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A CONVERGENCE THEOREM FOR APPROXIMATING MINIMIZATION AND FIXED POINT PROBLEMS FOR NON-SELF MAPPINGS IN HADAMARD SPACES

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Abstract. We propose a modified Halpern-type algorithm involving a Lipschitz hemicontractive non-self mapping and the resolvent of a convex function in a Hadamard space. We obtain a strong convergence of the proposed algorithm to a minimizer of a convex function which is also a fixed point of a Lipschitz hemicontractive non-self mapping. Furthermore, we give a numerical example to illustrate and support our method. Our proposed method improves and extends some recent works in the literature.

1. Introduction

Let (X, d) be a metric space and $x, y \in X$. A geodesic path c joining x to y is an isometry $c : I = [0, d(x, y)] \to X$ such that $c(0) = x$, $c(d(x, y)) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in I$. The image of a geodesic path is called the geodesic segment, which is denoted by $[x, y]$ whenever it is unique. We say that a metric space X is a geodesic space if for every pair of points $x, y \in X$, there is a minimal geodesic from x to y. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three vertices (points in X) and three geodesic segments joining each pair of vertices (the edges of Δ). For any geodesic triangle, there is a comparison (Alexandrov) triangle $\bar{\Delta} \subset \mathbb{R}^2$ such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic space X is a CAT(0) space if the distance between an arbitrary pair of points on a geodesic triangle Δ does not exceed the distance between its pair of corresponding points on its comparison triangle $\bar{\Delta}$. If Δ is a geodesic triangle and $\bar{\Delta}$ is its comparison triangle in X, then Δ is said to satisfy

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the CAT(0) inequality if for all points x, y of Δ and $\bar{x}, \bar{y}$ of $\bar{\Delta}$,

$$d(x, y) = d_{\mathbb{R}^2}(\bar{x}, \bar{y}).$$

Let x, y, z be points in X and let y_0 be the midpoint of the segment $[y, z]$; then the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d(y, z). \tag{1.1}$$

Inequality (1.1) is known as the CN inequality of Bruhat and Tits [8]. A geodesic space X is said to be a CAT(0) space if all geodesic triangles satisfy the CAT(0) inequality. Equivalently, X is called a CAT(0) space if and only if it satisfies the CN inequality. Examples of CAT(0) spaces include: Euclidean spaces $\mathbb{R}^n$, Hilbert spaces, simply connected Riemannian manifolds of nonpositive sectional curvature, $\mathbb{R}$-trees, Hilbert ball [16], hyperbolic spaces [38], among others. A complete CAT(0) space is called a Hadamard space.

Let D be a nonempty, closed, and convex subset of a Hadamard space X. A point $x \in X$ is called a fixed point of a non-self mapping $T : D \to X$ if $x = Tx$. We denote by $F(T)$ the set of fixed points of T. The mapping T is said to be:

(1) L-Lipschitz, if there exists $L > 0$ such that

$$d(Tx, Ty) \leq Ld(x, y), \quad \text{for all } x, y \in D;$$

if $L = 1$, then T is nonexpansive;

(2) quasi nonexpansive, if $F(T) \neq \emptyset$ and

$$d(Tx, p) \leq d(x, p), \quad \text{for all } x \in X \text{ and } p \in F(T);$$

(3) k-demicontractive, if $F(T) \neq \emptyset$ and there exists $k \in [0, 1)$ such that

$$d^2(Tx, p) \leq d^2(x, p) + k\ d^2(x, Tx), \quad \text{for all } x, y \in D \text{ and } p \in F(T);$$

(4) hemicontractive, if $F(T) \neq \emptyset$ and

$$d^2(Tx, p) \leq d^2(x, p) + d^2(x, Tx), \quad \text{for all } x, y \in D \text{ and } p \in F(T).$$

Clearly, every nonexpansive mapping with $F(T) \neq \emptyset$ is quasi nonexpansive. The class of 0-demicontractive mappings coincides with the class of quasi nonexpansive mappings. The class of hemicontractive mappings contains the class of demicontractive mappings and hence, the class of quasi nonexpansive mappings.

Approximation of fixed points of non-self mappings has mostly been studied in Hilbert spaces and Banach spaces with the use of projections P_D and sunny nonexpansive retractions, respectively (see [32, 49, 51, 52] and references therein). However, calculating the P_D (for instance) in an iterative algorithm is generally a very difficult task to accomplish because the expression of the P_D is usually not known and may require a self-approximating algorithm even when the P_D is the nearest point projection. To overcome this difficulty, Colao and Marino [10] proposed a method that is devoid of the P_D for a Krasnoselskii–Mann algorithm involving a non-self nonexpansive mapping T in a real Hilbert space. The authors imposed the conditions that the set D is strictly convex, T is inward, and the control sequence in the algorithm is calculated independently at each step. In fact, they proved the following convergence theorem.

Theorem 1.1. *Let D be a strictly convex, closed and nonempty subset of a real Hilbert space H, and let $T : D \to H$ be a nonexpansive mapping that satisfies the inward condition and $F(T) \neq \emptyset$. Then the algorithm*

$$\begin{cases} x_0 \in D, \\ \alpha_0 = \max\{\frac{1}{2}, h(x_0)\}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ \alpha_{n+1} = \max\{\alpha_n, h(x_{n+1})\} \end{cases}$$

is well defined and $\{x_n\}$ weakly converges to a point $p \in F(T)$. Moreover, if $\sum\limits_{n=0}^{\infty}(1-\alpha_n) < \infty$, then $\{x_n\}$ converges strongly.

Furthermore, the authors posed the following question: Will Theorem 1.1 still hold with all assumptions maintained if the nonexpansive mapping is replaced with a more general class of mappings? Colao et al. [11] provided an affirmative answer to this question by replacing the nonexpansive mapping with a k-strictly pseudocontractive mapping in Theorem 1.1. They also obtained strong convergence of the sequence $\{x_n\}$ to a fixed point of the k-strictly pseudocontractive mapping.

In 2017, Tufa and Zegeye [45] extended the work of Colao and Marino [10] to the framework of Hadamard spaces. They proposed and studied the following Krasnoloskii–Mann algorithm for multivalued nonexpansive non-self mappings: For an arbitrary initial point $x_0 \in D$, let the sequence $\{x_n\}$ be generated by

$$\begin{cases} \alpha_n = \max\{\frac{1}{2}, h_{y_0}(x_0)\}, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)y_n, \\ \alpha_{n+1} = \max\{\alpha_n, h_{y_{n+1}}(x_{n+1})\}, \end{cases} \tag{1.2}$$

where $y_n \in Tx_n$ is such that $d(y_n, y_{n+1}) \leq \mathcal{D}(Tx_n, Tx_{n+1})$ and $h_{y_n}(x_n) := \inf\{\lambda \geq 0 : \lambda x_n \oplus (1-\lambda)y_n \in D\}$. They established that the sequence generated from (1.2) converges to a fixed point of multivalued nonexpansive non-self mapping without the use of metric projections or sunny nonexpansive retraction mappings. The authors further studied (1.2) in [47] for multivalued demicontractive non-self mappings, which are more general than nonexpansive mappings. Recently, Tufa and Zegeye [46] studied an Ishikawa-type algorithm involving a non-self multivalued hemicontractive mapping T with the assumption that the mapping T is strongly inward. They proposed the following algorithm: For an arbitrary initial point $x_0 \in D$, let the sequence $\{x_n\}$ be generated by

