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THE CONVEX AND WEAK CONVEX DOMINATION NUMBER OF CONVEX POLYTOPES

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Abstract. This paper is devoted to solving the weakly convex dominating set problem and the convex dominating set problem for some classes of planar graphs—convex polytopes. We consider all classes of convex polytopes known from the literature and present exact values of weakly convex and convex domination number for all classes, namely A_n, B_n, C_n, D_n, E_n, R_n, R''_n, Q_n, S_n, S''_n, T_n, T''_n and U_n. When n is up to 26, the values are confirmed by using the exact method, while for greater values of n theoretical proofs are given.

1. Introduction

This paper is dedicated to solving the weak convex and convex dominating set problems (WCDSP/CDSP) for some special classes of graphs. A *dominating set* of a graph G can be defined as a set of vertices in G such that any vertex $u \in V(G)$ either belongs to the dominating set or is adjacent to some vertex that belongs to the dominating set. A closely related problem is that of finding the domination number, i.e. finding the dominating set with the minimum cardinality. The minimal cardinality of a dominating set will be denoted as $\gamma(G)$.

In this paper, we consider only simple graphs, i.e. graphs without loops or parallel edges. In order to introduce the convexity property in graphs let us define the distance $d(u,v)$ between any two vertices u and v of G as the length of the shortest path between them. Since the graph is not weighted, the distance $d(u,v)$ is equal to the number of edges in the shortest path.

We say that a set of vertices S, $S \subset V(G)$, is a *weakly convex* (or *isometric* [17]) set in G if for any two vertices $u, v \in S$, there exists at least one shortest u–v path that contains only vertices belonging to S. Now, a *weakly convex dominating* set is a set S that is both weakly convex and dominating. Similarly to the domination number, the *weakly convex domination number* $\gamma_{wconv}(G)$ is the minimal cardinality among all weakly convex dominating sets. The weakly convex dominating set problem (WCDSP) is the problem of finding such minimal cardinality.

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As expected, convex domination requires stricter conditions. We will say that a set of vertices S, $S \subset V(G)$, is a *convex* set in G if for any two vertices $u, v \in S$ all vertices in all shortest $u-v$ paths belong to S. A set S is *convex dominating* if it is convex and dominating at the same time. The *convex domination number* $\gamma_{conv}(G)$ is the minimal cardinality among all convex dominating sets. The convex dominating set problem (CDSP) can be now defined as the problem of determining such minimal cardinality. There are other domination related problems depending on additional conditions such as Roman domination, signed Roman domination[1], etc.

The convex domination number was defined during verbal communication between Jerzy Topp and Magdalena Lemanska in 2002 (stated in [15]). In [18] it was proven that decision problems of WCDSP and CDSP are NP-complete even for bipartite and split graphs. So, determining the weakly convex domination number and the convex domination number is NP-hard in a general case.

In [14], relations between γ_{conv} and γ_{wconv} were discussed for certain classes of graphs, and the following lemma was proposed for connected graphs:

Lemma 1.1 ([14]). *For any connected graph G, we have*

$$\gamma(G) \leq \gamma_{wconv}(G) \leq \gamma_{conv}(G).$$

In the paper [10] different bounds of the weakly convex domination and the convex domination number were obtained. It was shown in [4] that the convex domination number can be arbitrarily increased or decreased by an edge subdivision.

In this paper we will consider finding the weakly convex and the convex domination number for some classes of planar graphs. Some of these classes, called convex polytopes, were for the first time considered in [2], where they were denoted as R_n and Q_n. Other classes, such as $A_n, B_n, C_n, D_n, E_n, R''_n, S_n, S''_n, T_n$ and U_n, were introduced in [5, 6, 9, 8, 7]. Certain graph invariants of convex polytopes were considered in [11]. For all these we assume that $n \geq 5$, and all indices are taken modulo n.

For $n \geq 27$, the values of both the weak convex and the convex domination number are stated by theorems. For $n \leq 26$, the exact method from [12] is used.

The largest graph for which the exact method was used is R''_{26}, with 156 vertices and 234 edges. The exact method found that the weak convex and the convex domination numbers are equal to 156.

2. Convex polytopes D_n

The graph of convex polytope D_n, $n \geq 5$ (Figure 1), introduced in [1], consists of $2n$ 5-sided faces and a pair of n sided faces. We will use the standard notation $[n] = \{1, 2, \ldots, n\}$ and $[n]_0 = \{0, 1, 2, \ldots, n\}$.

[1] A. Kartelj, D. Matić and V. Filipović, Solving the signed Roman domination and signed total Roman domination problems by exact and heuristic methods (unpublished manuscript).

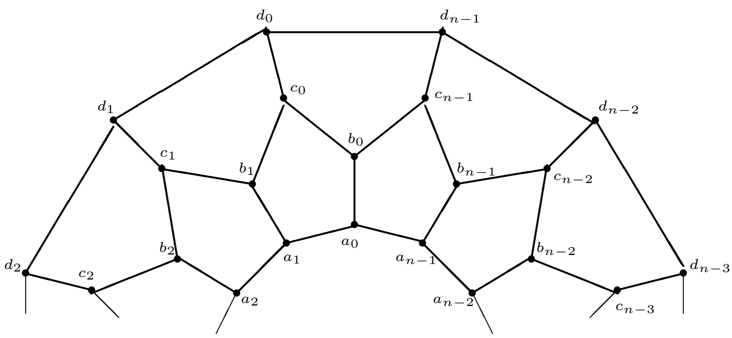

FIGURE 1. The graph of convex polytope D_n

We have

$$V(D_n) = \{a_i, b_i, c_i, d_i \mid i \in [n-1]_0\},$$
$$E(D_n) = \{a_i a_{i+1}, d_i d_{i+1}, a_i b_i, b_i c_i, c_i d_i, b_{i+1} c_i \mid i \in [n-1]_0\}.$$

Theorem 2.1. *For every convex polytope D_n with $n \geq 27$ it holds that $\gamma_{wconv}(D_n) = |V(D_n)| = 4n$.*

Proof. Let S be a weak convex dominating set of D_n. We will prove that $S = V(D_n)$. First we will show that $a_i \in S$ for every $i \in [n-1]_0$. From the fact that S is a dominating set, for arbitrary i it follows that there exist u from $\mathcal{N}[a_i] \cap S$ and w from $\mathcal{N}[a_{i+3}] \cap S$. Since $n \geq 27$, $\mathcal{N}[a_i] = \{a_i, a_{i-1}, a_{i+1}, b_i\}$ and $\mathcal{N}[a_{i+3}] = \{a_{i+3}, a_{i+2}, a_{i+4}, b_{i+3}\}$, we have the 16 subcases displayed in Table 1.

As it can be seen from Table 1, in all 16 subcases the shortest path is unique and all 16 shortest paths contain vertices a_{i+1} and a_{i+2}. Since for any $u \in \mathcal{N}[a_i]$ and any $w \in \mathcal{N}[a_{i+3}]$, vertex a_{i+1} belongs to a unique shortest u–w path and S is a weak convex dominating set, vertex a_{i+1} must belong to S. As i is arbitrary we have $a_i \in S$ for every $i \in [n-1]_0$.

