 1$. By Theorem 4.1 we know that v does not belong to any $\gamma_{t,coi}(H)$ -set. Let S_H be a $\gamma_{t,coi}(H)$ -set and let B be a $\beta(G)$ -set, which has cardinality $n - \alpha(G)$ by Theorem 1.1. We obtain the set S by union of B and a copy of S_H in every copy of H . Since $V(G) - B$ is independent, S is a TC-ID set of $G \circ_v H$ of cardinality $n\gamma_{t,coi}(H) + (n - \alpha(G)) = n(\gamma_{t,coi}(H) + 1) - \alpha(G)$. So $\gamma_{t,coi}(G \circ_v H) \leq n(\gamma_{t,coi}(H) + 1) - \alpha(G)$.

Observe also that $V(G) - D$ is an independent set, and so $|V(G) \cap D| \geq n - \alpha(G)$. Hence, by Lemma 4.2 we have

$$\begin{aligned} \gamma_{t,coi}(G \circ_v H) &\geq \sum_{x \in V(G) \cap D} \gamma_{nt,coi}(H; x) + \sum_{x \in V(G) - D} \gamma_{t,coi}(H) \\ &= \sum_{x \in V(G) \cap D} (\gamma_{t,coi}(H) + 1) + \sum_{x \in V(G) - D} \gamma_{t,coi}(H) \\ &= n\gamma_{t,coi}(H) + |V(G) \cap D| \\ &\geq n\gamma_{t,coi}(H) + (n - \alpha(G)) \\ &= n(\gamma_{t,coi}(H) + 1) - \alpha(G). \end{aligned}$$

Hence, $\gamma_{t,coi}(G \circ_v H) = n(\gamma_{t,coi}(H) + 1) - \alpha(G)$, and (iii) follows.

Therefore, $\gamma_{t,coi}(G \circ_v H) \in \{n(\gamma_{t,coi}(H) - 1), n\gamma_{t,coi}(H), n(\gamma_{t,coi}(H) + 1) - \alpha(G)\}$, which completes the proof. \square

A particular case of the total co-independent domination number of rooted product graphs $G \circ_v H$, specifically when H contains a universal vertex, is presented below.

Theorem 4.4. *Let G be a graph of order $n_G \geq 2$ with no isolated vertex and let H be a graph of order $n_H \geq 3$. If H has a universal vertex v , then*

$$\gamma_{t,coi}(G \circ_v H) = \begin{cases} n_G, & H \cong K_{1,t}; \\ n_G(n_H - \alpha(H)), & H \not\cong K_{1,t}. \end{cases}$$

Proof. If H is a star graph with universal vertex v , then it is straightforward to see that $\gamma_{t,coi}(G \circ_v H) = n_G$. From now on, we assume that H is a graph different from a star. So $\alpha(H) \leq n_H - 2$. As v is a universal vertex of H , there exists a $\gamma_{t,coi}(H)$ -set S containing v and by Lemma 1.3, $\gamma_{t,coi}(H) = |S| = n_H - \alpha(H)$. Also, we notice that every $\gamma_{nt,coi}(H; v)$ -set is a TC-ID set of H , and so $\gamma_{t,coi}(H) \leq \gamma_{nt,coi}(H; v)$. By Theorem 4.1 (a), it follows that $\gamma_{nt,coi}(H; v) = \gamma_{t,coi}(H)$ and consequently $\gamma_{t,coi}(G \circ_v H) = n_G \gamma_{t,coi}(H)$ by Theorem 4.3 (ii). \square

Let G and H be two graphs of order n_G and n_H , respectively. The *corona product* graph $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G and n_G copies of H and joining by an edge every vertex from the i th copy of H with the i th vertex of G . Notice that any corona graph $G \odot H$ can be presented as a rooted product graph $G \circ_v H'$, where $H' \cong K_1 \vee H$ and v is the vertex of K_1 . Also, observe that $\alpha(H') = \alpha(H)$. Hence, Lemma 1.3 and Theorem 4.4 lead to the following result.

Theorem 4.5. *If G is a graph of order $n_G \geq 2$ with no isolated vertex, then for every graph H of order $n_H \geq 2$,*

$$\gamma_{t,coi}(G \odot H) = \begin{cases} n_G, & H \cong \overline{K}_t; \\ n_G(n_H - \alpha(H) + 1), & H \not\cong \overline{K}_t. \end{cases}$$

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