

RECURRENT CURVATURE OVER FOUR-DIMENSIONAL HOMOGENEOUS MANIFOLDS

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ABSTRACT. Recurrent curvature properties are considered on four-dimensional pseudo-Riemannian homogeneous manifolds with non-trivial isotropy, and also on some geometric manifolds.

1. INTRODUCTION

It is well known that the most important geometric object living on a manifold is the curvature tensor. Many different conditions may be applied on the curvature tensor, and each of them is the geometric interpretation of an outer property. Being parallel is a famous condition on the curvature tensor, which determines the (locally) symmetric spaces. Locally symmetric manifolds have an important application in various fields of sciences like applied physics. In this way, manifolds with recurrent curvature, as a generalization of locally symmetric spaces, will find a special place. Many authors focused their studies on spaces with recurrent curvature. We present a brief survey about them.

Spaces with recurrent curvature were firstly introduced and characterized by H. S. Ruse [10]. He investigated these spaces and showed that the necessary and sufficient condition for a three-dimensional Riemannian manifold to have recurrent curvature is to accept a parallel vector field. Also in dimension 2, he showed that every Riemannian manifold has recurrent curvature.

A. G. Walker [13] made more extensive studies and proved several interesting results; probably the most important was the characterization of three-spaces with recurrent curvature, spaces known as strictly Walker manifolds.

In recent years, for the Lorentzian setting, several authors have studied three-manifolds with recurrent curvature; for example, E. García-Río et al. obtained a complete description of all locally homogeneous Lorentzian manifolds with recurrent curvature [5]. Using this classification, G. Calvaruso and A. Zaeim [1] investigated symmetries on this space and computed Ricci and curvature collineations

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on Lorentzian three-manifolds with recurrent curvature. A. Haji-Badali [7] investigated these spaces and showed that a Lorentzian three-manifold with recurrent curvature does not accept non-trivial proper gradient Ricci solitons; he also obtained a classification for Ricci almost solitons on these spaces.

Due to the previous research, we concentrate our work on the study of the recurrent curvature property of four-dimensional homogeneous pseudo-Riemannian manifolds. These examples contain important classes of pseudo-Riemannian manifolds and it is worthwhile to study different geometric properties. Our study is based on the classification of four-dimensional homogeneous manifolds with non-trivial isotropy, which was given by Komrakov in [9]. We complete this study by checking the recurrent condition for these spaces, obtaining a new classification for homogeneous four-dimensional pseudo-Riemannian manifolds with recurrent curvature.

This paper is organized in the following way. The second section contains preliminaries and some basic facts necessary for our study. Section 3 is devoted to the main theorem of this study, i.e., the classification of homogeneous four-spaces with recurrent curvature, and then we consider the geometry of each class in Section 4. In the last section, the recurrent curvature property is considered on some geometric manifolds with physical applications, e.g. Walker spaces.

2. PRELIMINARIES

A pseudo-Riemannian manifold (M, g) is called *homogeneous* if for any pair of points $p, q \in M$, there exists an isometry φ on M such that $\varphi(p) = q$. Homogeneous spaces with many interesting geometric properties are studied by several authors [2, 3, 5, 9]. It is well known that a homogeneous space may be realized and studied as a quotient space G/H , where G is the group of isometries, which acts transitively on (M, g) , and H is the isotropy subgroup.

We denote the Lie algebra of G and its subgroup H by \mathfrak{g} and \mathfrak{h} , respectively, and by \mathfrak{m} we denote the subspace of \mathfrak{g} complementary to \mathfrak{h} . One of the most important properties of homogeneous spaces is the existence of a one-to-one correspondence between invariant metrics on $M = G/H$ and non-degenerate symmetric bilinear forms on \mathfrak{m} . The pair $(\mathfrak{g}, \mathfrak{h})$ uniquely defines the following map:

$$\psi : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{m}), \quad \psi(x)(y) = [x, y]_{\mathfrak{m}}, \quad \text{for all } x \in \mathfrak{g}, y \in \mathfrak{m}.$$

By using this map, we can determine bilinear forms g on \mathfrak{m} with respect to the basis $\{e_1, \dots, e_r, u_1, \dots, u_n\}$, where $\{e_j\}_{j=1}^r$ and $\{u_i\}_{i=1}^n$ are bases for \mathfrak{h} and \mathfrak{m} , respectively. Now, the necessary and sufficient condition for the bilinear form g to be invariant is

$${}^t\psi(x) \circ g + g \circ \psi(x) = 0, \quad \text{for all } x \in \mathfrak{g}.$$

The Levi-Civita connection is computed by

$$\Lambda(x)(y_{\mathfrak{m}}) = \frac{1}{2}[x, y]_{\mathfrak{m}} + v(x, y), \quad \text{for all } x, y \in \mathfrak{g}, \quad (2.1)$$

where $v : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{m}$ is the \mathfrak{h} -invariant symmetric mapping uniquely determined by

$$2g(v(x, y), z_{\mathfrak{m}}) = g(x_{\mathfrak{m}}, [z, y]_{\mathfrak{m}}) + g(y_{\mathfrak{m}}, [z, x]_{\mathfrak{m}}), \quad \text{for all } x, y, z \in \mathfrak{g}.$$

Then, the curvature tensor is determined by

$$\begin{aligned}
 R : \mathfrak{m} \times \mathfrak{m} &\rightarrow \mathfrak{gl}(\mathfrak{m}) \\
 (x, y) &\rightarrow [\Lambda(x), \Lambda(y)] - \Lambda([x, y]),
 \end{aligned}
 \tag{2.2}$$

and the Ricci tensor ρ of the metric g with respect to the basis $\{u_i\}_{i=1}^4$ is

$$\rho(u_i, u_j) = \Sigma g(R(u_k, u_i)u_j, u_k).
 \tag{2.3}$$

We recall the following definition.