Note that the symmetry property holds for convex polytopes D_n, so vertices a_i, b_i, c_i, d_i can be relabeled as d_i, c_i, b_i, a_i, which gives the same graph D_n. Therefore, using the same argumentation as above, we have $d_i \in S$ for every $i \in [n-1]_0$.

As it can be shown, for arbitrary $i \in [n-1]_0$ the shortest path between a_i and d_i is unique (a_i–b_i–c_i–d_i) and has both endpoints in S ($a_i, d_i \in S$). From the fact that S is a weak convex dominating set it must hold that vertices b_i and c_i belong to S for all $i \in [n-1]_0$. Due to the inclusion $\{b_i, c_i \mid i \in [n-1]_0\} \subset S$ and previous facts, it stands that $\{a_i, b_i, c_i, d_i \mid i \in [n-1]_0\} \subseteq S$, implying that $S = V(D_n)$. □

Corollary 2.2. *For every convex polytope D_n with $n \geq 27$ it holds that $\gamma_{conv}(D_n) = |V(D_n)| = 4n$.*

Proof. Since for each graph G it holds that $\gamma_{wconv}(G) \leq \gamma_{conv}(G) \leq |V(G)|$ and $\gamma_{wconv}(D_n) = |V(D_n)| = 4n$, we have $\gamma_{conv}(D_n) = 4n$. □

Table 1. Shortest paths used in Theorems 2.1 and 3.1

u	w	*Shortest path*
a_i	a_{i+3}	$a_i - a_{i+1} - a_{i+2} - a_{i+3}$
	a_{i+2}	$a_i - a_{i+1} - a_{i+2}$
	a_{i+4}	$a_i - a_{i+1} - a_{i+2} - a_{i+3} - a_{i+4}$
	b_{i+3}	$a_i - a_{i+1} - a_{i+2} - a_{i+3} - b_{i+3}$
a_{i-1}	a_{i+3}	$a_{i-1} - a_i - a_{i+1} - a_{i+2} - a_{i+3}$
	a_{i+2}	$a_{i-1} - a_i - a_{i+1} - a_{i+2}$
	a_{i+4}	$a_{i-1} - a_i - a_{i+1} - a_{i+2} - a_{i+3} - a_{i+4}$
	b_{i+3}	$a_{i-1} - a_i - a_{i+1} - a_{i+2} - a_{i+3} - b_{i+3}$
a_{i+1}	a_{i+3}	$a_{i+1} - a_{i+2} - a_{i+3}$
	a_{i+2}	$a_{i+1} - a_{i+2}$
	a_{i+4}	$a_{i+1} - a_{i+2} - a_{i+3} - a_{i+4}$
	b_{i+3}	$a_{i+1} - a_{i+2} - a_{i+3} - b_{i+3}$
b_i	a_{i+3}	$b_i - a_i - a_{i+1} - a_{i+2} - a_{i+3}$
	a_{i+2}	$b_i - a_i - a_{i+1} - a_{i+2}$
	a_{i+4}	$b_i - a_i - a_{i+1} - a_{i+2} - a_{i+3} - a_{i+4}$
	b_{i+3}	$b_i - a_i - a_{i+1} - a_{i+2} - a_{i+3} - b_{i+3}$

3. Convex polytopes R''_n

The graph of convex polytope R''_n, $n \geq 5$ (Figure 2), introduced in [16], has the following vertex and edge sets:

$$V(R''_n) = \{a_i, b_i, c_i, d_i, e_i, f_i \mid i \in [n-1]_0\}$$

$$E(R''_n) = \{(a_i, a_{i+1}), (f_i, f_{i+1}), (a_i, b_i), (b_i, c_i), (c_i, d_i), (d_i, e_i), (e_i, f_i), (b_{i+1}, c_i), (d_{i+1}, e_i) \mid i \in [n-1]_0\}.$$

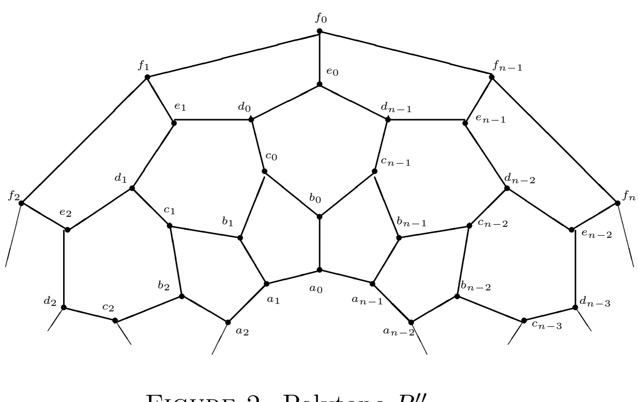

Figure 2. Polytope R''_n

Theorem 3.1. *For every convex polytope R''_n with $n \geq 27$ it holds that $\gamma_{wconv}(R''_n) = |V(R''_n)| = 6n$.*

Proof. Let S be a weak convex dominating set of R''_n. We will prove that $S = V(R''_n)$. First we will show that $a_i \in S$ for every $i \in [n-1]_0$. Similarly to Theorem 2.1, from the fact that S is a dominating set for an arbitrary i, it stands that there exist $u \in \mathcal{N}[a_i] \cap S$ and $w \in \mathcal{N}[a_{i+3}] \cap S$. Because $n \geq 27$, $\mathcal{N}[a_i] = \{a_i, a_{i-1}, a_{i+1}, b_i\}$ and $\mathcal{N}[a_{i+3}] = \{a_{i+3}, a_{i+2}, a_{i+4}, b_{i+3}\}$, we have 16 subcases, same as for D_n, displayed in Table 1.

Again, as it can be seen from Table 1, in all 16 subcases the shortest path is unique and all 16 shortest paths contain vertices a_{i+1} and a_{i+2}. Same as for D_n, a_{i+1} must belong to S. This is because S is a weak dominating set and some $u, w \in S$ contain a_{i+1} in its unique shortest $u{-}w$ path. As i is arbitrary we have $a_i \in S$ for every $i \in [n-1]_0$.

Note that the symmetry property also holds for convex polytopes R''_n, so vertices a_i, b_i, c_i, d_i, e_i, f_i can be relabeled as f_i, e_i, d_i, c_i, b_i, a_i, giving the same graph R''_n. Therefore, because of the same arguments used for a-vertices, we have that all f-vertices belong to S, i.e. $\{f_i \mid i \in [n-1]_0\} \subset S$.

