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THE WEAKLY ZERO-DIVISOR GRAPH OF A COMMUTATIVE RING

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Abstract. Let R be a commutative ring with identity, and let $Z(R)$ be the set of zero-divisors of R. The weakly zero-divisor graph of R is the undirected (simple) graph $W\Gamma(R)$ with vertex set $Z(R)^*$, and two distinct vertices x and y are adjacent if and only if there exist $r \in \operatorname{ann}(x)$ and $s \in \operatorname{ann}(y)$ such that $rs = 0$. It follows that $W\Gamma(R)$ contains the zero-divisor graph $\Gamma(R)$ as a subgraph. In this paper, the connectedness, diameter, and girth of $W\Gamma(R)$ are investigated. Moreover, we determine all rings whose weakly zero-divisor graphs are star. We also give conditions under which weakly zero-divisor and zero-divisor graphs are identical. Finally, the chromatic number of $W\Gamma(R)$ is studied.

1. Introduction

The theory of graphs associated with rings was started by Beck [9] in 1981 and has grown a lot since then. Anderson and Livingston [2] modified Beck's definition and introduced the notion of zero-divisor graph. Surely, this is the most important graph associated with a ring, and not only zero-divisor graphs but also various generalizations of it have attracted many researchers; see for instance [1, 7, 13, 8, 5, 4, 10, 16, 17]. Therefore, this paper is devoted to introducing and studying a super graph of zero-divisor graphs. First let us recall some necessary notation and terminology from ring theory and graph theory.

Throughout this paper, all rings are assumed to be commutative with identity and they are not fields. We denote by $\operatorname{Min}(R)$ and $\operatorname{Nil}(R)$ the set of all minimal prime ideals of R and the set of all nilpotent elements of R, respectively. For a subset A of a ring R, we let $A^* = A \setminus \{0\}$. For every subset I of R, we denote the *annihilator* of I by $\operatorname{ann}_R(I)$. The ring R is called *local* if it has a unique maximal ideal. Also, the ring R is said to be *reduced* if it has no non-zero nilpotent element. For any undefined notation or terminology in ring theory, we refer the reader to [6].

Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $\operatorname{diam}(G)$ and $\operatorname{girth}(G)$ we mean the diameter and the girth of G, respectively. We write $u - v$ to denote an edge with ends u, v. A graph

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$H = (V_0, E_0)$ is called a *subgraph of* G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph by* V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \mid u, v \in V_0\}$. Let G_1 and G_2 be two disjoint graphs. The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with the vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$. Also G is called a *null graph* if it has no edge. A complete bipartite graph with part sizes m, n is denoted by $K_{m,n}$. If $m = 1$, then the complete bipartite graph is called *star graph*. Also, a complete graph of n vertices is denoted by K_n. A *clique* of G is a maximal complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G. For a graph G, let $\chi(G)$ denote the *vertex chromatic number* of G, i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. For any undefined notation or terminology in graph theory, we refer the reader to [18].

The *zero-divisor graph* of a ring R, denoted by $\Gamma(R)$, is a graph with the vertex set $Z(R)^*$, and two distinct vertices x and y are adjacent if and only if $xy = 0$. The *weakly zero-divisor graph* of R is defined as the graph $W\Gamma(R)$ with the vertex set $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if there exist $r \in \mathrm{ann}(x)$ and $s \in \mathrm{ann}(y)$ such that $rs = 0$. In this paper, we study some connections between the graph-theoretic properties of $W\Gamma(R)$ and some algebraic properties of rings. Moreover, we investigate the affinity between weakly zero-divisor graph and zero-divisor graph associated with a ring. We focus especially on the conditions under which these two graphs are identical. Finally, the coloring of weakly zero-divisor graphs is studied.

2. Basic properties of weakly zero-divisor graphs

In this section, we study fundamental properties of $W\Gamma(R)$. It is shown that $W\Gamma(R)$ is always a connected graph and $\mathrm{diam}(W\Gamma(R)) \leq 2$. Moreover, we prove that if $W\Gamma(R)$ contains a cycle, then $\mathrm{girth}(W\Gamma(R)) \leq 4$. We begin with a lemma containing several useful properties of $W\Gamma(R)$.

Lemma 2.1. *Let R be a ring. Then the following statements hold:*

1. *If $x - y$ is an edge of $\Gamma(R)$, for some distinct elements $x, y \in Z(R)^*$, then $x - y$ is an edge of $W\Gamma(R)$.*
2. *If $x \in \mathrm{Nil}(R)^*$, then x is adjacent to all other vertices.*
3. *$\mathrm{Nil}(R)^*$ is a complete subgraph of $W\Gamma(R)$.*

Proof. (1) Suppose that $x - y$ is an edge of $\Gamma(R)$, for some distinct elements $x, y \in Z(R)^*$. Thus $xy = 0$ and clearly $x \in \mathrm{ann}(y)$ and $y \in \mathrm{ann}(x)$. Hence $x - y$ is an edge of $W\Gamma(R)$.

(2) Assume that $x \in \mathrm{Nil}(R)^*$, for some $x \in Z(R)^*$, and let $y \in V(W\Gamma(R))$ and $r \in \mathrm{ann}(y)$. Since $x \in \mathrm{Nil}(R)^*$, we deduce that there exists a positive integer $n \in \mathbb{N}$ such that $x^n = 0$ and $x^i \neq 0$, for all $1 \leq i \leq n - 1$. It is clear that $x^{n-1} \in \mathrm{ann}(x)$. If $x^{n-1}r = 0$, then $x - y$ is an edge of $W\Gamma(R)$. If $x^{n-1}r \neq 0$, then $x^{n-1}r \in \mathrm{ann}(x) \cap \mathrm{ann}(y)$ and $x^{n-1}rx^{n-1}r = 0$. Thus $x - y$ is an edge of $W\Gamma(R)$.

(3) It is clear, by part (2). □

By using Lemma 2.1, we give upper bounds for $\operatorname{diam}(W\Gamma(R))$ and $\operatorname{girth}(W\Gamma(R))$ (if $W\Gamma(R)$ contains a cycle).

