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MONSTER GRAPHS ARE DETERMINED BY THEIR LAPLACIAN SPECTRA

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Abstract. A graph G is determined by its Laplacian spectrum (DLS) if every graph with the same Laplacian spectrum is isomorphic to G. A multi-fan graph is a graph of the form $(P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_k}) \bigtriangledown K_1$, where K_1 denotes the complete graph of size 1, $P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_k}$ is the disjoint union of paths P_{n_i}, $n_i \geq 1$ and $1 \leq i \leq k$; and a starlike tree is a tree with exactly one vertex of degree greater than 2. If a multi-fan graph and a starlike tree are joined by identifying their vertices of degree more than 2, then the resulting graph is called a monster graph. In some earlier works, it was shown that all multi-fan and path-friendship graphs are DLS. The aim of this paper is to generalize these facts by proving that all monster graphs are DLS.

1. Basic definitions

As usual $G = (V, E)$ will denote a simple graph having n vertices and m edges, with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. The complement of G, denoted $\overline{G}$, is a graph having all the vertices of G and an edge between two vertices v and w if and only if there exists no edge between v and w in the original graph G. $\deg_G(v)$ ($d_G(v)$ for short) is the degree of a vertex v in G. In this paper, $G \cup H$ denotes the disjoint union of graphs G and H. The join $G \bigtriangledown H$ of the graphs G and H is obtained from $G \cup H$ by joining each vertex of G to each vertex of H.

A *multi-fan graph* is a graph of the form $(P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_k}) \bigtriangledown K_1$, where K_1 denotes the complete graph of size 1, $P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_k}$ is the disjoint union of paths P_{n_i}, $n_i \geq 1$ and $1 \leq i \leq k$; and a *starlike tree* is a tree with exactly one vertex of degree greater than 2. If a multi-fan graph and a starlike tree are joined by identifying their vertices of degree more than 2, then the resulting graph is called a *monster graph*. A monster graph with parameters $n = (n_1 + \cdots + n_k) + (n_{k+1} + \cdots + n_t) + 1$, k, $s = n_1 + \cdots + n_k$ and t vertices of degree 1 is denoted by $M(n, k, s, t)$. If $P_{n_1} = \cdots = P_{n_i} = P_2$, $n_i \geq 1$ and $1 \leq i \leq k$, then the monster

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graph is called *path-friendship* [13]. The authors in [13, 10] showed independently that all path-friendship graphs are DLS. Our proof is simpler and based on some facts in [17].

Spectral graph theory is the study of eigenvalues and eigenvectors of matrices associated with graphs. Suppose $V(G) = \{v_1, \ldots, v_n\}$. The adjacency and Laplacian matrices of G are denoted by $A(G)$ and $L(G)$, respectively. If $A(G) = [a_{ij}]$, then, by definition, $a_{ij} = 1$ if and only if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. On the other hand, if $L(G) = [l_{ij}]$, then $l_{ij} = a_{ij} - d_{ij}$, where d_{ii} is the degree of v_i and $d_{ij} = 0$ when $i \neq j$.

Let $\mu_1, \mu_2, \ldots, \mu_t$ be the distinct eigenvalues of $L(G)$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively. The *Laplacian spectrum* or *L-spectrum* of G is the multi-set of eigenvalues of $L(G)$ usually written in non-increasing order $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_t(G) = 0$. Van Dam and Haemers [16] conjectured that almost all graphs are determined by their spectrum, that is, up to isomorphism they are the only graph with that spectrum. However, very few graphs are known to have that property, and so discovering new classes of such graphs is an interesting problem. Formally, we define two graphs G and H to be *L-cospectral* if they have the same L-spectrum, and a graph G is *determined by its Laplacian spectrum*, abbreviated by DLS, if no other graphs are L-cospectral with G.

2. Preliminaries

In this section, some known results which are crucial throughout this paper are given. We also review the most important results on DLS-graphs. Let us start by the main properties of these graphs.

Theorem 2.1 ([12, 16, 17])**.** *The following can be obtained from the Laplacian spectrum of a graph:*

(i) *the number of vertices,*
(ii) *the number of edges,*
(iii) *the number of spanning trees,*
(iv) *the number of components,*
(v) *the sum of the squares of the degrees of the vertices.*

The next theorem relates the Laplacian spectra of complementary graphs.