As it can be shown, for arbitrary $i \in [n-1]_0$ the shortest path between a_{i+1} and f_i is unique ($a_{i+1}{-}b_{i+1}{-}c_i{-}d_i{-}e_i{-}f_i$), and has both endpoints in S ($a_{i+1}, f_i \in S$), so from the fact that S is a weak convex dominating set it must hold that for all $i \in [n-1]_0$ we have $b_{i+1}, c_i, d_i, e_i \in S$. Due to the inclusion $\{b_{i+1}, c_i, d_i, e_i \mid i \in [n-1]_0\} \subset S$ and previous facts, we have $\{a_{i+1}, b_{i+1}, c_i, d_i, e_i, f_i \mid i \in [n-1]_0\} \subseteq S$, implying that $V(R''_n) \subseteq S$. Therefore, it holds that $S = V(R''_n)$. □

Corollary 3.2. *For every convex polytope R''_n with $n \geq 27$ it holds that $\gamma_{conv}(R''_n) = |V(R''_n)| = 6n$.*

Proof. Since, for each graph G, it holds that $\gamma_{wconv}(G) \leq \gamma_{conv}(G) \leq |V(G)|$ and $\gamma_{wconv}(R''_n) = |V(R''_n)| = 6n$, we have $\gamma_{conv}(R''_n) = 6n$. □

4. Convex polytopes A_n and R_n

The classes of convex polytopes A_n and R_n (Figure 3) were introduced in [6] and [2] respectively. Their sets of vertices are

$$V(A_n) = V(R_n) = \{a_i, b_i, c_i \mid i \in [n-1]_0\},$$

while their sets of edges are

$$E(R_n) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, a_i b_i, b_i c_i, a_{i+1} b_i \mid i \in [n-1]_0\},$$
$$E(A_n) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, a_i b_i, b_i c_i, a_{i+1} b_i, b_{i+1} c_i \mid i \in [n-1]_0\}.$$

Theorem 4.1. *For every convex polytope A_n with $n \geq 27$ it holds that $\gamma_{wconv}(A_n) = \gamma_{conv}(A_n) = n$.*

Proof. First, we will prove that $\gamma_{conv}(A_n) \leq n$. Let $S = \{b_i \mid i \in [n-1]_0\}$. Now, we will prove that S is a convex dominating set of A_n. The set S is a dominating set of A_n, since for all $i \in [n-1]_0$ it holds that $a_i, b_i, c_i \in \mathcal{N}[b_i]$, so

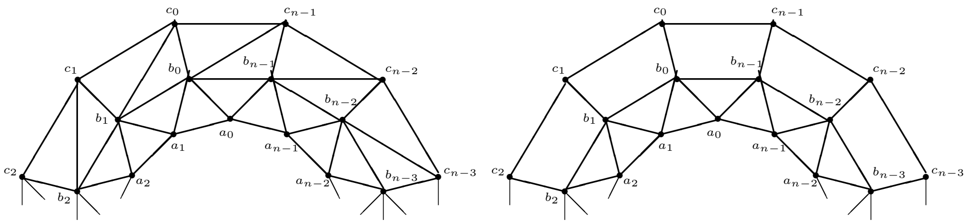

FIGURE 3. Polytopes A_n and R_n

$\bigcup_{i=0}^{n-1} \mathcal{N}[b_i] = \{a_i, b_i, c_i \mid i \in [n-1]_0\} = V(A_n)$. Also, S is a convex set, since all shortest paths between b-vertices contain only b-vertices. Therefore, S is a convex dominating set of A_n and $\gamma_{conv}(A_n) \leq |S| = n$. Note that the mentioned shortest paths are not always unique, so this argument can be used only for the convex domination, and not for the weak convex domination.

Next, we will prove that $\gamma_{wconv}(A_n) \geq n$. Let $k = \lfloor n/3 \rfloor$ and let S be a weak convex dominating set of A_n.

Let us observe vertices a_1 and c_2 from $V(A_n)$. Since S is a dominating set, there exist some neighbours of a_1 and c_2 which are in S, i.e. there are u and w such that $u \in S \cap \mathcal{N}[a_1]$ and $w \in S \cap \mathcal{N}[c_2]$. Since $\mathcal{N}[a_1] = \{a_1, a_0, a_2, b_0, b_1\}$ and $\mathcal{N}[c_2] = \{c_2, c_1, c_3, b_2, b_3\}$, it is easy to see that each shortest $u-w$ path contains some b-vertex, either b_0, b_1, b_2 or b_3. Since $u, w \in S$ and S is a weak convex set, one of the previously mentioned b-vertices is in S. Let $p \in \{0, 1, 2, 3\}$ be an index of the b-vertex which is in S.

If we observe vertices a_{k+1} and c_{k+2} from $V(A_n)$, using the same argumentation as before, there is $q \in \{k, k+1, k+2, k+3\}$ such that $b_q \in S$. And similarly, for vertices a_{2k+1} and c_{2k+2} from $V(A_n)$ there is $r \in \{2k, 2k+1, 2k+2, 2k+3\}$ such that $b_r \in S$.

For $n \geq 27$ it holds that $q - p \leq n/2 - 1$, $r - q \leq n/2 - 1$ and $n + p - r \leq n/2 - 1$, so all the shortest paths $b_p - b_q$, $b_q - b_r$ and $b_r - b_p$ are unique and each b-vertex is in one of them. Since S is a weak convex set, previously mentioned shortest paths are unique, and $b_p, b_q, b_r \in S$. So, every b-vertex is in S, i.e. for all i the vertex b_i belongs to S, which implies that $\{b_i \mid i \in [n-1]_0\} \subseteq S$. Therefore $|S| \geq n$.

Finally, since $\gamma_{wconv}(A_n) \leq \gamma_{conv}(A_n)$, $\gamma_{wconv}(A_n) \geq n$ and $\gamma_{conv}(A_n) \leq n$, we have $\gamma_{wconv}(A_n) = \gamma_{conv}(A_n) = n$. □

Theorem 4.2. *For every convex polytope R_n with $n \geq 27$ it holds that $\gamma_{conv}(R_n) = n$.*

Proof. The proof goes along similar lines as the proof of Theorem 4.1, and it will be omitted. □

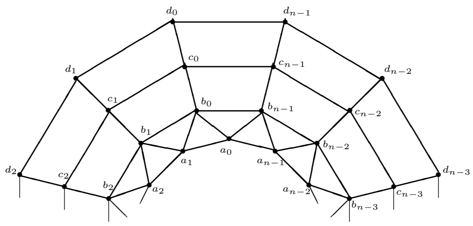

FIGURE 4. Polytope S_n

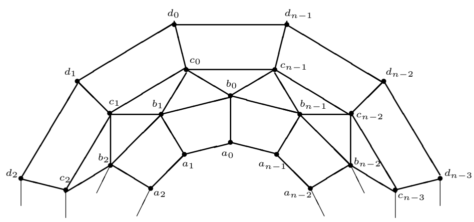

FIGURE 5. Polytope S''_n

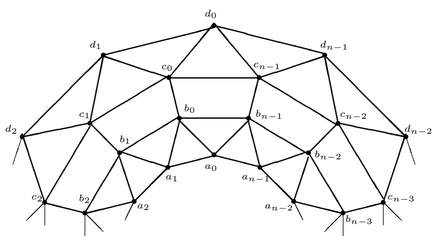

FIGURE 6. Polytope T_n

5. Convex polytopes S_n, S''_n and T_n

The class of convex polytopes S_n (Figure 4) was introduced in [6]. The sets of vertices and edges are

$$V(S_n) = \{a_i, b_i, c_i, d_i \mid i \in [n-1]_0\},$$
$$E(S_n) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, d_i d_{i+1}, a_i b_i, b_i c_i, c_i d_i, a_{i+1} b_i \mid i \in [n-1]_0\}.$$

The class of convex polytopes S''_n (Figure 5) was introduced in [8], where the authors also determined its metric dimension. The sets of vertices and edges defining these convex polytopes are

$$V(S''_n) = \{a_i, b_i, c_i, d_i \mid i \in [n-1]_0\},$$
$$E(S''_n) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, d_i d_{i+1}, a_i b_i, b_i c_i, c_i d_i, b_{i+1} c_i \mid i \in [n-1]_0\}.$$