Theorem 2.2. *Let R be a ring. Then $W\Gamma(R)$ is connected and* $\operatorname{diam}(W\Gamma(R)) \leq 2$. *Moreover, if $W\Gamma(R)$ contains a cycle, then* $\operatorname{girth}(W\Gamma(R)) \leq 4$.

Proof. By Lemma 2.1, every edge (path) of $\Gamma(R)$ is an edge (path) of $W\Gamma(R)$. Hence [2, Theorem 2.3] implies that $W\Gamma(R)$ is connected. Moreover, it follows from [15, p. 3541] that $\operatorname{girth}(W\Gamma(R)) \leq 4$. To complete the proof, we show that $\operatorname{diam}(W\Gamma(R)) \leq 2$. Suppose that $x - y$ is not an edge of $W\Gamma(R)$, for some distinct elements $x, y \in Z(R)^*$. Then $rs \neq 0$, for every $r \in \operatorname{ann}(x)$ and $s \in \operatorname{ann}(y)$. Since $rsx = 0$ and $rsy = 0$, we find the path $x - rs - y$ is in $W\Gamma(R)$. This completes the proof. □

The next result shows that $\operatorname{girth}(W\Gamma(R)) = 4$ may occur.

Theorem 2.3. *Let R be a ring and let $W\Gamma(R)$ contain a cycle. Then* $\operatorname{girth}(W\Gamma(R)) = 4$ *if and only if R is reduced with* $|\operatorname{Min}(R)| = 2$.

Proof. First suppose that $\operatorname{girth}(W\Gamma(R)) = 4$. If $\operatorname{Nil}(R) \neq (0)$, then by part (2) of Lemma 2.1, $\operatorname{girth}(W\Gamma(R)) = 3$, a contradiction. Hence $\operatorname{Nil}(R) = (0)$. We claim that $W\Gamma(R) = \Gamma(R)$. Assume, to the contrary, that $W\Gamma(R) \neq \Gamma(R)$. Then there are distinct elements $x, y \in Z(R)^*$ such that $x - y$ is an edge of $W\Gamma(R)$ which is not an edge of $\Gamma(R)$. Hence there are $r \in \operatorname{ann}(x)$ and $s \in \operatorname{ann}(y)$ such that $rs = 0$, $r \neq s$, $x \neq r \neq y$, and $y \neq s \neq x$.

We consider the following cases.

Case 1. $0 \neq b \in \operatorname{ann}(x) \cap \operatorname{ann}(y)$. Thus $b - x - y - b$ is a cycle in $W\Gamma(R)$ of length three. Hence $\operatorname{girth}(W\Gamma(R)) = 3$, a contradiction.

Case 2. $\operatorname{ann}(x) \cap \operatorname{ann}(y) = 0$. Then it is not hard to check that y, xy, x are pairwise distinct. Since $r \in \operatorname{ann}(x) \subseteq \operatorname{ann}(xy)$ and $rs = 0$, we deduce that $xy - y$ is an edge of $W\Gamma(R)$. Also $xy - x$ is an edge of $W\Gamma(R)$, as $s \in \operatorname{ann}(y) \subseteq \operatorname{ann}(xy)$ and $rs = 0$. Therefore, $xy - x - y - xy$ is a cycle in $W\Gamma(R)$ of length three, a contradiction, and so the claim is proved. This fact, together with $\operatorname{girth}(W\Gamma(R)) = 4$ and the fact that R is reduced, implies that $|\operatorname{Min}(R)| = 2$, by [3, Theorem 2.2]. Conversely, suppose that R is reduced and $\operatorname{Min}(R) = \{P_1, P_2\}$. Since R is reduced, $Z(R) = P_1 \cup P_2$ and $P_1 \cap P_2 = (0)$, by [12, Corollary 2.4]. It is enough to show that P_1, P_2 are independent sets of $W\Gamma(R)$. Let $x, y \in P_1$, $0 \neq r \in \operatorname{ann}(x)$, and $0 \neq s \in \operatorname{ann}(y)$. Then $r, s \in P_2$, as $P_1 \cap P_2 = 0$. If $rs = 0$, then either $r = 0$ or $s = 0$, a contradiction. Similarly, P_2 is independent. Then $W\Gamma(R) = K_{|P_1^*|,|P_2^*|}$. By hypothesis $W\Gamma(R)$ contains a cycle and so $\operatorname{girth}(\Gamma(R)) = 4$. □

The next result provides conditions under which $W\Gamma(R)$ contains a triangle.

Theorem 2.4. *Let R be a reduced ring and assume that $Z(R)^*$ is an ideal of R. Then $W\Gamma(R) \neq \Gamma(R)$ and* $\operatorname{girth}(W\Gamma(R)) = 3$.

Proof. Let $a \in Z(R)^*$ and $b \in \text{ann}(a) \setminus \{0\}$. Then $a + b \in Z(R)^*$, as $Z(R)$ is an ideal. Since $a(b+a) \neq 0$, we deduce that $a - a + b$ is not an edge of $\Gamma(R)$. A simple check yields $\text{ann}(a + b) \subseteq \text{ann}((b + a)a) = \text{ann}(a^2)$, and so $\text{ann}(a + b) \subseteq \text{ann}(a^2)$. Then there exists $m \in R$ such that $m \in \text{ann}(a+b)$ and $m \in \text{ann}(a^2)$. Thus $ma = 0$, since R is reduced. Hence $mb = 0$. This fact, together with $m \in \text{ann}(a + b)$ and $b \in \text{ann}(a)$, implies that $a+b-a$ is an edge of $W\Gamma(R)$. Since $a+b-a$ is an edge of $W\Gamma(R)$ that is not an edge of $\Gamma(R)$, we conclude that $W\Gamma(R) \neq \Gamma(R)$. To complete the proof, we show that $\text{girth}(\Gamma(R)) = 3$. We claim that $a + b \neq (a + b)a \neq a$. If $(a + b)a = a$, then $a^2 = a$ and so R is decomposable. Hence $Z(R)$ is not an ideal, a contradiction. Thus $(a + b)a \neq a$. Also if $a + b = (a + b)a$, then $a + b = a^2$ and $a^2 \neq a$. These imply that $a^2 = (a + b)a = a^2 \cdot a = a^3$ and so a^2 is idempotent. Again we get a contradiction. By the above assumptions, $m \in \text{ann}(a+b) \subseteq \text{ann}(a)$, $b \in \text{ann}(a) = \text{ann}(a^2) = \text{ann}((a+b)a)$, and $mb = 0$. Thus $a+b-a-(a+b)a-a+b$ is a triangle in $W\Gamma(R)$, as desired. □

In the following theorem we classify all rings with star weakly zero-divisor graphs.