$$\begin{cases} \beta_n \in [\max\{\beta_{n-1}, h_{u_n}(x_n)\}, 1), \\ y_n = \beta_n x_n \oplus (1-\beta_n)u_n, \\ \alpha_1 \in [\max\{\beta_1, f_{u_1 z_1}(x_1)\}, 1), \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)z_n, \quad n \geq 2, \\ \alpha_{n+1} \in [\max\{\beta_1, f_{u_n z_n}(x_n)\}, 1), \end{cases} \tag{1.3}$$

where $u_n \in Tx_n$ is such that $h_{u_n}(x_n) := \inf\{\lambda \geq 0 : \lambda x_n \oplus (1-\lambda)u_n \in D\}$, $z_n \in Ty_n$ is such that $d(u_n, z_n) \leq \mathcal{D}(Tx_n, Ty_n) + (1-\beta_n)d(u_n, x_n)$, and $f_{u_n z_n}(x_n) := \inf\{\mu \geq 0 : \mu x_n \oplus (1-\mu)y_n \in D\}$. They showed that (1.3) converges to a fixed point of the multivalued hemicontractive non-self mapping in a Hadamard space.

Very recently, Zegeye and Tufa [50] proposed a new Halpern–Ishikawa type algorithm for approximating fixed points of non-self mappings without the use of metric projections or sunny nonexpansive retractions. They proposed the following algorithm for an L-Lipschitz k-pseudocontractive mapping in a real Hilbert space: For an arbitrary $x_0 \in D$, the sequence $\{x_n\}$ generated by

$$\begin{cases} \lambda_n \in [\max\{\beta, h(x_n)\}, 1), \\ y_n = \lambda_n x_n \oplus (1-\beta_n)Tx_n, \\ \theta_n \in [\max\{\lambda_n, l(y_n)\}, 1), \\ x_{n+1} = \alpha_n u \oplus (1-\alpha_n)(\theta_n x_n + (1-\theta_n)Ty_n), \quad n \geq 0, \end{cases}$$

converges to a fixed point of the multivalued hemicontractive mapping, where $\beta \in \left(1 - \frac{1}{1+\sqrt{L^2+1}}\right)$, $h(x_n) := \inf\{\lambda \geq 0 : \lambda x_n + (1-\lambda)Tx_n \in D\}$, and $l(y_n) := \inf\{\theta \geq 0 : \theta x_n + (1-\theta)Ty_n \in D\}$.

Let X be a geodesic space and let f be a proper, convex, and lower semicontinuous functional (see details in Section 2) defined on X. If there exists a point $x \in X$ such that $f(x) = \min_{y \in X} f(y)$, then x is called a minimizer of f. The set of minimizers of f is denoted by $\arg\min_{x \in X} f(x)$ and the problem of finding such minimizers is called minimization problem (MP). MP remains one of the major problems in optimization theory that has been widely studied by many authors. MPs play vital role in nonlinear analysis and geometry. They have notable applications in computer vision, machine learning, electronic structure computation, system balancing and robot manipulation, and several iterative algorithms have been studied for solving MPs and related optimization problems (see [1, 4, 5, 19, 20, 22, 23, 24, 25, 26, 34, 33, 36, 42, 43, 44, 48] and references therein). The Proximal Point Algorithm (PPA) is one of the most popular and effective methods for solving MPs. The PPA was first introduced by Martinet [31] and was further developed by Rockafellar [39]. The latter proved that the PPA converges weakly to a minimizer of a proper, convex, and lower semicontinuous function in the framework of real Hilbert spaces. In 2013, Bačák [6] introduced and studied the PPA in the framework of Hadamard spaces, which he proposed as follows: For an arbitrary $x_1 \in X$, define a sequence iteratively by

$$x_{n+1} = J^f_{\mu_n}(x_n), \tag{1.4}$$

where $\mu_n > 0$ for all $n \geq 1$ and $J^f_{\mu_n} : X \to X$ is the Moreau–Yosida resolvent of a proper, convex, and lower semicontinuous function defined as

$$J^f_{\mu_n}(x) = \arg\min_{y \in X} \big(f(y) + \frac{1}{2\mu_n} d^2(y, x)\big),$$

for all $x \in X$. We note from [27] that $J^f_{\mu_n}$ is well defined. Under the conditions that f has a minimizer in X and $\sum_{n=1}^{\infty} \mu_n = \infty$, Bačák established that (1.4) Δ-converges to the minimizer of f.

The problem of finding minimizers of convex functions which are also fixed points of nonlinear mappings has recently been studied in the framework of Hadamard spaces (see [3, 9, 21, 29, 40] and references therein). However, the mappings used are mostly self mappings. In the case where the mappings are non-self, the metric projection or sunny nonexpansive retraction is involved in the algorithm, which usually affects the algorithm's efficiency (see [17, 53] and references therein). Based on this fact and motivations from Tufa and Zegeye [46, 50], the following research question arises:

Question: Can we construct an algorithm for finding a minimizer of a convex function which is also a fixed point of a non-self mapping in Hadamard spaces without the use of metric projection?

In this paper, we provide an affirmative answer to the research question above by proposing a modified Halpern-type algorithm involving a Lipschitz hemicontractive non-self mapping and a resolvent of a convex function in a Hadamard space. We obtain a strong convergence of the proposed algorithm to a minimizer of the convex function which is also a fixed point of the Lipschitz hemicontractive non-self mapping. Furthermore, we give a numerical example to support our method. Our proposed method improves and extends some recent works in the literature.

2. Preliminaries

In this section, we state some known and useful definitions and results which will be needed in the proof of our main theorem. In what follows, we denote strong and Δ-convergence by "$\to$" and "$\rightharpoonup$", respectively.

Definition 2.1. Let X be a Hadamard space. A function $f : X \to (-\infty, \infty]$ is said to be

(i) *convex*, if
$$f(\lambda x \oplus (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \quad \forall\, x, y \in X,\ \lambda \in (0,1);$$

(ii) *proper*, if $D := \{x \in X : f(x) < +\infty\} \neq \emptyset$, where D denotes the domain of f;

(iii) *lower semicontinuous* at a point $x \in D$, if
$$f(x) \leq \liminf_{n\to\infty} f(x_n),$$
for each sequence $\{x_n\}$ in D such that $\lim\limits_{n\to\infty} x_n = x$;

(iv) *lower semicontinuous* on D, if it is lower semicontinuous at every point in D.

Definition 2.2 ([27])**.** Let X be a Hadamard space and let $f : X \to (-\infty, \infty]$ be a proper, convex, and lower semicontinuous function. For $\mu > 0$, the *Moreau–Yosida* *resolvent* of f is defined as

$$J_\mu^f(x) = \underset{y\in X}{\arg\min}\big(f(y) + \frac{1}{2\mu}d^2(y,x)\big),$$

for all $x \in X$.

Lemma 2.3 ([27]). *Let X be a Hadamard space and let $f : X \to (-\infty,\infty]$ be proper, convex, and lower semicontinuous. For any $\mu > 0$, the resolvent J_μ^f of f is nonexpansive.*

Lemma 2.4 ([27]). *Let X be a Hadamard space and let $f : X \to (-\infty,\infty]$ be proper, convex, and lower semicontinuous. Then, the following identity holds:*

$$J_\mu^f x = J_\lambda^f\left(\frac{\mu-\lambda}{\mu}J_\mu^f x \oplus \frac{\lambda}{\mu}x\right)$$

for all $x \in X$ and $\mu \geq \lambda > 0$.