The graph of convex polytope T_n (Figure 6), introduced in [8], consists of $4n$ 3-sided faces, n 4-sided faces and a pair of n sided faces. In this case, the vertices and edges are

$$V(T_n) = \{a_i, b_i, c_i, d_i \mid i \in [n-1]_0\},$$
$$E(T_n) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, d_i d_{i+1}, a_i b_i, b_i c_i, c_i d_i, a_{i+1} b_i, c_{i+1} d_i \mid i \in [n-1]_0\}.$$

Theorem 5.1. *For every $n \geq 27$ it holds that $\gamma_{wconv}(S_n) = \gamma_{conv}(S_n) = 2n$.*

Proof. First let us prove that $\gamma_{conv}(S_n) \leq 2n$. In order to accomplish this let us prove that the set $S = \{b_i, c_i \mid i \in [n-1]_0\}$ is a convex dominating set for S_n. The set S is obviously a dominating set of S_n, since for all $i \in [n-1]_0$ it holds that $a_i, b_i, c_i, d_i \in \mathcal{N}[b_i] \cup \mathcal{N}[c_i]$, so $\bigcup_{i=0}^{n-1} \mathcal{N}[b_i] \cup \bigcup_{i=0}^{n-1} \mathcal{N}[c_i] = \{a_i, b_i, c_i, d_i \mid i \in [n-1]_0\} = V(S_n)$. Also, S is a convex set, since:

- All shortest paths between b-vertices contain only b-vertices;
- all shortest paths between c-vertices contain only c-vertices;
- all shortest paths between any b-vertex and any c-vertex contain only some b-vertices and some c-vertices.

Therefore, S is a convex dominating set for S_n and $\gamma_{conv}(S_n) \leq |S| = 2n$. Note that the shortest paths mentioned are not always unique, so this argument can be used only for the convex domination, and not for the weak convex domination.

Next, we will prove that $\gamma_{wconv}(S_n) \geq 2n$. Using the same argumentation as in the proof of Theorem 4.1, it can be concluded that every b-vertex is in S, i.e. for all $i \in [n-1]_0$ it holds that $b_i \in S$, which implies that $\{b_i \mid i \in [n-1]_0\} \subseteq S$.

Similarly, it can be concluded that c-vertices belong to the set S, i.e. $\{c_i \mid i \in [n-1]_0\} \subseteq S$. Therefore, $\{b_i, c_i \mid i \in [n-1]_0\} \subseteq S$ so $|S| \geq 2n$.

Finally, since $\gamma_{wconv}(S_n) \leq \gamma_{conv}(S_n)$, $\gamma_{wconv}(S_n) \geq 2n$ and $\gamma_{conv}(S_n) \leq 2n$, we have $\gamma_{wconv}(S_n) = \gamma_{conv}(S_n) = 2n$. □

Theorem 5.2. *For every $n \geq 27$ it holds that*

$$\gamma_{wconv}(S''_n) = \gamma_{conv}(S''_n) = \gamma_{wconv}(T_n) = \gamma_{conv}(T_n) = 2n.$$

Proof. The proof goes along similar lines as the proof of Theorem 5.1, so it will be omitted. □

6. Convex polytopes Q_n and T''_n

Convex polytopes of the class Q_n (Figure 7) were introduced in [2]. They are specified by these sets of vertices and edges:

$$V(Q_n) = \{a_i, b_i, c_i, d_i \mid i \in [n-1]_0\},$$
$$E(Q_n) = \{a_i a_{i+1}, b_i b_{i+1}, d_i d_{i+1}, a_i b_i, b_i c_i, c_i d_i, b_{i+1} c_i \mid i \in [n-1]_0\}.$$

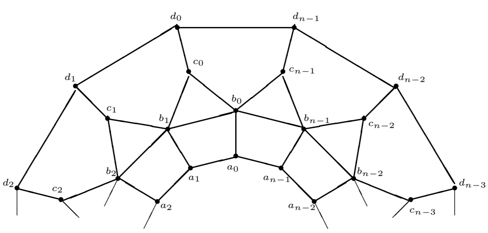

Figure 7. Polytope Q_n

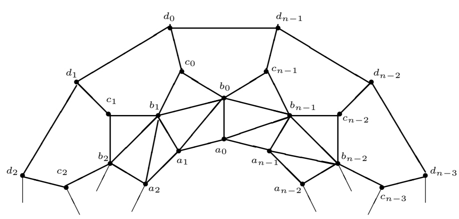

Figure 8. Polytope T''_n

Convex polytopes of the class T''_n (Figure 8) were introduced in [6]. They are specified by these sets of vertices and edges:

$$V(T''_n) = \{a_i, b_i, c_i, d_i \mid i \in [n-1]_0\},$$
$$E(T''_n) = \{a_i a_{i+1}, b_i b_{i+1}, d_i d_{i+1}, a_i b_i, b_i c_i, c_i d_i, b_{i+1} c_i, a_{i+1} b_i \mid i \in [n-1]_0\}.$$

Theorem 6.1. *For every $n \geq 27$ it holds that $\gamma_{wconv}(Q_n) = \gamma_{conv}(Q_n) = 2n$.*

Proof. We start by proving that $\gamma_{conv}(Q_n) \leq 2n$. Let $S = \{b_i, c_i \mid i \in [n-1]_0\}$. Now, we will prove that S is a convex dominating set of Q_n. The set S is obviously a dominating set of Q_n, since for all $i \in [n-1]_0$ it holds that $a_i, b_i, c_i, d_i \in \mathcal{N}[b_i] \cup \mathcal{N}[c_i]$, so $\bigcup_{i=0}^{n-1} \mathcal{N}[b_i] \cup \bigcup_{i=0}^{n-1} \mathcal{N}[c_i] = \{a_i, b_i, c_i, d_i \mid i \in [n-1]_0\} = V(Q_n)$. Also, S is a convex set, since:

- All shortest paths between b-vertices contain only b-vertices;
- all shortest paths between c-vertices, except endpoints, contain only b-vertices;
- all shortest paths between any b-vertex and any c-vertex, except c-endpoint, contain only b-vertices.

Therefore, S is a convex dominating set for Q_n and $\gamma_{conv}(Q_n) \leq |S| = 2n$. Note that the shortest paths mentioned are not always unique, so this argument can be used only for the convex domination, and not for the weak convex domination.

Next, we will prove that $\gamma_{wconv}(Q_n) \geq 2n$. Using the same argumentation as in the proof of Theorem 4.1, it can be concluded that every b-vertex is in S, i.e. for all $i \in [n-1]_0$ it holds that $b_i \in S$, which implies that $\{b_i \mid i \in [n-1]_0\} \subseteq S$.

Since vertices b_i such that $i \in [n-1]_0$ are in S, and S is a weak convex dominating set, from the uniqueness of the shortest path $b_j - c_j - d_j$ the following implication holds:

$$Q_n : \quad d_j \in S \Rightarrow c_j \in S. \tag{6.1}$$

Let $k = \lfloor \frac{n-1}{2} \rfloor$ and $i \in [n-1]_0$. Therefore, it is easy to see that $2k+1 \leq n \leq 2k+2$. To obtain domination for all d-vertices, S must contain some of c-vertices or/and d-vertices. Precisely, for $\mathcal{N}[d_i] = \{d_i, d_{i-1}, d_{i+1}, c_i\}$ and $\mathcal{N}[d_{i+k}] = \{d_{i+k}, d_{i+k-1}, d_{i+k+1}, c_{i+k}\}$, the following conditions must hold: $\mathcal{N}[d_i] \cap S \neq \emptyset$ and $\mathcal{N}[d_{i+k}] \cap S \neq \emptyset$.