Theorem 2.2 ([2])**.** *Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ and $\overline{\mu}_1 \geq \overline{\mu}_2 \geq \cdots \geq \overline{\mu}_n = 0$ be the Laplacian spectra of G and $\overline{G}$, respectively. Then $\overline{\mu}_i = n - \mu_{n-i}$ for $i = 1, 2, \ldots, n-1$.*

A *walk* is defined as a finite-length alternating sequence of vertices and edges. The total number of edges covered in a walk is called the *length* of the walk. A walk is said to be closed if the starting and ending vertices are identical, i.e. if the walk starts and ends at the same vertex. For graphs G and H, we let $N_G(H)$ be the number of subgraphs of G that are isomorphic to H. Further, let $W_G(i)$ be the number of closed walks of length i in G and let $W'_H(i)$ be the number of closed walks of length i in H that contain all the edges of H. Then $W_G(i) = \sum N_G(H) W'_H(i)$, where the sum is taken over all connected subgraphs H of G for which $W'_H(i) \neq 0$. This equation provides some formulas for calculating the number of some short closed walks in a graph. Note that if $\mathrm{tr}(M)$ denotes the trace of a matrix M, then $W_G(3) = \mathrm{tr}(A^3(G))$.

Theorem 2.3 ([15])**.** *Suppose G is a graph with exactly m edges. The number of closed walks of lengths 2, 3, and 4 in G can be computed by the following formulas:*

(i) $W_G(2) = 2m$,
(ii) $W_G(3) = \mathrm{tr}(A^3(G)) = 6N_G(C_3)$,
(iii) $W_G(4) = 2m + 4N_G(P_3) + 8N_G(C_4)$.

Suppose G is a simple graph with vertex set $\{v_1, \dots, v_n\}$. Turning to the degrees of the vertices in graphs, as before, we let $d_i = \deg_G(v_i)$, and assume that $d_1(G) \geq d_2(G) \geq \cdots \geq d_n(G)$. In addition, the eigenvalues of G are $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$.

Theorem 2.4 ([6])**.** *If G is a graph with at least one edge, then $\mu_1(G) \geq d_1(G)+1$. Moreover, if G is connected, then equality holds if and only if $d_1(G) = n-1$.*

The next result uses the quantity $\theta(v) = \sum \frac{\deg(u)}{\deg(v)}$, where the sum is taken over the neighbors u of the vertex v. A *semi-regular bipartite graph* is a bipartite graph G with bipartition (A, B) in which every two vertices in A (and also in B) have the same degree as each other.

Theorem 2.5 ([10, 12])**.** *If G is a connected graph, then $\mu_1(G) \leq \max_v(\deg(v) + \theta(v))$, with equality if and only if G is a regular or a semi-regular bipartite graph.*

Theorem 2.6 ([10])**.** *If G is a nontrivial graph, then $\mu_1(G) \leq d_1(G) + d_2(G)$; and if G is connected, then $\mu_2(G) \geq d_2(G)$.*

Theorem 2.7 ([14])**.** *The first four coefficients of the Laplacian polynomial of a graph G, $\varphi(G) = \sum l_i x^i$, are*

$$l_0 = 1, \quad l_1 = -2m, \quad l_2 = 2m^2 - m - \frac{1}{2}\sum_{i=1}^{n} d_i^2,$$

and

$$l_3 = \frac{1}{3}\left(-4m^3 + 6m^2 + 3m\sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i^3 - 3\sum_{i=1}^{n} d_i^2 + 6N_G(C_3)\right).$$

The following result is an immediate consequence of Theorem 2.7.

Corollary 2.8. *If G and H are L-cospectral graphs with the same degrees, then they have the same number of triangles.*

It follows from Theorems 2.1 and 2.7 that if G and G' are L-cospectral graphs with vertex degrees $d_1, d_2, \dots, d_n$ and $d'_1, d'_2, \dots, d'_n$, respectively, then

$$\mathrm{tr}(A^3(G)) - \sum_{i=1}^{n} d_i^3 = \mathrm{tr}(A^3(G')) - \sum_{i=1}^{n} d_i'^3.$$

Based on this result, Liu and Huang [8] defined the following graph invariant for a graph G:

$$\varepsilon(G) = \mathrm{tr}(A^3(G)) - \sum_{i=1}^{n} (d_i - 2)^3.$$

Theorem 2.9 ([8])**.** *If G and H are L-cospectral, then $\varepsilon(G) = \varepsilon(H)$.*

We end this section with the following useful result:

Theorem 2.10 ([11])**.** *If u is a vertex of G and $G - u$ is the subgraph obtained from G by deleting u, then*

$$\mu_i(G) \geq \mu_i(G-u) \geq \mu_{i+1}(G) - 1, \quad i = 1, 2, \ldots, n-1.$$

3. Main results

We start with some facts which have been proved so far.