Theorem 2.5. *Let R be a ring. Then $W\Gamma(R)$ is a star graph if and only if one of the following statements holds:*

(1) *$R \cong \mathbb{Z}_2 \times D$, where D is an integral domain.*
(2) $|\text{Nil}(R)| = |Z(R)| = 3$.

Proof. One side is clear. To prove the other side, suppose that $W\Gamma(R)$ is a star graph. By Lemma 2.1 (3), $|\text{Nil}(R)| \leq 3$. We consider the following cases.

Case 1. $|\text{Nil}(R)| = 1$ (i.e., R is reduced). Suppose that $a \in V(W\Gamma(R))$ is adjacent to all the other vertices. We claim that a is idempotent. For, if not, $\text{ann}(a) = \text{ann}(a^2)$, as R is reduced. This implies that a and a^2 are adjacent to all the other vertices. Then $Z(R) = \{0, a, a^2\}$, since $W\Gamma(R)$ is star. But it is clear that $a^2 \neq 0$, $a \cdot a^2 \neq 0$, and $(a^2)^2 \neq 0$ (since R is reduced), a contradiction, and so the claim is proved. Therefore, $R \cong R_1 \times R_2$, where R_1, R_2 are two rings. We show that $R_1 \cong \mathbb{Z}_2$ and $R_2 \cong D$, where D is an integral domain. If $R_1 \cong \mathbb{Z}_2$ and $R_2 \cong \mathbb{Z}_2$, then there is nothing to prove. Without loss of generality, suppose that $|R_2^*| \geq 2$ (i.e., $1 \neq b \in R_2^*$). For any $1 \neq r \in R_1$, $(r, 0)$ is a zero-divisor and so $(r, 0) - (0, 1) - (1, 0) - (0, b) - (r, 0)$ is a cycle in $W\Gamma(R)$, a contradiction unless $r = 0$. Hence, $R_1 \cong \mathbb{Z}_2$. If $x \in Z(R_2)^*$ and $a \in \text{ann}(x)$, then it is easily seen that the induced subgraph on the vertices $(1, 0)$, $(0, x)$, and $(0, a)$ forms a triangle in $W\Gamma(R)$, a contradiction. Thus $Z(R_2) = (0)$ and so $R \cong \mathbb{Z}_2 \times D$, where D is an integral domain.

Case 2. We show that $|\text{Nil}(R)| = 2$ does not happen. Suppose, to the contrary, that $|\text{Nil}(R)| = 2$. Since $\Gamma(R)$ is a star graph, [2, Theorem 2.5] implies that $\text{ann}(x) = Z(R)$, for some $x \in Z(R)^*$. We show that $W\Gamma(R)$ is complete. Suppose that z and y are two vertices of $W\Gamma(R)$ such that $y \neq x \neq z$. Since $x \in \text{ann}(y) \cap \text{ann}(z)$ and $x^2 = 0$, $y - z$ is an edge of $W\Gamma(R)$, i.e., $W\Gamma(R)$ is complete. This fact, together with $W\Gamma(R)$ being a star graph, implies that $W\Gamma(R) \cong K_2$. So

$Z(R) = \{0, x, b\}$. This yields $b^2 = b$ and hence $R \cong Rb \times R(1-b)$, i.e., $Z(R)$ is not an ideal, a contradiction. Therefore $|\operatorname{Nil}(R)| \neq 2$.

Case 3. $|\operatorname{Nil}(R)| = 3$. By Lemma 2.1 (2), we conclude that $W\Gamma(R) \cong K_2$ and so $|\operatorname{Nil}(R)| = |Z(R)| = 3$. □

The last result of this section is devoted to studying complete weakly zero divisor graphs. First, we fix a notation. Let $R \cong R_1 \times R_2 \times \cdots \times R_m$, where every R_i is a ring, for $1 \leq i \leq m$. By e_i we mean the i-th standard basis vector, for every $i = 1, \ldots, m$. Indeed, $e_i = (0, \ldots, 0, 1_{R_i}, 0, \ldots, 0)$.

Theorem 2.6. *Let R be an Artinian ring. Then $W\Gamma(R)$ is a complete graph if and only if one of the following statements holds:*

(1) $R \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.
(2) $R \cong R_1 \times \cdots \times R_m$*, where R_i is a non-reduced Artinian local ring, for every* $1 \leq i \leq m$.

Proof. First suppose that $W\Gamma(R)$ is a complete graph. By [6, Theorem 8.7], $R \cong R_1 \times \cdots \times R_m$, where R_i is an Artinian local ring, for every $1 \leq i \leq m$. If every R_i, $1 \leq i \leq m$, is non-reduced, then there is nothing to prove. So suppose that at least one of the R_i's is a field, say R_1 (obviously, every reduced local Artinian ring is a field). Consider the following two cases.

Case 1. $R_i \cong \mathbb{Z}_2$, for every $i \neq 1$. We show that $R \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. Suppose, to the contrary, that $R_1 \ncong \mathbb{Z}_2$. Let $1 \neq u \in R_1^*$. Then $x = (u, 1, \ldots, 1, 0)$, $y = (1, 1, \ldots, 1, 0) \in V(W\Gamma(R))$ and $\operatorname{ann}(x) = \operatorname{ann}(y) = (0, \ldots, 0, 1)$. Therefore, x, y are not adjacent, a contradiction.