There are 9 nontrivial possible cases. As it can be seen from Table 2, each of the 7 other possibilities can be directly reduced to one of the following cases.

Case 1: *There is i such that $d_i \in S$ and $c_{k+i} \in S$.* Let us consider the unique shortest path $d_i - d_{i+1} - \cdots - d_{k+i-1} - d_{k+i} - c_{k+i}$ whose endpoints are in S. Since S is a weak convex dominating set, we have that $d_i, d_{i+1}, \ldots, d_{k+i-1}, d_{k+i} \in S$. From (6.1) it follows that $c_i, c_{i+1}, \ldots, c_{k+i-1}, c_{k+i} \in S$. As S contains at least $k+1$ c-vertices and $k+1$ d-vertices and all n b-vertices, and $2k+2 \geq n$, we have $|S| \geq n + 2k + 2 \geq 2n$.

Case 2: *There is i such that $c_i \in S$ and $d_{k+i} \in S$.* Similarly to Case 1, we can consider the unique shortest path $c_i - d_i - d_{i+1} - \cdots - d_{k+i-1} - d_{k+i}$ whose endpoints are in S. Again, since S is a weak convex dominating set, $d_i, d_{i+1}, \ldots, d_{k+i-1}, d_{k+i} \in S$. In the same way, from (6.1) it follows that $c_i, c_{i+1}, \ldots, c_{k+i-1}, c_{k+i} \in S$, again giving $|S| \geq n + 2k + 2 \geq 2n$.

Case 3: *There is i such that $c_i \in S$ and $d_{k+i-1} \in S$.* Similarly to Case 2, we can consider the unique shortest path $c_i - d_i - d_{i+1} - \cdots - d_{k+i-2} - d_{k+i-1}$ whose endpoints are in S. Again, since S is a weak convex dominating set,

$d_i, d_{i+1}, \ldots, d_{k+i-2}, d_{k+i-1} \in S$. In the same way, from (6.1) it follows that $c_i, c_{i+1}, \ldots, c_{k+i-2}, c_{k+i-1} \in S$. Let $S' = \{b_l \mid l \in [n-1]_0\} \cup \{c_j, d_j \mid j \in [k+i-1]_0, j \geq i\}$; then $S' \subseteq S$ and $|S'| \geq 2n-2$. Since S is a dominating set, at least one vertex from $\mathcal{N}[d_{i-2}]$ and at least one vertex from $\mathcal{N}[d_{i-5}]$ must be in S. We have $\mathcal{N}[d_{i-2}] \cap \mathcal{N}[d_{i-5}] = \emptyset$, and $n \geq 27$, which implies that $\mathcal{N}[d_{i-2}] \cap S' = \emptyset$ and $\mathcal{N}[d_{i-5}] \cap S' = \emptyset$. Therefore, we have $|S| \geq |S'| + 2 = 2n$.

Case 4: *There is i such that $c_i \in S$ and $d_{k+i+1} \in S$.* If n is *odd* ($n = 2k+1$), we can consider the unique shortest path $d_{k+i+1} - d_{k+i+2} - \cdots - d_{i-1} - d_i - c_i$ whose endpoints are in S. Again, since S is a weak convex dominating set, $d_{k+i+1}, d_{k+i+2}, \ldots, d_{i-1}, d_i \in S$. In the same way, from (6.1) it follows that $c_{k+i+1}, c_{k+i+2}, \ldots, c_{i-1}, c_i \in S$, giving $|S| \geq n + 2(k+1) = n + 2k + 2 = 2n + 1 > 2n$. Otherwise, if n is *even* ($n = 2k+2$), we can consider the two shortest paths between vertices c_i and d_{k+i+1}:

- shortest path $c_i - d_i - d_{i+1} - \cdots - d_{k+i} - d_{k+i+1}$;
- shortest path $d_{k+i+1} - d_{k+i+2} - \cdots - d_{i-1} - d_i - c_i$.

Since both endpoints c_i and d_{k+i+1} are in S, and S is a weak convex dominating set, at least one shortest path must be in S. Therefore, we have that $d_i, d_{i+1}, \ldots, d_{k+i}, d_{k+i+1} \in S$ or $d_{k+i+1}, d_{k+i+2}, \ldots, d_{i-1}, d_i \in S$, which leads to $c_i, c_{i+1}, \ldots, c_{k+i}, c_{k+i+1} \in S$ or $c_{k+i+1}, c_{k+i+2}, \ldots, c_{i-1}, c_i \in S$. In both subcases, S must have at least n b-vertices, $k+2$ c-vertices and $k+2$ d-vertices, giving $|S| \geq n + 2(k+2) = n + 2k + 4 = 2n + 2 > 2n$.

Case 5: *There is i such that $d_{i+1} \in S$ and $c_{k+i} \in S$.* Similarly to Case 3, we can consider the unique shortest path $d_{i+1} - d_{i+2} - \cdots - d_{k+i-1} - d_{k+i} - c_{k+i}$ whose endpoints are in S. Again, since S is a weak convex dominating set, $d_{i+1}, d_{i+2}, \ldots, d_{k+i-1}, d_{k+i} \in S$. In the same way, (6.1) implies that $c_{i+1}, c_{i+2}, \ldots, c_{k+i-1}, c_{k+i} \in S$. Let $S' = \{b_l \mid l \in [n-1]_0\} \cup \{c_j, d_j \mid j \in [k+i], j \geq i+1\}$; it follows that $S' \subseteq S$ and $|S'| \geq 2n-2$. Since S is a dominating set, at least one vertex from $\mathcal{N}[d_{i-1}]$ and at least one vertex from $\mathcal{N}[d_{i-4}]$ must be in S. We have $\mathcal{N}[d_{i-1}] \cap \mathcal{N}[d_{i-4}] = \emptyset$ and $n \geq 27$, which imply that $\mathcal{N}[d_{i-1}] \cap S' = \emptyset$ and $\mathcal{N}[d_{i-4}] \cap S' = \emptyset$. Therefore, we have $|S| \geq |S'| + 2 = 2n$.

Case 6: *There is i such that $d_{i+1} \in S$ and $d_{k+i-1} \in S$.* Similarly to Cases 3 and 5, we can consider the unique shortest path $d_{i+1} - d_{i+2} - \cdots - d_{k+i-2} - d_{k+i-1}$ whose endpoints are in S. Again, since S is a weak convex dominating set, $d_{i+1}, d_{i+2}, \ldots, d_{k+i-2}, d_{k+i-1} \in S$. In the same way, (6.1) implies that $c_{i+1}, c_{i+2}, \ldots, c_{k+i-2}, c_{k+i-1} \in S$. Let $S' = \{b_l \mid l \in [n-1]_0\} \cup \{c_j, d_j \mid j \in [k+i-1], j \geq i+1\}$, which leads to $S' \subseteq S$ and $|S'| \geq 2n-4$. Since S is a dominating set, at least one vertex from neighbourhoods $\mathcal{N}[d_{i-1}]$, $\mathcal{N}[d_{i-4}]$, $\mathcal{N}[d_{i-7}]$ and $\mathcal{N}[d_{i-10}]$ must be in S. It is easy to see that all four mentioned neighbourhoods are mutually disjoint and, since $n \geq 27$, we conclude that each neighbourhood has an empty intersection with S', so $|S| \geq |S'| + 4 = 2n$.