Theorem 3.1 ([9])**.** *All friendship graphs are DLS.*

Theorem 3.2 ([5])**.** *All starlike trees are DLS.*

Theorem 3.3 ([10, 13])**.** *All path-friendship graphs are DLS.*

Theorem 3.4 ([9])**.** *All multi-fan graphs are DLS.*

Consider a monster graph $G = M(n,k,s,t)$. Clearly $d_2(G) \in \{1,2,3\}$. If $d_2(G) = 1$, then $G = K_{1,n-1}$, which is DLS. If $d_2(G) = 2$, then G is a path-friendship graph that is DLS [10, 13]. So, we consider $d_2(G) = 3$. In the following, we show that all the monster graphs G with $d_2(G) = 3$ are DLS.

Lemma 3.5. *Consider the monster graph $G = M(n,k,s,t)$ and let $d_2(G) = 3$. Then*

(i) $s + t + 1 \leq \mu_1(G) < s + t + 4$.
(ii) $\mu_2(G) < 5$.

Proof. (i) It follows from Theorems 2.4 and 2.5 that

$$\begin{aligned} s+t+1 \leq \mu_1(G) &\leq s+t+1+\frac{4k+3(s-2k)+2t}{s+t} \\ &= s+t+1+\frac{3s-2k+2t}{s+t} \\ &= s+t+1+\frac{(2s+2t)}{s+t}+\frac{(s-2k)}{s+t} \\ &< s+t+4. \end{aligned}$$

(ii) The proof follows immediately from Theorem 2.10 and the fact that the greatest Laplacian eigenvalue of a path is less than 4. ☐

Let G be a connected graph with n vertices and m edges. The graph G is called a k-cyclic graph if $m = n + k - 1$. Consider the monster graph G with $n = n(G) = (n_1 + \cdots + n_k) + (n_{k+1} + \cdots + n_t) + 1$ and thus $m = m(G) = (n_1 - 1 + \cdots + n_k - 1) + (n_{k+1} - 1 + \cdots + n_t - 1) + t - k + (n_1 + \cdots + n_k) = n - 1 - t + t - k + n_1 + \cdots + n_k = n - 1 - k + n_1 + \cdots + n_k$. It can be easily seen that $m = m(G) = n + (n_1 + \cdots + n_k - k) - 1$ and $m = m(G) = n + s - k - 1$, which means that G is a $(s-k)$-cyclic graph.

Theorem 3.6. *If H is L-cospectral with $G = M(n,k,s,t)$, then they have the same degree sequence.*

Proof. By Lemma 3.5, $\mu_2(H) < 5$, and thus it follows from Theorem 2.6 that $d_2(H) \leq 4$. Since H and G are L-cospectral, by Theorem 2.1, H is also connected, and has the same order, size, and sum of the squares of its degrees as G. Let n_i denote the number of vertices of degree i in H, for $i = 1, 2, \ldots, d_1(H)$. Then

$$\sum_{i=1}^{d_1(H)} n_i = n(G), \tag{3.1}$$

$$\sum_{i=1}^{d_1(H)} i n_i = 2m(G), \tag{3.2}$$

$$\sum_{i=1}^{d_1(H)} i^2 n_i = n'_1 + 4n'_2 + 9n'_3 + d_1^2(G), \tag{3.3}$$

where n'_i is the number of vertices of degree i $(i = 1, 2, 3)$ belonging to G.

