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ON BAER MODULES

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Abstract. A commutative ring R is said to be a Baer ring if for each $a \in R$, $\operatorname{ann}(a)$ is generated by an idempotent element $b \in R$. In this paper, we extend the notion of a Baer ring to modules in terms of weak idempotent elements defined in a previous work by Jayaram and Tekir. Let R be a commutative ring with a nonzero identity and let M be a unital R-module. M is said to be a Baer module if for each $m \in M$ there exists a weak idempotent element $e \in R$ such that $\operatorname{ann}_R(m)M = eM$. Various examples and properties of Baer modules are given. Also, we characterize a certain class of modules/submodules such as von Neumann regular modules/prime submodules in terms of Baer modules.

1. Introduction

In this paper, we focus only on commutative rings with nonzero identity and nonzero unital modules. Let R denote such a ring and let M denote such an R-module. For each $a \in R$, the principal ideal generated by a will be designated by (a). R is said to be a *von Neumann regular* (for short, *VN-regular*) *ring* if for each $a \in R$ there exists $x \in R$ such that $a = a^2x$ [20]. Note that R is a VN-regular ring if and only if for each $a \in R$, $(a) = (e)$ for some idempotent element $e \in R$. The concept of VN-regular ring and its generalizations have been widely studied in many papers and have some applications to other areas such as graph theory. See, for example, [13], [15] and [17]. In [14], Kist gave a generalization of VN-regular rings, called Baer rings, as follows: a ring R is said to be a *Baer ring* if for each $a \in R$, the annihilator $\operatorname{ann}(a) = \{r \in R : ra = 0\}$ of a is generated by an idempotent element $e \in R$. It is easy to see that every VN-regular ring is also a Baer ring, but the converse is not true in general. For instance, consider an integral domain R that is not a field. Then clearly R is a Baer ring but not a VN-regular ring. We note here that some authors studied Baer rings under different names such as *principally quasi Baer ring* and *P.P. ring.* For details, one can consult [6], [8] and [9].

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Our aim in this article is to extend the notion of a Baer ring to modules, and to investigate the relations between Baer modules and VN-regular modules. For the sake of completeness, we now give some notions and notations which will be frequently used in this paper. Let N be a submodule of M, let K be a nonempty subset of M and let J be a nonempty subset of R. The residuals of N by K and J are defined as $(N :_R K) = \{a \in R : aK \subseteq N\}$ and $(N :_M J) = \{m \in M : Jm \subseteq N\}$, respectively. In particular, we use $\operatorname{ann}_R(N)$ to denote $(0 :_R N)$. M is called a faithful module if $\operatorname{ann}_R(M) = (0)$. Also, M is said to be a *multiplication module* if each submodule N of M has the form $N = IM$ for some ideal I of R [5]. For more information about multiplication modules, we refer the reader to [1] and [7]. M is called a torsion free module if the set of all torsion elements $T(M) = \{m \in M : \operatorname{ann}_R(m) \neq 0\} = (0)$. Also, M is said to be a *torsion module* if $T(M) = M$. Otherwise, we say that M is a *non-torsion module*, that is, there exists $m \in M$ such that $\operatorname{ann}_R(m) = 0$. In [12], Jayaram and Tekir extended the notion of an idempotent element in rings to modules in terms of weak idempotent elements and they studied VN-regular modules. An element $e \in R$ is said to be a *weak idempotent element* if $em = e^2m$ for each $m \in M$, or equivalently, $e - e^2 \in \operatorname{ann}_R(M)$. Also, M is said to be a *VN-regular module* if for each $m \in M$ we have $Rm = aM = a^2M$ for some $a \in R$ [12]. The authors in [12, Lemma 3 and Theorem 2] showed that a finitely generated R-module M is a VN-regular module if and only if for each $m \in M$ there exists a weak idempotent element $e \in R$ such that $Rm = eM$.

An R-module M is said to be a *Baer module* if for each $m \in M$ there exists a weak idempotent element $e \in R$ such that $\operatorname{ann}_R(m)M = eM$. Among various results in this paper, in Section 2 we give basic properties of Baer modules. In particular, we show that simple modules, torsion free modules, second modules and finitely generated VN-regular modules are Baer modules (see Example 2.2, Example 2.4, Example 2.5 and Proposition 2.8). Also, we characterize Baer modules in terms of the Baer property of the factor ring $R/\operatorname{ann}_R(M)$ (see Theorem 2.14). Section 3 is dedicated to the study of σ-submodules, Baer submodules and m-submodules. Let N be a submodule of M. It is said that N is a *σ-submodule* of M if $m \in N$ implies that $(N :_R M) + \operatorname{ann}_R(m) = R$. It is easy to see that σ-submodules of the R-module R are the exactly pure ideals of R—recall that an ideal I of R is said to be a *pure ideal* if for each $a \in I$ there exists $b \in I$ such that $a = ab$. We give some characterizations of Baer modules and VN-regular modules in terms of σ-submodules (see Theorem 3.6 and Theorem 3.8). Also, we determine the conditions under which Baer modules and VN-regular modules are equivalent (see Theorem 3.10). Moreover, in Proposition 3.13, we characterize prime submodules in terms of Baer modules. Also, we study the prime submodules of Baer modules (see Proposition 3.14). Finally, we determine when the lattice $\sigma(M)$ of all σ-submodules of M is a Boolean lattice (see Proposition 3.20). In the last section, we investigate the Baer property of polynomial modules and power series modules (see Corollary 4.8).

2. Characterizations of Baer modules

Throughout this paper, R will always denote a commutative ring with nonzero identity and M will denote a unital R-module.

Definition 2.1. An R-module M is said to be a *Baer module* if for each $m \in M$, $\mathrm{ann}_R(m)M = eM$ for some weak idempotent $e \in R$.

Example 2.2. Every simple R-module M is a Baer module. If $m \in M$, then either $Rm = 0$ or $Rm = M$. This implies that $\mathrm{ann}_R(m) = R$ or $\mathrm{ann}_R(m) = \mathrm{ann}_R(M)$. Thus we have $\mathrm{ann}_R(m)M = 1M$ or $\mathrm{ann}_R(m)M = 0M$. In particular, the $\mathbb{Z}$-module $\mathbb{Z}_p$ is a Baer module for each prime number p.

Proposition 2.3. *Let $n > 1$ be an integer. Then $\mathbb{Z}_n$ is a Baer $\mathbb{Z}$-module if and only if n is square free.*

Proof. ($\Leftarrow$): Assume that n is a square free integer. Then there exist distinct prime numbers $p_1, p_2, \ldots, p_r$ such that $n = p_1 p_2 \cdots p_r$. Let $0 \neq \overline{m} \in \mathbb{Z}_n$. If $\gcd(m, n) = 1$, then $\mathrm{ann}_{\mathbb{Z}}(\overline{m}) = p_1 p_2 \cdots p_r \mathbb{Z}$ and so $\mathrm{ann}_{\mathbb{Z}}(\overline{m})\mathbb{Z}_n = 0\mathbb{Z}_n$. So suppose that $\gcd(m, n) \neq 1$. Then we may assume that $m = kp_1 p_2 \cdots p_t$ for some $k \in \mathbb{Z}$ with $\gcd(k, n) = 1$ and $1 \leq t < r$. Thus we have $\mathbb{Z}\overline{m} = \mathbb{Z}\overline{p_1 p_2 \cdots p_t}$ and so $\mathrm{ann}_{\mathbb{Z}}(\overline{m}) = p_{t+1} \cdots p_r \mathbb{Z}$, which implies that $\mathrm{ann}_{\mathbb{Z}}(\overline{m})\mathbb{Z}_n = p_{t+1} \cdots p_r \mathbb{Z}_n$. Now we show that $p_{t+1} \cdots p_r \mathbb{Z}_n = e\mathbb{Z}_n$ for some weak idempotent $e \in \mathbb{Z}$. Consider the following system of equations:

$$\begin{aligned} p_{t+1} \cdots p_r x &\equiv 1 \pmod{p_1} \\ p_{t+1} \cdots p_r x &\equiv 1 \pmod{p_2} \\ &\cdots \\ p_{t+1} \cdots p_r x &\equiv 1 \pmod{p_t}. \end{aligned}$$

By the Chinese remainder theorem, we can find a solution $s \in \mathbb{Z}$ for the above system. Note that $\gcd(s, p_i) = 1$ for each $1 \leq i \leq t$. Now, put $p_{t+1} \cdots p_r s = e$. Let $\mathrm{ord}(\overline{e})$ denote the order of $\overline{e}$ in the additive group of $\mathbb{Z}_n$. Then note that $\mathrm{ord}(\overline{e}) = \mathrm{ord}(\overline{p_{t+1} \cdots p_r})$ and so $p_{t+1} \cdots p_r \mathbb{Z}_n = e\mathbb{Z}_n$. Since $e \equiv 1 \pmod{p_i}$ for each $1 \leq i \leq t$, we have $e^2 - e \in \mathrm{ann}_{\mathbb{Z}}(\mathbb{Z}_n)$, showing that e is weak idempotent. Thus the $\mathbb{Z}$-module $\mathbb{Z}_n$ is a Baer module.