Definition 2.1. A pseudo-Riemannian manifold (M, g) is called a *recurrent curvature manifold* if there exists a 1-form ω such that the following relation holds:

$$\nabla \mathbf{R} = \omega \otimes \mathbf{R},
 \tag{2.4}$$

where \mathbf{R} is a $(0, 4)$ curvature tensor.

Generally, for any tensor field T on (M, g) we can consider the spaces with recurrent tensor field T by studying the existence of a 1-form ω such that $\nabla T = \omega \otimes T$. Trivially, a locally symmetric manifold, i.e., $\nabla \mathbf{R} = 0$, has recurrent curvature.

3. FOUR-DIMENSIONAL HOMOGENEOUS MANIFOLDS WITH NON-TRIVIAL ISOTROPY

Now, we start with the classification of four-dimensional homogeneous manifolds with non-trivial isotropy, which is given by Komrakov in [9], and from now on, in this section and the following one, we will assume that $(M = G/H, g)$ is an arbitrary pseudo-Riemannian four-dimensional homogeneous manifold with non-trivial isotropy, equipped with an invariant metric g .

Theorem 3.1. *The homogeneous manifold (M, g) has non-trivially recurrent curvature (not locally symmetric) if and only if one of the cases of Table 1 occurs, where $\{\theta^i\}_{i=1}^4$ is the dual basis of $\{u_i\}_{i=1}^4$ and ω_i are the components of the one-form ω with respect to the basis $\{\theta^i\}_{i=1}^4$.*

Proof. We start by the case-by-case study of homogeneous spaces with non-trivial isotropy in [9]. Referring to [3], where a complete list of locally symmetric examples is presented, we restrict our study to the non-locally symmetric cases.

We show the details for the case 1.1¹.01, and the other cases may be treated in a similar way. As stated in the preliminaries, every invariant metric on the homogeneous space $M = G/H$ is in one-to-one correspondence with an invariant inner product on \mathfrak{m} . For this case, there exists a basis $\{e_1, u_1, \dots, u_4\}$ of \mathfrak{g} ; the nonzero brackets are

$$[e_1, u_1] = u_1, \quad [e_1, u_3] = -u_3, \quad [u_1, u_3] = [u_2, u_4] = u_2, \quad [u_3, u_4] = u_3,$$

case	invariant metric	nonzero ω_i and conditions
1.1 ¹ : 1	$2a\theta^1\theta^3 + b\theta^2\theta^2 + 2c\theta^2\theta^4 + d\theta^4\theta^4,$ $a(c^2 - bd) \neq 0$	$b = 0, \omega_4 = -2$
1.1 ¹ : 2	"	$b = 0, \omega_4 = -2\lambda$
1.1 ² : 1	$a\theta^1\theta^1 + b\theta^2\theta^2 + 2c\theta^2\theta^3 + a\theta^3\theta^3 + d\theta^4\theta^4,$ $a(c^2 - bd) \neq 0$	$b = 0, \omega_4 = -4$
1.1 ² : 2	"	$b = 0, \omega_4 = -2p$
1.3 ¹ : 5	$-2a\theta^1\theta^4 + 2a\theta^2\theta^3 + b\theta^3\theta^3 + 2c\theta^3\theta^4 + d\theta^4\theta^4,$ $a \neq 0$	$\mu = 2, \lambda = 0, \omega_4 = -4$
"	"	$\mu = 0, \lambda = \lambda, 2c + d\lambda = 0,$ $\omega_3 = -2\lambda$
1.3 ¹ : 7	"	$\lambda = 0, d + 2c = 0, \omega_3 = -2$
1.3 ¹ : 9	"	$\lambda = 1, \omega_4 = -2$
1.3 ¹ : 12	"	$b = 0, \omega_4 = -2$
"	"	$\lambda = 1 - \mu, \omega_4 = -2$
"	"	$\mu = \frac{1}{2}, \lambda = \lambda, \omega_4 = -2$
"	"	$\lambda = \mu - 1, \omega_4 = -2\mu$
"	"	$\lambda = \mu + 1, \omega_4 = -2\mu - 2$
1.3 ¹ : 13	"	$\lambda = \frac{1}{2}, \omega_4 = -2$ $\lambda = \frac{1}{2}, \omega_4 = -1$ $\lambda = \frac{3}{2}, \omega_4 = -3$
1.3 ¹ : 14	"	$\lambda = \lambda, \omega_4 = -2$
1.3 ¹ : 19	"	$b = 0, \omega_4 = -2$
1.3 ¹ : 21	"	$\lambda = \lambda, b = 0, \omega_4 = -2$ $\lambda = \frac{1}{2}, \omega_4 = -2$ $\lambda = 2, \omega_4 = -4$
1.3 ¹ : 23	"	$\omega_4 = -2$
1.3 ¹ : 24	"	$\lambda = 2, \omega_3 = -6$
1.3 ¹ : 25	"	$\lambda = 2, \omega_3 = -6$
1.4 ¹ : 2	$-2a\theta^1\theta^3 + a\theta^2\theta^2 + b\theta^3\theta^3 + 2c\theta^3\theta^4 + d\theta^4\theta^4,$ $ad \neq 0$	$p = 1, \omega_4 = -2$
1.4 ¹ : 9	"	$a = -\frac{d}{(1+p)^2}, r = (1+p)(2+p)$ $\omega_3 = -2p - 4$
"	"	$a = -\frac{d}{4p^2}, r = p(-1+p)$ $\omega_3 = -2 - 2p$
"	"	$dp = -\frac{1}{2}, r = -\frac{a+4d}{4a}$ $\omega_3 = -2$
1.4 ¹ : 10	"	$\omega_3 = -2$
1.4 ¹ : 12 or 13	"	$\omega_3 = -2,$ or $r = 1, 4a + d = 0, \omega_3 = -2$
2.2 ¹ : 2 or 3	$2a\theta^1\theta^3 + 2a\theta^2\theta^4 + b\theta^2\theta^2, a \neq 0$	$\omega_2 = 2p$ or $\omega_2 = 2$
2.5 ¹ : 3	$2a\theta^1\theta^3 + 2a\theta^2\theta^4 + b\theta^3\theta^3, a \neq 0$	$h = 0, \omega_3 = -2$ $r = \frac{h^2}{4} - \frac{1}{2}h,$ $k = 0, \omega_4 = -2 - h$
"	"	
2.5 ¹ : 4	"	$\omega_3 = -2$
2.5 ² : 2	$2a\theta^1\theta^3 + a\theta^2\theta^2 + b\theta^3\theta^3 + a\theta^4\theta^4, a \neq 0$	$s = 0, \omega_3 = -2$ $\omega_3 = -2$
3.3 ¹ : 1	$2a\theta^1\theta^3 + 2a\theta^2\theta^4 + b\theta^3\theta^3, a \neq 0$	$\omega_3 = -2$
3.3 ² : 1	$2a\theta^1\theta^3 + a\theta^2\theta^2 + b\theta^3\theta^3 + a\theta^4\theta^4, a \neq 0$	$\omega_3 = -2$