Set $r = n_{k+1} + \cdots + n_t$ and $n = s + r + 1$. Clearly, $n(G) = n$, $m(G) = n + s - k - 1 = 2s + r - k$, $n'_1 = t$, $n'_2 = 2k + r - t$, $n'_3 = n - 1 - 2k - r = s - 2k$ and $d_1(G) = s + t$. By adding up Eqs. (3.1), (3.2) and (3.3) with coefficients 2, -3, 1, respectively we get

$$\sum_{i=1}^{d_1(H)} (i^2 - 3i + 2) n_i = s(s + 2t - 1) + (t-1)(t-2) - 4k. \tag{3.4}$$

By Lemma 3.5, $s + t + 1 \leq \mu_1(G) < s + t + 4$. It follows from Theorem 2.4 that $d_1(H) + 1 \leq \mu_1(H) = \mu_1(G) < s + t + 4$, which leads to $d_1(H) \leq s + t + 2$. On the other hand, by Lemma 3.5 and Theorem 2.6 one can conclude that $s + t + 1 \leq \mu_1(G) = \mu_1(H) \leq d_1(H) + d_2(H) \leq d_1(H) + 4$, which leads to $d_1(H) \geq s + t - 3$. Therefore, we have $s + t - 3 \leq d_1(H) \leq s + t + 2$. It follows from Theorem 2.9 that

$$6N_H(C_3) - \sum_{i=1}^{n} (d_i(H) - 2)^3 = 6(s-k) - (-t + s - 2k + (s + t - 2)^3). \tag{3.5}$$

Our main proof will consider some cases as follows:

1. $d_1(H) = s + t - 3$.
   1.1. We first assume that $n_{s+t-3} = 1$. In this subcase, $s+t-3 = d_1(H) > 3 \geq d_2(H)$. From (3.4) and by a straightforward calculation, we get

$$((s+t-3)^2 - 3(s+t-3) + 2) + 2n_3 = s(s+2t-1) + (t-1)(t-2) - 4k.$$

   It follows from Eqs. (3.1), (3.2) and (3.3) that $n_3 = 4s + 3t - 2k - 9$, $n_1 = 3s + 4t - 12$ and $n_2 = -6s + r + 2k - 7t + 21$. Consequently by (3.5) we have $N_H(C_3) = \frac{-3s^2+23s-38-3t^2+21t-6st-2k}{2}$. Obviously, for $s \geq 4$, $N_H(C_3) < 0$ and this is clearly a contradiction. Consider the following cases:
   (a) $s = 1$. Then for any natural number t we always have $N_H(C_3) = -9 - k + \frac{-3t^2+15t}{2} < 0$, an impossibility.
   (b) $s = 2$. Then for any natural number $t \geq 2$ we always have $N_H(C_3) = -2 - k + \frac{-3t^2+9t}{2} < 0$ and this is a contradiction. If $t = 1$ and $k \geq 2$, then $N_H(C_3) < 0$, a contradiction. If $t = k = 1$, then $0 = d_1(H) > 3$, a contradiction.
   (c) $s = 3$. If $t = 1$, then $1 = d_1(H) > 3$, a contradiction. Now, if $t \geq 2$, then for any natural number $t \geq 2$ we get $N_H(C_3) = 2 - k + \frac{-3t^2+3t}{2} < 0$ and this is an impossibility.

   1.2. $n_{s+t-3} \geq 2$. Then $s+t-3 = d_1(H) = d_2(H) \leq 3$, which implies that the pair (s,t) equals $(3,1)$, $(3,2)$, $(3,3)$, $(4,1)$, or $(4,2)$. So we need to consider the following five subcases:
   (a) $(s,t) = (3,1)$. Therefore, $1 = d_1(H) = d_2(H)$. Therefore, $H = G = P_2$. On the other hand, $n'_1 = 1$, which is a contradiction.
   (b) $(s,t) = (3,2)$. Therefore, $2 = d_1(H) = d_2(H)$. If H is a path or a cycle, then $H = G$ and so $n'_3 = n_3 = 0$ or $3 - 2k = 0$ or $k = 1.5$, a contradiction.
   (c) $(s,t) = (3,3)$. Therefore, $3 = d_1(H) = d_2(H)$. Hence $n'_3 = 3 - 2k$ and so $k = 1$. So, $n_3 = 11$ and by (3.1) and (3.2) we have $n_1 = 5$ and $n_2 = r - 12$. Consequently, by (3.5) we get $N_H(C_3) = -\frac{22}{3}$, a contradiction.
   (d) $(s,t) = (4,1)$. As a result, $2 = d_1(H) = d_2(H)$ and this means that H is a path or a cycle, since H is a connected graph. Hence $H = G = P_n$ or $H = G = C_n$, for some natural number $n \geq 3$. On the other hand, $n'_1 = 1$ and this is obviously a contradiction.
   (e) $(s,t) = (4,2)$. Thus, $3 = d_1(H) = d_2(H)$. On the other hand, $2 \leq n_3 = 14 - 2k$. Obviously, for $3 \leq k \leq 6$ we will have $n'_3 < 0$ and this is impossible. Consider the following two cases:
      (i) $k = 1$. This means that $n_3 = 12$, $n_1 = 8$ and $n_2 = r - 15$ and so $N_H(C_3) = -7$, a contradiction.
      (ii) $k = 2$. Therefore, $n_3 = 10$, $n_1 = 8$ and $n_2 = r - 13$. This means that $N_H(C_3) = -8$, which is impossible.

