REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 62, No. 2, 2021, Pages 365–383
Published online: November 4, 2021
https://doi.org/10.33044/revuma.1722

SIMPLE, LOCAL AND SUBDIRECTLY IRREDUCIBLE STATE RESIDUATED LATTICES

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Abstract. This paper is devoted to investigating the notions of simple, local and subdirectly irreducible state residuated lattices and some of their related properties. The filters generated by a subset in state residuated lattices are characterized and it is shown that the lattice of filters of a state residuated lattice forms a complete Heyting algebra. Maximal, prime and minimal prime filters of a state residuated lattice are investigated and it is shown that any filter of a state residuated lattice contains a minimal prime filter. Finally, the relevant notions are discussed and characterized.

1. Introduction

The notion of a state is an analogue of probability measure. Such a notion plays a crucial role in the theory of quantum structures which generalizes the Kolmogorov probabilistic space. Forty years after the appearance of MV-algebras [2], states of MV-algebras (MV-states) have been introduced by [17] as maps from any MV-algebra to the real unit interval $[0, 1]$ satisfying a normalization condition and a generalized version of the finite additivity law of classical probability measures. This notion was introduced as an averaging process for formulas in Łukasiewicz logic and also as a special case of a state on a D-poset [14]. Since MV-algebras with state are not universal algebras, they do not automatically induce an assertional logic. The papers [7] and [8] presented an algebraizable logic using a probabilistic approach, and its equivalent algebraic semantics is precisely the variety of state MV-algebras. The concept of a state BL-algebra was introduced by [4], as an extension of the concept of a state MV-algebra. The paper [11] introduced the notion of state operators on residuated lattices and investigated some related properties of such operators. Recently, [13] and [19] defined and studied some properties of generalized state operators on residuated lattices and non-commutative residuated lattices, respectively. The present work is greatly motivated by the the ones above and a desire to extend these investigations to residuated lattices. Our findings

2020 *Mathematics Subject Classification.* 06F99, 06D20.

Key words and phrases. residuated lattice; state residuated lattice; state filter; simple state residuated lattice; local state residuated lattice; subdirectly irreducible state residuated lattice.

show that the results obtained by [3] can also be reproduced and improved via state residuated lattices.

This paper is organized in four sections. In Section 2, some definitions, properties and results relative to residuated lattices and state residuated lattices are recalled. We illustrate them by some examples of residuated lattices which will be used in the following sections of the paper. Also, it is shown that the filter lattice of a state residuated lattice forms a complete Heyting algebra and the set of principal filters of a state residuated lattice $\mathfrak{A}$ is a sublattice of the filter lattice of $\mathfrak{A}$. In Section 3, the notions of maximal, prime and minimal prime filter of a state residuated lattice are investigated and two fundamental theorem are given for prime and minimal prime filters (Theorems 3.8 and 3.11). Furthermore, it is observed that any filter of a state residuated lattice contains a minimal prime filter (Corollary 3.12). In Section 4, the notions of simple, local and subdirectly irreducible state residuated lattices are investigated. It is proved that a good state residuated lattice $\mathfrak{A}_\nu$ is simple if and only if $\mathfrak{A}$ is simple as a residuated lattice (Proposition 4.1). Local state residuated lattices are characterized by means of nilpotent elements (Corollary 4.9) and it is shown that a good and faithful state residuated lattice $\mathfrak{A}_\nu$ is local if and only if $\mathfrak{A}$ is local as a residuated lattice (Corollary 4.11). Finally, subdirectly irreducible state residuated lattices are characterized and it is shown that a good and faithful state residuated lattice $\mathfrak{A}_\nu$ is subdirectly irreducible if and only if $\mathfrak{A}$ is subdirectly irreducible as a residuated lattice (Corollary 4.21).

2. Preliminaries

In this section, we recall some definitions, properties and results relative to residuated lattices and state residuated lattices which will be used in the following. For a survey of residuated lattices we refer to [12] and for a survey of state residuated lattices we refer to [19]. The results in this section are original, unless otherwise stated.

2.1. **Residuated lattices.** An algebra $\mathfrak{A} = (A; \vee, \wedge, \odot, \to, 1)$ is called a *commutative residuated lattice* if $\ell(\mathfrak{A}) = (A; \vee, \wedge)$ is a lattice, $(A; \odot, 1)$ is a commutative monoid and $(\odot, \to)$ is an adjoint pair, i.e., $a \odot b \leq c$ iff $a \leq b \to c$, for all $a, b, c \in A$. A commutative residuated lattice with a constant 0 (which can denote any element) is called a *pointed commutative residuated lattice* or a *commutative full Lambek algebra* (*FL_e algebra*). If 1 is a top element of $\ell(\mathfrak{A})$, then $\mathfrak{A}$ is called a *commutative integral residuated lattice*. An FL_e algebra $\mathfrak{A}$ in which $(A; \vee, \wedge, 0, 1)$ is a bounded lattice is called a *FL_{ew} algebra*. An FL_{ew} algebra is sometimes called a *commutative bounded integral residuated lattice*. In this paper, by a *residuated lattice* we mean an FL_{ew} algebra. A residuated lattice $\mathfrak{A}$ is called *nondegenerate* if $0 \neq 1$. For a residuated lattice $\mathfrak{A}$ and $a \in A$ we put $\neg a := a \to 0$. We denote by $\mathcal{RL}$ the class of residuated lattices. Following the results of [1], we deduce that the class $\mathcal{RL}$ is equational, hence it forms a variety. A residuated lattice $\mathfrak{A}$ is called a divisible residuated lattice if it satisfies the divisibility condition

(div) $x \odot (x \to y) = x \wedge y$.

We denote by $\mathcal{DRL}$ the class of divisible residuated lattices. Obviously, the class $\mathcal{DRL}$ is equational. Hence it forms a subvariety of the variety $\mathcal{RL}$. A residuated lattice $\mathfrak{A}$ in which $x \odot y = x \wedge y$ (or equivalently, $x^2 = x$) for all $x, y \in A$ is called a Heyting algebra or pseudo-Boolean algebra [20]. Obviously, any Heyting algebra is a divisible residuated lattice.

The following remark provides some rules of calculus in a residuated lattice which will be used in this paper.

Remark 2.1 ([12, Proposition 2.2])**.** Let $\mathfrak{A}$ be a residuated lattice. The following conditions are satisfied for any $x, y, z \in A$:

(r_1) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$;

(r_2) $x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$.

Example 2.2. Let $A_6 = \{0, a, b, c, d, 1\}$ be a lattice whose Hasse diagram is given by Figure 1. Routine calculation shows that $\mathfrak{A}_6 = (A_6; \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a residuated lattice where the commutative operation "$\odot$" is given by Table 1 and the operation "$\rightarrow$" is defined by $x \rightarrow y = \bigvee\{a \in A_6 \mid x \odot a \leq y\}$, for any $x, y \in A_6$.

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a		a	a	0	a	a
b			a	0	a	b
c				c	c	c
d					d	d
1						1

TABLE 1

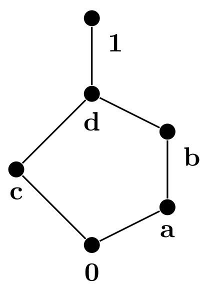

FIGURE 1

Example 2.3. Let $C_6 = \{0, a, b, c, d, 1\}$ be a lattice whose Hasse diagram is given by Figure 2. Routine calculation shows that $\mathfrak{C}_6 = (C_6; \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a residuated lattice where the commutative operation "$\odot$" is given by Table 2 and the operation "$\rightarrow$" is defined by $x \rightarrow y = \bigvee\{a \in C_6 \mid x \odot a \leq y\}$, for any $x, y \in C_6$.

Example 2.4. Let $A_7 = \{0, a, b, c, d, e, 1\}$ be a lattice whose Hasse diagram is given by Figure 3. Routine calculation shows that $\mathfrak{A}_7 = (A_7; \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a residuated lattice where the commutative operation "$\odot$" is given by Table 3 and the operation "$\rightarrow$" is defined by $x \rightarrow y = \bigvee\{a \in A_7 \mid x \odot a \leq y\}$, for any $x, y \in A_7$.