($\Rightarrow$): Let $n > 1$ be a nonsquare free integer. Without loss of generality, we may assume that $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ for some prime numbers $p_1, p_2, \ldots, p_r$ such that $\alpha_1 \geq 2$ and $\alpha_i \geq 1$ for each $2 \leq i \leq r$. Put $\overline{m} = \overline{p_1^{\alpha_1 - 1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}}$. Then $\mathrm{ann}_{\mathbb{Z}}(\overline{m}) = p_1\mathbb{Z}$ and so $\mathrm{ann}_{\mathbb{Z}}(\overline{m})\mathbb{Z}_n = p_1\mathbb{Z}_n$. Assume that $p_1\mathbb{Z}_n = e\mathbb{Z}_n$ for some weak idempotent $e \in \mathbb{Z}$. This implies that $e = kp_1$ for some $k \in \mathbb{Z}$. Since $\mathrm{ord}(\overline{e}) = \mathrm{ord}(\overline{p_1})$, we have $\gcd(e, n) = p_1$ and so $\gcd(k, p_i) = 1$ for every $1 \leq i \leq r$. Since $e - e^2 = kp_1(1 - kp_1) \equiv 0 \pmod{n}$, we conclude that p_1 divides $(1 - kp_1)$, which is a contradiction. Thus the $\mathbb{Z}$-module $\mathbb{Z}_n$ is not a Baer module. $\square$

Example 2.4. Every torsion free R-module M is a Baer module. Choose a nonzero element $m \in M$; then it is clear that $\operatorname{ann}_R(m)M = 0 = 0M$.

Recall from [21] that a nonzero submodule N of M is said to be a *second submodule* of M if, for each $a \in R$, either $aN = N$ or $aN = (0)$. In particular, an R-module M is called a *second module* if it is a second submodule of itself.

Example 2.5. (i) Let $\operatorname{ann}_R(M)$ be a maximal ideal of R and let $m \in M$. Since $\operatorname{ann}_R(M) \subseteq \operatorname{ann}_R(m)$, we conclude that either $\operatorname{ann}_R(m) = \operatorname{ann}_R(M)$ or $\operatorname{ann}_R(m) = R$. This implies that $\operatorname{ann}_R(m)M = 0M$ or $\operatorname{ann}_R(m)M = 1M$. Hence M is a Baer module. For instance, the $\mathbb{Z}$-module $\mathbb{Z}_5 \times \mathbb{Z}_5$ is a Baer module, and also it is neither a simple nor a torsion free module.

(ii) Every second R-module is a Baer module.

An R-module M is said to be a *λ_0-module* if for each finite number of ideals $I_1, I_2, \ldots, I_n$ we have $\bigcap_{i=1}^n (I_i M) = \left[\bigcap_{i=1}^n (I_i + \operatorname{ann}_R(M))\right] M$. By [7, Corollary 1.7], every multiplication module is an example of λ_0-module. Also note that every vector space is a λ_0-module. Thus the class of λ_0-modules properly contains the class of multiplication modules.

Proposition 2.6. *Let M be a λ_0-module. The following statements are equivalent.*

(i) *M is a Baer module.*

(ii) *For any finitely generated submodule N of M, $\operatorname{ann}_R(N)M = eM$ for some weak idempotent $e \in R$.*

Proof. (i) ⇒ (ii): Suppose that M is a Baer module and N is a finitely generated submodule of M. Then we can write $N = Rm_1 + \cdots + Rm_n$ for some $m_1, m_2, \ldots, m_n \in M$. Since M is Baer, there exist weak idempotents $e_i \in R$ such that $\operatorname{ann}_R(m_i)M = e_i M$ for each $i = 1, 2, \ldots, n$. This implies that $\operatorname{ann}_R(N)M = \left(\bigcap_{i=1}^n \operatorname{ann}_R(m_i)\right)M$. Since M is a λ_0-module, we get

$$\left(\bigcap_{i=1}^{n} \operatorname{ann}_R(m_i)\right)M = \bigcap_{i=1}^{n} [\operatorname{ann}_R(m_i)M] = \bigcap_{i=1}^{n} e_i M = e_1 e_2 \cdots e_n M$$

by [12, Lemma 1 (iii)]. Thus we have $\operatorname{ann}_R(N)M = eM$, where $e = e_1 e_2 \cdots e_n$ is a weak idempotent element of R.

(ii) ⇒ (i): It is clear. □

Recall that an R-module M is said to be a *reduced module* if, for $a \in R$ and $m \in M$, whenever $am = 0$ one has $aM \cap Rm = 0$. It is clear that an R-module M is a reduced module if and only if $a^2 m = 0$ implies that $am = 0$ for each $a \in R$ and $m \in M$ [16].

Proposition 2.7. *Every finitely generated Baer module is a reduced module.*

Proof. Suppose that M is a finitely generated Baer module and $a^2 m = 0$ for some $a \in R$ and $m \in M$. Then $a \in \operatorname{ann}_R(am)$ and so $aM \subseteq \operatorname{ann}_R(am)M = eM$ for some weak idempotent $e \in R$, since M is a Baer module. This yields $am = em' = e^2 m'$ for some $m' \in M$. Then we get $am = e^2 m' = e(em') = e(am)$ and so $(1 - e) \in$

$\operatorname{ann}_R(am)$, which implies that $(1-e)M \subseteq \operatorname{ann}_R(am)M \subseteq eM$. Thus we have $eM = eM+(1-e)M = M$ and so $\operatorname{ann}_R(am)M = eM = M$. Now we will show that $\operatorname{ann}_R(am) = R$. Suppose to the contrary that $\operatorname{ann}_R(am) = R$. Then there exists a maximal ideal Q containing $\operatorname{ann}_R(am)$. Thus we get $\operatorname{ann}_R(am)M = M \subseteq QM$ and so $QM = M$. Since M is finitely generated, by [4, Corollary 2.5] we get $1-r \in \operatorname{ann}_R(M)$ for some $r \in Q$. As $\operatorname{ann}_R(M) \subseteq Q$, it follows that $1 \in Q$, which is a contradiction. Therefore $\operatorname{ann}_R(am) = R$ and hence $am = 0$. □

Proposition 2.8. *Let M be a finitely generated R-module. If M is a VN-regular module, then M is a Baer module.*

Proof. Suppose that M is a finitely generated VN-regular module. Choose $m \in M$. Since M is finitely generated VN-regular, $Rm = eM$ for some weak idempotent $e \in R$. As $e - e^2 \in \operatorname{ann}_R(M)$, we get $(1-e)Rm = (1-e)eM = 0$ and so $(1-e) \in \operatorname{ann}_R(Rm)$. This yields $(1-e)M \subseteq \operatorname{ann}_R(m)M$. To show the reverse inclusion, take $r \in \operatorname{ann}_R(m) = \operatorname{ann}_R(Rm)$. Then we have $r(Rm) = r(eM) = (re)M = 0$, which implies that $rem' = 0$ for each $m' \in M$. Let $m' \in M$. Then we conclude that $rm' = rm' - rem' = (1-e)(rm') \in (1-e)M$, which yields $\operatorname{ann}_R(m)M \subseteq (1-e)M$. Thus we have $\operatorname{ann}_R(m)M = (1-e)M$, where $1-e$ is a weak idempotent in R. Consequently, M is a Baer module. □

Definition 2.9. An R-module M is said to be an *annihilator multiplication module* if for each $m \in M$, $\operatorname{ann}_R(m) = \operatorname{ann}_R(IM)$ for some finitely generated ideal I of R.

Note that every multiplication module is an annihilator multiplication module. But the following example shows that an annihilator multiplication module is not necessarily a multiplication module.

Example 2.10. (i) Every torsion free module is an annihilator multiplication module. Let M be a torsion free R-module and let $m \in M$. If $m = 0$, then $\operatorname{ann}_R(m) = R = \operatorname{ann}_R((0)M)$. Otherwise, we would have $\operatorname{ann}_R(m) = 0 = \operatorname{ann}_R((1)M)$. Also note that a torsion free module need not be a multiplication module. For instance, the $\mathbb{Z}$-module $\mathbb{Z}[i]$ (Gaussian integers) is an annihilator multiplication module but not a multiplication module.

(ii) Every simple module is an annihilator multiplication module.