Definition 2.5 ([47]). A subset D of a Hadamard space X is *convex* if D contains every geodesic segment joining any two points. A convex subset D of X is said to be *strictly convex* if for $x, y \in \partial D$ and $t \in (0,1)$ we have

$$tx \oplus (1-t)y \in \mathring{D},$$

where ∂D and $\mathring{D}$ denote boundary and interior points of D, respectively.

Definition 2.6 ([47]). Let D be a nonempty subset of a Hadamard space X and let $x \in D$. A set $I_D(x)$ is said to be *inward* if for any $x \in D$,

$$I_D(x) = \left\{w \in X : w = x \text{ or } y = \left(1-\frac{1}{\lambda}\right)x \oplus \frac{1}{\lambda}w, \text{ for some } y \in D,\ \lambda \geq 1\right\}.$$

The mapping $T : D \to X$ is called an *inward mapping* on D if for any $x \in D$ one has $Tx \in I_D(x)$.

Lemma 2.7 ([46]; see also [50]). *Let D be a nonempty, closed, and convex subset of a Hadamard space and let $T : D \to X$ be a mapping. Define $h : D \to \mathbb{R}$ by $h(x) = \inf\{\lambda \geq 0 : \lambda x \oplus (1-\lambda)Tx \in D\}$. Then, for any $x \in D$, the following hold:*

(i) *$h(x) \in [0,1]$ and $h(x) = 0$ if and only if $Tx \in D$;*
(ii) *if $\beta \in [h(x), 1]$, then $\beta x \oplus (1-\beta)Tx \in D$;*
(iii) *if T is inward, then $h(x) < 1$;*
(iv) *if $Tx \notin D$, then $h(x) \oplus (1-h(x))Tx \in \partial D$.*

Definition 2.8 ([7]). Let X be a Hadamard space and let $(a,b) \in X \times X$ be denoted by $\overrightarrow{ab}$ and called a vector in $X \times X$. A *quasilinearization map* $\langle\cdot,\cdot\rangle$: $(X\times X)\times(X\times X) \to \mathbb{R}$ is defined by

$$\langle\overrightarrow{ab},\overrightarrow{cd}\rangle = \frac{1}{2}\left(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)\right), \quad \forall\, a,b,c,d \in X.$$

It is easy to see that $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$, and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$ for all $a, b, c, d, e \in X$. Recall that a geodesic space X is said to satisfy the Cauchy–Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a,b)d(c,d), \quad \forall\, a,b,c,d \in X.$$

Definition 2.9. Let $\{x_n\}$ be a bounded sequence in a geodesic metric space X. Then, the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is defined by

$$A(\{x_n\}) = \{\bar{v} \in X : \limsup_{n\to\infty} d(\bar{v}, x_n) = \inf_{v\in X} \limsup_{n\to\infty} d(v, x_n)\}.$$

It is generally known that in a Hadamard space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in X is said to be Δ-convergent to a point $\bar{v} \in X$ if $A(\{x_{n_k}\}) = \{\bar{v}\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = \bar{v}$ (see [14]). The concept of Δ-convergence in metric spaces was first introduced and studied by Lim [30]. Kirk and Panyanak [28] later introduced and studied this concept in CAT(0) spaces, and proved that it is very similar to weak convergence in a Banach space setting.

Definition 2.10. Let D be a nonempty, closed, and convex subset of a Hadamard space X. A mapping $T : D \to D$ is said to be Δ-*demiclosed* if, for any bounded sequence $\{x_n\}$ in X such that $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, one has $x = Tx$.

Lemma 2.11 ([13]). *Let X be a Hadamard space and let $T : X \to X$ be a nonexpansive mapping. Then T is Δ-demiclosed.*

Lemma 2.12 ([15]). *Every bounded sequence in a Hadamard space has a Δ-convergent subsequence.*

Lemma 2.13. *Let X be a Hadamard space, and let $x, y, z \in X$ and $t, s \in [0,1]$. Then,*

(i) $d(tx \oplus (1-t)y, z) \leq td(x,z) + (1-t)d(y,z)$ *(see [15]);*
(ii) $d^2(tx \oplus (1-t)y, z) \leq td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y)$ *(see [15]);*
(iii) $d^2(tx \oplus (1-t)y, z) \leq t^2 d^2(x,z) + (1-t)^2 d^2(y,z) + 2t(1-t)\langle \overrightarrow{xz}, \overrightarrow{yz} \rangle$ *(see [12]).*

Lemma 2.14 ([2]). *Let X be a Hadamard space, $\{x_n\}$ a sequence in X, and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if*

$$\limsup_{n\to\infty} \langle \overrightarrow{x_n x}, \overrightarrow{yx} \rangle \leq 0, \quad \textit{for all } y \in X.$$

Lemma 2.15 ([37]). *Let X be a Hadamard space and $\{x_n\}$ a sequence in X. If there exists a nonempty subset D such that*

(i) $\lim_{n\to\infty} d(x_n, z)$ *exists for every $z \in D$, and*
(ii) *if $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ which is Δ-convergent to x, then $x \in D$,*

then there is a $p \in D$ such that $\{x_n\}$ is Δ-convergent to p in X.

Lemma 2.16 ([35])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n + \gamma_n, \quad n \geq 0,$$

where $\{\alpha_n\}$, $\{\delta_n\}$, and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$;

(iii) $\gamma_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.17 ([41])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers, such that there exists a subsequence $\{n_j\}$ of $\{n\}$ with $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$, such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{i \leq k : a_i < a_{i+1}\}$.

3. Main result

We begin this section with the hypotheses that will be needed in the construction of our algorithm.

Let D be a nonempty, closed, and convex subset of a Hadamard space X and let $T : D \to X$ be an inward mapping. We define a Halpern-type proximal point algorithm as follows: Choose $u, x_0 \in D$ and let

$$\begin{aligned} h(x_0) &:= \inf\{\gamma \geq 0 : \gamma x_0 \oplus (1-\gamma)J^f_{\mu_n} x_0 \in D\} \text{ and } \gamma_0 \in \big[\max\{\beta, h(x_0)\}, 1\big); \\ l(u_0) &:= \inf\{\lambda \geq 0 : \lambda u_0 \oplus (1-\lambda)Tu_0 \in D\} \text{ and } \lambda_0 \in \big[\max\{\gamma_0, l(u_0)\}, 1\big); \\ \kappa(y_0) &:= \inf\{\theta \geq 0 : \theta x_0 \oplus (1-\theta)Ty_0 \in D\} \text{ and } \theta_0 \in \big[\max\{\lambda_0, \kappa(y_0)\}, 1\big). \end{aligned}$$

Then, by Lemma 2.7, we have that $u_0 := \gamma_0 x_0 \oplus (1-\gamma_0)J^f_\mu x_0 \in D$, $y_0 := \lambda_0 u_0 \oplus (1-\lambda_0)Tu_0 \in D$, and $\theta_0 u_0 \oplus (1-\theta_0)Ty_0 \in D$. Hence, it follows that

$$x_1 := \alpha_0 u \oplus (1-\alpha_0)(\theta_0 u_0 \oplus (1-\theta_0)Ty_0) \in D.$$