Let $\mathfrak{A}$ be a residuated lattice. A non-void subset F of A is called a filter of $\mathfrak{A}$ if $x, y \in F$ implies $x \odot y \in F$ and $x \vee y \in F$ for any $x \in F$ and $y \in A$. The set of filters of $\mathfrak{A}$ is denoted by $\mathscr{F}(\mathfrak{A})$. A filter F of $\mathfrak{A}$ is called proper if $F \neq A$. Clearly, F is a proper filter if and only if $0 \notin F$. For any subset X of A the filter of $\mathfrak{A}$ generated by X is denoted by $\mathscr{F}(X)$. For each $x \in A$ the filter generated by $\{x\}$ is denoted

⊙	0	a	b	c	d	1
0	0	0	0	0	0	0
a		0	0	0	0	a
b			0	0	0	b
c				0	0	c
d					0	d
1						1

TABLE 2

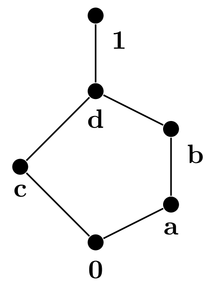

FIGURE 2

⊙	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a		a	a	a	a	a	a
b			b	a	b	a	b
c				a	a	c	c
d					b	c	d
e						e	e
1							1

TABLE 3

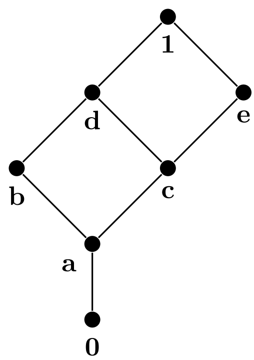

FIGURE 3

by $\mathscr{F}(x)$ and called principal filter. The set of principal filters of $\mathfrak{A}$ is denoted by $\mathscr{P}\mathscr{F}(\mathfrak{A})$. Recall [10, Definition 41] that if $\mathfrak{A}$ is a complete lattice, an element $a \in A$ is called *compact* provided that $a \leq \bigvee X$, for any $X \subseteq A$, implies that $a \leq \bigvee Y$ for some finite $Y \subseteq X$. According to [5], $(\mathscr{F}(\mathfrak{A}); \cap, \veebar, \mathbf{1}, A)$ is a complete lattice whose compact elements are exactly the principal filters of $\mathfrak{A}$.

Remark 2.5. Recall [10, §5.7] that a complete lattice $\mathfrak{A}$ is called a *frame* if it satisfies the join infinite distributive law (JID), i.e., for any $a \in A$ and $S \subseteq A$, $a \wedge \bigvee S = \bigvee\{a \wedge s \mid s \in S\}$. It is well known that a complete lattice is a Heyting algebra if and only if it is a frame. According to [9], for any residuated lattice $\mathfrak{A}$, the complete lattice $\mathscr{F}(\mathfrak{A})$ is a frame. So $(\mathscr{F}(\mathfrak{A}); \cap, \veebar, \hookrightarrow, 0, 1)$ is a Heyting algebra where $F \hookrightarrow G = \veebar\{H \in \mathscr{F}(\mathfrak{A}) \mid F \cap H \subseteq G\}$, for any $F, G \in \mathscr{F}(\mathfrak{A})$.

The following proposition has a routine verification.

Proposition 2.6. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of A. The following assertions hold for any $x, y \in A$ and $X \subseteq A$:*

(1) $\mathscr{F}(F, X) := F \veebar \mathscr{F}(X) = \{a \in A \mid f \odot x_1 \odot \cdots \odot x_n \leq a,\ f \in F,\ x_1, \ldots, x_n \in X,\ n \geq 1\}$;
(2) $\mathscr{F}(F, x) \cap \mathscr{F}(F, y) = \mathscr{F}(F, x \vee y)$;
(3) $x \leq y$ *implies* $\mathscr{F}(F, y) \subseteq \mathscr{F}(F, x)$;
(4) $\mathscr{F}(F, x) \veebar \mathscr{F}(F, y) = \mathscr{F}(F, x \wedge y) = \mathscr{F}(F, x \odot y) = \mathscr{F}(F, x, y)$;
(5) $\mathscr{PF}(\mathfrak{A})$ *is a sublattice of* $\mathscr{F}(\mathfrak{A})$.

Let $\mathfrak{A}$ be a residuated lattice. We define the distance functions $d(a, b) = (a \to b) \odot (b \to a)$, for any $a, b \in A$. With any filter of a residuated lattice $\mathfrak{A}$ we associate a binary relation $\equiv_F$ on A by

$$(a, b) \in \equiv_F \text{ if and only if } d(a, b) \in F.$$

The binary relation $\equiv_F$ is an equivalence relation on A. $\equiv_F$ is called the equivalence relation induced by F. In the following, for any $a \in A$, the equivalence class $a/{\equiv_F}$ is denoted by $[a]_F$.

2.2. **State residuated lattices.**

Definition 2.7 ([19])**.** Let $\mathfrak{A}$ be a residuated lattice. A mapping $\nu : A \to A$ is called a *state operator* on $\mathfrak{A}$ if it satisfies the following assertions:

($\mathfrak{s}_1$) $\nu(0) = 0$;
($\mathfrak{s}_2$) ν is monotone;
($\mathfrak{s}_3$) $\nu(x \to y) = \nu(x) \to \nu(x \wedge y)$;
($\mathfrak{s}_4$) $\nu(\nu(x) \odot \nu(y)) = \nu(x) \odot \nu(y)$;
($\mathfrak{s}_5$) $\nu(\nu(x) \vee \nu(y)) = \nu(x) \vee \nu(y)$;
($\mathfrak{s}_6$) $\nu(\nu(x) \wedge \nu(y)) = \nu(x) \wedge \nu(y)$.

If $\mathfrak{A}$ is a residuated lattice and ν is a state operator on $\mathfrak{A}$, then the pair $\mathfrak{A}_\nu = (\mathfrak{A}; \nu)$ is called a *state residuated lattice* or, more precisely, a *residuated lattice with internal state* ν. We denote by $\mathcal{SRL}$ the class of state residuated lattices. By [19, Lemma 3.1], it follows that the class $\mathcal{SRL}$ is equational and so it is a variety. For any state operator ν on a residuated lattice $\mathfrak{A}$, we set $\ker(\nu) = \nu^{\leftarrow}(1)$. A state operator ν is called *faithful* if $\ker(\nu) = \{1\}$.

Example 2.8. Let $\mathfrak{A}$ be a residuated lattice. Clearly i_A is a state operator. So $\mathfrak{A}_{i_A}$ is a state residuated lattice.

Example 2.9. Let $\mathfrak{A}$ be a residuated lattice and ν be an idempotent endomorphism of $\mathfrak{A}$. Clearly ν is a state operator. So $\mathfrak{A}_\nu$ is a state residuated lattice.

Example 2.10. Consider Example 2.4. One can check that the mapping $\nu : \mathfrak{A}_7 \longrightarrow \mathfrak{A}_7$ defined by $\nu(x) = 0$ for $x \in \{0, a, b, c\}$ and $\nu(x) = 1$ for $x \in \{d, e, 1\}$ is a state operator.

Proposition 2.11. *Let $\mathfrak{A}_\nu$ be a state residuated lattice. The following assertions hold for any $x, y \in A$:*

($\mathfrak{s}_7$) $\nu(\neg x) = \neg\nu(x)$;

($\mathfrak{s}_8$) $\nu(x) \odot \nu(y) \leq \nu(x \odot y)$;

($\mathfrak{s}_9$) $\nu^2(x) = \nu(x)$;

($\mathfrak{s}_{10}$) *if x and y are comparable, then $\nu(x \to y) = \nu(x) \to \nu(y)$.*

Proof. It follows by [19, Proposition 3.6]. □

Lemma 2.12. *Let $\mathfrak{A}_\nu$ be a state residuated lattice. If ν is faithful, then $\nu(x) = 0$ implies that $x = 0$.*

Proof. Let ν be a faithful operator and $\nu(x) = 0$, for some $x \in A$. By ($\mathfrak{s}_7$) it follows that $\nu(\neg x) = 1$ and this implies that $\neg x = 1$. So we have $0 \leq x \leq \neg\neg x = 0$ and thus the result holds. □

Let $\mathfrak{A}$ be a residuated lattice. Following [19, Definition 3.7], a mapping $\nu : A \to A$ is called a *good state operator* on $\mathfrak{A}$ if it satisfies ($\mathfrak{s}_1$), ($\mathfrak{s}_4$), ($\mathfrak{s}_5$), ($\mathfrak{s}_6$) and the following assertion:

($\mathfrak{gs}$) $\nu(x \to y) = \nu(x) \to \nu(y)$.