Case 2. $R_i \ncong \mathbb{Z}_2$, for some $i \neq 1$. We show that this case does not occur. Without loss of generality, suppose that $R_m \ncong \mathbb{Z}_2$. Let $x = (0, 1, \ldots, 1, u)$, $y = (0, 1, \ldots, 1, 1) \in V(W\Gamma(R))$, and $1 \neq u \in R_m \setminus Z(R)$. Then $\operatorname{ann}(x) = \operatorname{ann}(y) = \{(r, 0, \ldots, 0) \mid r \in R^*{}_1\}$. This implies that x is not adjacent to y, a contradiction.

To prove the other side, first suppose that $R \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. One may easily check that $V(W\Gamma(R)) = \{(x_1, \ldots, x_m) \in R \mid x_i = 0 \text{ for some } 1 \leq i \leq m\}$. We show that $W\Gamma(R)$ is complete. To see this, suppose that $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ are two distinct arbitrary elements of $V(W\Gamma(R))$. Then there exist $1 \leq i, j \leq m$ such that $i \neq j$, $x_i = 0$, and $x_j = 0$. Since $e_i \in \operatorname{ann}(X)$, $e_j \in \operatorname{ann}(Y)$, and $e_i e_j = 0$, we conclude that x is adjacent to y, as desired.

Now suppose that $R \cong R_1 \times \cdots \times R_m$, where R_i is an non-reduced Artinian local ring, for every $1 \leq i \leq m$. We put $A = \{(x_1, \ldots, x_m) \in R \mid x_i \in \operatorname{Nil}(R_i)^* \text{ for some } 1 \leq i \leq m\}$ and $B = \{(x_1, \ldots, x_m) \in R \mid x_i \notin \operatorname{Nil}(R_i)^*$ for all $1 \leq i \leq m$ and $x_i = 0$ for some $1 \leq i \leq m\}$. One may easily check that $V(W\Gamma(R)) = A \cup B$, $A \cap B = \varnothing$. We show that $W\Gamma(R)$ is a complete graph. To see this, consider the following cases.

Case 1. Let $X_1 = (x_1, \ldots, x_m)$ and $X_2 = (x'_1, \ldots, x'_m)$ be two distinct elements of A. Then $x_i \in \operatorname{Nil}(R_i)^*$ for some $1 \leq i \leq m$ and $x'_j \in \operatorname{Nil}(R_j)^*$ for some $1 \leq j \leq m$, and hence there exist two positive integers n, m such that $x_i^n = 0$,

$x_i^{n-1} \neq 0$ and ${x'_j}^m = 0$, ${x'_j}^{m-1} \neq 0$ (fix i and j). We have the following two subcases.

Subcase A. If $i \neq j$, then $(x_i^{n-1} \cdot e_i)({x'_j}^{m-1} \cdot e_j) = 0$. Since $(x_i^{n-1} \cdot e_i) \in \operatorname{ann}(X_1)$ and $({x'_j}^{m-1} \cdot e_j) \in \operatorname{ann}(X_2)$, X_1 is adjacent to X_2.

Subcase B. If $i = j$, then either $x_i^{n-1} \cdot {x'_i}^{m-1} = 0$ or $x_i^{n-1} \cdot {x'_i}^{m-1} \neq 0$. If $x_i^{n-1} \cdot {x'_i}^{m-1} = 0$, then $(x_i^{n-1} \cdot e_i) \cdot ({x'_i}^{m-1} \cdot e_i) = 0$. Hence X_1 is adjacent to X_2. If $x_i^{n-1} \cdot {x'_i}^{m-1} \neq 0$, then $r = x_i^{n-1} \cdot {x'_i}^{m-1} \cdot e_i \in \operatorname{ann}(x_1) \cap \operatorname{ann}(x_2)$. Since $r^2 = 0$, X_1 is adjacent to X_2.

Case 2. Let $Y_1 = (y_1, \ldots, y_m)$ and $Y_2 = (y'_1, \ldots, y'_m)$ be two distinct elements of B. We can suppose that the i-th component of Y_1 is zero, for some $1 \leq i \leq m$, and also that the j-th component of Y_2 is zero, for some $1 \leq j \leq m$. We consider the following two subcases.

Subcase A. Let $i \neq j$. Since $e_i \in \operatorname{ann}(Y_1)$, $e_j \in \operatorname{ann}(y_2)$, and $e_i e_j = 0$, we conclude that Y_1 is adjacent to Y_2.

Subcase B. Let $i = j$. Since R_i is non-reduced, for every $1 \leq i \leq m$, there exists a non-zero nilpotent element r_i in $\operatorname{Nil}(R_j)^*$ such that $r_i^n = 0$ and $r_i^{n-1} \neq 0$, where n is a positive integer. It is clear that $r_i^{n-1} \cdot e_i \in \operatorname{ann}(Y_1)$, $r_i \cdot e_i \in \operatorname{ann}(Y_2)$, and $(r_i^{n-1} \cdot e_i) \cdot (r_i \cdot e_i) = 0$. This implies that Y_1 is adjacent to Y_2.

Case 3. Let $X_1 \in A$ and $Y_1 \in B$. Then we have the following two subcases.

Subcase A. If $i \neq j$, then $x_i^{n-1} e_i \cdot e_j = 0$. Since $x_i^{n-1} e_i \in \operatorname{ann}(x_1)$ and $e_j \in \operatorname{ann}(Y_1)$, we conclude that X_1 is adjacent to Y_1.

Subcase B. If $i = j$, then $r = x_i^{n-1} \cdot e_i \in \operatorname{ann}(x_1)$ and $s = x_i \cdot e_i \in \operatorname{ann}(y_1)$. Hence X_1 is adjacent to Y_1, since $rs = 0$.

Therefore $W\Gamma(R)$ is a complete graph. □

3. When is $W\Gamma(R)$ identical to $\Gamma(R)$?

As we have seen in Section 2, $\Gamma(R)$ is a subgraph of $W\Gamma(R)$. A natural question is posed: When are $W\Gamma(R)$ and $\Gamma(R)$ identical? In this section, we completely answer this question.