Lemma 2.11. *Every finitely generated Baer module is an annihilator multiplication module.*

Proof. Let M be a finitely generated Baer module and let $m \in M$. Then we have $\operatorname{ann}_R(m)M = eM$ for some weak idempotent $e \in R$. This implies that $eM + (1-e)M = \operatorname{ann}_R(m)M + (1-e)M = M$. Now we will show that $\operatorname{ann}_R(m) + (1-e) = R$. Suppose to the contrary that $\operatorname{ann}_R(m) + (1-e) = R$. Then there exists a maximal ideal Q containing $\operatorname{ann}_R(m) + (1-e)$. Then we have $QM = M$. By [4, Corollary 2.5], we get $(1-r)M = 0$ for some $r \in Q$. Since $1-r \in \operatorname{ann}_R(M) \subseteq Q$, we get $1 \in Q$, which is a contradiction. Thus we have $\operatorname{ann}_R(m)+(1-e) = R$. Then $r'+(1-e)x = 1$ for some $r' \in \operatorname{ann}_R(m)$ and $x \in R$. This implies that $e = er' + e(1-e)x$ and so $e \in \operatorname{ann}_R(m)+\operatorname{ann}_R(M) = \operatorname{ann}_R(m)$. Then we conclude that $(e) \subseteq \operatorname{ann}_R(m)$. Now take $r'' \in \operatorname{ann}_R(m)$. Thus we have $r''(1-e)M \subseteq \operatorname{ann}_R(m)(1-e)M = 0$ and hence $r''(1-e) \in \operatorname{ann}_R(M)$. This implies that $r'' = r''e + r''(1-e) \in (e) + \operatorname{ann}_R(M)$. Thus we have $\operatorname{ann}_R(m) = (e) + \operatorname{ann}_R(M)$. Since $\operatorname{ann}_R((1-e)M) = (e) + \operatorname{ann}_R(M)$, we get $\operatorname{ann}_R(m) = \operatorname{ann}_R((1-e)M)$. Therefore, M is an annihilator multiplication module. ☐

Proposition 2.12. *Let M be a finitely generated R-module. If M is a Baer module, then M is an annihilator multiplication module and $R/\operatorname{ann}_R(M)$ is a Baer ring.*

Proof. Suppose that M is a finitely generated Baer module. Then by Lemma 2.11, M is an annihilator multiplication module. Let $a + \operatorname{ann}_R(M) \in R/\operatorname{ann}_R(M)$. To prove that $R/\operatorname{ann}_R(M)$ is a Baer ring, we need to show that $\operatorname{ann}(a + \operatorname{ann}_R(M)) = (\overline{e})$ for some idempotent $\overline{e} \in R/\operatorname{ann}_R(M)$. It is easy to see that $\operatorname{ann}(a + \operatorname{ann}_R(M)) = \operatorname{ann}_R(aM)/\operatorname{ann}_R(M)$. Since M is finitely generated, it follows that $M = Rm_1 + Rm_2 + \cdots + Rm_n$ for some $m_1, m_2, \ldots, m_n \in M$ and so $aM = R(am_1) + \cdots + R(am_n)$. This implies that $\operatorname{ann}_R(aM) = \bigcap_{i=1}^n \operatorname{ann}_R(am_i)$. As M is a finitely generated Baer module, a similar argument as in the proof of Lemma 2.11 shows that $\operatorname{ann}_R(am_i) = (e_i) + \operatorname{ann}_R(M)$ for some weak idempotent $e_i \in R$. This implies that $\operatorname{ann}_R(am_i)/\operatorname{ann}_R(M) = (e_i + \operatorname{ann}_R(M))$. Then note that $e_i + \operatorname{ann}_R(M)$ is an idempotent of $R/\operatorname{ann}_R(M)$ and so

$$\begin{aligned}
\operatorname{ann}_R(aM)/\operatorname{ann}_R(M) &= \bigcap_{i=1}^n \operatorname{ann}_R(am_i)/\operatorname{ann}_R(M) \\
&= \bigcap_{i=1}^n [\operatorname{ann}_R(am_i)/\operatorname{ann}_R(M)] \\
&= \bigcap_{i=1}^n [(e_i + \operatorname{ann}_R(M))] \\
&= (e_1e_2\cdots e_n + \operatorname{ann}_R(M)) = (e + \operatorname{ann}_R(M)),
\end{aligned}$$

where $e = e_1e_2\cdots e_n$ and $e + \operatorname{ann}_R(M)$ is an idempotent of $R/\operatorname{ann}_R(M)$. ☐

Proposition 2.13. *Let M be an annihilator multiplication R-module and let $R/\operatorname{ann}_R(M)$ be a Baer ring. Then M is a Baer module.*

Proof. Let $m \in M$. Since M is an annihilator multiplication module, we have $\operatorname{ann}_R(m) = \operatorname{ann}_R(IM)$ for some finitely generated ideal $I = (a_1, a_2, \ldots, a_n)$ of R. This implies that $\operatorname{ann}_R(m) = \operatorname{ann}_R(a_1M + \cdots + a_nM) = \bigcap_{i=1}^n \operatorname{ann}_R(a_iM)$. Since $R/\operatorname{ann}_R(M)$ is a Baer ring, for each $a_i \in R$ we have $\operatorname{ann}(a_i + \operatorname{ann}_R(M)) = \operatorname{ann}_R(a_iM)/\operatorname{ann}_R(M) = (e_i) + \operatorname{ann}_R(M)/\operatorname{ann}_R(M)$ for some weak idempotent $e_i \in R$. This yields $\operatorname{ann}_R(a_iM) = (e_i) + \operatorname{ann}_R(M)$. Then we have $\operatorname{ann}_R(m) = \bigcap_{i=1}^n \operatorname{ann}_R(a_iM) = \bigcap_{i=1}^n((e_i) + \operatorname{ann}_R(M))$. Since $\bigcap_{i=1}^n((e_i) + \operatorname{ann}_R(M)) = (e) + \operatorname{ann}_R(M)$, where $e = e_1e_2\cdots e_n$ is a weak idempotent of R, we conclude that $\operatorname{ann}_R(m)M = eM$, which completes the proof. ☐

In the following theorem, we give a characterization of Baer modules in terms of Baer rings.

Theorem 2.14. (i) *Let M be a finitely generated R-module. Then M is a Baer module if and only if M is an annihilator multiplication module and $R/\operatorname{ann}_R(M)$ is a Baer ring.*

(ii) *Let M be a finitely generated multiplication R-module. Then M is a Baer module if and only if $R/\operatorname{ann}_R(M)$ is a Baer ring.*

Proof. The proof follows from Proposition 2.12 and Proposition 2.13. □

Proposition 2.15. *Suppose that M_i's are finitely generated R_i-modules for all $i \in \Delta$. Then $M = \prod_{i\in\Delta} M_i$ is a Baer $R = \prod_{i\in\Delta} R_i$-module if and only if M_i is a Baer R_i-module for each $i \in \Delta$.*

Proof. Let $\prod_{i\in\Delta} M_i$ be a Baer $\prod_{i\in\Delta} R_i$-module and let $m_j \in M_j$ for some $j \in \Delta$. Consider the sequence

$$(n_i)_{i\in\Delta} = \begin{cases} m_j, & i = j; \\ 0, & i \neq j. \end{cases}$$

Then note that $\operatorname{ann}_R((n_i)_{i\in\Delta}) = \prod_{i\in\Delta} \operatorname{ann}_{R_i}(n_i)$, where $\operatorname{ann}_{R_i}(n_i) = R_i$ for all $i \neq j$. Since M is a Baer R-module, we have

$$\begin{aligned} \operatorname{ann}_R((n_i)_{i\in\Delta})M &= \prod_{i\in\Delta} \operatorname{ann}_{R_i}(n_i) \prod_{i\in\Delta} M_i \\ &= (e_i)_{i\in\Delta} \prod_{i\in\Delta} M_i \\ &= \prod_{i\in\Delta} [e_i M_i] \end{aligned}$$

for some weak idempotent $(e_i)_{i\in\Delta}$ of R. Also, it can be easily shown that e_j is a weak idempotent of R_j for all $j \in \Delta$ and $\prod_{i\in\Delta}[e_i M_i] = \prod_{i\in\Delta} \operatorname{ann}_{R_i}(n_i) \prod_{i\in\Delta} M_i \subseteq \prod_{i\in\Delta}[\operatorname{ann}_{R_i}(n_i)M_i]$. This implies that $e_j M_j \subseteq \operatorname{ann}_{R_j}(m_j)M_j$. Let $r \in \operatorname{ann}_{R_j}(m_j)$ and put

$$(r_i)_{i\in\Delta} = \begin{cases} r_i = r, & i = j; \\ 1, & i \neq j. \end{cases}$$

Then $(r_i)_{i\in\Delta} \in \operatorname{ann}_R((n_i)_{i\in\Delta})$ and so

$$(r_i)_{i\in\Delta} M = \prod_{i\in\Delta} [r_i M_i] \subseteq \operatorname{ann}_R((n_i)_{i\in\Delta})M = \prod_{i\in\Delta} [e_i M_i],$$

so we have $rM_j \subseteq e_j M_j$, which implies that $\operatorname{ann}_{R_j}(m_j)M_j \subseteq e_j M_j$. Thus we have $\operatorname{ann}_{R_j}(m_j)M_j = e_j M_j$. Conversely, assume that M_i is a Baer R_i-module for each $i \in \Delta$. Let $(m_i)_{i\in\Delta} \in \prod_{i\in\Delta} M_i$. Then for each $i \in \Delta$, $\operatorname{ann}_{R_i}(m_i)M_i = e_i M_i$ for some weak idempotent $e_i \in R_i$. Since M_i is finitely generated, we have

$\operatorname{ann}_{R_i}(m_i) = (e_i) + \operatorname{ann}_{R_i}(M_i)$ and so $(e_i) \subseteq \operatorname{ann}_{R_i}(m_i)$. Also note that

$$\begin{aligned}\operatorname{ann}_R((m_i)_{i\in\Delta})M &= \prod_{i\in\Delta} \operatorname{ann}_{R_i}(m_i) \prod_{i\in\Delta} M_i \\ &\subseteq \prod_{i\in\Delta} [\operatorname{ann}_{R_i}(m_i)M_i] = \prod_{i\in\Delta} [e_i M_i] \\ &= (e_i)_{i\in\Delta} \prod_{i\in\Delta} M_i.\end{aligned}$$

On the other hand, since $(e_i)_{i\in\Delta} \subseteq \prod_{i\in\Delta} \operatorname{ann}_{R_i}(m_i) = \operatorname{ann}_R((m_i)_{i\in\Delta})$, we have that $(e_i)_{i\in\Delta} \prod_{i\in\Delta} M_i \subseteq \operatorname{ann}_R((m_i)_{i\in\Delta})M$. Thus, $\operatorname{ann}_R((m_i)_{i\in\Delta})M = (e_i)_{i\in\Delta}M$. Since $(e_i)_{i\in\Delta}$ is a weak idempotent of R, $\prod_{i\in\Delta} M_i$ is a Baer $\prod_{i\in\Delta} R_i$-module. □

Recall from [3] that a submodule N of M is said to be a *pure submodule* if $IM \cap N = IN$ for each ideal I of R.