Let $\mathfrak{A}$ be a residuated lattice and ν be a good state operator on $\mathfrak{A}$. The pair $\mathfrak{A}_\nu = (\mathfrak{A}; \nu)$ is called a *good state residuated lattice*. By [19, Proposition 3.8] it follows that if $\mathfrak{A}_\nu$ is a good state residuated lattice, then it satisfies ($\mathfrak{s}_2$) and ($\mathfrak{s}_3$), and so $\mathfrak{A}_\nu$ is a state residuated lattice.

Proposition 2.13. *Let $\mathfrak{A}$ be a chain residuated lattice. Then any state operator on $\mathfrak{A}$ is good.*

Proof. It is obvious by ($\mathfrak{s}_{10}$). □

Lemma 2.14. *Let $\mathfrak{A}_\nu$ be a good state residuated lattice. The state operator ν is faithful if and only if it is injective.*

Proof. Let ν be a faithful operator and $\nu(x) = \nu(y)$, for some $x, y \in A$. By ($\mathfrak{gs}$) we have $\nu(x \to y) = \nu(x) \to \nu(y) = \nu(y) \to \nu(y) = 1$ and this ensures that $x \to y = 1$ by the hypothesis of faithfulness of ν. Analogously, we can obtain that $y \to x = 1$ and this shows that $x = y$. The converse is evident. □

Proposition 2.15. *Let $\mathfrak{A}_\nu$ be a good state residuated lattice. The following assertion holds for any $x, y \in A$:*

$$\nu(x \odot y) = \nu(x) \odot \nu(y).$$

Proof. It follows by [19, Proposition 3.9]. □

Let $\mathfrak{A}_\nu$ be a state residuated lattice. A subset F of A is called a filter of $\mathfrak{A}_\nu$ if F is a filter of $\mathfrak{A}$ and $\nu(F) \subseteq F$. The set of filters of $\mathfrak{A}_\nu$ is denoted by $\mathscr{F}(\mathfrak{A}_\nu)$. It is obvious that $\{1\}, \ker(\nu), A \in \mathscr{F}(\mathfrak{A}_\nu)$. One can easily check that if F is a filter of $\mathfrak{A}$ contained in $\ker(\nu)$, then F is a filter of $\mathfrak{A}_\nu$. For any subset X of A the filter of $\mathfrak{A}_\nu$ generated by X is denoted by $\mathscr{F}^{\mathfrak{A}_\nu}(X)$, and when there is no ambiguity it is denoted by $\mathscr{F}^{\nu}(X)$. For any $x \in A$, the filter of $\mathfrak{A}_{\nu}$ generated by $\{x\}$ is denoted by $\mathscr{F}^{\nu}(x)$ and called *principal filter* of $\mathfrak{A}_{\nu}$. The set of principal filters of $\mathfrak{A}_{\nu}$ is denoted by $\mathscr{PF}(\mathfrak{A}_{\nu})$. A congruence $\mathfrak{R}$ of $\mathfrak{A}$ is called a *congruence* of $\mathfrak{A}_{\nu}$ (or, ν*-congruence*) if $(a_1, a_2) \in \mathfrak{R}$ implies $(\nu(a_1), \nu(a_2)) \in \mathfrak{R}$. The set of all ν-congruences will be denoted by $\mathrm{Con}(\mathfrak{A}_{\nu})$. It is obvious that $\Delta_A, \kappa(\nu) \in \mathrm{Con}(\mathfrak{A}_{\nu})$, where $\Delta_A = \{(a,a) \mid a \in A\}$ and $\kappa(\nu) = \{(a_1, a_2) \in A^2 \mid \nu(a_1) = \nu(a_2)\}$. By [19, Proposition 4.10] it follows that if $\mathfrak{A}_{\nu}$ is a state residuated lattice then there is a lattice isomorphism between ν-filters and ν-congruences of $\mathfrak{A}_{\nu}$.

Example 2.16. Consider the state residuated lattices $(\mathfrak{A}_6; i_{A_6})$ from Example 2.2, $(\mathfrak{C}_6; i_{C_6})$ from Example 2.3 and $(\mathfrak{A}_7; i_{A_7})$ from Example 2.4. The set of their filters is presented in Table 4.

	filters
$\mathfrak{A}_6$	$\{1\}, \{d,1\}, \{a,b,d,1\}, \{c,d,1\}, A_6$
$\mathfrak{C}_6$	$\{1\}, C_6$
$\mathfrak{A}_7$	$\{1\}, \{b,d,1\}, \{e,1\}, \{a,b,c,d,e,1\}, A_7$

TABLE 4

By [19, Proposition 3.6 $\mathfrak{s}_{18}$] it follows that if $\mathfrak{A}_{\nu}$ is a state residuated lattice, then $\nu(\mathfrak{A})$ is a subalgebra of $\mathfrak{A}$. Let $\mathfrak{A}_{\nu}$ be a state residuated lattice and F be a ν-filter of $\mathfrak{A}_{\nu}$. In the following, we set $\nu^{\leftarrow}(F) = \{a \in A \mid \nu(a) \in F\}$.

Proposition 2.17. *Let $\mathfrak{A}_{\nu}$ be a state residuated lattice. The following assertions hold:*

(1) $F \in \mathscr{F}(\mathfrak{A})$ *implies* $\nu^{\leftarrow}(F) \in \mathscr{F}(\mathfrak{A}_{\nu})$*;*
(2) $F \in \mathscr{F}(\mathfrak{A}_{\nu})$ *implies* $F \cap \nu(A) = \nu(F) \in \mathscr{F}(\nu(A))$*;*
(3) $F \in \mathscr{F}(\nu(A))$ *implies* $\nu^{\leftarrow}(F) \in \mathscr{F}(\mathfrak{A}_{\nu})$*.*

Proof. It follows by [19, Proposition 4.2]. □

Proposition 2.18. *Let $\mathfrak{A}_{\nu}$ be a state residuated lattice and F be a filter of $\mathfrak{A}_{\nu}$. The mapping $\nu/F : A/F \longrightarrow A/F$ defined by $\nu/F(a/F) = \nu(a)/F$ is a state operator on $\mathfrak{A}/F$.*

Proof. It follows by [19, Proposition 4.3]. □

Remark 2.19. The state residuated lattice $(\mathfrak{A}/F; \nu/F)$ in the above proposition shall be called the *quotient state residuated lattice* and denoted by $\mathfrak{A}_{\nu}/F$.

Proposition 2.20. *Let $\mathfrak{A}_{\nu}$ be a state residuated lattice and X be a subset of A. Then $\mathscr{F}^{\nu}(X) = \mathscr{F}(X \cup \nu(X))$.*

Proof. It follows by [19, Proposition 4.4]. □

Corollary 2.21. *Let $\mathfrak{A}_\nu$ be a state residuated lattice, F be a filter of $\mathfrak{A}_\nu$ and $x, y \in A$. The following assertions hold:*

(1) $\mathscr{F}^\nu(x) = \mathscr{F}(x \odot \nu(x)) = \mathscr{F}(x \wedge \nu(x)) = \mathscr{F}(x, \nu(x)) = \mathscr{F}(x) \veebar \mathscr{F}(\nu(x))$;
(2) $\mathscr{F}^\nu(F, x) = \mathscr{F}(F \cup (x \odot \nu(x))) = \{a \in A \mid f \odot (x \odot \nu(x))^n \leq a,\ f \in F,\ n \geq 1\}$;
(3) *if* $x \leq y$ *then* $\mathscr{F}^\nu(y) \subseteq \mathscr{F}^\nu(x)$;
(4) $\mathscr{F}^\nu(\nu(x)) \subseteq \mathscr{F}^\nu(x)$.