Therefore, by mathematical induction, we have

$$\begin{cases} \gamma_n \in \big[\max\{\beta, h(x_n)\}, 1\big), \\ u_n = \gamma_n x_n \oplus (1-\gamma_n)J^f_{\mu_n} x_n, \\ \lambda_n \in \big[\max\{\gamma_n, l(u_n)\}, 1\big), \\ y_n = \lambda_n u_n \oplus (1-\lambda_n)Tu_n, \\ \theta_n \in \big[\max\{\lambda_n, \kappa(y_n)\}, 1\big), \\ x_{n+1} = \alpha_n u \oplus (1-\alpha_n)(\theta_n u_n \oplus (1-\theta_n)Ty_n), \quad n \geq 0, \end{cases} \tag{3.1}$$

where $h(x_n) := \inf\{\gamma \geq 0 : \gamma x_n \oplus (1-\gamma)J_{\mu_n}^f x_n \in D\}$, $l(u_n) := \inf\{\lambda \geq 0 : \lambda u_n \oplus (1-\lambda)Tu_n \in D\}$, $\kappa(y_n) := \inf\{\theta \geq 0 : \theta u_n \oplus (1-\theta)Ty_n \in D\}$, $\beta \in (0,1)$, and $\{\alpha_n\}$ is a sequence in $(0,1)$.

Remark 3.1. The choice of the parameters γ_n, λ_n, and θ_n is not made a priori but determined at the intervals obtained at the n-th step. These intervals are determined by the values of the resolvent of f for γ_n, the mapping T, and the geometry of the set D for λ_n and θ_n. This is different from what we have in [10, 11], where the parameters are uniquely determined and, consequently, take a particular value at each step. In our case, we allow the parameters to take any point on the intervals obtained at each step. This same idea was used in [46, 50].

Lemma 3.2. *Let D be a nonempty, closed, and convex subset of a Hadamard space X and let $T : D \to X$ be an L-Lipschitz hemicontractive inward mapping. Let $f : X \to (-\infty, \infty]$ be a proper, convex, and lower semicontinuous function such that $\beta \in \left(1 - \frac{1}{1+\sqrt{L^2+1}}, 1\right)$. Suppose that $\Gamma := F(T) \cap \arg\min_{u\in X} f(y) \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by (3.1) is bounded.*

Proof. Let $p \in \Gamma$. Then by (3.1), Lemma 2.13 (ii), and the nonexpansivity of $J_{\mu_n}^f$, we have

$$\begin{aligned}
d^2(u_n,p) &\leq \gamma_n d^2(x_n,p) + (1-\gamma_n)d^2(J_{\mu_n}^f x_n, p) - \gamma_n(1-\gamma_n)d^2(J_{\mu_n}^f x_n, x_n) \\
&\leq \gamma_n d^2(x_n,p) + (1-\gamma_n)d^2(x_n,p) - \gamma_n(1-\gamma_n)d^2(J_{\mu_n}^f x_n, x_n) \qquad (3.2)\\
&\leq d^2(x_n,p) - \gamma_n(1-\gamma_n)d^2(J_{\mu_n}^f x_n, x_n).
\end{aligned}$$

From (3.1), (3.2), and the fact that T is hemicontractive, we obtain

$$\begin{aligned}
d^2(y_n,p) &\leq \lambda_n d^2(u_n,p) + (1-\lambda_n)d^2(Tu_n,p) - \lambda_n(1-\lambda_n)d^2(u_n,Tu_n) \\
&\leq \lambda_n d^2(u_n,p) + (1-\lambda_n)\big[d^2(u_n,p) + d^2(u_n,Tu_n)\big] - \lambda_n(1-\lambda)d^2(u_n,Tu_n) \\
&\leq d^2(x_n,p) + (1-\lambda_n)^2 d^2(u_n,Tu_n).
\end{aligned} \qquad (3.3)$$

From (3.1), (3.2), and (3.3), we have

$$\begin{aligned}
d^2(x_{n+1},p) &\le \alpha_n d^2(u,p) + (1-\alpha_n)\\
&\quad \times \big[\theta_n d^2(u_n,p) + (1-\theta_n)d^2(Ty_n,p) - \theta_n(1-\theta_n)d^2(u_n,Ty_n)\big]\\
&\le \alpha_n d^2(u,p) + (1-\alpha_n)\\
&\quad \times \big[\theta_n d^2(x_n,p) + (1-\theta_n)\big(d^2(y_n,p) + d^2(Ty_n,y_n)\big)\\
&\qquad - \theta_n(1-\theta_n)d^2(u_n,Ty_n)\big] \qquad (3.4)\\
&\le \alpha_n d^2(u,p) + (1-\alpha_n)\\
&\quad \times \big[\theta_n d^2(x_n,p)\\
&\qquad + (1-\theta_n)\big(d^2(x_n,p) + (1-\lambda_n)^2 d^2(u_n,Tu_n) + d^2(Ty_n,y_n)\big)\\
&\qquad - \theta_n(1-\theta_n)d^2(u_n,Ty_n)\big]\\
&= \alpha_n d^2(u,p) + (1-\alpha_n)\\
&\quad \times \big[d^2(x_n,p) + (1-\theta_n)\big((1-\lambda_n)^2 d^2(u_n,Tu_n) + d^2(Ty_n,y_n)\big)\\
&\qquad - \theta_n(1-\theta_n)d^2(u_n,Ty_n)\big]. \qquad (3.5)
\end{aligned}$$

Observe that by using the Lipschitz property of T, we have

$$\begin{aligned}
d^2(y_n,Ty_n) &\le \lambda_n d^2(u_n,Ty_n) + (1-\lambda_n)d^2(Tu_n,Ty_n) - \lambda_n(1-\lambda_n)d^2(u_n,Tu_n)\\
&\le \lambda_n d^2(u_n,Ty_n) + (1-\lambda_n)L^2 d^2(u_n,y_n) - \lambda_n(1-\lambda_n)d^2(u_n,Tu_n).
\end{aligned} \qquad (3.6)$$

Substituting (3.6) in (3.5), we obtain

$$\begin{aligned}
&d^2(x_{n+1},p)\\
&\le \alpha_n d^2(u,p)\\
&\quad + (1-\alpha_n)\big[d^2(x_n,p) + (1-\theta_n)\big((1-\lambda_n)^2 d^2(u_n,Tu_n) + \lambda_n d^2(u_n,Ty_n)\\
&\qquad + (1-\lambda_n)L^2 d^2(u_n,y_n) - \lambda_n(1-\lambda_n)d^2(u_n,Tu_n)\big) - \theta_n(1-\theta_n)d^2(u_n,Ty_n)\big]\\
&= \alpha_n d^2(u,p) + (1-\alpha_n)d^2(x_n,p) + (1-\alpha_n)(1-\theta_n)(\lambda_n-\theta_n)d^2(u_n,Ty_n)\\
&\quad + (1-\alpha_n)\big[(1-\theta_n)\big((1-\lambda_n)^2 d^2(u_n,Tu_n) + (1-\lambda_n)L^2 d^2(u_n,y_n)\\
&\qquad - \lambda_n(1-\lambda_n)d^2(u_n,Tu_n)\big)\big]\\
&\le \alpha_n d^2(u,p) + (1-\alpha_n)d^2(x_n,p) + (1-\alpha_n)(1-\theta_n)(\lambda_n-\theta_n)d^2(u_n,Ty_n)\\
&\quad + (1-\alpha_n)(1-\theta_n)\big[\big((1-\lambda_n)^2 d^2(u_n,Tu_n) + (1-\lambda_n)^3 L^2 d^2(u_n,Tu_n)\\
&\qquad - \lambda_n(1-\lambda_n)d^2(u_n,Tu_n)\big)\big]\\
&= \alpha_n d^2(u,p) + (1-\alpha_n)d^2(x_n,p) - (1-\alpha_n)(1-\theta_n)(\theta_n-\lambda_n)d^2(u_n,Ty_n)\\
&\quad - (1-\alpha_n)(1-\theta_n)(1-\lambda_n)\big[1-2(1-\lambda_n) - L^2(1-\lambda_n)^2\big]d^2(u_n,Tu_n).
\end{aligned} \qquad (3.7)$$