TABLE 2. Other possibilities in Theorem 6.1

Possibility	*Reduced to case*	*Reason*
$(\exists i)\ (d_i \in S \wedge d_{k+i} \in S)$	1	$d_{k+i} \in S \Rightarrow c_{k+i} \in S$
$(\exists i)\ (d_i \in S \wedge d_{k+i-1} \in S)$	3	$d_i \in S \Rightarrow c_i \in S$
$(\exists i)\ (d_i \in S \wedge d_{k+i+1} \in S)$	4	$d_i \in S \Rightarrow c_i \in S$
$(\exists i)\ (d_{i-1} \in S \wedge d_{k+i} \in S)$	7	$d_{k+i} \in S \Rightarrow c_{k+i} \in S$
$(\exists i)\ (d_{i-1} \in S \wedge d_{k+i-1} \in S)$	1	$i' = i-1,\ d_{k+i} \in S \Rightarrow c_{k+i} \in S$
$(\exists i)\ (d_{i+1} \in S \wedge d_{k+i} \in S)$	5	$d_{k+i} \in S \Rightarrow c_{k+i} \in S$
$(\exists i)\ (d_{i+1} \in S \wedge d_{k+i+1} \in S)$	1	$i' = i+1,\ d_{k+i} \in S \Rightarrow c_{k+i} \in S$

Case 7: *There is i such that $d_{i-1} \in S$ and $c_{k+i} \in S$.* Similarly to Case 4, for *odd* n ($n = 2k+1$) we have the unique shortest path $c_{k+i} - d_{k+i} - d_{k+i+1} - \cdots - d_{i-2} - d_{i-1}$ whose endpoints are in S. Again, since S is a weak convex dominating set, $d_{k+i}, d_{k+i+1}, \ldots, d_{i-2}, d_{i-1} \in S$. In the same way, (6.1) implies that $c_{k+i}, c_{k+i+1}, \ldots, c_{i-2}, c_{i-1} \in S$, giving $|S| \geq n+2(k+1) = n+2k+2 = 2n+1 > 2n$. Otherwise, for *even* n ($n = 2k+2$), we can consider the two shortest paths between vertices d_{i-1} and c_{k+i}:

- shortest path $d_{i-1} - d_i - \cdots - d_{k+i-1} - d_{k+i} - c_{k+i}$;
- shortest path $c_{k+i} - d_{k+i} - d_{k+i+1} - \cdots - d_{i-2} - d_{i-1}$.

Since both endpoints d_{i-1} and c_{k+i} are in S, and S is a weak convex dominating set, at least one shortest path must be in S. Therefore, $d_{i-1}, d_i, \ldots, d_{k+i-1}, d_{k+i} \in S$ or $d_{k+i}, d_{k+i+1}, \ldots, d_{i-2}, d_{i-1} \in S$, which leads to $c_{i-1}, c_i, \ldots, c_{k+i-1}, c_{k+i} \in S$ or $c_{k+i}, c_{k+i+1}, \ldots, c_{i-2}, c_{i-1} \in S$. In both subcases, S must have at least n b-vertices, $k+2$ c-vertices and $k+2$ d-vertices, giving $|S| \geq n + 2(k+2) = n + 2k + 4 = 2n + 2 > 2n$.

Case 8: *There is i such that $d_{i-1} \in S$ and $d_{k+i+1} \in S$.* Let us consider the unique shortest path $d_{k+i+1} - d_{k+i+2} - \cdots - d_{i-2} - d_{i-1}$ whose endpoints are in S. Since S is a weak convex dominating set, $d_{k+i+1}, d_{k+i+2}, \ldots, d_{i-2}, d_{i-1} \in S$. From (6.1) it follows that $c_{k+i+1}, c_{k+i+2}, \ldots, c_{i-2}, c_{i-1} \in S$. Let $S' = \{b_l \mid l \in [n-1]_0\} \cup \{c_j, d_j \mid j \in [n+i-1],\ j \geq k+i+1\}\}$. First, S' is a subset of S. Second, since $n+i-1-(k+i+1)+1 = n-k-1$, S' must have n b-vertices, $n-k-1$ c-vertices and $n-k-1$ d-vertices, giving $|S'| = n + 2(n-k-1) = 3n - 2k - 2 \geq 2n - 1$. Since S is a dominating set, at least one vertex from $\mathcal{N}[d_{i+1}]$ must be in S. From $n \geq 27$ we have $\mathcal{N}[d_{i+1}] \cap S' = \emptyset$. Therefore, we have $|S| \geq |S'| + 1 \geq 2n$.

Case 9: *For all i it holds that $c_i \in S$ and $c_{k+i} \in S$.* Since $i \in [n-1]_0$, it holds that all c-vertices are in S. As all b-vertices are already in S, we have $|S| \geq 2n$.

Finally, since $|S| \geq 2n$ for all nine cases, $\gamma_{wconv}(Q_n) \geq 2n$. Since $\gamma_{wconv}(Q_n) \leq \gamma_{conv}(Q_n)$, $\gamma_{wconv}(Q_n) \geq 2n$ and $\gamma_{conv}(Q_n) \leq 2n$, we have that $\gamma_{wconv}(Q_n) = \gamma_{conv}(Q_n) = 2n$. □

Theorem 6.2. *For every $n \geq 27$ it holds that $\gamma_{wconv}(T''_n) = \gamma_{conv}(T''_n) = 2n$.*

Proof. The proof goes along similar lines as the proof of Theorem 6.1, so it will be omitted. □

7. Convex polytopes B_n, C_n and E_n

The graph of convex polytope B_n (Figure 9) was introduced in [2] and consists of $2n$ 4-sided faces, n 3-sided faces, n 5-sided faces and a pair of n-sided faces. The set of vertices is

$$V(B_n) = \{a_i, b_i, c_i, d_i, e_i \mid i \in [n-1]_0\}$$

and the set of edges is

$$E(B_n) = \{a_i a_{i+1}, b_i b_{i+1}, d_i d_{i+1}, e_i e_{i+1}, a_i b_i, b_i c_i, b_{i+1} c_i, c_i d_i, d_i e_i \mid i \in [n-1]_0\}.$$

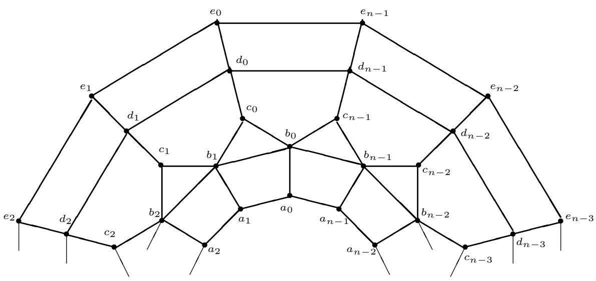

Figure 9. Polytope B_n

Convex polytopes C_n (Figure 10) were introduced in [9] and consist of $3n$ 3-sided faces, n 4-sided faces, n 5-sided faces and a pair of n-sided faces. Their sets of vertices and edges are

$$V(C_n) = \{a_i, b_i, c_i, d_i, e_i \mid i \in [n-1]_0\}$$

and

$$E(C_n) = \{a_i a_{i+1}, b_i b_{i+1}, d_i d_{i+1}, e_i e_{i+1}, a_i b_i, b_i c_i, c_i d_i, d_i e_i, b_{i+1} c_i, d_{i-1} e_i \mid i \in [n-1]_0\}.$$