Proof. It follows by [19, Corollary 4.5]. □

Proposition 2.22. *Let $\mathfrak{A}_\nu$ be a state residuated lattice. Then $(\mathscr{F}(\mathfrak{A}_\nu); \cap, \veebar, \{1\}, A)$ is a complete sublattice of $\mathscr{F}(\mathfrak{A})$.*

Proof. Let $\mathcal{F}$ be a subfamily of $\mathscr{F}(\mathfrak{A}_\nu)$. Obviously we have $\cap\mathcal{F} \in \mathscr{F}(\mathfrak{A}_\nu)$. Also, we have $\nu(\cup\mathcal{F}) = \cup_{F \in \mathcal{F}} \nu(F) \subseteq \cup\mathcal{F}$ and so by Proposition 2.20 it follows that

$$\begin{aligned}
\nu(\veebar\mathcal{F}) &= \nu(\mathscr{F}(\cup\mathcal{F})) \\
&= \nu(\mathscr{F}(\cup\mathcal{F} \cup \nu(\cup\mathcal{F}))) \\
&= \nu(\mathscr{F}^\nu(\cup\mathcal{F})) \\
&\subseteq \mathscr{F}^\nu(\cup\mathcal{F}) = \mathscr{F}(\cup\mathcal{F}) = \veebar\mathcal{F}.
\end{aligned}$$

This shows that $\mathscr{F}(\mathfrak{A}_\nu)$ is a complete sublattice of the complete lattice $\mathscr{F}(\mathfrak{A})$. □

Corollary 2.23. *Let $\mathfrak{A}_\nu$ be a state residuated lattice. $\mathscr{F}(\mathfrak{A}_\nu)$ is a frame.*

Proof. It follows by Remark 2.5 and Proposition 2.22. □

Corollary 2.24. *Let $\mathfrak{A}_\nu$ be a state residuated lattice and $x, y \in A$. The following assertions hold:*

(1) $\mathscr{F}^\nu(x) \cap \mathscr{F}^\nu(y) = \mathscr{F}(x \vee y, x \vee \nu(y), \nu(x) \vee y, \nu(x) \vee \nu(y))$;
(2) $\mathscr{F}^\nu(x) \veebar \mathscr{F}^\nu(y) = \mathscr{F}(x, y, \nu(x), \nu(y))$;
(3) $\mathscr{F}^\nu(x \odot y) \subseteq \mathscr{F}^\nu(x) \veebar \mathscr{F}^\nu(y)$.

Proof. (1): By distributivity of the lattice $\mathscr{F}(\mathfrak{A})$, Proposition 2.6 (4) and Corollary 2.21 (1), we have the following sequence of formulas:

$$\begin{aligned}
\mathscr{F}^\nu(x) \cap \mathscr{F}^\nu(y) &= \mathscr{F}(x \odot \nu(x)) \cap \mathscr{F}(y \odot \nu(y)) \\
&= (\mathscr{F}(x) \veebar \mathscr{F}(\nu(x))) \cap (\mathscr{F}(y) \veebar \mathscr{F}(\nu(y))) \\
&= (\mathscr{F}(x) \cap \mathscr{F}(y)) \veebar (\mathscr{F}(x) \cap \mathscr{F}(\nu(y))) \\
&\quad \veebar (\mathscr{F}(\nu(x)) \cap \mathscr{F}(y)) \veebar (\mathscr{F}(\nu(x)) \cap \mathscr{F}(\nu(y))) \\
&= \mathscr{F}(x \vee y) \veebar \mathscr{F}(x \vee \nu(y)) \veebar \mathscr{F}(\nu(x) \vee y) \veebar \mathscr{F}(\nu(x) \vee \nu(y)) \\
&= \mathscr{F}(x \vee y, x \vee \nu(y), \nu(x) \vee y, \nu(x) \vee \nu(y)).
\end{aligned}$$

(2): By Proposition 2.6 (4) and Corollary 2.21 (1), we have the following sequence of formulas:

$$\begin{aligned}
\mathscr{F}^\nu(x) \veebar \mathscr{F}^\nu(y) &= \mathscr{F}(x, \nu(x)) \veebar \mathscr{F}(y, \nu(y)) \\
&= \mathscr{F}(x, y, \nu(x), \nu(y)).
\end{aligned}$$

(3): It is a direct consequence of Corollary 2.21 (1) and (2). □

Corollary 2.25. *Let $\mathfrak{A}_\nu$ be a state residuated lattice and $x, y \in A$. The following assertions hold:*

(1) $\mathscr{F}^\nu(x) \cap \mathscr{F}^\nu(y) = \mathscr{F}^\nu((x \odot \nu(x)) \vee (y \odot \nu(y)))$;
(2) $\mathscr{F}^\nu(x) \veebar \mathscr{F}^\nu(y) = \mathscr{F}^\nu(x \odot y)$.

Proof. (1): By Proposition 2.21 (1) we have $\mathscr{F}^\nu((x \odot \nu(x)) \vee (y \odot \nu(y))) = \mathscr{F}((x \odot \nu(x)) \vee (y \odot \nu(y))) \veebar \mathscr{F}(\nu((x \odot \nu(x)) \vee (y \odot \nu(y))))$. By Proposition 2.6 (2) and (4) we have

$$\begin{aligned}\mathscr{F}((x \odot \nu(x)) \vee (y \odot \nu(y))) &= \mathscr{F}(x \odot \nu(x)) \cap \mathscr{F}(y \odot \nu(y)) \\ &= (\mathscr{F}(x) \veebar \mathscr{F}(\nu(x))) \cap (\mathscr{F}(y) \veebar \mathscr{F}(\nu(y))) \\ &= (\mathscr{F}(x) \cap \mathscr{F}(y)) \veebar (\mathscr{F}(x) \cap \mathscr{F}(\nu(y))) \\ &\quad \veebar (\mathscr{F}(\nu(x)) \cap \mathscr{F}(y)) \veebar (\mathscr{F}(\nu(x)) \cap \mathscr{F}(\nu(y))) \\ &= \mathscr{F}(x \vee y) \veebar \mathscr{F}(x \vee \nu(y)) \veebar \mathscr{F}(\nu(x) \vee y) \\ &\quad \veebar \mathscr{F}(\nu(x) \vee \nu(y)) \\ &= \mathscr{F}(x \vee y, x \vee \nu(y), \nu(x) \vee y, \nu(x) \vee \nu(y)).\end{aligned}$$

On the other hand, by Proposition 2.6 (3) we have $\mathscr{F}(\nu((x \odot \nu(x)) \vee (y \odot \nu(y)))) \subseteq \mathscr{F}(\nu(x) \vee \nu(y))$. So the result holds by Corollary 2.24 (1).

(2): It follows by Corollaries 2.21 (3) and 2.24 (3). □

Corollary 2.26. *Let $\mathfrak{A}_\nu$ be a state residuated lattice. Then $\mathscr{PF}(\mathfrak{A}_\nu)$ is a sublattice of $\mathscr{F}(\mathfrak{A}_\nu)$.*

Proof. It follows immediately by Corollary 2.25. □

3. Maximal, prime and minimal prime filters

In this section, the notions of maximal, prime and minimal prime filters of a state residuated lattice are investigated. Let $\mathfrak{A}_\nu$ be a state residuated lattice. Recall that a proper filter M of $\mathfrak{A}_\nu$ is called *maximal* if it is not strictly contained in any filter of $\mathfrak{A}_\nu$ [19]. We use $\mathrm{Max}(\mathfrak{A}_\nu)$ to denote the set of all maximal filters of $\mathfrak{A}_\nu$.

Proposition 3.1. *Any proper ν-filter of a state residuated lattice $\mathfrak{A}_\nu$ can be extended to a maximal ν-filter.*

Proof. It follows by [19, Proposition 4.12]. □

The following proposition characterizes maximal filters of a residuated lattice.