Lemma 2.16. *Every pure submodule of a Baer module is also a Baer module.*

Proof. It is obvious. □

Corollary 2.17. *Let $\{M_i\}_{i\in\Delta}$ be a family of R-modules. Consider the following cases:*

(i) *$\prod_{i\in\Delta} M_i$ is a Baer R-module.*
(ii) *$\bigoplus_{i\in\Delta} M_i$ is a Baer R-module.*
(iii) *M_i is a Baer R-module for each $i \in \Delta$.*

Then (i) ⇒ (ii) ⇒ (iii) *always holds.*

Proof. (i) ⇒ (ii) and (ii) ⇒ (iii) follow from Lemma 2.16. □

The following example shows that in Corollary 2.17 the implication (iii) ⇒ (i) is not always true.

Example 2.18. Consider the simple $\mathbb{Z}$-module $\mathbb{Z}_{p_i}$, where p_i is the i-th prime number. Then by Example 2.2, the $\mathbb{Z}$-module $\mathbb{Z}_{p_i}$ is a Baer module. Let $M = \prod_i \mathbb{Z}_{p_i}$ and let

$$(m_i) = \begin{cases} \overline{1}, & i = 1; \\ \overline{0}, & i \neq 1. \end{cases}$$

Then it is easy to see that $\operatorname{ann}_{\mathbb{Z}}((m_i)) = 2\mathbb{Z}$ and also $\operatorname{ann}_{\mathbb{Z}}((m_i)) \prod_i \mathbb{Z}_{p_i} = 2\prod_i \mathbb{Z}_{p_i}$. Since M is a faithful $\mathbb{Z}$-module, note that the only weak idempotents of $\mathbb{Z}$ are 0 and 1. As $\operatorname{ann}_{\mathbb{Z}}((m_i))M \neq 0M$ and $\operatorname{ann}_{\mathbb{Z}}((m_i))M \neq 1M$, it follows that M is not a Baer module.

Let M be an R-module and let S be a multiplicatively closed subset of R. Then $S^{-1}M$ denotes the quotient module of M over the quotient ring $S^{-1}R$. In particular, if we take $S = R - P$ for some prime ideal P of R, then we use M_P (resp. R_P) to denote $S^{-1}M$ (resp. $S^{-1}R$).

Proposition 2.19. *If M is a Baer R-module, then $S^{-1}M$ is a Baer $S^{-1}R$-module.*

Proof. Let $\frac{m}{s} \in S^{-1}M$, where $s \in S$, $m \in M$. It is easily seen that $\mathrm{ann}_{S^{-1}R}(\frac{m}{s}) = S^{-1}(\mathrm{ann}_R(m))$. Since M is a Baer module, we get $\mathrm{ann}_R(m)M = eM$ for some weak idempotent $e \in R$. This implies that $\mathrm{ann}_{S^{-1}R}(\frac{m}{s})S^{-1}M = S^{-1}(\mathrm{ann}_R(m))\, S^{-1}M = S^{-1}[\mathrm{ann}_R(m)M] = S^{-1}(eM) = \frac{e}{1}S^{-1}M$. Since e is weak idempotent in R, $\frac{e}{1}$ is weak idempotent in $S^{-1}R$. Thus, $S^{-1}M$ is a Baer $S^{-1}R$-module. □

Corollary 2.20. *Let M be a Baer R-module. Then M_P is a Baer R_P-module for each prime ideal P of R.*

Let M be an R-module. $R \ltimes M = R \oplus M$, the *idealization* of the R-module M, or the *trivial extension* of R by M, is a commutative ring with componentwise addition and multiplication $(r, m)(s, m') = (rs, rm' + sm)$ for each $r, s \in R$, $m, m' \in M$ [18]. Also the set of all nilpotent elements in $R \ltimes M$ is characterized as follows:

$$\sqrt{0_{R\ltimes M}} = \sqrt{0} \ltimes M$$

(see [2] and [10]). So it is easy to see that $R \ltimes M$ is a reduced ring if and only if R is a reduced ring and $M = 0$. In this case, $R \ltimes M$ is isomorphic to R.

Corollary 2.21. *Let M be an R-module. Then*

(i) *$R \ltimes M$ is a VN-regular ring if and only if R is a VN-regular ring and $M = 0$.*
(ii) *$R \ltimes M$ is a Baer ring if and only if R is a Baer ring and $M = 0$.*

Proof. Since VN-regular rings and Baer rings are reduced, the results follow from the isomorphism $R \ltimes 0 \cong R$. □

3. σ-submodules, Baer submodules and m-submodules

In this section, we characterize Baer modules and VN-regular modules in terms of Baer submodules and σ-submodules.

Definition 3.1. Let M be an R-module and let N be a submodule of M. Then N is said to be a *σ-submodule* if $m \in N$ implies that $\mathrm{ann}_R(m) + (N :_R M) = R$. In particular, an ideal I of R is called a *σ-ideal* if I is a σ-submodule of the R-module R.

Note that the σ-ideals of R are precisely the pure ideals of R. It is easy to verify that the set of all σ-submodules is closed under arbitrary sum and finite intersection.

Lemma 3.2. *Every σ-submodule is a pure submodule.*

Proof. Let I be an ideal of R and let N be a σ-submodule of M. Choose $x \in IM \cap N$. Then we can write $x = r_1m_1 + \cdots + r_nm_n$ for some $r_i \in I$ and $m_i \in M$. Since N is a σ-submodule of M, we have $\mathrm{ann}_R(x) + (N :_R M) = R$. This implies that $1 = y + s$ for some $y \in \mathrm{ann}_R(x)$ and $s \in (N :_R M)$. Then we conclude that $x = yx + sx = sx = r_1(sm_1) + \cdots + r_n(sm_n) \in IN$. Thus $IM \cap N \subseteq IN$. The reverse inclusion always holds. □

Example 3.3 (A pure submodule that is not a σ-submodule)**.** Consider $R = \mathbb{Z}$, $M = \mathbb{Z}_2 \times \mathbb{Z}$ and $N = \mathbb{Z}_2 \times 0$. First note that $(N :_R M) = 0$. Let $m = (1, 0) \in N$. Then $\operatorname{ann}_R(m) + (N :_R M) = 2\mathbb{Z} \neq R$ and so N is not a σ-submodule. Let $r \in R$. If r is even, then $(r)M \cap N = \{(0, 0)\} = (r)N$. If r is odd, we have $(r)M \cap N = N = (r)N$, that is, N is a pure submodule of M.

Let M be an R-module and let $m \in M$. Then one can easily see that $m \in \operatorname{ann}_M(\operatorname{ann}_R(m))$ and $\operatorname{ann}_R(\operatorname{ann}_M(\operatorname{ann}_R(m))) = \operatorname{ann}_R(m)$.

Definition 3.4. Let M be an R-module. A submodule N of M is called a *Baer submodule* if $m \in N$ implies that $\operatorname{ann}_M(\operatorname{ann}_R(m)) \subseteq N$.

A Baer submodule of the R-module R is exactly a Baer ideal of R. The reader may consult [11] for details on Baer ideals of commutative rings.

Lemma 3.5. *Let M be an R-module. Then $\operatorname{ann}_M(\operatorname{ann}_R(m))$ is a Baer submodule of M.*

Proof. Let $x \in \operatorname{ann}_M(\operatorname{ann}_R(m))$ for some $x \in M$. Then we have $\operatorname{ann}_R(m) = \operatorname{ann}_R(\operatorname{ann}_M(\operatorname{ann}_R(m))) \subseteq \operatorname{ann}_R(x)$ and so $\operatorname{ann}_M(\operatorname{ann}_R(x)) \subseteq \operatorname{ann}_M(\operatorname{ann}_R(m))$. Thus $\operatorname{ann}_M(\operatorname{ann}_R(m))$ is a Baer submodule of M. □

We now characterize Baer modules in terms of Baer submodules and σ-submodules.