TABLE 1. (M, g) with recurrent curvature tensor

and the isotropy subalgebra is generated by $\mathfrak{h} = \text{span}\{e_1\}$. If we take $\mathfrak{m} = \text{span}\{u_1, \dots, u_4\}$, then the matrix related to $\psi(e_1)$, which was defined in the previous section, and the invariant metrics with respect to $\{u_i\}_{i=1}^4$ are

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & c \\ a & 0 & 0 & 0 \\ 0 & c & 0 & d \end{pmatrix},$$

and for the metric g to be nondegenerate it must satisfy $a^2(c^2 - bd) \neq 0$. Therefore, using equation (2.1), we have the following components of the Levi-Civita connection:

$$\Lambda_1 = \begin{pmatrix} 0 & -\frac{b}{2a} & 0 & \frac{a-c}{2a} \\ 0 & 0 & \frac{bd+ac-c^2}{2(bd-c^2)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{ab}{2(bd-c^2)} & 0 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} -\frac{b}{2a} & 0 & 0 & 0 \\ 0 & \frac{bc}{bd-c^2} & 0 & \frac{bd}{bd-c^2} \\ 0 & 0 & \frac{b}{2a} & 0 \\ 0 & \frac{-b^2}{bd-c^2} & 0 & \frac{bc}{bd-c^2} \end{pmatrix},$$

$$\Lambda_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{bd+ac+c^2}{2(bd-c^2)} & 0 & 0 & 0 \\ 0 & \frac{b}{2a} & 0 & \frac{a+c}{2a} \\ -\frac{ab}{2(bd-c^2)} & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_4 = \begin{pmatrix} \frac{a-c}{2a} & 0 & 0 & 0 \\ 0 & \frac{c^2}{bd-c^2} & 0 & \frac{dc}{bd-c^2} \\ 0 & 0 & -\frac{a-c}{2a} & 0 \\ 0 & -\frac{bc}{bd-c^2} & 0 & -\frac{c^2}{bd-c^2} \end{pmatrix}.$$

So, by a direct computation over (2.2), the components of curvature tensor are obtained as

$$R_{12} = \begin{pmatrix} 0 & -\frac{b^2(2a^2+bd-c^2)}{4a^2(bd-c^2)} & 0 & -\frac{b((bd-c^2)(a+c)+2a^2c)}{4a^2(bd-c^2)} \\ 0 & 0 & \frac{(bd-ac-c^2+2a^2)b}{4a(bd-c^2)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{b^2}{4(bd-c^2)} & 0 \end{pmatrix},$$

$$R_{13} = \begin{pmatrix} \frac{b(3bd-3c^2-a^2)}{4a(bd-c^2)} & 0 & 0 & 0 \\ 0 & -\frac{cb}{2(bd-c^2)} & 0 & -\frac{db}{2(bd-c^2)} \\ 0 & 0 & \frac{b(-3bd+3c^2+a^2)}{4a(bd-c^2)} & 0 \\ 0 & \frac{b^2}{2(bd-c^2)} & 0 & \frac{cb}{2(bd-c^2)} \end{pmatrix},$$

$$R_{14} = \begin{pmatrix} 0 & -\frac{b((c^2-bd)(a-c)+2a^2c)}{4a^2(bd-c^2)} & 0 & -\frac{c^2(a^2-c^2)+bd(a^2+c^2)}{4a^2(bd-c^2)} \\ 0 & 0 & -\frac{bad+cbd+a^2c-c^3}{4a(bd-c^2)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{b(a+c)}{4(bd-c^2)} & 0 \end{pmatrix},$$