Proposition 3.2. *Let $\mathfrak{A}_\nu$ be a state residuated lattice. For any proper ν-filter M, the following assertions are equivalent:*

(1) *M is a maximal ν-filter;*
(2) *if $x \notin M$, there exists an integer n such that $\neg(\nu(x))^n \in M$.*

Proof. It follows by [19, Theorem 4.14]. □

Corollary 3.3. *Let $\mathfrak{A}_\nu$ be a state residuated lattice and M be a maximal filter of $\mathfrak{A}_\nu$. Then the state operator ν/M is faithful.*

Proof. Let $\nu/M(x/M) = 1/M$. So $\nu(x)/M = 1/M$ and this implies that $\nu(x) \in M$. Assume that $x \notin M$. Then by Proposition 3.2 it follows that $(\neg\nu(x))^n \in M$, for some integer n and so $\neg\nu(x) \in M$. This means that $0 \in M$ and this is a contradiction. So we have $x \in M$ and so $x/M = 1/M$. This shows that the state operator ν/M is faithful. □

Let $\mathfrak{A}_\nu$ be a state residuated lattice and α be a cardinal. A proper filter G of $\mathfrak{A}_\nu$ is called *α-irreducible* if for any family of filters $\mathcal{F}$ of cardinal α, $G = \bigcap \mathcal{F}$ implies $G = F$, for some $F \in \mathcal{F}$. A filter G is called *(finite) irreducible* if it is α-irreducible for any (finite) cardinal α. A filter P is called *prime* if it is finite irreducible. It is obvious that a proper filter P of $\mathfrak{A}_\nu$ is prime if and only if P is 2-irreducible. The set of prime filters of $\mathfrak{A}_\nu$ is called the *prime spectrum* of $\mathfrak{A}_\nu$ and denoted by $\mathrm{Spec}(\mathfrak{A}_\nu)$. Clearly, any maximal filter of $\mathfrak{A}_\nu$ is irreducible and hence prime. The next proposition characterizes prime filters of a state residuated lattice.

Proposition 3.4. *Let $\mathfrak{A}_\nu$ be a state residuated lattice. For any proper filter P of $\mathfrak{A}_\nu$, the following assertions are equivalent:*

(1) P is prime;
(2) $F_1 \cap F_2 \subseteq P$ implies $F_1 \subseteq P$ or $F_2 \subseteq P$, for any $F_1, F_2 \in \mathscr{F}(\mathfrak{A}_\nu)$;
(3) $(x \odot \nu(x)) \vee (y \odot \nu(y)) \in P$ implies $x \in P$ or $y \in P$, for any $x, y \in A$.

Proof. It follows by [19, Proposition 4.18]. □

Proposition 3.5. *Let $\mathfrak{A}_\nu$ be a state residuated lattice and P be a proper ν-filter of $\mathfrak{A}_\nu$. If $\{F \in \mathscr{F}(\mathfrak{A}_\nu) \mid P \subseteq F\}$ is a chain, then P is ν-prime.*

Proof. Let $\Sigma = \{F \in \mathscr{F}(\mathfrak{A}_\nu) \mid P \subseteq F\}$ be a chain. Assume that P is not ν-prime. So there exist $x, y \in A$ such that $(x \odot \nu(x)) \vee (y \odot \nu(y)) \in P$, $x \notin P$ and $y \notin P$. Since $\mathscr{F}^\nu(P, x), \mathscr{F}^\nu(P, y) \in \Sigma$, without loss of generality let $\mathscr{F}^\nu(P, x) \subseteq \mathscr{F}^\nu(P, y)$. By distributivity of $\mathscr{F}(\mathfrak{A}_\nu)$ and Corollary 2.25 (1), we have the following sequence of formulas:

$$\begin{aligned} P = P \veebar \mathscr{F}^\nu((x \odot \nu(x)) \vee (y \odot \nu(y))) &= P \veebar (\mathscr{F}^\nu(x) \cap \mathscr{F}^\nu(y)) \\ &= \mathscr{F}^\nu(P, x) \cap \mathscr{F}^\nu(P, y) \\ &= \mathscr{F}^\nu(P, x). \end{aligned}$$

This shows that $x \in P$; a contradiction. Hence P must be a prime filter. □

Let $\mathfrak{A}_\nu$ be a state residuated lattice. A nonempty subset $\mathscr{C}$ of A is called a *$\vee$-closed subset* of $\mathfrak{A}_\nu$ if $x, y \in \mathscr{C}$ implies that $(x \odot \nu(x)) \vee (y \odot \nu(y)) \in \mathscr{C}$. The set of $\vee$-closed subsets of $\mathfrak{A}_\nu$ shall be denoted by $\mathscr{C}(\mathfrak{A}_\nu)$. Clearly, $\{1\}, A \in \mathscr{C}(\mathfrak{A}_\nu)$.

Remark 3.6. (1) It is obvious that a filter P is prime if and only if $P^c := A \backslash P$ is $\vee$-closed. Also, if Γ is a family of prime filters, then $(\cup\Gamma)^c$ is $\vee$-closed.

(2) A $\vee$-closed set $\mathscr{C}$ in [6, Theorem 3.8] and [13, Lemma 1] is defined as a subset in which $x, y \in \mathscr{C}$ implies $x \vee y \in \mathscr{C}$, but in Theorem 3.11 we show that the complement of a prime filter must be a $\vee$-closed set.

Let $\mathfrak{A}_\nu$ be a state residuated lattice. It is obvious that $(A;\mathscr{C}(\mathfrak{A}_\nu))$ is a closed set system. The closure operator associated with this system shall be denoted by $\mathscr{C}^\nu$.

Lemma 3.7. *Let $\mathfrak{A}_\nu$ be a state residuated lattice, F be a filter of $\mathfrak{A}_\nu$ and $x \in A$. If $\mathscr{C}^\nu(x) \cap F \neq \emptyset$, then $x \in F$.*

Proof. It is obvious that $y \in \mathscr{C}^\nu(x)$ implies $y \leq x$. This proves the result. ☐

Theorem 3.8 (Prime filter theorem)**.** *If $\mathscr{C}$ is a $\vee$-closed subset of a state residuated lattice $\mathfrak{A}_\nu$ which does not meet the filter F, then F is contained in a filter P which is maximal with respect to the property of not meeting $\mathscr{C}$; furthermore P is prime.*

Proof. Let $\Sigma = \{G \in \mathscr{F}(\mathfrak{A}_\nu) \mid F \subseteq G, G \cap \mathscr{C} = \emptyset\}$. We can easily find that Σ satisfies the conditions of Zorn's lemma. Let P be a maximal element of Σ. Let $(x \odot \nu(x)) \vee (y \odot \nu(y)) \in P$ and neither $x \notin P$ nor $y \notin P$. By maximality of P we have $\mathscr{F}^\nu(P,x) \cap \mathscr{C} \neq \emptyset$ and $\mathscr{F}^\nu(P,y) \cap \mathscr{C} \neq \emptyset$. Suppose that $a_x \in \mathscr{F}^\nu(P,x) \cap \mathscr{C}$ and $a_y \in \mathscr{F}^\nu(P,y) \cap \mathscr{C}$. By Corollary 2.25 (1), we get that $(a_x \odot \nu(a_x)) \vee (a_y \odot \nu(a_y)) \in P \cap \mathscr{C}$; a contradiction. ☐

Corollary 3.9. *Let F be a filter of a state residuated lattice $\mathfrak{A}_\nu$ and X be a subset of A. The following assertions hold:*

(1) if $X \nsubseteq F$, there exists a prime filter P which is maximal with respect to the property of containing F and $X \nsubseteq P$;
(2) $\mathscr{F}^\nu(X) = \bigcap\{P \in \mathrm{Spec}(\mathfrak{A}_\nu) \mid X \subseteq P\}$.

Proof. (1): Let $x \in X - F$. By taking $\mathscr{C}^\nu(x)$ and using Lemma 3.7 it follows that $\mathscr{C}^\nu(x) \cap F = \emptyset$. So the result holds by Theorem 3.8.

(2): Set $\sigma = \{P \in \mathrm{Spec}(\mathfrak{A}_\nu) \mid X \subseteq P\}$. Obviously, we have $\mathscr{F}^\nu(X) \subseteq \bigcap \sigma$. Let $a \notin \mathscr{F}^\nu(X)$. By (1) it follows that there exits a prime filter P containing $\mathscr{F}^\nu(X)$ such that $a \notin P$. This shows that $a \notin \bigcap \sigma$. ☐

Let $\mathfrak{A}_\nu$ be a state residuated lattice and X be a subset of A. A prime filter P is called a *minimal prime filter belonging to* X, or an *X-minimal prime filter*, if $\{Q \in \mathrm{Spec}(\mathfrak{A}_\nu) \mid X \subseteq Q \subseteq P\} = P$. The set of all X-minimal prime filters of $\mathfrak{A}_\nu$ is denoted by $\mathrm{Min}_X(\mathfrak{A}_\nu)$. A prime filter P is called a *minimal prime filter* if $P \in \mathrm{Min}_1(\mathfrak{A}_\nu)$. The following result is an easy consequence of Zorn's lemma.