From the hypothesis, we have

$$1-2(1-\lambda_n)-L^2(1-\lambda_n)^2 \geq 1-2(1-\gamma_n)-L^2(1-\gamma_n)^2,$$
$$1-2(1-\gamma_n)-L^2(1-\gamma_n)^2 \geq 1-2(1-\beta)-L^2(1-\beta)^2>0,$$
$$\theta_n \geq \lambda_n.$$

Then

$$d^2(x_{n+1},p) \leq \alpha_n d^2(u,p)+(1-\alpha_n)d^2(x_n,p).$$

Thus, by mathematical induction,

$$d^2(x_{n+1},p) \leq \max\{d^2(u,p)+d^2(x_0,p)\} \quad \text{for all } n \geq 0.$$

Therefore, $\{x_n\}$ is bounded and consequently $\{u_n\}$, $\{y_n\}$, and $\{Ty_n\}$ are bounded. □

Theorem 3.3. *Let D be a nonempty, closed, and convex subset of a Hadamard space X and let $T : D \to X$ be an L-Lipschitz hemicontractive inward mapping such that T is Δ-demiclosed with $\beta \in \left(1-\frac{1}{1+\sqrt{L^2+1}}, 1\right)$. Let $f : X \to (-\infty,\infty]$ be a proper, convex, and lower semicontinuous function. Suppose that $\Gamma := F(T) \cap \underset{u\in X}{\arg\min} f(y) \neq \emptyset$, $\{\mu_n\}$ is a sequence such that $\mu_n \geq \mu > 0$ for all $n \geq 1$, and the sequence $\{x_n\}$ is generated by (3.1) such that the following conditions hold:*

- (A1) $\lim_{n\to\infty} \alpha_n = 0$,
- (A2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (A3) $\sum_{n=1}^{\infty} (1-\theta_n) < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to an element of Γ.

Proof. We divide our proof into two cases:

Case 1. Assume that $\{d^2(x_n,p)\}$ is a monotonically decreasing sequence. Then

$$\left[d^2(x_n,p)-d^2(x_{n+1},p)\right] \to 0, \quad \text{as } n \to \infty.$$

Let $K_n = (1-\alpha_n)(1-\theta_n)(1-\lambda_n)[1-2(1-\lambda_n)-L^2(1-\lambda_n)]$. Then from (3.7) we obtain

$$\begin{aligned} K_n d^2(u_n,Tu_n) &\leq \alpha_n d^2(u,p)+(1-\alpha_n)d^2(x_n,p)-d^2(x_{n+1},p) \\ &= \alpha_n\left[d^2(u,p)-d^2(x_n,p)\right]+d^2(x_n,p)-d^2(x_{n+1},p). \end{aligned}$$

Clearly $K_n > 0$; hence by condition (A1) we have that

$$d(u_n,Tu_n) \to 0, \quad \text{as } n \to \infty. \tag{3.8}$$

Similarly, from (3.7) we obtain

$$d(u_n,Ty_n) \to 0, \quad \text{as } n \to \infty. \tag{3.9}$$

From (3.1) and (3.8), we have that

$$d(u_n,y_n) \leq (1-\lambda_n)d(u_n,Tu_n) \to 0, \quad \text{as } n \to \infty. \tag{3.10}$$

Using (3.9) and (3.10), we obtain that

$$d(y_n, Ty_n) \to 0, \quad \text{as } n \to \infty. \tag{3.11}$$

Again, from (3.2), (3.3), and (3.4), we obtain

$$\begin{aligned}
&d^2(x_{n+1,}p) \\
&\leq \alpha_n d^2(u,p) + (1-\alpha_n)\big[\theta_n d^2(u_n,p) + (1-\theta_n)d^2(Ty_n,p)\big] \\
&\leq \alpha_n d^2(u,p) + (1-\alpha_n)\theta_n d^2(u_n,p) + (1-\alpha_n)(1-\theta_n)\big[d^2(y_n,p) + d^2(Ty_n,y_n)\big] \\
&\leq \alpha_n d^2(u,p) + (1-\alpha_n)\theta_n\big[d^2(x_n,p) - \gamma_n(1-\gamma_n)d^2(J^f_{\mu_n}x_n,x_n)\big] \\
&\quad + (1-\alpha_n)(1-\theta_n)\big[d^2(x_n,p) + (1-\lambda_n)^2 d^2(u_n,Tu_n) + d^2(Ty_n,y_n)\big] \\
&= \alpha_n d^2(u,p) + (1-\alpha_n)d^2(x_n,p) \\
&\quad + (1-\alpha_n)(1-\theta_n)\big[(1-\lambda_n)^2 d^2(u_n,Tu_n) + d^2(y_n,Ty_n)\big] \\
&\quad - \theta_n\gamma_n(1-\gamma_n)(1-\alpha_n)d^2(J^f_{\mu_n}x_n,x_n),
\end{aligned}$$

which implies

$$\begin{aligned}
\theta_n\gamma_n(1-\gamma_n)(1-\alpha_n)&d^2(J^f_{\mu_n}x_n,x_n) \\
&\leq \alpha_n\big(d^2(u,p) - d^2(x_n,p)\big) + d^2(x_n,p) - d^2(x_{n+1},p) \\
&\quad + (1-\alpha_n)(1-\theta_n)\big[(1-\lambda_n)^2 d^2(u_n,Tu_n) + d^2(y_n,Ty_n)\big].
\end{aligned}$$

Then by condition (A1), (3.8), and (3.11), we obtain that

$$d(J^f_{\mu_n}x_n, x_n) \to 0, \quad \text{as } n \to \infty. \tag{3.12}$$

Since $\mu_n \geq \mu > 0$ for all $n \geq 1$, from Lemma 2.4, Lemma 2.13 (i), (3.12), and the nonexpansivity of J^f_μ, we obtain that

$$\begin{aligned}
d(J^f_\mu x_n, J^f_{\mu_n}x_n) &= d\left(J^f_\mu x_n, J^f_\mu\left(\frac{\mu_n-\mu}{\mu_n}J^f_{\mu_n}x_n \oplus \frac{\mu}{\mu_n}x_n\right)\right) \\
&\leq d\left(x_n, \left(1-\frac{\mu}{\mu_n}J^f_{\mu_n}x_n \oplus \frac{\mu}{\mu_n}x_n\right)\right) \\
&= \left(1-\frac{\mu}{\mu_n}\right)d(x_n, J^f_{\mu_n}x_n) \to 0, \quad \text{as } n \to \infty.
\end{aligned} \tag{3.13}$$

Thus, by (3.12) and (3.13), we obtain

$$d(J^f_\mu x_n, x_n) \leq d(J^f_\mu x_n, J^f_{\mu_n}x_n) + d(J^f_{\mu_n}x_n, x_n) \to 0, \quad \text{as } n \to \infty. \tag{3.14}$$