Theorem 3.1. *Let R be a reduced ring that is not an integral domain. Then $W\Gamma(R) = \Gamma(R)$ if and only if $|\operatorname{Min}(R)| = 2$.*

Proof. Suppose that $W\Gamma(R) = \Gamma(R)$. If $|\operatorname{Min}(R)| \geq 3$, then by [14, Theorem 2.6], $\operatorname{diam}(\Gamma(R)) = 3$. This contradicts Theorem 2.2. Hence $|\operatorname{Min}(R)| = 2$, as $|\operatorname{Min}(R)| = 1$ means that R is an integral domain. Conversely, suppose that P_1 and P_2 are two distinct minimal prime ideals of R. It is not hard to check that $W\Gamma(R) = \Gamma(R) = K_{|P_1^*|,|P_2^*|}$. □

Next, we study non-reduced rings R whose weakly zero-divisor graphs and zero-divisor graphs are identical.

Theorem 3.2. *Let R be a non-reduced ring. Then the following statements are equivalent:*

(1) $W\Gamma(R) = \Gamma(R)$.

(2) $Z(R)^2 = 0$.
(3) $\Gamma(R)$ *is a complete graph.*

Proof. (1) $\Longrightarrow$ (2). Let $x \in \mathrm{Nil}(R)^*$. Then by part (2) of Lemma 2.1, x is adjacent to all the other vertices in $W\Gamma(R)$. This fact, together with $W\Gamma(R) = \Gamma(R)$, implies that $\mathrm{ann}(x) = Z(R)$, by [2, Theorem 2.5]. Thus $W\Gamma(R)$ is a complete graph, and so is $\Gamma(R)$. Hence by [2, Theorem 2.8], the result holds.

(2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1) are clear. □

Theorem 3.2 leads to the following corollary.

Corollary 3.3. *Let R be a non-reduced ring. Then the following statements are equivalent:*

(1) $W\Gamma(R)$ *is a star graph.*
(2) $\mathrm{girth}(W\Gamma(R)) = \infty$.
(3) $W\Gamma(R) = \Gamma(R)$ *and* $\mathrm{girth}(\Gamma(R)) = \infty$.
(4) $|Z(R)^*| = |\mathrm{Nil}(R)^*| = 2$.
(5) $W\Gamma(R) = \Gamma(R) = K_{1,1}$.

Proof. (1) $\Longrightarrow$ (2). It is clear.

(2) $\Longrightarrow$ (3). If $a \in \mathrm{Nil}(R)^*$, then a is adjacent to all the other vertices in $W\Gamma(R)$. Since $\mathrm{girth}(W\Gamma(R)) = \infty$ and $\Gamma(R)$ is a connected subgraph of $W\Gamma(R)$, we conclude that $W\Gamma(R) = \Gamma(R)$, and so $\mathrm{girth}(\Gamma(R)) = \infty$.

(3) $\Longrightarrow$ (4). If $W\Gamma(R) = \Gamma(R)$, then $W\Gamma(R) = \Gamma(R)$ is a complete graph, by Theorem 3.2. Since $\mathrm{girth}(\Gamma(R)) = \infty$ and R is non-reduced, we have that $|Z(R)^*| = |\mathrm{Nil}(R)^*| = 2$.

(4) $\Longrightarrow$ (5) and (5) $\Longrightarrow$ (1) are clear. □

4. Coloring of $W\Gamma(R)$

In this section, we study the coloring of $W\Gamma(R)$. First, we state the following lemma.

Lemma 4.1. *Let $R \cong D_1 \times D_2 \times \cdots \times D_n$, where $n \geq 3$ is a positive integer and D_i is an integral domain, for every $1 \leq i \leq n$. Then $W\Gamma(R) = K_m \bigvee H_n$, where H_n is a complete n-partite graph and K_m is a complete graph.*

Proof. Let $A = \{X = (x_1, \ldots, x_n) \in R \mid \text{only one of } x_i\text{'s is zero}\}$ and $B = \{X = (x_1, \ldots, x_n) \in R \mid \text{at least two of the } x_i\text{'s are zero}\}$. It is clear that $V(W\Gamma(R)) = A \cup B$. Suppose that $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ are elements of A, where $x_i, y_i \in D_i$, for every $1 \leq i \leq n$. Define the relation $\sim$ on A as follows: $X \sim Y$ whenever $x_i = 0$ if and only if $y_i = 0$, for every $1 \leq i \leq n$. It is easily seen that $\sim$ is an equivalence relation on A. By $[X_i]$, we mean the equivalence class of X_i, where $X_i = (1, 1, \ldots, 1, 0, 1, \ldots, 1)$ such that only the i-th component is zero, for every $1 \leq i \leq n$. It is clear that $A = \bigcup_{i=1}^{n} [X_i]$. We claim that $W\Gamma(R)[A]$ is a complete n-partite subgraph of $W\Gamma(R)$. First we show that there is no adjacency between elements of $[X_i]$, for every $1 \leq i \leq n$. To see this, suppose that $X = (x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_n)$