Lemma 3.10. *Let F be a filter of a state residuated lattice $\mathfrak{A}_\nu$. If $\mathscr{C}$ is a $\vee$-closed subset of $\mathfrak{A}_\nu$ which does not meet F, then $\mathscr{C}$ is contained in a state $\vee$-closed subset $\mathscr{C}$ of $\mathfrak{A}_\nu$ which is maximal with respect to the property of not meeting F.*

The following theorem gives a fundamental characterization of minimal prime filters in a state residuated lattice.

Theorem 3.11 (Minimal prime filter theorem)**.** *Let F be a filter of a state residuated lattice $\mathfrak{A}_\nu$. A subset P of A is an F-minimal prime filter of $\mathfrak{A}_\nu$ if and only if P^c (the complement of P in A) is a $\vee$-closed subset of $\mathfrak{A}_\nu$ which is maximal with respect to the property of not meeting F.*

Proof. Let P be a subset of A such that P^c is a $\vee$-closed subset of $\mathfrak{A}_\nu$ which is maximal with respect to the property of not meeting F. By Proposition 3.8, there exists a prime filter Q containing F which does not meet P^c. So Q is contained in P. By Remark 3.6 (1), Q^c is a $\vee$-closed subset of $\mathfrak{A}_\nu$ containing P^c and we have $Q^c \cap F = \emptyset$. So by maximality of P^c we deduce that $P^c = Q^c$ and this means that $P = Q$. This shows that P is a prime filter and moreover that P is an F-minimal prime filter.

Conversely, let P be an F-minimal prime filter of $\mathfrak{A}_\nu$. By Remark 3.6 (1), P^c is a $\vee$-closed subset of $\mathfrak{A}_\nu$. By using Lemma 3.10, we can get a $\vee$-closed subset $\mathscr{C}$ of $\mathfrak{A}_\nu$ which is maximal with respect to the property of not meeting F. By the case just proved, $\mathscr{C}^c$ is an F-minimal prime filter which does not meet P^c and this implies that $\mathscr{C}^c \subseteq P$. By hypothesis $\mathscr{C} = P^c$, and this shows that P^c is a $\vee$-closed subset of $\mathfrak{A}_\nu$ which is maximal with respect to the property of not meeting F. □

Corollary 3.12. *Let $\mathfrak{A}_\nu$ be a state residuated lattice, X be a subset of A and P be a prime filter containing X. There exists an X-minimal prime filter which is contained in P.*

Proof. By Remark 3.6 (1), P^c is a $\vee$-closed subset of $\mathfrak{A}_\nu$ such that $P^c \cap \mathscr{F}^\nu(X) = \emptyset$. By Lemma 3.10, we can obtain a $\vee$-closed subset $\mathscr{C}$ of $\mathfrak{A}_\nu$ containing P^c which is maximal with respect to the property of not meeting $\mathscr{F}^\nu(X)$. By Theorem 3.11, $\mathscr{C}^c$ is an $\mathscr{F}^\nu(X)$-minimal prime filter which is contained in P. Consequently, $\mathscr{C}^c$ is an X-minimal prime filter which is contained in P. □

The following corollary should be compared with Corollary 3.9 (2).

Corollary 3.13. *Let F be a filter of a state residuated lattice $\mathfrak{A}_\nu$ and X be a subset of A. The following assertions hold:*

(1) If $X \nsubseteq F$, there exists an F-minimal prime filter $\mathfrak{m}$ such that $X \nsubseteq \mathfrak{m}$;
(2) $\mathscr{F}^\nu(X) = \bigcap \mathrm{Min}_X(\mathfrak{A}_\nu)$.

Proof. (1): It is a direct consequence of Corollary 3.9 (1) and Corollary 3.12.

(2): Set $\sigma_X = \{P \in \mathrm{Spec}(\mathfrak{A}_\nu) \mid X \subseteq P\}$. By Corollary 3.9 (2), it is sufficient to show that $\bigcap \mathrm{Min}_X(\mathfrak{A}_\nu) = \bigcap \sigma_X$. It is obvious that $\bigcap \sigma_X \subseteq \bigcap \mathrm{Min}_X(\mathfrak{A}_\nu)$. Let $a \in \bigcap \mathrm{Min}_X(\mathfrak{A}_\nu)$ and P be an arbitrary element of σ_X. By Corollary 3.12, there exists an X-minimal prime filter $\mathfrak{m}$ containing in P. Hence, $a \in \mathfrak{m} \subseteq P$ and this states that $\bigcap \mathrm{Min}_X(\mathfrak{A}_\nu) \subseteq \bigcap \sigma_X$. □

Example 3.14. Consider the state residuated lattices $(\mathfrak{A}_6; i_{A_6})$ from Example 2.2, $(\mathfrak{C}_6; i_{C_6})$ from Example 2.3 and $(\mathfrak{A}_7; i_{A_7})$ from Example 2.4. The set of their maximal, prime and minimal prime filters is presented in Table 5.

	prime filters		
	Maximal		**Minimal**
$\mathfrak{A}_6$	{a,b,d,1},{c,d,1}	{d,1}	{1}
$\mathfrak{C}_6$	{1}		{1}
$\mathfrak{A}_7$	{a,b,c,d,e,1}		{b,d,1},{e,1}

Table 5

4. Simple, local and subdirectly irreducible state residuated lattices

In this section we study the notion of simple, local and subdirectly irreducible state residuated lattices and find relations between them. Recall that for a residuated lattice $\mathfrak{A}$ and $a \in A$ we put $a^n := a \odot \cdots \odot a$ (n times), for any integer n. An element $a \in A$ is called *nilpotent* if $a^n = 0$, for some integer n. The set of nilpotent elements of $\mathfrak{A}$ is denoted by $\mathfrak{ni}(\mathfrak{A})$. It is easy too see that $\mathfrak{ni}(\mathfrak{A})$ is an ideal of $\ell(\mathfrak{A})$. In the following, we set $\mathfrak{in}(\mathfrak{A}) = A \setminus \mathfrak{ni}(\mathfrak{A})$.

A residuated lattice $\mathfrak{A}$ is called *simple* if $\mathscr{F}(\mathfrak{A}) = \{\{1\}, A\}$ (see Figure 4). A residuated lattice $\mathfrak{A}$ is simple if and only if $\mathfrak{ni}(\mathfrak{A}) = A \setminus \{1\}$, [15, Lemma 1.1]. Following [11, Definition 3.20] a state residuated lattice $\mathfrak{A}_\nu$ is called *simple* if $\mathscr{F}(\mathfrak{A}_\nu) = \{\{1\}, A\}$.

Proposition 4.1. *Let $\mathfrak{A}_\nu$ be a state residuated lattice. $\mathfrak{A}_\nu$ is simple if and only if $\nu(A)$ is a simple residuated lattice and ν is faithful.*

Proof. It follows by [19, Corollary 4.7]. □

Figure 4. The filter lattice of a simple state residuated lattice

Corollary 4.2. *Let $\mathfrak{A}_\nu$ be a simple state residuated lattice. Then for any $1 \neq a \in A$, $\nu(a) \in \mathfrak{ni}(\mathfrak{A})$.*

Proof. Let $1 \neq a \in A$. By Proposition 4.1 it follows that $\nu(a) \neq 1$ and so there exists an integer n such that $(\nu(a))^n = 0$. □

Proposition 4.3. *Let $\mathfrak{A}_\nu$ be a state residuated lattice and M be a proper filter of $\mathfrak{A}_\nu$. The following assertions are equivalent:*

(1) $\mathfrak{A}_\nu/M$ is simple;
(2) M is maximal.