$$\begin{aligned}
 R_{23} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{(bd+ac+2a^2-c^2)b}{4a(bd-c^2)} & 0 & 0 & 0 \\ 0 & \frac{b^2(2a^2+bd-c^2)}{4a^2(bd-c^2)} & 0 & \frac{b((c^2-bd)(a-c)+2a^2c)}{4a^2(bd-c^2)} \\ \frac{b^2}{4(bd-c^2)} & 0 & 0 & 0 \end{pmatrix}, \\
 R_{24} &= \begin{pmatrix} \frac{b}{2a} & 0 & 0 & 0 \\ 0 & -\frac{cb}{bd-c^2} & 0 & -\frac{bd}{bd-c^2} \\ 0 & 0 & -\frac{b}{2a} & 0 \\ 0 & \frac{b^2}{bd-c^2} & 0 & \frac{bc}{bd-c^2} \end{pmatrix}, \quad \text{and} \\
 R_{34} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{bd(a+c)+a^2c-c^3}{4a(bd-c^2)} & 0 & 0 & 0 \\ 0 & -\frac{b((bd-c^2)(a+c)+2a^2c)}{4a^2(bd-c^2)} & 0 & -\frac{c^2(a^2-c^2)+bd(a^2+c^2)}{4a^2(bd-c^2)} \\ \frac{(a-c)b}{4(bd-c^2)} & 0 & 0 & 0 \end{pmatrix},
 \end{aligned}$$

where $R_{ij} = R(u_i, u_j)$. Now, let $\omega = \omega_1\theta^1 + \omega_2\theta^2 + \omega_3\theta^3 + \omega_4\theta^4$; a direct calculation using (2.4) gives $(\nabla\mathbf{R} - \omega \otimes \mathbf{R})_{3142i} = \frac{1}{2}b\omega_i$. Since $\omega \neq 0$, we immediately get $b = 0$ (the case $\omega = 0$ means that (M, g) is locally symmetric). Applying $b = 0$ for the remaining terms of $(\nabla\mathbf{R} - \omega \otimes \mathbf{R})_{ijklm}$ gives

$$\begin{cases} \omega_i c^2 (a^2 - c^2) = 0, & i = 1, \dots, 3; \\ (\omega_4 + 2)(a^2 - c^2) = 0. \end{cases}$$

As $a \neq \pm c$ (the flat case), we have $\omega_1 = \omega_2 = \omega_3 = 0$ and $\omega_4 = -2$. □

4. GEOMETRY OF THEOREM 3.1

An Einstein-like metric and a commutative curvature operator over the homogenous four-manifold (M, g) were presented in [14] and [8], respectively. Here, we introduce a large class of some geometric examples in our classification of Theorem 3.1. The Einstein manifold is one of the most important manifolds in geometry and physics. It is well known that the manifold (M, g) is called *Einstein* if $\rho = \eta g$ for a real constant η . Also, it is obvious that the manifold is flat if the curvature tensor vanishes identically.

The Weyl conformal tensor W is a $(0, 4)$ tensor field on (M, g) which is completely determined by its components in the following way:

$$W_{ijkh} = R_{ijkh} + \frac{1}{2}(g_{ik}\rho_{jh} + g_{jh}\rho_{ik} - g_{ih}\rho_{jk} - g_{jk}\rho_{ih}) - \frac{Sc}{6}(g_{ik}g_{jh} - g_{ih}g_{jk}), \quad (4.1)$$

where Sc is the scalar curvature. A four-dimensional pseudo-Riemannian manifold is called *conformally flat* if its Weyl conformal tensor W vanishes identically.

Theorem 4.1. *If the homogeneous manifold (M, g) has non-trivially recurrent curvature (not locally symmetric), as in Table 1, then some of their geometric properties (flatness, Ricci flatness non-flat, being Ricci parallel non-locally symmetric, being Einstein non-Ricci-flat, and conformal flatness) are as in Table 2.*