are two distinct arbitrary elements of $[X_i]$. Then we have $\operatorname{ann}(X) = \operatorname{ann}(Y) = \{(0,\dots,0,a_i,0,\dots,0) \mid a_i \in D_i\}$. This implies that there are no elements r, s of $\operatorname{ann}(X) = \operatorname{ann}(Y)$ such that $rs = 0$, and so X is not adjacent to Y. Now, suppose that $[X_i]$ and $[X_j]$ are two distinct arbitrary equivalence classes of A. We show that each element of $[X_i]$ is adjacent to each element of $[X_j]$. Let $X = (x_1, x_2, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$ be an element of $[X_i]$ and $Y = (y_1, y_2, \dots, y_{j-1}, 0, y_{j+1}, \dots, y_n)$ be an element of $[X_j]$. Then $e_i \in \operatorname{ann}(X)$ and $e_j \in \operatorname{ann}(Y)$, where e_i, e_j, are the ith and jth standard basis vectors. Since $e_ie_j = 0$, we conclude that X is adjacent to Y. Therefore $W\Gamma(R)[A] = H_n$, where H_n is a complete n-partite graph. In what follows, we show that $W\Gamma(R)[B] = K_m$, where $m = |B|$. Let $X = (x_1, x_2, \dots, x_{k-1}, 0, x_{k+1}, \dots, x_{l-1}, 0, x_{l+1}, \dots, x_n) \in B$ and $Y = (y_1, y_2, \dots, y_{i-1}, 0, y_{i+1}, \dots, y_{j-1}, 0, y_j, \dots, y_n) \in B$. Then either $k \neq i$ or $k \neq j$. With no loss of generality, assume that $i \neq k$. Then $e_k \in \operatorname{ann}(X)$, $e_i \in \operatorname{ann}(Y)$, and $e_ke_i = 0$. Hence X is adjacent to Y and thus $W\Gamma(R)[B] = K_m$. To complete the proof, we show that every vertex contained in B is adjacent to every vertex contained in A. Let $X = (x_1, x_2, \dots, x_{k-1}, 0, x_{k+1}, \dots, x_{l-1}, 0, x_{l+1}, \dots, x_n) \in B$ and $Y = (y_1, y_2, \dots, y_{i-1}, 0, y_{i+1}, \dots, y_n) \in [x_i] \subset A$. Then $i \neq k$ or $i \neq l$. With no loss of generality, assume that $i \neq k$. Since $e_k \in \operatorname{ann}(X)$, $e_i \in \operatorname{ann}(Y)$, and $e_ke_i = 0$, we conclude that X is adjacent to Y. Therefore $W\Gamma(R) = K_m \bigvee H_n$. □

To state our main result in this section, we need to fix a notation.

Notation. Let $R \cong F_1 \times \cdots \times F_k \times R_1 \times \cdots \times R_n$, where F_i is a field for every $1 \leq i \leq k$ and R_j is a non-field Artinian local ring, for every $1 \leq j \leq n$. Set $A = \bigcup_{i=1}^{k} A_i$, where $A_i = \{(x_1, \dots, x_k, y_1, \dots, y_n) \mid x_i = 0$ for exactly one $1 \leq i \leq k$, and y_j is a unit of R_j for all $1 \leq j \leq n\}$. Moreover, put $M = |Z(R)^*| - |A|$.

Theorem 4.2. *Let $R \cong F_1 \times \cdots \times F_k \times R_1 \times \cdots \times R_n$, where F_i is a field for every $1 \leq i \leq k$ and R_j is an Artinian local ring with $|\operatorname{Nil}(R_j)^*| \neq 0$ for every $1 \leq j \leq n$. Then $\omega(W\Gamma(R)) = \chi(W\Gamma(R)) = M + k$.*

Proof. We put $A = \bigcup_{i=1}^{k} A_i$, where

$$A_i = \{(x_1, \dots, x_k, y_1, \dots, y_n) \mid x_i = 0 \text{ for exactly one } 1 \leq i \leq k, \text{ and } y_j \text{ is a unit of } R_j \text{ for all } 1 \leq j \leq n\}$$

and $B = \bigcup_{i=1}^{3} B_i$, where

$$B_1 = \{(x_1, \dots, x_k, y_1, \dots, y_n) \mid y_j \in \operatorname{Nil}(R_j)^* \text{ for some } 1 \leq j \leq n\},$$

$$B_2 = \{(x_1, \dots, x_k, y_1, \dots, y_n) \mid x_i \neq 0 \text{ for all } 1 \leq i \leq k, y_j \notin \operatorname{Nil}(R_j)^* \text{ for all } 1 \leq j \leq n, \text{ and only one of } y_j\text{'s is zero}\}$$

and

$$B_3 = \{(x_1, \dots, x_k, y_1, \dots, y_n) \mid y_j \notin \operatorname{Nil}(R_j)^* \text{ for all } 1 \leq j \leq n, \text{ and at least two components are zero}\}.$$

One may check that $V(W\Gamma(R)) = A \cup B$, $A \cap B = \varnothing$, and so $\{A, B\}$ is a partition of $V(W\Gamma(R))$. We note that $B_1 \cap B_2 = B_1 \cap B_3 = B_2 \cap B_3 = \varnothing$. First we show that $W\Gamma(R) = W\Gamma(R)[A] \bigvee W\Gamma(R)[B]$. Indeed, we have the following claims:

Claim 1. $W\Gamma(R)[A]$ is a complete K-partite subgraph of $W\Gamma(R)$.

Suppose that A_i and A_j are two distinct arbitrary sets. It is enough to show that there is no adjacency between two vertices of A_i and that every vertex of A_i is adjacent to all the vertices of A_j. To see this, let X_1 and X_2 be two vertices of A_i and Y_1 a vertex of A_j. So $X_1 = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_k, y_1, \ldots, y_n)$, $X_2 = (x'_1, \ldots, x'_{i-1}, 0, x'_{i+1}, \ldots, x'_k, y'_1, \ldots, y'_n)$, and $Y_1 = (x''_1, \ldots, x''_{j-1}, 0, x''_{j+1}, \ldots, x''_k, y''_1, \ldots, y''_n)$, where $i \neq j$. Then $\operatorname{ann}(X_1) = \operatorname{ann}(X_2) = \{(0, \ldots, 0, a_i, 0, \ldots, 0) \mid a_i \in F_i\}$, and so there are no elements r, s of $\operatorname{ann}(X_1) = \operatorname{ann}(X_2)$ such that $rs = 0$. This implies that X_1 and X_2 are not adjacent. Also $e_i \in \operatorname{ann}(X_1)$ and $e_j \in \operatorname{ann}(Y)$. Since $i \neq j$, we obtain $e_i e_j = 0$. Therefore X_1 is adjacent to Y, as desired.

Claim 2. $W\Gamma(R)[B]$ is a complete subgraph of $W\Gamma(R)$.

Suppose that $X = (x_1, \ldots, x_k, y_1, \ldots, y_n)$ and $Y = (x'_1, \ldots, x'_k, y'_1, \ldots, y'_n)$ are two vertices of $W\Gamma(R)[B]$. Then we have the following cases.