Proof. (1)⇒(2): Let $x \notin M$. So $x/M \neq 1/M$. By Proposition 2.18 and Corollary 4.2 it follows that $(\nu(x))^n/M = (\nu(x)/M)^n = (\nu/M(x/M))^n = 0/M$, for some integer n. This implies that $\neg(\nu(x))^n \in M$ and so M is a maximal filter by Proposition 3.2.

(2)⇒(1): By Corollary 3.3 it follows that the state operator ν/M is faithful. Now, let $\nu(x)/M = \nu/M(x/M) \neq 1/M$. So $\nu(x) \notin M$. By $(\mathfrak{s}_9)$ and Proposition 3.2 it follows that $\neg(\nu(x))^n = \neg(\nu(\nu(x)))^n \in M$. Thus $(\nu/M(x/M))^n = (\nu(x)/M)^n = 0/M$ and this shows that $\nu/M(\mathfrak{A}_\nu/M)$ is simple. So the result holds by Proposition 4.1. □

Let $\mathfrak{A}_\nu$ be a state residuated lattice. It is obvious that if $\mathfrak{A}$ is a simple residuated lattice, then $\mathfrak{A}_\nu$ is a simple state residuated lattice. The following proposition gives a sufficient condition for a simple state residuated lattice to be a simple residuated lattice.

Proposition 4.4. *Let $\mathfrak{A}_\nu$ be a good and simple state residuated lattice. Then $\mathfrak{A}$ is simple.*

Proof. Let $\mathfrak{A}_\nu$ be simple and $1 \neq x \in A$. By Corollary 4.2 it follows that $(\nu(x))^n = 0$, for some integer n. Since $\mathfrak{A}_\nu$ is good, by Proposition 2.15 it follows that $\nu(x^n) = 0$, and since ν is faithful (thanks to Proposition 4.1), by Lemma 2.12 it follows that $x^n = 0$. Thus $x \in \mathfrak{ni}(\mathfrak{A})$ and this shows that $\mathfrak{A}$ is simple. □

Following [3, Definition 4.1], a residuated lattice $\mathfrak{A}$ is called *local* if it has exactly one maximal filter, denoted by $M_{\mathfrak{A}}$ (see Figure 5). In [3, Proposition 4.4] it was shown that $\mathfrak{A}$ is a local residuated lattice if and only if $\mathfrak{in}(\mathfrak{A})$ is a filter of $\mathfrak{A}$.

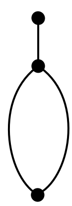

FIGURE 5. The filter lattice of a local state residuated lattice

Recall that a local state residuated lattice is a state residuated lattice with exactly one state maximal filter. This class of state residuated lattices are characterized in [11, Proposition 3.24].

Corollary 4.5. *Let $\mathfrak{A}_\nu$ be a state residuated lattice. The following assertions are equivalent:*

(1) $\mathfrak{A}_\nu$ is local;
(2) $\nu(A)$ is local.

Proposition 4.6. *Let $\mathfrak{A}_\nu$ be a state residuated lattice. The following assertions are equivalent:*

(1) $\mathfrak{in}(\nu(A))$ *is a filter of* $\nu(A)$*;*
(2) $\mathfrak{A}_\nu$ *is a local state residuated lattice;*
(3) $x, y \in \mathfrak{in}(\nu(A))$ *implies* $x \odot y \in \mathfrak{in}(\nu(A))$*, for any* $x, y \in \nu(A)$*.*

Proof. (1)⇒(2): It is obvious that $\mathfrak{in}(\nu(A))$ is a proper filter. Let F be a proper filter of $\nu(A)$ and $x \in F$. So we have $\mathscr{F}^{\nu(A)}(x) \subseteq F$ and this implies that $x \in \mathfrak{in}(\nu(A))$. Thus $F \subseteq \mathfrak{in}(\nu(A))$ and this shows that $\mathfrak{in}(\nu(A))$ is the unique maximal filter of $\nu(A)$. So $\nu(A)$ is a local residuated lattice. By Corollary 4.5 it follows that $\mathfrak{A}_\nu$ is a local state residuated lattice.

(2)⇒(1): By Corollary 4.5 it follows that $\mathfrak{A}_\nu$ is a local state residuated lattice. Let $x \in \mathfrak{in}(\nu(A))$. So $\mathscr{F}^{\nu(A)}(x)$ is a proper filter of $\nu(A)$. By Proposition 3.1 it follows that $x \in \mathscr{F}^{\nu(A)}(x) \subseteq M_{\mathfrak{A}_\nu}$. This shows that $\mathfrak{in}(\nu(A)) \subseteq M_{\mathfrak{A}_\nu}$. The inverse inclusion is evident and so we have $\mathfrak{in}(\nu(A)) = M_{\mathfrak{A}_\nu}$. Thus the result holds.

(1)⇒(3): It is evident.

(3)⇒(1): Since $1 \in \mathfrak{in}(\nu(A))$, $\mathfrak{in}(\nu(A))$ is nonempty. Let $x \in \mathfrak{in}(\nu(A))$ and $y \in \nu(A)$. Thus for any integer n we have $0 < x^n \leq (x \vee y)^n$ and this shows that $x \vee y \in \mathfrak{in}(\nu(A))$. □

Corollary 4.7. *Any simple state residuated lattice is local.*

Proof. Let $\mathfrak{A}_\nu$ be a simple state residuated lattice. By Corollary 4.2 it follows that $\mathfrak{in}(\nu(A)) = \{1\}$ and so the result holds by Proposition 4.6. □

Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. We set $F^\top = \{a \in A \mid a \odot x = 0, \exists x \in F\}$. Elements of $F^\top$ are called *orthogonal* elements to F.

Lemma 4.8. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. The following assertions hold:*

(1) $F^\top = \{a \in A \mid \neg a \in F\}$*;*
(2) $F^\top$ *is an ideal of* $\ell(\mathfrak{A})$*.*

Proof. (1): Set $\Sigma = \{a \in A \mid \neg a \in F\}$. If $a \in F^\top$, then $a \odot x = 0$, for some $x \in F$. This implies that $x \leq \neg a$ and so $a \in \Sigma$. Conversely, $a \in \Sigma$ implies that $\neg a = x$, for some $x \in F$, and this implies that $a \odot x = 0$. So we have $a \in F^\top$.

(2): It is evident by (r_1). □

Corollary 4.9. *Let $\mathfrak{A}_\nu$ be a local state residuated lattice. The following assertions hold:*

(1) *For any* $x \in \nu(A)$*,* $x \in \mathfrak{ni}(\nu(A))$ *or* $\neg x \in \mathfrak{ni}(\nu(A))$*;*
(2) $(\mathfrak{in}(\nu(A)))^\top \subseteq \mathfrak{ni}(\nu(A))$*;*
(3) $(\mathfrak{in}(\nu(A)))^\top \cap \mathfrak{in}(\nu(A)) = \emptyset$*.*

Proof. (1): Since $x \odot \neg x \in \mathfrak{ni}(\nu(A))$, it follows by Proposition 4.6(3).

(2): Let $a \in (\mathfrak{in}(\nu(A)))^\top$. By Lemma 4.8 (1) it follows that $\neg a \in \mathfrak{in}(\nu(A))$ and so by (1) we obtain that $a \in \mathfrak{ni}(\nu(A))$.

(3): It is evident. □

Lemma 4.10. *Let $\mathfrak{A}_\nu$ be a good state residuated lattice and ν be faithful. We have*

$$\mathfrak{in}(\mathfrak{A}) = \nu^{\leftarrow}(\mathfrak{in}(\nu(A))).$$

Proof. It is an immediate consequence of Lemma 2.12, Lemma 2.14 and Proposition 2.15. □

Corollary 4.11. *Let $\mathfrak{A}_\nu$ be a good state residuated lattice and ν be faithful. The following assertions are equivalent:*

(1) *$\mathfrak{A}_\nu$ is local;*
(2) *$\mathfrak{A}$ is local.*

Proof. (1)⇒(2): By Proposition 4.6, $\mathfrak{in}(\nu(A))$ is a filter of $\nu(A)$. So by Proposition 2.17 (3) it follows that $\nu^{\leftarrow}(\mathfrak{in}(\nu(A)))$ is a filter of $\mathfrak{A}$. Consequently, by Lemma 4.10, $\mathfrak{in}(\mathfrak{A})$ is a filter of $\mathfrak{A}$ and so $\mathfrak{A}$ is a local residuated lattice.