Also from (3.1) and (3.12), we have

$$d(u_n, x_n) \leq (1-\gamma_n)d(J^f_{\mu_n}x_n, x_n) \to 0, \quad \text{as } n \to \infty. \tag{3.15}$$

Hence from (3.10) and (3.15), we obtain

$$d(x_n, y_n) \to 0, \quad \text{as } n \to \infty.$$

Let $c_n = \theta_n u_n \oplus (1-\theta_n)Ty_n$. By Lemma 2.13 (i), (3.9), and (3.15), we have

$$\begin{aligned} d(c_n, x_n) &\leq \theta_n d(u_n, x_n) + (1-\theta_n) d(Ty_n, x_n) \\ &\leq \theta_n d(u_n, x_n) + (1-\theta_n)\big[d(Ty_n, u_n) + d(u_n, x_n)\big] \to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{3.16}$$

Thus by condition (A1) and (3.16), we obtain

$$d(x_{n+1}, x_n) \leq \alpha_n d(u, x_n) + (1-\alpha_n) d(c_n, x_n) \to 0, \quad \text{as } n \to \infty. \tag{3.17}$$

Since T is L-Lipschitz, we have

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, u_n) + d(u_n, Tu_n) + d(Tu_n, Tx_n) \\ &\leq d(x_n, x_{n+1}) + \alpha_n d(u, u_n) + (1-\alpha_n) d(c_n, u_n) \\ &\quad + d(u_n, Tu_n) + L d(u_n, x_n) \\ &\leq d(x_n, x_{n+1}) + \alpha_n d(u, u_n) + (1-\alpha_n)\big(d(c_n, x_n) \\ &\quad + d(x_n, u_n)\big) + d(u_n, Tu_n) + L d(u_n, x_n). \end{aligned}$$

Therefore by condition (A1), (3.17), (3.16), (3.15), and (3.8), we obtain

$$d(x_n, Tx_n) \to 0, \quad \text{as } n \to \infty. \tag{3.18}$$

By Lemma 2.12 and Lemma 3.2, we have that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ Δ-converges to z. Then, from (3.18) and the assumption that T is Δ-demiclosed, we obtain that $z \in F(T)$. Furthermore, it follows from (3.15) that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ Δ-converges to z. Also, since J_μ^f is nonexpansive, we obtain from (3.14) and Lemma 2.11 that $z \in F(J_\mu^f)$. Therefore, we conclude that $z \in \Gamma$.

It follows from Lemma 2.15 that there exists $z \in \Gamma$ such that $\{x_n\}$ Δ-converges to z. Thus, by Lemma 2.14, we obtain

$$\limsup_{n\to\infty} \langle \overrightarrow{uz}, \overrightarrow{x_n z} \rangle \leq 0, \quad \forall\, u \in X. \tag{3.19}$$

Furthermore, we use the quasilinearization properties to obtain

$$\begin{aligned} \langle \overrightarrow{uz}, \overrightarrow{c_n z} \rangle &= \langle \overrightarrow{uz}, \overrightarrow{c_n x_n} \rangle + \langle \overrightarrow{uz}, \overrightarrow{x_n z} \rangle \\ &\leq d(u, z) d(c_n, x_n) + \langle \overrightarrow{uz}, \overrightarrow{x_n z} \rangle. \end{aligned} \tag{3.20}$$

Hence, we obtain from (3.16) and (3.19) that

$$\limsup_{n\to\infty} \langle \overrightarrow{uz}, \overrightarrow{c_n z} \rangle \leq 0. \tag{3.21}$$

Also, by condition (A1) and inequality (3.21), we get

$$\limsup_{n\to\infty} \big(\alpha_n d^2(u, z) + 2(1-\alpha_n)\langle \overrightarrow{uz}, \overrightarrow{c_n z} \rangle\big) \leq 0. \tag{3.22}$$

Next, we show that $\{x_n\}$ converges strongly to z. By (3.2), (3.3), and Lemma 2.13 (iii), we have

$$\begin{aligned}
&d^2(x_{n+1},z)\\
&\quad\le \alpha_n^2 d^2(u,z)+(1-\alpha_n)^2d^2(c_n,z)+2\alpha_n(1-\alpha_n)\langle\overrightarrow{uz},\overrightarrow{c_nz}\rangle\\
&\quad\le \alpha_n^2 d^2(u,z)+(1-\alpha_n)^2\big[\theta_n d^2(u_n,z)+(1-\theta_n)d^2(Ty_n,z)\big]\\
&\qquad+2\alpha_n(1-\alpha_n)\langle\overrightarrow{uz},\overrightarrow{c_nz}\rangle\\
&\quad\le \alpha_n^2 d^2(u,z)+(1-\alpha_n)^2\big[\theta_n d^2(x_n,z)+(1-\theta_n)\big(d^2(y_n,z)+d^2(y_n,Ty_n)\big)\big]\\
&\qquad+2\alpha_n(1-\alpha_n)\langle\overrightarrow{uz},\overrightarrow{c_nz}\rangle\\
&\quad\le \alpha_n^2 d^2(u,z)+(1-\alpha_n)^2\\
&\qquad\times\big[\theta_n d^2(x_n,z)+(1-\theta_n)\big(d^2(x_n,z)+(1-\lambda_n)^2d^2(u_n,Tu_n)+d^2(y_n,Ty_n)\big)\big]\\
&\qquad+2\alpha_n(1-\alpha_n)\langle\overrightarrow{uz},\overrightarrow{c_nz}\rangle\\
&\quad\le \alpha_n^2 d^2(u,z)+(1-\alpha_n)^2d^2(x_n,z)+(1-\alpha_n)^2(1-\theta_n)\\
&\qquad\times\big[(1-\lambda_n)^2d^2(u_n,Tu_n)+d^2(y_n,Ty_n)\big]\\
&\qquad+2\alpha_n(1-\alpha_n)\langle\overrightarrow{uz},\overrightarrow{c_nz}\rangle\\
&\quad\le (1-\alpha_n)d^2(x_n,z)+\alpha_n\big(\alpha_n d^2(u,z)+2(1-\alpha_n)\langle\overrightarrow{uz},\overrightarrow{c_nz}\rangle\big)\\
&\qquad+(1-\alpha_n)(1-\theta_n)\big[(1-\lambda_n)d^2(u_n,Tu_n)+d^2(y_n,Ty_n)\big]\\
&\quad\le (1-\alpha_n)d^2(x_n,z)+\alpha_n\big(\alpha_n d^2(u,z)+2(1-\alpha_n)\langle\overrightarrow{uz},\overrightarrow{c_nz}\rangle\big)+(1-\theta_n)M,
\end{aligned}\tag{3.23}$$

where

$$M:=\sup_{n\ge1}\big\{d^2(u_n,Tu_n)+d^2(y_n,Ty_n)\big\}.$$

Hence, by (3.22), condition (A3), and applying Lemma 2.16 to (3.23), we conclude that $\{x_n\}$ converges strongly to z.