Case 1. Let X and Y be two vertices of B_1. Then $y_i \in \operatorname{Nil}(R_i^*)$ for some $1 \leq i \leq n$, and $y'_j \in \operatorname{Nil}(R_j^*)$ for some $1 \leq j \leq n$. Hence there exist two positive integers n, m such that $y_i^n = 0$, $y_i^{n-1} \neq 0$ and ${y'_j}^m = 0$, ${y'_j}^{m-1} \neq 0$. Fix i and j and consider the following two subcases.

Subcase A. If $i = j$, then either $y_i^{n-1}{y'_i}^{m-1} = 0$ or $y_i^{n-1}{y'_i}^{m-1} \neq 0$. If $y_i^{n-1}{y'_i}^{m-1} = 0$, then $(0, \ldots, 0, y_i^{n-1}, 0, \ldots, 0)(0, \ldots, 0, {y'_i}^{m-1}, 0, \ldots, 0) = 0$. Hence X is adjacent to Y, since $(0, \ldots, 0, y_i^{n-1}, 0, \ldots, 0) \in \operatorname{ann}(X)$ and $(0, \ldots, 0, {y'_i}^{m-1}, 0, \ldots, 0) \in \operatorname{ann}(Y)$. If $y_i^{n-1}{y'_i}^{m-1} \neq 0$, then $a = (0, \ldots, 0, y_i^{n-1}{y'_i}^{m-1}, 0, \ldots, 0) \in \operatorname{ann}(X) \cap \operatorname{ann}(Y)$. Hence X is adjacent to Y, since $a^2 = 0$.

Subcase B. If $i \neq j$, then $(0, \ldots, 0, y_i^{n-1}, 0, \ldots, 0)(0, \ldots, 0, {y'_j}^{m-1}, 0, \ldots, 0) = 0$. Hence X is adjacent to Y.

Case 2. Let X and Y be two vertices of B_2. We can suppose that the $(i+k)$-th component of X is zero, for some $1 \leq i \leq n$, and also that the $(j+k)$-th component of Y is zero, for some $1 \leq j \leq n$. We have the following two subcases.

Subcase A. Let $i = j$. Since R_i is non-reduced for every $1 \leq i \leq n$, there exists a non-zero nilpotent element y_i in $\operatorname{Nil}(R_i)^*$ such that $y_i^n = 0$ and $y_i^{n-1} \neq 0$, where n is a positive integer. It is clear that $(0, \ldots, 0, y_i^{n-1}, 0, \ldots, 0) \in \operatorname{ann}(X)$, $(0, \ldots, 0, y_i, 0, \ldots, 0) \in \operatorname{ann}(Y)$, and $(0, \ldots, 0, y_i^{n-1}, 0, \ldots, 0)(0, \ldots, 0, y_i, 0, \ldots, 0) = 0$. This implies that X is adjacent to Y.

Subcase B. Let $i \neq j$. Since $e_{k+i} \in \operatorname{ann}(X)$, $e_{k+j} \in \operatorname{ann}(Y)$, and $e_{k+i}e_{k+j} = 0$, we conclude that X is adjacent to Y.

Case 3. Let X and Y be two vertices of B_3. Since $X \in B_3$, two components of X are zero. We can suppose that the i-th and j-th components are the zero of X, for some $1 \leq i \leq k+n$ and $1 \leq j \leq k+n$. Similarly, since $Y \in B_3$, we can suppose that the l-th and h-th components are the zero of Y, for some $1 \leq l \leq k+n$ and $1 \leq h \leq k+n$. It is clear that either $i \neq l$ or $i \neq h$. Without loss of generality, take $i \neq l$. It is easily seen that $e_i \in \operatorname{ann}(X)$, $e_l \in \operatorname{ann}(Y)$, and $e_i e_l = 0$. Hence X is adjacent to Y.

Case 4. Let X be a vertex of B_1 and Y be a vertex of B_2. Since $X \in B_1$, we have $y_i \in \operatorname{Nil}(R_i^*)$, for some $1 \leq i \leq n$, and there exists a positive integer n such that $y_i^n = 0$, $y_i^{n-1} \neq 0$. Then $(0, \dots, 0, y_i^{n-1}, 0, \dots, 0) \in \operatorname{ann}(X)$. On the other hand, since $Y \in B_2$, for the component y'_j, $1 \leq j \leq n$, we have $y'_j = 0$. We consider the following two subcases.

Subcase A. Let $i = j$. It is clear that $(0, \dots, 0, y_i^{n-1}, 0, \dots, 0) \in \operatorname{ann}(X)$, $(0, \dots, 0, y_i, 0, \dots, 0) \in \operatorname{ann}(Y)$, and $(0, \dots, 0, y_i^{n-1}, 0, \dots, 0)(0, \dots, 0, y_i, 0, \dots, 0) = 0$. This implies that X is adjacent to Y.

Subcase B. Let $i \neq j$. Clearly, $(0, \dots, 0, y_i^{n-1}, 0, \dots, 0) \in \operatorname{ann}(X)$, $e_{k+j} \in \operatorname{ann}(Y)$, and $(0, \dots, 0, y_i^{n-1}, 0, \dots, 0)e_{k+j} = 0$ imply that X is adjacent to Y.

Case 5. Let X be a vertex of B_1 and Y be a vertex of B_3. Since $Y \in B_3$, two components of Y are zero. We can suppose that the i-th and j-th components are the zero of Y, for $1 \leq i \leq k+n$ and $1 \leq j \leq k+n$. So e_j and $e_i \in \operatorname{ann}(Y)$. Also, by an argument similar to that in Case 4, we can suppose that $(0, \dots, 0, y_l^{n-1}, 0, \dots, 0) \in \operatorname{ann}(X)$ such that $y_l^n = 0$ for $1 \leq l \leq n$. Clearly, either $i \neq l$ or $j \neq l$. Without loss of generality, take $i \neq l$. This implies that $e_i(0, \dots, 0, y_l^{n-1}, 0, \dots, 0) = 0$, as desired.

Case 6. Let X be a vertex of B_1 and Y be a vertex of B_3. The proof is similar to that of Case 5. Therefore $W\Gamma(R)[B]$ is a complete subgraph of $W\Gamma(R)$.