Theorem 3.6. *Let M be a finitely generated R-module. Then M is a Baer module if and only if every Baer submodule is a σ-submodule.*

Proof. Suppose that M is a Baer module and N is a Baer submodule of M. Let $m \in N$. Since N is a Baer submodule, we have $\operatorname{ann}_M(\operatorname{ann}_R(m)) \subseteq N$. As M is a Baer module, we have $\operatorname{ann}_R(m)M = eM$ for some weak idempotent $e \in R$. Therefore $\operatorname{ann}_R(m) = (e) + \operatorname{ann}_R(M)$. Also note that $(1 - e)\operatorname{ann}_R(m)m' = 0$ for all $m' \in M$ and so $(1 - e)M \subseteq \operatorname{ann}_M(\operatorname{ann}_R(m)) \subseteq N$. This implies that $(1 - e) \subseteq (N :_R M)$ and so $(e) + \operatorname{ann}_R(M) + (1 - e) = R \subseteq \operatorname{ann}_R(m) + (N :_R M)$. Thus we have $\operatorname{ann}_R(m) + (N :_R M) = R$, that is, N is a σ-submodule of M. Now assume that every Baer submodule is a σ-submodule. For the converse, take $m \in M$. By Lemma 3.5, $\operatorname{ann}_M(\operatorname{ann}_R(m))$ is a Baer submodule and also $m \in \operatorname{ann}_M(\operatorname{ann}_R(m))$. Then by assumption, $\operatorname{ann}_M(\operatorname{ann}_R(m))$ is a σ-submodule and so $\operatorname{ann}_R(m) + (\operatorname{ann}_M(\operatorname{ann}_R(m)) :_R M) = R$. Since $\operatorname{ann}_R(m)(\operatorname{ann}_M(\operatorname{ann}_R(m)) :_R M) \subseteq \operatorname{ann}_R(M)$, we have $\operatorname{ann}_R(m)M = eM$ for some weak idempotent $e \in R$ by [12, Lemma 2]. Thus M is a Baer module. □

Theorem 3.7. *Let M be a finitely generated R-module. Then M is a Baer R-module if and only if every submodule N of M is a Baer R-module.*

Proof. The "only if" part is clear. Assume that M is a Baer module and N is a proper submodule of M. Let $m \in N$. Then $\operatorname{ann}_R(m)M = eM$ for some weak idempotent $e \in R$. Note that $e \in R$ is also a weak idempotent with respect to the R-module N. Since M is finitely generated, $\operatorname{ann}_R(m) = (e) + \operatorname{ann}_R(M)$ and so $\operatorname{ann}_R(m)N = eN$, which completes the proof. □

In the following theorem, we give a new characterization of VN-regular modules in terms of σ-submodules.

Theorem 3.8. *Let M be a finitely generated R-module. The following statements are equivalent.*

(i) *Every proper submodule is a σ-submodule.*
(ii) *Every proper cyclic submodule is a σ-submodule.*
(iii) *M is a VN-regular module.*

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii): Suppose (ii) holds. Let $m \in M$. By assumption, we have $\operatorname{ann}_R(m) + (Rm :_R M) = R$. Since $\operatorname{ann}_R(m)(Rm :_R M) \subseteq \operatorname{ann}_R(M)$, by [12, Lemma 2] we have $\operatorname{ann}_R(m) = (e) + \operatorname{ann}_R(M)$ and $(Rm :_R M) = (1 - e) + \operatorname{ann}_R(M)$ for some weak idempotent $e \in R$. So $(1 - e)M = (Rm :_R M)M \subseteq Rm$. Also $m = 1.m = em + (1 - e)m = (1 - e)m$ as $em = 0$, which implies that $Rm \subseteq (1 - e)M$. Therefore, $Rm = (1 - e)M$ and hence M is a VN-regular module.

(iii) $\Rightarrow$ (i): Suppose that M is a VN-regular module and N is a submodule of M. Let $m \in N$. Then we have $Rm = eM$ for some weak idempotent $e \in R$. Since $(1 - e)Rm = 0$, we conclude that $(1 - e) \subseteq \operatorname{ann}_R(m)$. On the other hand, it is clear that $(e) \subseteq (Rm :_R M) \subseteq (N :_R M)$. Thus we have $(1 - e) + (e) = R \subseteq \operatorname{ann}_R(m) + (N :_R M)$. Therefore, N is a σ-submodule of M. □

Lemma 3.9. *Let M be an R-module and let N be a submodule of M. The following statements are equivalent.*

(i) *N is a Baer submodule of M.*
(ii) $\operatorname{ann}_R(m) \subseteq \operatorname{ann}_R(m')$*, with $m \in N$, implies that $m' \in N$.*
(iii) $N = \bigcup_{m \in N} \operatorname{ann}_M(\operatorname{ann}_R(m))$.

Proof. (i) $\Rightarrow$ (ii): Suppose that N is a Baer submodule of M and $\operatorname{ann}_R(m) \subseteq \operatorname{ann}_R(m')$ with $m \in N$. Then we have

$$m' \in \operatorname{ann}_M(\operatorname{ann}_R(m')) \subseteq \operatorname{ann}_M(\operatorname{ann}_R(m)) \subseteq N,$$

since N is a Baer submodule.

(ii) $\Rightarrow$ (iii): Let $x \in \operatorname{ann}_M(\operatorname{ann}_R(m))$ for some $m \in N$. Then we have $\operatorname{ann}_R(m) \subseteq \operatorname{ann}_R(x)$; by (ii) we get $x \in N$. Thus we have $\bigcup_{m \in N} \operatorname{ann}_M(\operatorname{ann}_R(m)) \subseteq N$. The reverse inclusion always holds.

(iii) $\Rightarrow$ (i): It is clear. □

By Proposition 2.8, we know that a finitely generated VN-regular module is also a Baer module. But the converse is not true in general. For example, the $\mathbb{Z}$-module $\mathbb{Z}_5 \times \mathbb{Z}_5$ is a Baer module which is not a VN-regular module. One can ask the conditions under which a Baer module is a VN-regular module. The following theorem gives an answer to this question.

Theorem 3.10. *Suppose that M is a finitely generated R-module. The following statements are equivalent.*

(i) *M is a reduced multiplication module in which every submodule is a Baer submodule.*
(ii) *M is a reduced multiplication module and for each $m, m' \in M$, $\mathrm{ann}_R(m) = \mathrm{ann}_R(m')$ implies that $Rm = Rm'$.*
(iii) *M is a VN-regular module.*
(iv) *M is a Baer module and every submodule is a Baer submodule.*

Proof. (i) $\Rightarrow$ (ii) follows from Lemma 3.9.

(ii) $\Rightarrow$ (iii): Suppose (ii) holds. By [12, Theorem 1 and Theorem 2], it is sufficient to show that $rM = r^2M$ for each $r \in R$. Let $r \in R$. Since M is a reduced module, we have $\mathrm{ann}_R(rm) = \mathrm{ann}_R(r^2m)$ for all $m \in M$. Then by (ii), we have $Rrm = Rr^2m$ for all $m \in M$ and so we get $rM = r^2M$, which completes the proof.

(iii) $\Rightarrow$ (i): Suppose (iii) holds. Clearly, M is a reduced multiplication module. Let N be a submodule of M and let $m \in N$. Since M is VN-regular, $Rm = eM$ for some weak idempotent $e \in R$. This implies that $\mathrm{ann}_M(\mathrm{ann}_R(m)) = eM = Rm \subseteq N$. Thus N is a Baer submodule of M.

(iii) $\Rightarrow$ (iv) follows from (i) and Proposition 2.8.

(iv) $\Rightarrow$ (iii): Assume that M is a Baer module and every submodule of M is a Baer submodule. Let $m \in M$. Since M is a Baer module, we have $\mathrm{ann}_R(m)M = eM$ for some weak idempotent $e \in R$. Since M is finitely generated, we conclude that $\mathrm{ann}_R(m) = (e) + \mathrm{ann}_R(M)$. As Rm is a Baer submodule, we conclude that $Rm \subseteq \mathrm{ann}_M(\mathrm{ann}_R(m)) = (1-e)M \subseteq Rm$ and so $Rm = e'M$, where $e' = 1 - e$ is a weak idempotent element of R. Therefore, M is a VN-regular module. □

Lemma 3.11. *Let M be an R-module. If N is a prime submodule of M, then M/N is a Baer R-module.*

Proof. Assume that N is a prime submodule of M. Let $m \in M$. Then it is easy to see that $\mathrm{ann}_R(m+N) = (N :_R m)$. If $m \in N$, then $\mathrm{ann}_R(m+N)(M/N) = (N :_R m)(M/N) = R(M/N) = 1(M/N)$. Assume that $m \notin N$. Since N is a prime submodule, we conclude that $(N :_R m) = (N :_R M)$ and so $\mathrm{ann}_R(m+N)(M/N) = (N :_R m)(M/N) = (N :_R M)(M/N) = 0_{M/N} = 0(M/N)$. Thus M/N is a Baer R-module. □

Example 3.12. The converse of Lemma 3.11 is not necessarily true. By Proposition 2.3, the $\mathbb{Z}$-module $\mathbb{Z}_6$ is a Baer module. But $6\mathbb{Z}$ is not a prime submodule of the $\mathbb{Z}$-module $\mathbb{Z}$.

In the following Proposition 3.13, we characterize prime submodules of finitely generated modules in terms of Baer modules.

Proposition 3.13. *Let M be a finitely generated R-module and let N be a proper submodule of M. The following statements are equivalent.*

(i) *N is a prime submodule of M.*
(ii) *$(N :_R M)$ is a prime ideal and M/N is a Baer R-module.*

Proof. (i) $\Rightarrow$ (ii) follows from Lemma 3.11.