case	flat	Ricci-flat non-flat	Ricci parallel non-locally symmetric	Einstein non-Ricci-flat	conformally flat
1.1 ¹ : 1	$b = 0, a = \pm c$	X	X	$b \neq 0$ and $bd + (a^2 - c^2) = 0$	$b = 0$
1.1 ¹ : 2	$b = 0, \lambda = \frac{1}{2}$	X	X	$b \neq 0, \lambda = \frac{1}{2}$ $4\eta bd - 4\eta c^2 + 3b = 0$	$b = 0$ or $\lambda = 0, \frac{1}{2}$
1.1 ² : 1	X	X	X	$bd - c^2 - 4a^2 = 0$	$b = 0$
1.1 ² : 2	$b = 0, p = 1$	X	X	$\eta bd - \eta c^2 + 3b = 0$ $b \neq 0, p = 1$	$b = 0$ or $p = 0, 1$
1.3 ¹ : 5	$b + d = 0,$ $\mu = 2, \lambda = 0$	$b + d \neq 0,$ $\mu = 2, \lambda = 0$	$b + d \neq 0,$ $\mu = 2, \lambda = 0$	X	$b\mu(\mu - 1) - 2c\lambda\mu$ $+ d\lambda^2 + d\mu + 2c\lambda = 0$ or $\lambda = \mu = 0,$ or $2c + d\lambda = 0, \mu = 0$
1.3 ¹ : 7	X	X	X	X	$d - b\lambda + 2c = 0$
1.3 ¹ : 9	$b = 0, \lambda = 1$	$b \neq 0, \lambda = 1$	$\lambda \neq 0, -1, b \neq 0$	X	$\lambda = 0, -1$ or $b = 0$
1.3 ¹ : 12	$\mu = 0, \lambda = 1,$ or $\mu = \frac{1}{2}, \lambda = \frac{3}{2},$ or $\mu = \frac{1}{2}, \lambda = -\frac{1}{2},$ or $\mu = 1, \lambda = 0,$ or $b = 0,$ $\lambda - \mu \pm 1 = 0$	$b \neq 0,$ $\lambda - \mu + 1 = 0$ or $b \neq 0,$ $\lambda - \mu - 1 = 0$	$a \neq 0,$ $\lambda - \mu + 1 = 0$ or $a \neq 0,$ $\lambda - \mu - 1 = 0$	X	$b = 0,$ $\lambda - \mu + 1 = 0$ $\mu = \frac{1}{2}$
1.3 ¹ : 13	X	$\lambda = -\frac{1}{2}, \frac{3}{2}$	$\lambda = -\frac{1}{2}, \frac{3}{2}$	X	X
1.3 ¹ : 14	X	$\lambda = 0$ or 1	$\lambda = 0$ or 1	X	X
1.3 ¹ : 19	X	X	X	X	$b = 0$
1.3 ¹ : 21	$b = 0, \lambda = 0, 2$	$b \neq 0, \lambda = 2$	$b \neq 0, \lambda = 2$	X	$b = 0$ or $\lambda = 0, \frac{1}{2}$
1.3 ¹ : 23	X	✓	✓	X	X
1.3 ¹ : 24	$a - 4d = 0, \lambda = 2$	$a - 4d \neq 0, \lambda = 2$	$a - 4d \neq 0, \lambda = 2$	X	$a - 2d\lambda^2 + 2d\lambda = 0,$ or $\lambda = \frac{2}{3}$
1.3 ¹ : 25	$a + 4d = 0, \lambda = 2$	$a + 4d \neq 0, \lambda = 2$	$a + 4d \neq 0, \lambda = 2$	X	$a + 2d\lambda^2 - 2d\lambda = 0,$ $\lambda = \frac{2}{3}$
1.4 ¹ : 2	$b = 0, p = 1$	X	$p = \frac{5}{3}$	$p = \frac{5}{3}, 3\tau d + 4 = 0$ or $p \neq 1, b = 0, \tau d + 3p^2 - 6p + 3 = 0$	$b = 0$ or $p = 3$
1.4 ¹ : 9	$a + d = 0, r = \frac{3}{4},$ $p = -\frac{1}{2}$	$d + 2ra + 2pa$ $+ 2p^2a = 0$	$d + 2ra + 2pa$ $+ 2p^2a = 0$	X	$4ra + a + 4d = 0$ $p = -\frac{1}{2}$
1.4 ¹ : 10	$r = p = 0,$ or $r = 0, p = -1$	$p^2 + p - r = 0$	$p^2 + p + r = 0$	X	$p^2 + p - r = 0$
1.4 ¹ : 12	$r = 0$	X	X	X	$r = 0$
1.4 ¹ : 13	X	$2ar + 2a + d = 0$	✓	X	X
2.2 ¹ : 2	$p = -2, 2$	X	X	X	✓
2.2 ¹ : 3	X	X	X	X	✓
2.5 ¹ : 3	X	$r - \frac{h^2}{4} + \frac{h}{2} = 0$	$r - \frac{h^2}{2} + \frac{h}{2} = 0$	X	X
2.5 ¹ : 4	$r = \frac{h^2}{4} + \frac{h}{2}$	X	X	X	✓
2.5 ² : 2	$r^2 + p = 0 = s$	$r^2 + p = 0 \neq s$	$r^2 + p = 0 \neq s$	X	$s = 0$
3.3 ¹ : 1	$p = 0$	X	X	X	✓
3.3 ² : 1	$p = 0$	X	X	X	✓

TABLE 2. Geometric properties of Theorem 4.1

Proof. We proceed similarly to the proof of Theorem 3.1; so, we present the details for the cases 1.3¹.09 and 2.5².02, and other cases may be considered in a similar

way. In Theorem 3.1, every invariant metric on the homogeneous space $M = G/H$ is in a one-to-one correspondence with an invariant inner product on \mathfrak{m} . First, we consider the case 1.3¹.09. For this case, there exists a basis $\{e_1, u_1, \dots, u_4\}$ of \mathfrak{g} , the nonzero brackets are

$$\begin{aligned} [e_1, u_3] &= u_1, & [e_1, u_4] &= u_2, & [u_2, u_3] &= \lambda u_1, \\ [u_2, u_4] &= -\lambda e_1 + (\lambda + 1)u_2, & [u_3, u_4] &= -\lambda u_3, & \lambda &\in \mathbb{R}, \end{aligned}$$

and the isotropy subalgebra is generated by $\mathfrak{h} = \text{span}\{e_1\}$. If we take $\mathfrak{m} = \text{span}\{u_1, \dots, u_4\}$, then the matrix H_1 related to $\psi(e_1)$ and the invariant metric with respect to $\{u_i\}_{i=1}^4$ are

$$H_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 0 & 0 & 0 & -a \\ 0 & 0 & a & 0 \\ 0 & a & b & c \\ -a & 0 & c & d \end{pmatrix},$$

and from the nondegeneracy of the metric g we have $a^4 \neq 0$. So, the Levi-Civita connection (2.1) and the curvature operator are as follows:

$$\begin{aligned} \Lambda_1 &= 0, & \Lambda_2 &= \begin{pmatrix} 0 & 0 & \frac{\lambda}{2} + \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{\lambda}{2} + \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \Lambda_3 &= \begin{pmatrix} 0 & -\frac{\lambda}{2} + \frac{1}{2} & -\frac{\lambda b}{a} & \frac{-c(\lambda-1)}{2a} \\ 0 & 0 & 0 & \frac{-b(\lambda+1)}{2a} \\ 0 & 0 & 0 & -\frac{\lambda}{2} + \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \Lambda_4 &= \begin{pmatrix} 0 & 0 & \frac{-c(\lambda-1)}{2a} & 0 \\ 0 & -\frac{\lambda}{2} - \frac{1}{2} & \frac{-b(\lambda+1)}{2a} & -\frac{\lambda c}{a} \\ 0 & 0 & \frac{\lambda}{2} + \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

and by using equation (2.2),

$$\begin{aligned} R_{24} &= \begin{pmatrix} 0 & 0 & -\frac{\lambda^2}{4} + \frac{\lambda}{2} - \frac{1}{4} & 0 \\ 0 & 0 & 0 & -\frac{\lambda^2}{4} + \frac{\lambda}{2} - \frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ R_{34} &= \begin{pmatrix} -\frac{\lambda^2}{4} + \frac{\lambda}{2} - \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-b\lambda(\lambda+1)}{a} \\ -\frac{b(5\lambda^2+2\lambda+1)}{4a} & 0 & 0 & -\frac{\lambda^2}{4} + \frac{\lambda}{2} - \frac{1}{4} \\ \frac{-c(\lambda-1)^2}{4a} & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Thus, by direct calculations, this case will be flat if the following equations are satisfied:

$$\begin{cases} \frac{\lambda^2}{4} - \frac{\lambda}{2} + \frac{1}{4} = 0, \\ b\lambda(\lambda + 1) = 0, \\ b(5\lambda^2 + 2\lambda + 1) = 0, \\ c(\lambda - 1)^2 = 0. \end{cases}$$

Therefore, this case can be flat just if $b = 0$ and $\lambda = 1$ as it appeared in Table 2.

According to equation (2.3), the corresponding Ricci tensor is

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\lambda^2}{2} + \lambda - \frac{1}{2} \end{pmatrix}.$$

So, this case is Ricci-flat if $\lambda = 1$, but if also $b = 0$ then the corresponding space is flat, and also by taking direct covariant derivative over ρ , this case is Ricci parallel if $\lambda \neq 0, -1$ and $b \neq 0$. Then, this case is Ricci parallel non-locally symmetric. Now, the Einstein condition yields

$$d\eta - \frac{\lambda^2}{2} + \lambda - \frac{1}{2} = a\eta = b\eta = c\eta = 0,$$

which means that $b \neq 0, \lambda = 1$, and $\eta = 0$, which is also Ricci-flat. So, in this case, there is no Einstein non-Ricci-flat example.

Conformal flatness is obtained if the equation (4.1) vanishes. So, we have

$$b\lambda(\lambda + 1) = 0;$$

therefore, this case is conformally flat if $\lambda = 0$ or $\lambda = -1$ or $b = 0$.

Now, we consider the case 2.5².02, where there exists a basis $\{e_1, e_2, u_1, \dots, u_4\}$ of \mathfrak{g} , the nonzero brackets are

$$\begin{aligned} [e_1, u_2] &= u_1, & [e_1, u_3] &= -u_2, & [e_2, u_3] &= u_4, & [e_2, u_4] &= -u_1, & [u_1, u_3] &= u_1, \\ [u_2, u_3] &= (p + s)e_1 + re_2 + u_2 - 2ru_4, & [u_2, u_4] &= 2ru_1, \\ [u_3, u_4] &= -re_1 + (p - s)e_2 - 2ru_2 - u_4, & p, r, s &\in \mathbb{R}, & r, s &\geq 0, \end{aligned}$$

and the isotropy subalgebra is generated by $\mathfrak{h} = \text{span}\{e_1, e_2\}$. If we take $\mathfrak{m} = \text{span}\{u_1, \dots, u_4\}$, then the matrices H_1 and H_2 related to $\psi(e_i)$ and the invariant metric with respect to $\{u_i\}_{i=1}^4$ are

$$H_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ a & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix},$$

and for the metric g to be nondegenerate we must have $a^4 \neq 0$. Using equation (2.1) gives

$$\begin{aligned} \Lambda_1 &= 0, & \Lambda_2 &= \begin{pmatrix} 0 & -1 & 0 & r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \end{pmatrix}, \\ \Lambda_3 &= \begin{pmatrix} -1 & 0 & -\frac{b}{a} & 0 \\ 0 & 0 & 0 & -r \\ 0 & 0 & 1 & 0 \\ 0 & r & 0 & 0 \end{pmatrix}, & \Lambda_4 &= \begin{pmatrix} 0 & -r & 0 & -1 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \end{aligned}$$

and also by (2.2) we have the curvature operators

$$\begin{aligned} R_{23} &= \begin{pmatrix} 0 & -r^2 - p - s & 0 & 0 \\ 0 & 0 & r^2 + p + s & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ R_{34} &= \begin{pmatrix} 0 & 0 & 0 & r^2 + p - s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -r^2 - p + s & 0 \end{pmatrix}. \end{aligned}$$

It is obvious that if $r^2 + p + s = 0$ and $r^2 + p - s = 0$ then the metric is flat. The corresponding Ricci tensor is

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2r^2 + 2p & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

therefore, this case is Ricci-flat if $r^2 + p = 0$, and to have non-flat examples we must have $s \neq 0$. The same computation shows that this condition also holds for the Ricci parallel non-flat case.