The graph of convex polytope E_n (Figure 11), similar to the C_n, introduced in [9], consists of $5n$ 3-sided faces, n 5-sided faces and a pair of n-sided faces, where

$$V(E_n) = \{a_i, b_i, c_i, d_i, e_i \mid i \in [n-1]_0\}$$
$$E(E_n) = \{a_i a_{i+1}, b_i b_{i+1}, d_i d_{i+1}, e_i e_{i+1}, a_i b_i, b_i c_i, c_i d_i, d_i e_i, b_{i+1} c_i, d_{i-1} e_i, a_{i-1} b_i \mid i \in [n-1]_0\}.$$

Theorem 7.1. *For every $n \geq 27$ it holds that $\gamma_{wconv}(B_n) = \gamma_{conv}(B_n) = 3n$.*

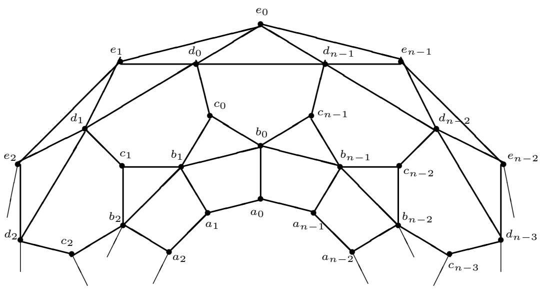

FIGURE 10. Polytope C_n

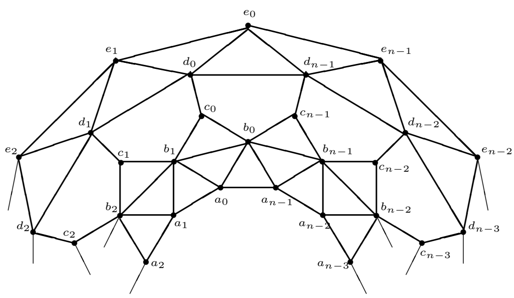

FIGURE 11. Polytope E_n

Proof. First we will prove that $\gamma_{conv}(B_n) \leq 3n$. Let $S = \{b_i, c_i, d_i \mid i \in [n-1]_0\}$. Now, we will prove that S is a convex dominating set for B_n. The set S is obviously a dominating set of B_n, since for all $i \in [n-1]_0$ it holds that $a_i, b_i, c_i, d_i, e_i \in \mathcal{N}[b_i] \cup \mathcal{N}[c_i] \cup \mathcal{N}[d_i]$, so $\bigcup_{i=0}^{n-1} \mathcal{N}[b_i] \cup \bigcup_{i=0}^{n-1} \mathcal{N}[c_i] \cup \bigcup_{i=0}^{n-1} \mathcal{N}[d_i] = \{a_i, b_i, c_i, d_i, e_i \mid i \in [n-1]_0\} = V(B_n)$. Also, S is a convex set, since each shortest path between two vertices from $\{b_i, c_i, d_i \mid i \in [n-1]_0\}$ contains only b-vertices, c-vertices or d-vertices.

Therefore, S is a convex dominating set for B_n and $\gamma_{conv}(B_n) \leq |S| = 3n$. Note that, as before, shortest paths are not always unique, so this argument can be used only for the convex domination, and not for the weak convex domination.

Next, we will prove that $\gamma_{wconv}(B_n) \geq 3n$. Using the same argumentation as in the proof of Theorem 4.1, it can be concluded that every b-vertex is in S, i.e. $\{b_i \mid i \in [n-1]_0\} \subseteq S$.

Using the same argumentation as in the proof of Theorem 4.1, it can be concluded that $\{d_i \mid i \in [n-1]_0\} \subseteq S$.

For an arbitrary $i \in [n-1]_0$ the shortest path between b_i and d_i is unique $(b_i - c_i - d_i)$, and has both endpoints in S $(b_i, d_i \in S)$, so from the fact that S is a weak convex dominating set it must hold that for all $i \in [n-1]_0$, $c_i \in S$. Due to the inclusion $\{c_i \mid i \in [n-1]_0\} \subset S$ and previous facts, we have $\{b_i, c_i, d_i \mid i \in [n-1]_0\} \subseteq S \Rightarrow |S| \geq 3n$.

Finally, since $\gamma_{wconv}(B_n) \leq \gamma_{conv}(B_n)$, $\gamma_{wconv}(B_n) \geq 3n$ and $\gamma_{conv}(B_n) \leq 3n$, we have that $\gamma_{wconv}(B_n) = \gamma_{conv}(B_n) = 3n$. □

Theorem 7.2. *For every $n \geq 27$ it holds that*

$$\gamma_{wconv}(C_n) = \gamma_{conv}(C_n) = \gamma_{wconv}(E_n) = \gamma_{conv}(E_n) = 3n.$$

Proof. The proof goes along similar lines as the proof of Theorem 7.1, so it will be omitted. □

8. Convex polytopes U_n

The convex polytopes U_n, $n \geq 5$ (Figure 12), were introduced in [8]. They consist of n 4-sided faces, $2n$ 5-sided faces and a pair of n-sided faces, having these vertex and edge sets:

$$V(U_n) = \{a_i, b_i, c_i, d_i, e_i \mid i \in [n-1]_0\},$$
$$E(U_n) = \{a_i a_{i+1}, b_i b_{i+1}, e_i e_{i+1}, a_i b_i, b_i c_i, c_i d_i, d_i e_i, c_{i+1} d_i \mid i \in [n]\}.$$

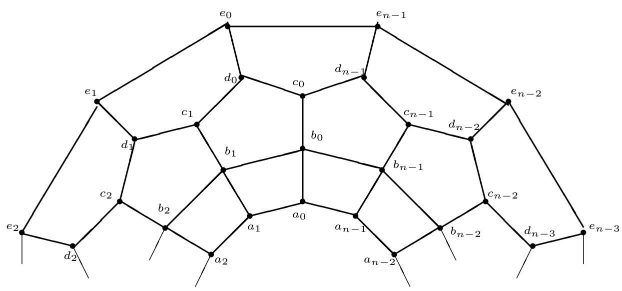

FIGURE 12. Polytope U_n

Theorem 8.1. *For every $n \geq 27$ it holds that $\gamma_{wconv}(U_n) = \gamma_{conv}(U_n) = 4n$.*

Proof. First, we will prove that $\gamma_{conv}(U_n) \leq 4n$. Let $S = \{b_i, c_i, d_i, e_i \mid i \in [n-1]_0\}$. Now, we will prove that S is a convex dominating set for U_n. The set S is obviously a dominating set of U_n, since for all $i \in [n-1]_0$ it holds that $a_i, b_i, c_i, d_i, e_i \in \mathcal{N}[b_i] \cup \mathcal{N}[c_i] \cup \mathcal{N}[d_i] \cup \mathcal{N}[e_i]$, so $\bigcup_{i=0}^{n-1} \mathcal{N}[b_i] \cup \bigcup_{i=0}^{n-1} \mathcal{N}[c_i] \cup \bigcup_{i=0}^{n-1} \mathcal{N}[d_i] \cup \bigcup_{i=0}^{n-1} \mathcal{N}[e_i] = \{a_i, b_i, c_i, d_i, e_i \mid i \in [n-1]_0\} = V(U_n)$. Also, S is a convex set, since each shortest path between two vertices from $\{b_i, c_i, d_i, e_i \mid i \in [n-1]_0\}$ contains only b-vertices, c-vertices, d-vertices or e-vertices.