(ii) $\Rightarrow$ (i): Suppose (ii) holds. Let $rm \in N$ with $m \notin N$. Since M/N is a Baer R-module, $\operatorname{ann}_R(m+N)(M/N) = e(M/N)$ for some $e \in R$ such that $e - e^2 \in \operatorname{ann}_R(M/N) = (N :_R M)$. Since M/N is finitely generated and $\operatorname{ann}_R(m+N)(M/N) = e(M/N)$, we get $\operatorname{ann}_R(m+N) = (e) + \operatorname{ann}_R(M/N)$. This implies that $(N :_R m) = (e) + (N :_R M)$. Since $e(1-e) \in (N :_R M)$ and $(N :_R M)$ is a prime ideal, we get either $e \in (N :_R M)$ or $1 - e \in (N :_R M)$. If $1 - e \in (N :_R M)$, then $1 \in (e) + (1-e) \subseteq (e) + (N :_R M) = (N :_R m)$ and so $m \in N$, a contradiction. Thus we get $e \in (N :_R M)$ and hence $(e) + (N :_R M) = (N :_R M) = (N :_R m)$, which implies that $r \in (N :_R m) = (N :_R M)$. □

Recall that a submodule N of M is said to be an *essential* (or *large*) *submodule* if for each nonzero submodule N' of M, $N' \cap N \neq 0$; or, equivalently, $N' \cap N = 0$ implies $N' = 0$. Also a submodule N of M has a *complement* (or N is called a *complemented submodule*) if there exists a submodule K of M such that $N + K = M$ and $N \cap K = 0$.

Proposition 3.14. *Let M be a finitely generated multiplication Baer R-module. Then every prime submodule is either essential or has a complement.*

Proof. Suppose that N is a prime submodule of a finitely generated multiplication Baer module M. Assume that N is not essential. Then there exists a nonzero submodule N' of M such that $N \cap N' = 0$. This implies that $N \cap aM = 0$ for all $a \in (N' :_R M)$. Since $N' \neq 0$, there exists $a' \in (N' :_R M)$ such that $a'M \neq 0$. As $N \cap a'M = 0$, we have $a'M \nsubseteq N$. This implies that $a'm' \notin N$ for some $m' \in M$. Note that $(N :_R M) \cap (a'M :_R M) \subseteq \operatorname{ann}_R(M)$ and so $(a')(N :_R M) \subseteq \operatorname{ann}_R(M)$. Then we conclude that $(N :_R M)a'm' = 0$, which implies that $(N :_R M) \subseteq \operatorname{ann}_R(a'm')$. Since M is a finitely generated Baer module, we have $\operatorname{ann}_R(a'm') = (e) + \operatorname{ann}_R(M)$ for some weak idempotent $e \in R$. Thus we have $N = (N :_R M)M \subseteq eM$. Since $e \in \operatorname{ann}_R(a'm')$, we have $e(a'm') = 0$. As N is a prime submodule, we have $e \in (N :_R M)$ and so $N = eM$. Hence N has a complement by [12, Lemma 1]. □

Proposition 3.15. *Let M be a finitely generated Baer module and let N be a σ-submodule of M. Then M/N is a Baer R-module.*

Proof. Suppose that N is a σ-submodule of M. Let $m \in M$. Now we will show that $\operatorname{ann}_R(m+N)(M/N) = e(M/N)$ for some element $e \in R$ such that $e - e^2 \in (N :_R M)$. Since M is a Baer module, we have $\operatorname{ann}_R(m)M = eM$ for some weak idempotent e of R. As M is finitely generated, it follows that $\operatorname{ann}_R(m) = (e) + \operatorname{ann}_R(M)$. Now we show that $\operatorname{ann}_R(m+N) = (N :_R m) = (e) + (N :_R M)$. It is easy to see that $(e) + (N :_R M) \subseteq (N :_R m)$. Let $a \in (N :_R m)$. Then we have $am \in N$. Since N is a σ-submodule, we get $\operatorname{ann}_R(am) + (N :_R M) = R$ and so $x + y = 1$ for some $x \in \operatorname{ann}_R(am)$ and $y \in (N :_R M)$. This implies that $a = ax + ay$. Since $axm = 0$, we have $ax \in \operatorname{ann}_R(m) = (e) + \operatorname{ann}_R(M) \subseteq (e) + (N :_R M)$. Thus we have $a = ax + ay \in (e) + (N :_R M)$. Therefore $\operatorname{ann}_R(m+N) = (e) + (N :_R M)$, which implies that $\operatorname{ann}_R(m+N)(M/N) = e(M/N) + (N :_R M)(M/N) = e(M/N)$. □

Corollary 3.16. (i) *Let M be a finitely generated Baer module and let N be a Baer submodule of M. Then M/N is a Baer R-module.*

(ii) *Let M be a finitely generated Baer R-module and let P be a prime ideal of R with $(PM :_R M) = P$. If PM is a Baer submodule of M, then PM is a prime submodule of M.*

Proof. (i) follows from Theorem 3.6 and Proposition 3.15, while (ii) follows from Theorem 3.6, Proposition 3.13 and Proposition 3.15. □

Definition 3.17. Let M be an R-module and let N be a submodule of M. Then N is called an *m-submodule* if $N = (N :_R M)M$.

Note that M is a multiplication module if and only if every submodule is an m-submodule (see [7]). Clearly, the set of all m-submodules of M is closed under arbitrary sum. If M is a λ_0-module, then this set is closed under finite intersection. For each $a \in R$, aM and IM (with I an ideal of R) are examples of m-submodules. So if M is a λ_0-module, then the set of all m-submodules forms a lattice.

Proposition 3.18. *Let M be an R-module and let N be a submodule of M. Then*

(i) *If N is a σ-submodule, then N is an m-submodule.*
(ii) *If N is a finitely generated σ-submodule, then N is a complemented m-submodule.*
(iii) *If N is an m-submodule of M and $(N :_R M)$ is a σ-ideal of R, then N is a σ-submodule of M.*
(iv) *Let M be a non-torsion module and let N be a σ-submodule of M. Then N is an m-submodule of M and $(N :_R M)$ is a σ-ideal of R.*
(v) *Let M be a finitely generated Baer module which is also non-torsion. Then N is a prime submodule and σ-submodule if and only if N is an m-submodule, $(N :_R M)$ is a σ-ideal and $(N :_R M)$ is a prime ideal.*
(vi) *Let M be a Baer R-module. If N is an m-submodule of M, then N is a Baer R-module.*
(vii) *Let M be a Baer R-module. If M is finitely generated and N is a Baer submodule of M, then N is an m-submodule.*

Proof. (i) Suppose that N is a σ-submodule. Let $m \in N$. Then $\operatorname{ann}_R(m) + (N :_R M) = R$, which implies that $a + b = 1$ for some $a \in \operatorname{ann}_R(m)$ and $b \in (N :_R M)$. Again $m = am + bm = bm \in (N :_R M)M$. Therefore, N is contained in $(N :_R M)M$ and hence $N = (N :_R M)M$.

(ii) Let N be a finitely generated σ-submodule. Suppose $N = Rm_1 + Rm_2 + \cdots + Rm_n$. Then $\operatorname{ann}_R(m_i) + (N :_R M) = R$ for each $i = 1, \dots, n$. Let $I = \bigcap_{i=1}^{n} \operatorname{ann}_R(m_i)$. Then, clearly we have $I + (N :_R M) = R$. Also $I(N :_R M)$ is contained in $\operatorname{ann}_R(M)$, since $I(N :_R M)M \subseteq IN = 0$. By [12, Lemma 2], $I = (e) + \operatorname{ann}_R(M)$ and $(N :_R M) = (1 - e) + \operatorname{ann}_R(M)$ for some weak idempotent e of R. Note that $(N :_R M)M = (1 - e)M$ is contained in N. Also, if $n \in N$ then $en = 0$, so $n = (1 - e)n \in (1 - e)M$ and hence $N = (1 - e)M$. Therefore, by (i) and [12, Lemma 1], N is a complemented m-submodule.

(iii) Assume that N is an m-submodule and $(N :_R M)$ is a σ-ideal of R. Let $m \in N$. Then $m = a_1m_1 + \cdots + a_nm_n$ for some $a_i \in (N :_R M)$ and $m_i \in M$. Since $(N :_R M)$ is a σ-ideal of R, $\operatorname{ann}(a_i) + (N :_R M) = R$ for each $i = 1, 2, \ldots, n$. Then $\bigcap_{i=1}^{n} \operatorname{ann}(a_i) + (N :_R M) = R$, which implies that $\operatorname{ann}_R(m) + (N :_R M) = R$. Therefore, N is a σ-submodule.

(iv) Suppose that $T(M) \neq M$ and N is a σ-submodule of M. Then by (i), N is an m-submodule. Let $a \in (N :_R M)$. Since $M \neq T(M)$, there exists $m \in M$ such that $\operatorname{ann}_R(m) = 0$. As $am \in N$ and N is a σ-submodule, we conclude that $\operatorname{ann}_R(am) + (N :_R M) = R$. Thus $\operatorname{ann}(a) + (N :_R M) = R$. Therefore, $(N :_R M)$ is a σ-ideal of R.

(v) This follows from Proposition 3.13, Proposition 3.15, (iii) and (iv).