2. $d_1(H) = s+t-2$.
   2.1. $n_{s+t-2} = 1$. In this case, $s+t-2 = d_1(H) > 3 \geq d_2(H)$. From (3.4) and by a straightforward calculation, we get

$$((s+t-2)^2 - 3(s+t-2)+2) + 2n_3 = s(s+2t-1) + (t-1)(t-2) - 4k,$$

from which it follows that $n_3 = 3s+2t-2k-5$. It now follows from (3.2) and (3.3) that $n_2 = -5t-4s+r+2k+12$ and $n_1 = 3t+2s-7$. Set $f_1(t) = -(t-3)^2$ and $f_2(t,s,k) = -s^2+7s-2st-k$. From (3.5) we deduce that $N_H(C_3) = f_1(t)+f_2(t,s,k)$. If $t \geq 4$, then $f_1(t)+f_2(t,s,k) < 0$. Consider the following three cases:
   (a) $t=1$. Then $N_H(C_3) = -(s-1)(s-4)-k$. For $s \geq 4$, $N_H(C_3) < 0$ and this is a contradiction. If $(s,t) = (2,1)$, then $d_1(H) = 1 > 3$, which is impossible. If $(s,t) = (3,1)$, then $d_1(H) = 2 > 3$, which is impossible.
   (b) $t=2$. Then $N_H(C_3) = -s^2+3s-1-k$. For $s \geq 3$, $N_H(C_3) < 0$ and this is a contradiction. If $(s,t) = (2,2)$, then $d_1(H) = 2 > 3$, a contradiction. If $(s,t) = (1,2)$, then $d_1(H) = 1 > 3$, a contradiction.
   (c) $t=3$. Then $N_H(C_3) = -s^2+s-k$ and so we always have $N_H(C_3) < 0$, which is impossible.

   2.2 $n_{s+t-2} \geq 2$. Then $s+t-2 = d_1(H) = d_2(H) \leq 3$, which implies that the pair (s,t) equals $(3,1)$, $(3,2)$, or $(4,1)$. So we need to consider the following three subcases:
   (a) $(s,t) = (3,1)$. Therefore, $2 = d_1(H) = d_2(H)$. If H is a path, then $H = G$ is a path and so $n'_3 = n_3 = 0$ or $3-2k = 0$ or $k = \frac{3}{2}$, a contradiction. If H is a cycle, then $H = G$. On the other hand, $0 = n_1 = n'_1 = 1$, which is impossible.
   (b) $(s,t) = (3,2)$. Therefore, $3 = d_1(H) = d_2(H)$. It follows from (3.4) that $n_3 = 9-2k$. By (3.1) and (3.2) we get $n_1 = 5$ and $n_2 = r+2k-10$. Consequently, $N_H(C_3) = -k-1 < 0$, a contradiction.
   (c) $(s,t) = (4,1)$. In this subcase, $3 = d_1(H) = d_2(H)$. Since $0 \leq n'_3 = 4-2k$, $k \in \{1,2\}$. If $k=1$, then it follows from (3.4) that $n_3 = 8$ and by (3.1) and (3.2) we get $n_2 = -7+r$ and $n_1 = 4$. By (3.5), $N_H(C_3) = -\frac{7}{3}$, which is impossible. If $k=2$, then it follows from (3.4), (3.1) and (3.2) that $n_3 = 6$, $n_2 = -5+r$ and $n_1 = 4$. Now it follows from (3.5) that $N_H(C_3) = -\frac{8}{3}$, a contradiction.