Case 2. Suppose that $\{d^2(x_n,p)\}$ is not a monotone decreasing sequence. Then, there exists a subsequence $\{d^2(x_{n_i},p)\}$ of $\{d^2(x_n,p)\}$ such that

$$\{d^2(x_{n_i},p)\}<\{d^2(x_{n_i+1},p)\}\quad\text{for all } i\in\mathbb{N}.$$

Then, by Lemma 2.17, there exists a nondecreasing sequence $\{m_k\}\subset\mathbb{N}$ such that $m_k\to\infty$ and

$$\{d^2(x_{m_k},p)\}<\{d^2(x_{m_{k+1}},p)\}\quad\text{and}\quad\{d^2(x_k,p)\}<\{d^2(x_{k+1},p)\},\tag{3.24}$$

for all $k \in \mathbb{N}$. Thus, by (3.2), (3.3), (3.8), and (3.10), we have

$$\begin{aligned}
0 &\leq \lim_{k\to\infty} \big(d^2(x_{m_{k+1}}, p) - d^2(x_{m_k}, p)\big) \\
&\leq \limsup_{n\to\infty} \big(d^2(x_{n+1}, p) - d^2(x_n, p)\big) \\
&\leq \limsup_{n\to\infty} \big(\alpha_n d^2(u,p) + (1-\alpha_n) d^2(c_n, p) - d^2(x_n, p)\big) \\
&\leq \limsup_{n\to\infty} \big(\alpha_n d^2(u,p) + (1-\alpha_n)\big(\theta_n d^2(u_n, p) + (1-\theta_n) d^2(Ty_n, p)\big) - d(x_n, p)\big) \\
&\leq \limsup_{n\to\infty} \big(\alpha_n d^2(u,p) + (1-\alpha_n) \\
&\qquad\qquad \times \big(\theta_n d^2(x_n,p) + (1-\theta_n)\big(d^2(y_n,p) + d^2(Ty_n, y_n)\big)\big) - d^2(x_n,p)\big) \\
&\leq \limsup_{n\to\infty} \big(\alpha_n d^2(u,p) + (1-\alpha_n) \\
&\qquad\qquad \times \big(\theta_n d^2(x_n,p) + (1-\theta_n)\big(d^2(x_n,p) + (1-\lambda_n)^2 d^2(u_n, Tu_n) \\
&\qquad\qquad + d^2(Ty_n, y_n)\big)\big) - d^2(x_n,p)\big) \\
&\leq \limsup_{n\to\infty} \big(\alpha_n d^2(u,p) + (1-\alpha_n) d^2(x_n,p) - d^2(x_n,p) \\
&\qquad\qquad + (1-\lambda_n)^2 d^2(u_n, Tu_n) + d^2(Ty_n, y_n)\big) \\
&\leq \limsup_{n\to\infty} \big(\alpha_n d^2(u,p) + (1-\alpha_n) d^2(x_n,p) - d^2(x_n,p)\big),
\end{aligned}$$

which implies that

$$\lim_{k\to\infty} \big(d^2(x_{m_{k+1}}, p) - d^2(x_{m_k}, p)\big) = 0. \tag{3.25}$$

Following the same argument as in Case 1, we see that

$$\lim_{k\to\infty} \big(\alpha_{m_k} d^2(u,z) + 2(1-\alpha_{m_k})\langle \overrightarrow{uz}, \overrightarrow{c_{m_k} z}\rangle\big) \leq 0. \tag{3.26}$$

Also by (3.23), we have

$$\begin{aligned}
d^2(x_{m_{k+1}}, z) \leq{}& (1-\alpha_{m_k}) d^2(x_{m_k}, z) + \alpha_{m_k}(\alpha_{m_k} d^2(u,z) + 2(1-\alpha_{m_k})\langle \overrightarrow{uz}, \overrightarrow{c_{m_k} z}\rangle) \\
&+ (1-\theta_{m_k})\big[d^2(u_{m_k}, Tu_{m_k}) + d^2(y_{m_k}, Ty_{m_k})\big].
\end{aligned}$$

Since $d^2(x_{m_k}, z) < d^2(x_{m_{k+1}}, z)$, we obtain

$$\begin{aligned}
d^2(x_{m_k}, z) \leq{}& \alpha_{m_k} d^2(u,z) + 2(1-\alpha_{m_k})\langle \overrightarrow{uz}, \overrightarrow{c_{m_k} z}\rangle \\
&+ (1-\theta_{m_k})\big[d^2(u_{m_k}, Tu_{m_k}) + d^2(y_{m_k}, Ty_{m_k})\big],
\end{aligned}$$

which implies from (3.26) that

$$d(x_{m_k}, z) \to 0, \quad \text{as } k \to \infty. \tag{3.27}$$

It follows from (3.24), (3.25), and (3.27) that $\lim_{k\to 0} d(x_k, z) = 0$. Therefore, we conclude from both cases that $\{x_n\}$ converges strongly to $z \in \Gamma$. $\square$

If we set $J^f_{\mu_n} \equiv I$, where I is an identity mapping in Theorem 3.3, we obtain the following results.

Corollary 3.4. *Let D be a nonempty, closed, and convex subset of a Hadamard space X. Let $T : D \to X$ be an L-Lipschitz hemicontractive inward mapping such that T is Δ-demiclosed and $\Gamma := F(T)$. For arbitrary $u, x_0 \in D$, let the sequence $\{x_n\}$ be generated as follows:*

$$\begin{cases} \gamma_n \in [\beta, 1), \\ u_n = \gamma_n x_n \oplus (1-\gamma_n) x_n, \\ \lambda_n \in \big[\max\{\gamma_n, l(x_n)\}, 1\big), \\ y_n = \lambda_n x_n \oplus (1-\lambda_n) T u_n, \\ \theta_n \in \big[\max\{\lambda_n, \kappa(y_n)\}, 1\big), \\ x_{n+1} := \alpha_n u \oplus (1-\alpha_n)(\theta_n u_n \oplus (1-\theta_n) T y_n), \quad n \geq 0, \end{cases}$$

where $l(x_n) := \inf\{\lambda \geq 0 : \lambda x_n \oplus (1-\lambda) T x_n \in D\}$, $\kappa(y_n) := \inf\{\theta \geq 0 : \theta x_n \oplus (1-\theta) T y_n \in D\}$, $\beta \in \Big(1 - \frac{1}{1+\sqrt{L^2+1}}, 1\Big)$, and $\{\alpha_n\}$ is a sequence in $(0,1)$ such that conditions (A1)–(A3) are satisfied. Then $\{x_n\}$ converges strongly to an element of Γ.

Corollary 3.5. *Let D be a nonempty, closed, and convex subset of a Hadamard space X. Let $T : D \to X$ be a nonexpansive inward mapping and let $f : X \to (-\infty, \infty]$ be a proper, convex, and lower semicontinuous function. Suppose that $\Gamma := F(T) \cap \operatorname*{arg\,min}_{u \in X} f(y) \neq \emptyset$, and for arbitrary $u, x_0 \in D$, let the sequence $\{x_n\}$ be generated as follows:*

$$\begin{cases} \gamma_n \in \big[\max\{\beta, h(x_n)\}, 1\big), \\ u_n = \gamma_n x_n \oplus (1-\gamma_n) J^f_{\mu_n} x_n, \\ \lambda_n \in \big[\max\{\gamma_n, l(u_n)\}, 1\big), \\ y_n = \lambda_n u_n \oplus (1-\lambda_n) T u_n, \\ \theta_n \in \big[\max\{\lambda_n, \kappa(y_n)\}, 1\big), \\ x_{n+1} := \alpha_n u \oplus (1-\alpha_n)(\theta_n u_n \oplus (1-\theta_n) T y_n), \quad n \geq 0, \end{cases}$$

where $h(x_n) := \inf\{\gamma \geq 0 : \gamma x_n \oplus (1-\gamma) J^f_{\mu_n} x_n \in D\}$, $l(u_n) := \inf\{\lambda \geq 0 : \lambda u_n \oplus (1-\lambda) T u_n \in D\}$, $\kappa(y_n) := \inf\{\theta \geq 0 : \theta x_n \oplus (1-\theta) T y_n \in D\}$, $\beta \in \Big(\frac{\sqrt{2}}{1+\sqrt{2}}, 1\Big)$, and $\{\alpha_n\}$ is a sequence in $(0,1)$ such that conditions (A1)–(A3) are satisfied. Then $\{x_n\}$ converges strongly to an element of Γ.