(vi) Suppose that N is an m-submodule of M. Let $m \in N$. Since M is a Baer module, $\operatorname{ann}_R(m)M = eM$ for some weak idempotent $e \in R$. This implies that $eN = e(N :_R M)M = (N :_R M)eM = (N :_R M)\operatorname{ann}_R(m)M = \operatorname{ann}_R(m)(N :_R M)M = \operatorname{ann}_R(m)N$.

(vii) This follows from Theorem 3.6 and (i). □

Let M be an R-module. Then the set of all σ-submodules of M, denoted by $\sigma(M)$, is a lattice. Suppose that N_1, N_2 are σ-submodules of M. Then we define their product as follows:

$$N_1.N_2 = (N_1 :_R M)(N_2 :_R M)M.$$

Since every σ-submodule is pure and an m-submodule, it is easily seen that $N_1.N_2 = N_1 \cap N_2$ for all $N_1, N_2 \in \sigma(M)$.

Recall from [12] that an R-module M is said to be a *colon distributive* module if $(N + K :_R M) = (N :_R M) + (K :_R M)$ for each submodule N, K of M. We note that this notion was first studied by P. F. Smith in [19, Lemma 3.1] as a μ-module. Note that by [19, Theorem 3.8], a finitely generated module M is a colon distributive module if and only if it is a multiplication module. In [12, Lemma 3 and Theorem 2], it is shown that a finitely generated R-module M is a VN-regular module if and only if M is colon distributive and for each $m \in M$ there exists a weak idempotent element $e \in R$ such that $Rm = eM$.

Lemma 3.19. *Let M be a colon distributive module. Then $\sigma(M)$ is a distributive lattice.*

Proof. Let K, L and N be σ-submodules of M. Then $L+N$ is a σ-submodule of M. Thus $K.(L + N) = (K :_R M)(L + N :_R M)M$. Since M is a colon distributive module, $(L+N :_R M) = (L :_R M)+(N :_R M)$, and so $K.(L+N) = (K :_R M)(L :_R M)M + (K :_R M)(N :_R M)M = K.L + K.N$. Therefore $(K + L) \cap (K + N) = (K + L).(K + N) = K.K + K.L + K.N + L.N \subseteq K + (L.N) = K + (L \cap N)$, so $K + (L \cap N) = (K + L) \cap (K + N)$. □

The following proposition gives an equivalent condition for $\sigma(M)$ to be a Boolean lattice.

Proposition 3.20. *Let M be a colon distributive module. If M is a finitely generated module, then $\sigma(M)$ is a Boolean lattice if and only if every σ-submodule is finitely generated.*

Proof. Since M is a colon distributive module, by Lemma 3.19, $\sigma(M)$ is a distributive lattice. Assume that every σ-submodule is finitely generated. Then by Proposition 3.18, every σ-submodule is complemented and so $\sigma(M)$ is a Boolean lattice. Conversely, assume that M is finitely generated and $\sigma(M)$ is a Boolean lattice. Let $N \in \sigma(M)$. Then N is a complemented submodule. Since M is colon distributive, by [12, Lemma 3], $N = eM$ for some weak idempotent $e \in R$. Since M is finitely generated, $M = Rm_1 + Rm_2 + \cdots + Rm_n$ for some $m_1, m_2, \ldots, m_n \in M$. Then $N = eM = R(em_1) + R(em_2) + \cdots + R(em_n)$. Hence N is finitely generated. $\square$

4. Extension of Baer modules

In this section, we study polynomial modules and power series modules over a Baer module.

Proposition 4.1. *Let M be a reduced λ_0-module. If M is a Baer R-module, then $M[X]$ is a Baer $R[X]$-module.*

Proof. Suppose M is a Baer R-module. Let $m(x) = m_0 + m_1X + \cdots + m_nX^n \in M[X]$, where $m_i \in M$ and $0 \leq i \leq n$. Suppose that $r(x) = r_0 + r_1X + \cdots + r_kX^k \in \operatorname{ann}_{R[X]}(m(x))$. Then we have the following system of equations:

$$\begin{aligned} r_0m_0 &= 0 \\ r_0m_1 + r_1m_0 &= 0 \\ r_0m_2 + r_1m_1 + r_2m_0 &= 0 \\ &\cdots \\ r_km_n &= 0. \end{aligned}$$

Since $r_0m_0 = 0 = r_0m_1 + r_1m_0$, we have $r_0^2m_1 + r_1(r_0m_0) = 0$ and so $r_0^2m_1 = 0$. Since M is reduced, we have $r_0m_1 = 0$. A similar argument shows that $r_0 \in \bigcap_{j=0}^{n} \operatorname{ann}_R(m_j)$. Similarly, we have $r_i \in \bigcap_{j=0}^{n} \operatorname{ann}_R(m_j)$ for each $i = 1, 2, \ldots, k$. This implies that $\operatorname{ann}_{R[X]}(m(x)) = \left[\bigcap_{j=0}^{n} \operatorname{ann}_R(m_j)\right][X]$. Now put $I = \bigcap_{j=0}^{n} \operatorname{ann}_R(m_j)$. Since M is a λ_0-module, we have $IM = \bigcap_{j=0}^{n}[\operatorname{ann}_R(m_j)M]$. As M is a Baer R-module, for all $j = 0, 1, 2, \ldots, n$ we get $\operatorname{ann}_R(m_j)M = e_jM$ for some weak idempotent $e_j \in R$. This implies that $IM = (e_0e_1 \cdots e_n)M = eM$, where $e = e_0e_1 \cdots e_n$ is a weak idempotent of the R-module M. Thus we have

$$\begin{aligned} \operatorname{ann}_{R[X]}(m(x))M[X] &= I[X]M[X] = (IM)[X] \\ &= (eM)[X] = eM[X]. \end{aligned}$$

Since $e - e^2 \in \operatorname{ann}_R(M)$, we have $e - e^2 \in \operatorname{ann}_{R[X]}(M[X])$ and so the $R[X]$-module $M[X]$ is a Baer module. $\square$

Lemma 4.2. *Let M be an R-module. Consider the $R[X]$-module $M[X]$. Then*

(i) $\mathrm{ann}_{R[X]}(M[X]) = (\mathrm{ann}_R(M))[X]$.

(ii) *If $e(X) = e_0 + e_1X + e_2X^2 + \cdots + e_kX^k \in R[X]$ is a weak idempotent of the $R[X]$-module $M[X]$, then e_0 is a weak idempotent of the R-module M and $e_1, e_2, \ldots, e_k \in \mathrm{ann}_R(M)$.*

Proof. (i): It is clear.

(ii): Let $e(X)$ be a weak idempotent of the $R[X]$-module $M[X]$. First note that $e^2(X) = e_0^2 + (2e_0e_1)X + (2e_0e_2 + e_1^2)X^2 + (2e_0e_3 + 2e_1e_2)X^3 + (2e_0e_4 + 2e_1e_3 + e_2^2)X^4 + \cdots + e_k^2X^{2k}$ and thus $e(X) - e^2(X) = (e_0 - e_0^2) + (e_1 - 2e_0e_1)X + (e_2 - 2e_0e_2 - e_1^2)X^2 + (e_3 - 2e_0e_3 - 2e_1e_2)X^3 + \cdots - e_k^2X^{2k} \in (\mathrm{ann}_R(M))[X]$ by (i). This implies that $e_0 - e_0^2 \in \mathrm{ann}_R(M)$ and so e_0 is a weak idempotent of the R-module M. Since $e_1 - 2e_0e_1 \in \mathrm{ann}_R(M)$, we get $e_0e_1 - 2e_0^2e_1 = e_1(e_0 - e_0^2) - e_0^2e_1 \in \mathrm{ann}_R(M)$; thus we have $e_0^2e_1 \in \mathrm{ann}_R(M)$ and this yields $e_0e_1 \in \mathrm{ann}_R(M)$. Then we have $e_1 \in \mathrm{ann}_R(M)$. Similarly, we get $e_2, \ldots, e_k \in \mathrm{ann}_R(M)$. □

Proposition 4.3. *If $M[X]$ is a Baer $R[X]$-module, then M is a Baer R-module.*

Proof. Suppose that $M[X]$ is a Baer $R[X]$-module and $m \in M$. First note that $\mathrm{ann}_{R[X]}(m) = (\mathrm{ann}_R(m))[X]$. Since $M[X]$ is a Baer $R[X]$-module, we have that $\mathrm{ann}_{R[X]}(m)M[X] = e(X)M[X]$ for some weak idempotent $e(X) = e_0 + e_1X + e_2X^2 + \cdots + e_kX^k \in R[X]$. By Lemma 4.2 (ii), we get $e(X)M[X] = e_0M[X] = (e_0M)[X]$. Thus we get

$$\begin{aligned} \mathrm{ann}_{R[X]}(m)M[X] &= (\mathrm{ann}_R(m))[X]M[X] \\ &= (\mathrm{ann}_R(m)M)[X] \\ &= (e_0M)[X]. \end{aligned}$$

Then we have $\mathrm{ann}_R(m)M = e_0M$. Hence M is a Baer R-module. □

Theorem 4.4. (i) *Let M be a reduced λ_0-module. Then M is a Baer R-module if and only if $M[X]$ is a Baer $R[X]$-module.*

(ii) *Let M be a finitely generated λ_0-module. Then M is a Baer R-module if and only if $M[X]$ is a Baer $R[X]$-module.*

Proof. The proof of (i) follows from Proposition 4.1 and Proposition 4.3. The proof of (ii) follows from Proposition 2.7, Proposition 4.1 and Proposition 4.3. □

Let $M[[X]]$ denote the set of all formal power series in X with coefficients in M. Then $M[[X]]$ becomes an $R[[X]]$-module with scalar multiplication

$$\left(\sum_{i=0}^{\infty} a_iX^i\right)\left(\sum_{i=0}^{\infty} m_iX^i\right) = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} a_jm_{i-j}\right)X^i,$$

where $\sum_{i=0}^{\infty} a_iX^i \in R[[X]]$ and $\sum_{i=0}^{\infty} m_iX^i \in M[[X]]$. To prove when the $R[[X]]$-module $M[[X]]$ is a Baer module, first we give the following result. We omit the proof, since it is similar to that of Lemma 4.2.