Claim 3. Every vertex of $W\Gamma(R)[B]$ is adjacent to every vertex of $W\Gamma(R)[A]$.

Let $X = (x_1, \dots, x_k, y_1, \dots, y_n)$ be a vertex of $W\Gamma(R)[B]$ and $Y = (x'_1, \dots, x'_k, y'_1, \dots, y'_n)$ be a vertex of $W\Gamma(R)[A]$. Then there exists a positive integer m such that $Y \in A_m$, $1 \leq m \leq k$. Since $X \in B = \bigcup_{i=1}^{3} B_i$, either $X \in B_1$, $X \in B_2$, or $X \in B_3$. The following three cases complete the proof.

Case 1. Let $X \in B_1$. This implies that $y_i \in \operatorname{Nil}(R_i)^*$, for some $1 \leq i \leq n$ such that $y_i^n = 0$, $y_i^{n-1} \neq 0$, where n is a positive integer. Now, $(0, \dots, 0, y_i^{n-1}, 0, \dots, 0) \in \operatorname{ann}(X)$ and $e_m \in \operatorname{ann}(Y)$. Thus X is adjacent to Y, since $e_m(0, \dots, 0, y_i^{n-1}, 0, \dots, 0) = 0$.

Case 2. Let $X \in B_2$. Then the $(i+k)$-th component is zero for $1 \leq i \leq n$, and so $e_{i+k} \in \operatorname{ann}(X)$. Since $e_{i+k} e_m = 0$, we conclude that X is adjacent to Y.

Case 3. Let $X \in B_3$. The proof is similar to that of Case 3 in Claim 2. Therefore $W\Gamma(R) = K_M \bigvee H_k$, where $M = |B| = |B_1| + |B_2| + |B_3|$, and so $\omega(W\Gamma(R)) = \chi(W\Gamma(R)) = M + k$. □

In Theorems 4.3 and 4.4, we study weakly zero-divisor graphs with finite chromatic number.

Theorem 4.3. *Let R be a ring that is not an integral domain and suppose that $\chi(W\Gamma(R)) < \infty$. Then the following statements are equivalent.*

(1) $Z(R) = \operatorname{Nil}(R)$.
(2) *R is an Artinian local ring.*

Proof. (1) $\implies$ (2). Let $Z(R) = \mathrm{Nil}(R)$. Then $W\Gamma(R)$ is a complete graph, by Lemma 2.1 (3). Since $\chi(W\Gamma(R)) < \infty$, we have $2 \leq |Z(R)| = |\mathrm{Nil}(R)| < \infty$ and so $|R| < \infty$, by [11, Theorem 1]. This, together with $Z(R) = \mathrm{Nil}(R)$, implies that R is an Artinian local ring.

The converse is trivial. □

Following [11], we know that $Z(R)$ is finite if and only if either R is finite or an integral domain. So, for an Artinian local ring R, if $|\mathrm{Nil}(R)| \neq 1$ then R is finite if and only if $\mathrm{Nil}(R)$ is finite. We use these facts to prove the last result of this paper.

Theorem 4.4. *Let R be an Artinian ring. Then $\omega(W\Gamma(R)) = \chi(W\Gamma(R)) < \infty$ if and only if one of the following statements holds:*

1. *$R \cong F$, where F is a field.*
2. *R is a finite ring.*
3. *$R \cong F_1 \times F_2$, where F_i is a field, for $i = 1, 2$.*

Proof. Suppose that $\chi(W\Gamma(R)) = \omega(W\Gamma(R)) < \infty$. If $\chi(W\Gamma(R)) = \omega(W\Gamma(R)) = 0$, then R is an integer domain and so R is a field. Also, if $0 < \chi(W\Gamma(R)) = \omega(W\Gamma(R)) < \infty$, then we show that either $|R| < \infty$ or $R \cong F_1 \times F_2$. By [6, Theorem 8.7], $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i is an Artinian local ring, for every $1 \leq i \leq n$. We have the following two cases.

Case 1. If at least one of the R_i's is non-reduced, then we claim that $|R_i| < \infty$, for every $1 \leq i \leq n$. Let $\mathrm{Nil}(R_k) \neq 0$ (fixed k). Since $W\Gamma(R)[(0, \dots, \mathrm{Nil}(R_k), 0, \dots, 0)]$ is a complete subgraph of $W\Gamma(R)$ (by Lemma 2.1), $|Z(R_k)| = |\mathrm{Nil}(R_k)| < \infty$. Thus $|R_k| < \infty$. Also, let $A = \{(x_1, \dots, x_n) \mid x_i \in R_i$ with $i \neq k$ and $x_k \in \mathrm{Nil}(R_k)\}$. Then $W\Gamma(R)[A]$ is a complete subgraph of $W\Gamma(R)$, by an argument similar to that used in Case 1 of Claim 2 in Theorem 4.2. Since $\omega(W\Gamma(R)) = \chi(W\Gamma(R)) < \infty$, $|R_i| < \infty$ and so $|R| < \infty$.

Case 2. If R_i is reduced for every $1 \leq i \leq n$, then we have the following two subcases.

Subcase A. Let $n \geq 3$. We show that $|R| < \infty$. It is sufficient to show that $|R_i| < \infty$. Put $B = \{(x_1, \dots, x_n) \mid x_1 = x_2 = 0$ and $x_k \in R_k\}$, $A = \{(x_1, \dots, x_n) \mid x_2 = x_3 = 0$ and $x_k \in R_k\}$, and $C = \{(x_1, \dots, x_n) \mid x_1 = x_3 = 0$ and $x_k \in R_k\}$. Hence $W\Gamma(R)[B]$, $W\Gamma(R)[A]$, and $W\Gamma(R)[C]$ are complete subgraphs of $W\Gamma(R)$, by an argument similar to that used in Case 3 of Claim 2 in Theorem 4.2. Then $|R_i| < \infty$, and hence $|R| < \infty$.

Subcase B. Let $2 \geq n$. Since $0 < \omega(W\Gamma(R)) = \chi(W\Gamma(R))$, $n \neq 1$ and so $R \cong F_1 \times F_2$. □

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