Therefore, S is a convex dominating set for U_n and $\gamma_{conv}(U_n) \leq |S| = 4n$. Note that, as previously, mentioned shortest paths are not always unique, so this argument can be used only for the convex domination, and not for the weak convex domination.

Next, we will prove that $\gamma_{wconv}(U_n) \geq 4n$. Using the same argumentation as in the proof of Theorem 4.1, it can be concluded that every b-vertex is in S, i.e. for all $i \in [n-1]_0$ it holds that $b_i \in S$ implies that $\{b_i \mid i \in [n-1]_0\} \subseteq S$.

Also, we will prove that all e-vertices must be in the weak convex dominating set of U_n, i.e. for all $i \in [n-1]_0$ it holds that $e_i \in S$. We can notice that by removing vertices $\{a_i \mid i \in [n-1]_0\}$ from U_n we obtain D_n, and $\{b_i \mid i \in [n-1]_0\} \subseteq S$. Using the same argumentation for vertices $\{d_i \mid i \in [n-1]_0\}$ as in Theorem 2.1, we can conclude that $\{e_i \mid i \in [n-1]_0\} \subseteq S$.

Similarly to the proof of Theorem 2.1, for an arbitrary $i \in [n-1]_0$ the shortest path between b_i and e_i is unique $(b_i - c_i - d_i - e_i)$, and has both endpoints in S $(b_i, e_i \in S)$. From the fact that S is a weak convex dominating set it must hold that for all $i \in [n-1]_0$ the vertices b_i and d_i belong to S. Since $\{c_i, d_i \mid i \in [n-1]_0\} \subset S$, and from previous facts, it stands that $\{b_i, c_i, d_i, e_i \mid i \in [n-1]_0\} \subseteq S$, implying that $|S| \geq 4n$.

Finally, since $\gamma_{wconv}(U_n) \leq \gamma_{conv}(U_n)$, $\gamma_{wconv}(U_n) \geq 4n$ and $\gamma_{conv}(U_n) \leq 4n$, we have that $\gamma_{wconv}(U_n) = \gamma_{conv}(U_n) = 4n$. □

9. Conclusion

This paper presents values of the weak convex and the convex domination numbers for all known convex polytopes. Validity of these values is proven for $n \geq 27$, while for $n \leq 26$ the values are computed using the exact method. Values obtained by the exact method satisfy formulas obtained theoretically, except in some particular cases which will be mentioned below.

The results obtained are summarized in Table 3. Each row describes the weak convex and the convex dominating sets with minimal cardinality, and the corresponding weak convex and convex domination numbers for a certain polytope. The first column, labeled with G, contains the type of polytope. The second and third columns, labeled with $|V(G)|$ and $|E(G)|$, contain the number of vertices and edges, respectively. The fourth column, labeled *Cond.*, denotes the polytopes for which a certain result applies. In this column, 'gc' denotes that the result stands in the general case, while $n = 5$ means that the result presented with given row stands for the polytope where $n = 5$. The fifth column, labeled *Type*, describes the type of domination: 'wconv' for the weak convex domination and 'conv' for the convex domination, while 'wconv, conv' in a certain row means that the result stands for both weak convex and convex domination. The sixth column, labeled *Card.*, contains the weak convex and convex domination numbers. The seventh column, labeled *Basis*, contains the weak convex and the convex dominating set with minimal cardinality. For example, polytope B_n, in general, has the weak convex and convex dominating sets $\{b_i, c_i, d_i \mid i \in [n-1]_0\}$ with $3n$ elements, with one exception for $n = 5$ where the weak convex dominating set with minimal cardinality 14 is $\{b_i, c_i, d_i, e_i \mid i \in \{0, 1, 2\}\} \cup \{b_3, b_4\}$.

Future work could be directed to obtaining the weakly convex and convex domination numbers of some other classes of graphs. Another promising direction for future research would be to calculate other graph invariants of convex polytopes.

TABLE 3. Summary of results

G	$\|V(G)\|$	$\|E(G)\|$	*Cond.*	*Type*	*Card.*	*Basis*
A_n	$3n$	$7n$	gc	wconv, conv	n	$\{b_i \mid i \in [n-1]_0\}$
B_n	$5n$	$9n$	gc	wconv, conv	$3n$	$\{b_i, c_i, d_i \mid i \in [n-1]_0\}$
			$n=5$	wconv	14	$\{b_i, c_i, d_i, e_i \mid i = 0, 1, 2\} \cup \{b_3, b_4\}$
C_n	$5n$	$10n$	gc	wconv, conv	$3n$	$\{b_i, c_i, d_i \mid i \in [n-1]_0\}$
D_n	$4n$	$6n$	gc	wconv, conv	$4n$	$\{a, b_i, c_i, d_i \mid i \in [n-1]_0\}$
			$n=5$	wconv	10	$\{a_2, a_3, a_4, b_2, b_4, c_1, c_4, d_0, d_1, d_4\}$
E_n	$5n$	$11n$	gc	wconv, conv	$3n$	$\{b_i, c_i, d_i \mid i \in [n-1]_0\}$
Q_n	$4n$	$7n$	gc	wconv, conv	$2n$	$\{b_i, c_i \mid i \in [n-1]_0\}$
R_n	$3n$	$6n$	gc	wconv, conv	n	$\{b_i \mid i \in [n-1]_0\}$
R''_n	$6n$	$9n$	gc	wconv, conv	$6n$	$\{a, b_i, c_i, d_i, e_i, f_i \mid i \in [n-1]_0\}$
			$n=5$	wconv	20	$\{b_i, c_i, d_i, e_i \mid i \in \{0, 1, 2, 3, 4\}\}$
S_n	$4n$	$8n$	gc	wconv, conv	$2n$	$\{b_i, c_i \mid i \in [n-1]_0\}$
S''_n	$4n$	$8n$	gc	wconv, conv	$2n$	$\{b_i, c_i \mid i \in [n-1]_0\}$
T_n	$4n$	$9n$	gc	wconv, conv	$2n$	$\{b_i, c_i \mid i \in [n-1]_0\}$
			$n=5$	wconv	8	$\{a_1, b_0, b_3, b_4, c_0, c_3, c_4, d_0\}$
T''_n	$4n$	$8n$	gc	wconv, conv	$2n$	$\{b_i, c_i \mid i \in [n-1]_0\}$
U_n	$5n$	$8n$	gc	wconv, conv	$4n$	$\{b_i, c_i, d_i, e_i \mid i \in [n-1]_0\}$
			$n=5$	wconv	13	$\{a_0, a_3, a_4, b_0, b_3, b_4, c_0, c_3, d_0, d_2, e_0, e_1, e_2\}$

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