Lemma 4.5. *Let M be an R-module. Then the following statements are satisfied for the $R[[X]]$-module $M[[X]]$:*

(i) $\operatorname{ann}_{R[[X]]}(M[[X]]) = (\operatorname{ann}_R(M))[[X]]$.

(ii) *If $e(X) = \sum_{i=0}^{\infty} e_i X^i \in R[[X]]$ is a weak idempotent of the $R[[X]]$-module $M[[X]]$, then e_0 is a weak idempotent of the R-module M and $e_i \in \operatorname{ann}_R(M)$ for all $i \neq 0$. In this case, $e(X)M[[X]] = (e_0 M)[[X]]$.*

Proposition 4.6. *Let M be an R-module. If $M[[X]]$ is a Baer $R[[X]]$-module, then M is a Baer R-module.*

Proof. Suppose that $M[[X]]$ is a Baer $R[[X]]$-module. Let $m \in M$. First note that $\operatorname{ann}_{R[[X]]}(m) = (\operatorname{ann}_R(m))[[X]]$. Since $M[[X]]$ is a Baer $R[[X]]$-module, we have $(\operatorname{ann}_R(m))[[X]]M[[X]] = e(X)M[[X]]$ for some weak idempotent $e(X) \in R[[X]]$, where $e(X) = \sum_{i=0}^{\infty} e_i X^i$. By Lemma 4.5, $(\operatorname{ann}_R(m))[[X]]M[[X]] = (e_0 M)[[X]]$. Since $\operatorname{ann}_R(m)[[X]]M[[X]] \subseteq (\operatorname{ann}_R(m)M)[[X]]$, we conclude that $e_0 M \subseteq \operatorname{ann}_R(m)M$. Let $m' \in \operatorname{ann}_R(m)M$. Then $m' = r_1 m_1 + r_2 m_2 + \cdots + r_n m_n$ for some $r_i \in \operatorname{ann}_R(m)$ and $m_i \in M$. For any $r_i \in \operatorname{ann}_R(m)$, we have $r_i m_i = (r_i + 0X + 0X^2 + \cdots + 0X^n + \cdots)(m_i + 0X + 0X^2 + \cdots + 0X^n + \cdots) \in (\operatorname{ann}_R(m))[[X]]M[[X]] = (e_0 M)[[X]]$. Therefore $r_i m_i \in e_0 M$ and so $m' = r_1 m_1 + r_2 m_2 + \cdots + r_n m_n \in e_0 M$. This implies that $\operatorname{ann}_R(m)M = e_0 M$ and so M is a Baer R-module. □

Proposition 4.7. *Suppose that M is a finitely generated Baer module and that $R/\operatorname{ann}_R(M)$ has only finitely many idempotent elements. Then $M[[X]]$ is a Baer $R[[X]]$-module.*

Proof. Let M be a finitely generated Baer module and assume that $R/\operatorname{ann}_R(M)$ has only finitely many idempotent elements. Then M is a reduced module by Proposition 2.7. Let $m(x) = \sum_{i=0}^{\infty} m_i X^i \in M[[X]]$. Then note that

$$\operatorname{ann}_{R[[X]]}(m(x)) = \left[\bigcap_{i=0}^{\infty} \operatorname{ann}_R(m_i)\right][[X]].$$

Since M is a finitely generated Baer module and $R/\operatorname{ann}_R(M)$ has finitely many idempotents, we have that $\operatorname{ann}_R(m_i) = (e_i) + \operatorname{ann}_R(M)$ for some weak idempotent $e_i \in R$, where $1 \leq i \leq n$. Then $\bigcap_{i=0}^{\infty} \operatorname{ann}_R(m_i) = (e) + \operatorname{ann}_R(M)$, where $e = e_1 e_2 \cdots e_n$ is a weak idempotent element of R. This implies that

$$\begin{aligned} \operatorname{ann}_{R[[X]]}(m(x))M[[X]] &= ((e) + \operatorname{ann}_R(M))[[X]]M[[X]] \\ &= (eM)[[X]]. \end{aligned}$$

Put $e(X) = e + 0X + 0X^2 + \cdots + 0X^n + \cdots$. Then $e(X)$ is a weak idempotent of the $R[[X]]$-module $M[[X]]$ and also $\operatorname{ann}_{R[[X]]}(m(x))M[[X]] = e(X)M[[X]]$. Thus $M[[X]]$ is a Baer $R[[X]]$-module. □

As a consequence of Theorem 4.4, Proposition 4.6 and Proposition 4.7, we have the following corollary.

Corollary 4.8. *Let M be a finitely generated λ_0-module and let $R/\operatorname{ann}_R(M)$ have only finitely many idempotent elements. The following statements are equivalent.*

(i) *M is a Baer R-module.*
(ii) *$M[X]$ is a Baer $R[X]$-module.*
(iii) *$M[[X]]$ is a Baer $R[[X]]$-module.*

References

[1] D. D. Anderson, T. Arabaci, Ü. Tekir and S. Koç, On S-multiplication modules, *Comm. Algebra* **48** (2020), no. 8, 3398–3407. MR 4115356.

[2] D. D. Anderson and M. Winders, Idealization of a module, *J. Commut. Algebra* **1** (2009), no. 1, 3–56. MR 2462381.

[3] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Graduate Texts in Mathematics, Vol. 13, Springer-Verlag, New York, 1974. MR 0417223.

[4] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, MA, 1969. MR 0242802.

[5] A. Barnard, Multiplication modules, *J. Algebra* **71** (1981), no. 1, 174–178. MR 0627431.

[6] G. F. Birkenmeier, J. Y. Kim and J. K. Park, Principally quasi-Baer rings, *Comm. Algebra* **29** (2001), no. 2, 639–660. MR 1841988.

[7] Z. A. El-Bast and P. F. Smith, Multiplication modules, *Comm. Algebra* **16** (1988), no. 4, 755–779. MR 0932633.

[8] M. W. Evans, On commutative P. P. rings, *Pacific J. Math.* **41** (1972), 687–697. MR 0314827.

[9] E. Hashemi and A. Moussavi, Polynomial extensions of quasi-Baer rings, *Acta Math. Hungar.* **107** (2005), no. 3, 207–224. MR 2148584.

[10] J. A. Huckaba, *Commutative rings with zero divisors*, Monographs and Textbooks in Pure and Applied Mathematics, 117, Marcel Dekker, New York, 1988. MR 0938741.

[11] C. Jayaram, Baer ideals in commutative semiprime rings, *Indian J. Pure Appl. Math.* **15** (1984), no. 8, 855–864. MR 0757963.

[12] C. Jayaram and Tekir, von Neumann regular modules, *Comm. Algebra* **46** (2018), no. 5, 2205–2217. MR 3799203.

[13] C. Jayaram, Tekir and S. Koç, Quasi regular modules and trivial extension, *Hacet. J. Math. Stat.* **50** (2021), no. 1, 120–134. MR 4227913.

[14] J. Kist, Minimal prime ideals in commutative semigroups, *Proc. London Math. Soc.* (3) **13** (1963), 31–50. MR 0143837.

[15] S. Koç, On strongly π-regular modules, *Sakarya Univ. J. Sci.* **24** (2020), no. 4, 675–684.

[16] T.-K. Lee and Y. Zhou, Reduced modules, in *Rings, Modules, Algebras, and Abelian Groups*, 365–377, Lecture Notes in Pure and Appl. Math., 236, Dekker, New York, 2004. MR 2050725.

[17] R. Levy and J. Shapiro, The zero-divisor graph of von Neumann regular rings, *Comm. Algebra* **30** (2002), no. 2, 745–750. MR 1883021.

[18] M. Nagata, *Local Rings*, Interscience Tracts in Pure and Applied Mathematics, 13, Interscience Publishers, New York, 1962. MR 0155856.

[19] P. F. Smith, Complete homomorphisms between module lattices, *Int. Electron. J. Algebra* **16** (2014), 16–31. MR 3239068.

[20] J. Von Neumann, On regular rings. *Proc. Natl. Acad. Sci. USA* **22** (1936), no. 12, 707–713.

[21] S. Yassemi, The dual notion of prime submodules, *Arch. Math. (Brno)* **37** (2001), no. 4, 273–278. MR 1879449.

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