The Einstein condition in this case translates to

$$a\eta - 2r^2 + 2p = 0, \quad a\eta = 0, \quad b\eta = 0,$$

so, as $a \neq 0$, we must have $r^2 + p = 0$; so the metric will be flat. The conformal flatness in this case means that

$$as = 0;$$

since $a \neq 0$, this case is conformally flat if $s = 0$. □

If in (2.4), instead of the curvature tensor we use the Ricci tensor, the space is called *Ricci recurrent*.

Corollary 4.2. *The homogenous manifold $(M = G/H, g)$ is non-trivially Ricci recurrent if it is of the cases $1.3^1 : 21$ and $1.4^1 : 11$ with $\omega_1 = \omega_2 = \omega_3 = 0$, $\omega_4 = -2$, and $\omega_1 = \omega_2 = 0$, $\omega_3 = -2$, $\omega_4 = 0$, respectively.*

5. STRICT WALKER RECURRENT CURVATURE MANIFOLDS

The existence of parallel distributions on manifolds is a strong condition to simplify their geometric study. Of course, in the Riemannian geometry, the restriction of the metric to the mentioned distribution is non-degenerate, but the condition will be confined to the pseudo-Riemannian geometry when the metric restriction to the parallel distribution is degenerate. Walker manifolds were firstly introduced by A. G. Walker in 1950. He could find a canonical form for a Walker metric, by introducing local coordinates where the metric takes a very special shape [11, 12]. Following Walker’s studies, there exist local coordinates (x_1, x_2, x_3, x_4) on the four-dimensional manifold M , such that the Walker metric is of indefinite signature $(- - ++)$ with the metric

$$g_{(x_1, x_2, x_3, x_4)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(x_1, x_2, x_3, x_4) & c(x_1, x_2, x_3, x_4) \\ 0 & 1 & c(x_1, x_2, x_3, x_4) & b(x_1, x_2, x_3, x_4) \end{pmatrix}, \tag{5.1}$$

where $a(x_1, x_2, x_3, x_4)$, $b(x_1, x_2, x_3, x_4)$, $c(x_1, x_2, x_3, x_4)$ are arbitrary smooth functions on M , and the parallel distribution is generated by $\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\}$. Chaichi et al. have considered in [4] curvature properties of four-dimensional Walker metrics.

Furthermore, note that the plane field is strictly parallel if and only if the defining functions are independent from the coordinates (x_1, x_2) , i.e.,

$$\begin{aligned} a(x_1, x_2, x_3, x_4) &= a(x_3, x_4), \\ b(x_1, x_2, x_3, x_4) &= b(x_3, x_4), \\ c(x_1, x_2, x_3, x_4) &= c(x_3, x_4); \end{aligned}$$

then the corresponding metric will be called *strict Walker* metric. The Levi-Civita connection of the strict Walker manifolds is as follows:

$$\begin{aligned} \nabla_{\partial_{x_3}} \partial_{x_3} &= \frac{1}{2} \frac{\partial a}{\partial x_3} \partial_{x_1} + \left(\frac{\partial c}{\partial x_3} - \frac{1}{2} \frac{\partial a}{\partial x_4} \right) \partial_{x_2}, \\ \nabla_{\partial_{x_3}} \partial_{x_4} &= \nabla_{\partial_{x_4}} \partial_{x_3} = \frac{1}{2} \left(\frac{\partial a}{\partial x_4} \partial_{x_1} + \frac{\partial b}{\partial x_3} \partial_{x_2} \right), \\ \nabla_{\partial_{x_4}} \partial_{x_4} &= \left(\frac{\partial c}{\partial x_4} - \frac{1}{2} \frac{\partial b}{\partial x_3} \right) \partial_{x_1} + \frac{1}{2} \frac{\partial b}{\partial x_4} \partial_{x_2}. \end{aligned}$$

According to [4], a strict Walker metric defined globally on \mathbb{R}^4 has vanishing Christoffel symbols, so it is geodesically complete (cf. [6, Lemma 2.1]).

The nonzero components of the curvature tensor of any strict Walker metric are given by

$$R(\partial_{x_4}, \partial_{x_3})\partial_{x_4} = 2\mathcal{A} \partial_{x_1} \quad \text{and} \quad R(\partial_{x_3}, \partial_{x_4})\partial_{x_3} = 2\mathcal{A} \partial_{x_2},$$

where $\mathcal{A} = \frac{\partial^2 b}{\partial x_3^2} - 2 \frac{\partial^2 c}{\partial x_3 \partial x_4} + \frac{\partial^2 a}{\partial x_4^2}$.