4. Numerical example

In this section, we give a numerical example to illustrate our result in a Hadamard space (non-Hilbert space).

Example 4.1 ([18]). Let $X = \mathbb{R}^2$ be endowed with a metric $d_X : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, -\infty)$ defined by

$$d_X(x, y) = \sqrt{(x_1 - y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2}.$$

Then, $(\mathbb{R}^2, d_X)$ is a Hadamard space with the geodesic joining x to y given by

$$(1-t)x \oplus ty = \big((1-t)x_1 + ty_1, ((1-t)x_1 + ty_1)^2 - (1-t)(x_1^2 - x_2) - t(y_1^2 - y_2)\big).$$

Let $D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 - \frac{1}{2})^2 \leq 1\}$. Clearly, D is a nonempty, closed, and convex subset of $\mathbb{R}^2$. Now, define $T : D \to \mathbb{R}^2$ by $T(x_1, x_2) = (x_1, 2x_1^2 - x_2)$; then, T is nonexpansive in (D, d_X) with $F(T) = \{(0,0), (1,1)\}$. Hence, T is hemicontractive. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x_1, x_2) = 100((x_2+1) - (x_1+1)^2)^2 + x_1^2$. Then f is a proper, convex, and lower semicontinuous function in $(\mathbb{R}^2, d_X)$ but not convex in the classical sense.

Take $\beta = \frac{7}{10}$, $\mu_n = \frac{n+4}{n+1}$, and $\alpha_n = \frac{1}{n+3}$, for $n \geq 1$. Then the hypotheses hold for

$$\begin{aligned} h(x_n) &= \inf\{\gamma \geq 0 : \gamma x_n \oplus (1-\gamma) J^f_{\mu_n} x_n \in D\}, \\ l(u_n) &= \inf\{\lambda \geq 0 : \lambda u_n \oplus (1-\lambda) T u_n \in D\}, \\ \kappa(y_n) &= \inf\{\theta \geq 0 : \theta x_n \oplus (1-\theta) T y_n \in D\}. \end{aligned}$$

Hence, Algorithm (3.1) becomes

$$\begin{cases} \gamma_n \in \big[\max\{\dfrac{7}{10}, h(x_n)\}, 1\big), \\ u_n = \gamma_n x_n \oplus (1-\gamma_n) J^f_{\mu_n} x_n, \\ \lambda_n \in \big[\max\{\gamma_n, l(u_n)\}, 1\big), \\ y_n = \lambda_n u_n \oplus (1-\lambda_n) T u_n, \\ \theta_n \in \big[\max\{\lambda_n, \kappa(y_n)\}, 1\big), \\ x_{n+1} := \frac{1}{n+3} u \oplus \Big(1 - \frac{1}{n+3}\Big)(\theta_n u_n \oplus (1-\theta_n) T y_n). \end{cases}$$

Case 1A: $u = (1, -1)^T$, $x_0 = (\frac{1}{2}, \frac{1}{4})^T$;
Case 1B: $u = (-1, 1)^T$, $x_0 = (-\frac{1}{2}, \frac{1}{4})^T$;
Case 2A: $u = (-1, 2)^T$, $x_0 = (\frac{1}{4}, 3)^T$;
Case 2B: $u = \big(1, \frac{1}{2}\big)^T$, $x_0 = \big(\frac{1}{4}, -\frac{1}{2}\big)^T$.

MATLAB version R2019a is used to obtain the graphs of errors against number of iterations. Using different choices of the initial points u and x_1 (that is, **Case 1A**–**Case 2B**), we obtain the numerical results in Figure 1. We see that the error values converge to 0, which implies that choosing arbitrary starting points the sequence $\{x_n\}$ converges to an element in the solution set Γ.

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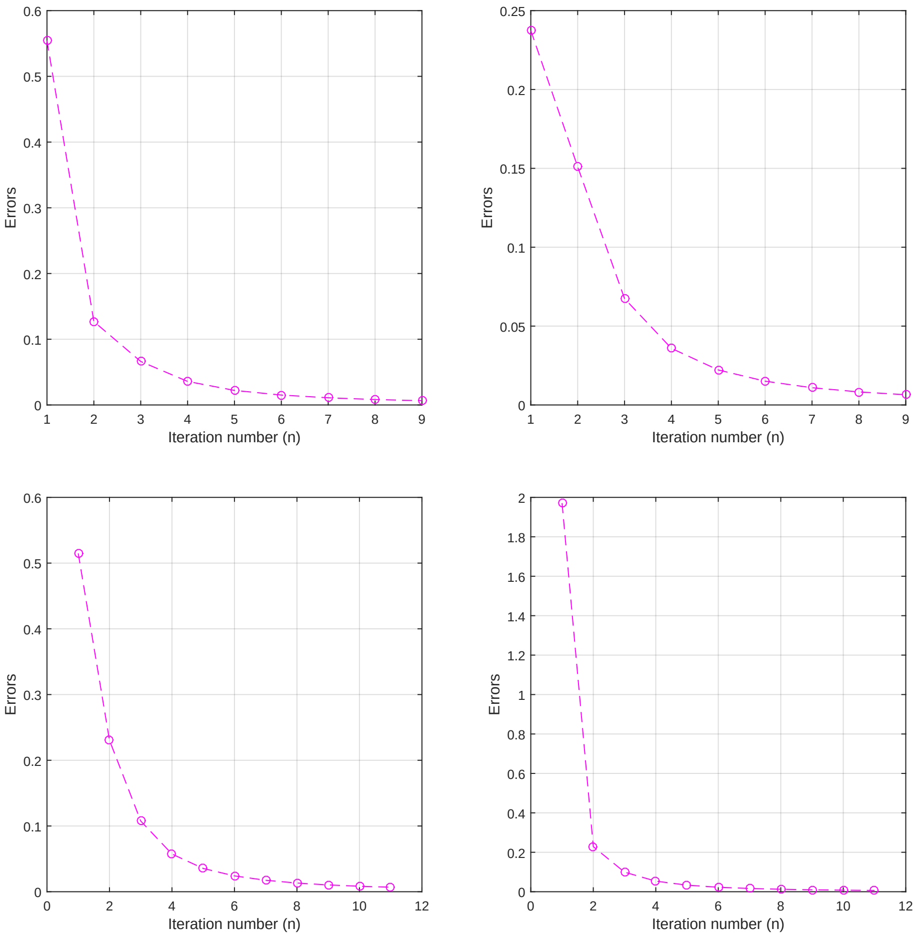

FIGURE 1. Errors vs Iteration number (n): **Case 1A** (top left); **Case 1B** (top right); **Case 2A** (bottom left); **Case 2B** (bottom right).

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