(2)⇒(1): Since $\mathfrak{A}$ is local, $\mathfrak{in}(\mathfrak{A})$ is a filter of $\mathfrak{A}$. By Lemma 4.10 it follows that $\nu^{\leftarrow}(\mathfrak{in}(\nu(A)))$ is a filter of $\mathfrak{A}$. Let $y_1, y_2 \in \mathfrak{in}(\nu(A))$. So there exist $x_1, x_2 \in A$ such that $\nu(x_1) = y_1$ and $\nu(x_2) = y_2$. This implies that $x_1, x_2 \in \mathfrak{in}(\mathfrak{A})$ and so $x_1 \odot x_2 \in \mathfrak{in}(\mathfrak{A})$. By Proposition 2.15 we get that $y_1 \odot y_2 = \nu(x_1) \odot \nu(x_2) = \nu(x_1 \odot x_2) \in \mathfrak{in}(\nu(A))$ and this means that $\mathfrak{A}_\nu$ is a local state residuated lattice by Proposition 4.6. □

Proposition 4.12 ([3, Proposition 4.9])**.** *Any chain residuated lattice is local.*

Corollary 4.13. *Any chain state residuated lattice is local.*

Proof. It is an immediate consequence of Proposition 2.13, Corollary 4.11 and Proposition 4.12. □

Let $\mathfrak{A}_\nu$ and $\mathfrak{B}_\mu$ be state residuated lattices. A mapping $h : A \longrightarrow B$ is called a *morphism*, in symbols $h : \mathfrak{A}_\nu \longrightarrow \mathfrak{B}_\mu$, if it preserves the fundamental operations. If $h : \mathfrak{A}_\nu \longrightarrow \mathfrak{B}_\mu$ is a morphism we put $\operatorname{coker}(h) = h^{\leftarrow}(1)$. It is easy to check that $\operatorname{coker}(h)$ is a filter of $\mathfrak{A}_\nu$. Also, it is obvious that h is injective if and only if $\operatorname{coker}(h) = \{1\}$. For a family of state residuated lattices $\{\mathfrak{A}_\nu\} \cup \{\mathfrak{A}^i_{\nu_i}\}_{i \in I}$ a *subdirect representation* of $\mathfrak{A}_\nu$ with factors $\mathfrak{A}^i_{\nu_i}$ is an embedding $h : \mathfrak{A}_\nu \longrightarrow \prod_{i \in I} \mathfrak{A}^i_{\nu_i}$ such that each $\pi_i \circ h$ is surjective. $\mathfrak{A}$ is called a *subdirect product* of a family $\{\mathfrak{A}^i_{\nu_i}\}_{i \in I}$ if there exists a subdirect representation of $\mathfrak{A}_\nu$ with factors $\mathfrak{A}^i_{\nu_i}$.

Lemma 4.14. *Let $\mathfrak{A}_\nu$ be a state residuated lattice and $\mathcal{F}$ be a family of filters of $\mathfrak{A}_\nu$. If $\bigcap \mathcal{F} = \{1\}$, then $\mathfrak{A}_\nu$ is a subdirect product of the family $\{\mathfrak{A}_\nu / F\}_{F \in \mathcal{F}}$.*

Proof. Define $h : \mathfrak{A}_\nu \longrightarrow \prod_{F \in \mathcal{F}} \mathfrak{A}_\nu / F$, by $h(a)(F) = a/F$, for any $a \in A$ and $F \in \mathcal{F}$. It is easy to see that h is a morphism and $\operatorname{coker}(h) = \bigcap \mathcal{F}$. So h is an embedding. Also, for any $F \in \mathcal{F}$, $\pi_F \circ h$ is the canonical projection, which is surjective. Therefore, h is a subdirect representation. □

Corollary 4.15. *Any state residuated lattice $\mathfrak{A}_\nu$ is a subdirect product of the family $\{\mathfrak{A}_\nu / \mathfrak{m}\}_{\mathfrak{m} \in \mathrm{Min}(\mathfrak{A})}$.*

Proof. It follows by Corollary 3.13 (2) and Lemma 4.14. □

Proposition 4.16. *Any simple state residuated lattice is subdirectly irreducible.*

Proof. It is evident. ☐

The following theorem characterizes subdirectly irreducible state residuated lattices by means of subdirectly irreducible residuated lattices.

Theorem 4.17. *Let $\mathfrak{A}_\nu$ be a faithful state residuated lattice. The following assertions are equivalent:*

(1) $\mathfrak{A}_\nu$ *is subdirectly irreducible;*
(2) $\nu(A)$ *is subdirectly irreducible.*

Proof. It follows by [19, Theorem 4.22]. ☐

Proposition 4.18 ([18, Lemmas 4.1 & 4.2])**.** *Let $\mathfrak{A}$ be a residuated lattice. $\mathfrak{A}$ is subdirectly irreducible if and only if there exists an element $1 \neq a \in A$ such that for any $1 \neq x \in A$ there exists a positive integer n for which $x^n \leq a$.*

Corollary 4.19. *Let $\mathfrak{A}_\nu$ be a faithful state residuated lattice. The following assertions are equivalent:*

(1) $\mathfrak{A}_\nu$ *is subdirectly irreducible;*
(2) there exists an element $1 \neq a \in A$ such that for any $1 \neq x \in A$ there exists a positive integer n for which $(\nu(x))^n \leq \nu(a)$.

Proof. It is an immediate consequence of Theorem 4.17 and Proposition 4.18. ☐

Corollary 4.20. *Let $\mathfrak{A}_\nu$ be a faithful subdirectly irreducible residuated lattice and $1 \neq x \in A$. If $\nu(x) \vee \nu(y) = 1$, for some $y \in A$, then $y = 1$.*

Proof. Let $1 \neq y$ be an element of A such that $\nu(x) \vee \nu(y) = 1$. Since $\mathfrak{A}_\nu$ is subdirectly irreducible and ν is faithful, there exists an element $1 \neq a \in A$ and an integer n such that $1 = (\nu(x))^n \vee (\nu(y))^n \leq \nu(a)$; a contradiction. So $\nu(y) = 1$ and this implies that $y = 1$. ☐

Corollary 4.21. *Let $\mathfrak{A}_\nu$ be a good state residuated lattice and ν be faithful. The following assertions are equivalent:*

(1) $\mathfrak{A}_\nu$ *is subdirectly irreducible;*
(2) $\mathfrak{A}$ *is subdirectly irreducible.*

Proof. (1)⇒(2): By Corollary 4.19, there exists an element $1 \neq a \in A$ such that for any $1 \neq x \in A$ there exists a positive integer n for which $(\nu(x))^n \leq \nu(a)$. Consider $a \in A$ and let $1 \neq x \in A$. By Proposition 2.15 it follows that $\nu(x^n) = (\nu(x))^n \leq \nu(a)$ and this implies that $x^n \leq a$ by Lemma 2.14. So the result holds by Proposition 4.18.

(2)⇒(1): By Proposition 4.18, there exists an element $1 \neq a \in A$ such that for any $1 \neq x \in A$ there exists a positive integer n for which $x^n \leq a$. Consider $a \in A$ and let $1 \neq x \in A$. So we have $(\nu(x))^n \leq \nu(x^n) \leq \nu(a)$ and this implies that $\mathfrak{A}_\nu$ is subdirectly irreducible by Corollary 4.19. ☐

Example 4.22. Consider the state residuated lattices $(\mathfrak{A}_6; i_{A_6})$ from Example 2.2, $(\mathfrak{C}_6; i_{C_6})$ from Example 2.3 and $(\mathfrak{A}_7; i_{A_7})$ from Example 2.4. We have as shown in Table 6.

	Simple	**Subdirectly irreducible**	**Local**
$\mathfrak{A}_6$		✓	
$\mathfrak{C}_6$	✓	✓	✓
$\mathfrak{A}_7$			✓

Table 6

Acknowledgments

The authors are highly grateful to the referees and editors for their valuable comments and suggestions which were helpful in improving this paper.

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Received: August 6, 2019
Accepted: December 17, 2019