Then, we consider the recurrent curvature condition for a strict Walker metric. Let

$$\omega = \omega_1(x_1, x_2, x_3, x_4)dx_1 + \omega_2(x_1, x_2, x_3, x_4)dx_2 + \omega_3(x_1, x_2, x_3, x_4)dx_3 + \omega_4(x_1, x_2, x_3, x_4)dx_4.$$

Now, by applying the recurrent curvature condition (2.4), we have

$$\begin{cases} \mathcal{A}\omega_1(x_1, x_2, x_3, x_4) = 0, \\ \mathcal{A}\omega_2(x_1, x_2, x_3, x_4) = 0, \\ \frac{\partial \mathcal{A}}{\partial x_3} - \mathcal{A}\omega_3(x_1, x_2, x_3, x_4) = 0, \\ \frac{\partial \mathcal{A}}{\partial x_4} - \mathcal{A}\omega_4(x_1, x_2, x_3, x_4) = 0. \end{cases}$$

Since $\mathcal{A} \neq 0$ (the case $\mathcal{A} = 0$ implies the trivial flat condition), satisfying the recurrent curvature condition is equivalent to establishing the following:

$$\omega_1 = \omega_2 = 0, \quad \omega_3 = \frac{1}{\mathcal{A}} \frac{\partial \mathcal{A}}{\partial x_3}, \quad \omega_4 = \frac{1}{\mathcal{A}} \frac{\partial \mathcal{A}}{\partial x_4}. \tag{5.2}$$

So, we have proved the next theorem.

Theorem 5.1. *A geodesically complete four-dimensional strict Walker manifold which is globally defined on \mathbb{R}^4 has recurrent curvature if and only if it satisfies the system of equations (5.2).*

In the same coordinate system, the functions a and c can be further specialized to satisfy $a(x_3, x_4) \equiv 0$, $c(x_3, x_4) \equiv 0$ (cf. [11]). So, \mathcal{A} reduces to $\mathcal{A} = \frac{\partial^2 b}{\partial x_3^2}$. Therefore, for a special non-flat example, we present the following one.

Example 5.2. The geodesically complete strict Walker metric in \mathbb{R}^4 , with vanishing a and c and b as

$$b = f(x_4)x_3^2 + g(x_4)x_3 + h(x_4),$$

for arbitrary smooth functions $f \neq 0$, g , and h , has recurrent curvature ($\nabla R = \omega \otimes R$) with the following 1-form ω on \mathbb{R}^4 :

$$\omega = \frac{f'(x_4)}{f(x_4)} dx_4.$$

6. CONCLUSION

Homogeneous examples of pseudo-Riemannian manifolds which have recurrent curvature without Riemannian counterpart have been presented. Especially, there are some open questions like the existence of some common underlying structure (presumably the existence of a parallel field of degenerate lines/planes). Here just for an example we investigate the existence of recurrent manifolds within the framework of strictly Walker metrics in dimension four. This allows us the construction of simple examples which are not locally homogeneous in signature $(2, 2)$, but a full description is still an open question.

REFERENCES

- [1] G. Calvaruso and A. Zaeim, Symmetries of Lorentzian three-manifolds with recurrent curvature, *SIGMA Symmetry Integrability Geom. Methods Appl.* **12** (2016), Paper No. 063, 12 pp. MR 3514944.
- [2] G. Calvaruso and A. Zaeim, Four-dimensional homogeneous Lorentzian manifolds, *Monatsh. Math.* **174** (2014), no. 3, 377–402. MR 3223494.
- [3] G. Calvaruso and A. Zaeim, Four-dimensional pseudo-Riemannian g.o. spaces and manifolds, *J. Geom. Phys.* **130** (2018), 63–80. MR 3807119.
- [4] M. Chaichi, E. García-Río and Y. Matsushita, Curvature properties of four-dimensional Walker metrics, *Classical Quantum Gravity* **22** (2005), no. 3, 559–577. MR 2115361.
- [5] E. García-Río, P. B. Gilkey and S. Nikčević, Homogeneity of Lorentzian three-manifolds with recurrent curvature, *Math. Nachr.* **287** (2014), no. 1, 32–47. MR 3153924.
- [6] P. Gilkey and S. Nikčević, Complete k -curvature homogeneous pseudo-Riemannian manifolds, *Ann. Global Anal. Geom.* **27** (2005), no. 1, 87–100. MR 2130535.
- [7] A. Haji-Badali, Ricci almost solitons on three-dimensional manifolds with recurrent curvature, *Mediterr. J. Math.* **14** (2017), no. 1, Paper No. 4, 9 pp. MR 3589920.
- [8] A. Haji-Badali and A. Zaeim, Commutative curvature operators over four-dimensional homogeneous manifolds, *Int. J. Geom. Methods Mod. Phys.* **12** (2015), no. 10, 1550123, 17 pp. MR 3415933.
- [9] B. Komrakov, Jr., Einstein-Maxwell equation on four-dimensional homogeneous spaces, *Lobachevskii J. Math.* **8** (2001), 33–165. MR 1846120.
- [10] H. S. Ruse, Three-dimensional spaces of recurrent curvature, *Proc. London Math. Soc. (2)* **50** (1948), 438–446. MR 0029250.
- [11] A. G. Walker, Canonical form for a Riemannian space with a parallel field of null planes, *Quart. J. Math. Oxford Ser. (2)* **1** (1950), no. 1, 69–79. MR 0035085.
- [12] A. G. Walker, Canonical forms. II. Parallel partially null planes, *Quart. J. Math. Oxford Ser. (2)* **1** (1950), no. 1, 147–152. MR 0037045.
- [13] A. G. Walker, On Ruse’s spaces of recurrent curvature, *Proc. London Math. Soc. (2)* **52** (1950), 36–64. MR 0037574.
- [14] A. Zaeim and A. Haji-Badali, Einstein-like pseudo-Riemannian homogeneous manifolds of dimension four, *Mediterr. J. Math.* **13** (2016), no. 5, 3455–3468. MR 3554319.

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