3. $d_1(H) = s+t-1$.
   3.1. $n_{s+t-1} = 1$. By an argument similar to that for (3.5), we get

$$((s+t-1)^2 - 3(s+t-1)+2) + 2n_3 = s(s+2t-1) + (t-1)(t-2) - 4k.$$

Hence, $n_3 = 2s+t-2k-2$. Combining (3.2) and (3.3), we find that the roots are $n_1 = 2t+s-3$ and $n_2 = -3t-2s+r+2k+5$. Set $f_1(t) = -(t-2)(t-3)$ and $f_2(t,s,k) = -s^2+7s-2st-2k$. From

(3.5) we deduce that $N_H(C_3) = \frac{f_1(t)+f_2(t,s,k)}{2}$. For $t \geq 3$, $N_H(C_3) < 0$ and this is a contradiction. Now, we consider the following subcases:

(a) $t = 1$. Then $N_H(C_3) = \frac{-s^2+5s-2k-2}{2}$. For $s \geq 5$ we have a contradiction. Consider the following subcases:
  (i) $s = 1$. Then $d_1(H) = 1 > 3$, which is impossible.
  (ii) $s = 2$. Then $d_1(H) = 2 > 3$, a contradiction.
  (iii) $s = 3$. Then $d_1(H) = 3 > 3$, an impossibility.
  (iv) $s = 4$. For $k \geq 2$, $N_H(C_3) < 0$ and this is a contradiction. If $k = 1$, $d_1(H) = 4$ and $n_4 = 1$. It follows from (3.1), (3.2) and (3.3) that $n_1 = 3$, $n_2 = r - 4$, $n_3 = 5$. On the other hand, H is a 3-cyclic graph and also $N_H(C_3) = 0$. By Theorem 2.9 $0 - (8 + 2) = 3 - (27 + 1)$, a contradiction.

(b) $t = 2$. Then $N_H(C_3) = \frac{-s^2+3s-2k}{2}$. If $(s, k) = (1, 1)$, then $d_1(H) = 2 > 3$, which is impossible. If $(s, k) = (2, 1)$, then $d_1(H) = 3 > 3$, which is impossible. For other cases $N_H(C_3) < 0$ and this is a contradiction.

3.2 $n_{s+t-1} \geq 2$. Then $s + t - 1 = d_1(H) = d_2(H) \leq 3$ and so $(s, t) \in \{(1, 1), (2, 1), (3, 1), (1, 3), (1, 2), (2, 2)\}$.

(a) $(s, t) = (1, 1)$. Then $d_1(H) = d_2(H) = 1$. This means that $H = K_2$ and so $G = K_2$. But $(s, t) = (1, 1)$ implies that $n'_1 = 1$, an impossibility.

(b) $(s, t) = (2, 1)$. Then $d_1(H) = d_2(H) = 2$. If H is a path, then $H = G$ and so $n_1 = n'_1 = 1$, a contradiction. If H is a cycle, then $H = G$. On the other hand, $0 = n_1 = n'_1 = 1$, which is impossible.

(c) $(s, t) = (3, 1)$. Therefore, $3 = d_1(H) = d_2(H)$. It follows from (3.4) that $2 \leq n_3 = 6 - 2k$. Therefore, $k \in \{1, 2\}$. If $k = 1$, then $n_3 = 4$ and so $n_1 = 2$ and $n_2 = r - 2$. By Theorem 2.9 $-6 = -1$, a contradiction. If $k = 2$, then $n'_3 < 0$, a contradiction.

(d) $(s, t) = (1, 3)$. Clearly, $0 \leq n'_3 = 1 - 2k$ and so $k = 0$, a contradiction.

(e) $(s, t) = (1, 2)$. Then $d_1(H) = d_2(H) = 2$. If H is a path, then $H = G$ is a path and so $1 - 2k = 0$ or $k = \frac{1}{2}$, which is impossible. If H is a cycle, then $H = G$. On the other hand, $0 = n_1 = n'_1 = 2$, a contradiction.

(f) $(s, t) = (2, 2)$. Then $d_1(H) = d_2(H) = 3$. Since $n'_3 = 2 - 2k$, $k = 1$. This implies that $n'_3 = 0$, a contradiction, since we consider the graph G with $d_2(G) = 3$.

4. $d_1(H) = s + t$. By (3.4) one can deduce that

$$((s + t)^2 - 3(s + t) + 2) + 2n_3 = s(s + 2t - 1) + (t - 1)(t - 2) - 4k.$$

From this it follows that $n_3 = s - 2k$. Combining (3.2) and (3.3), we find that $n_1 = t$ and $n_2 = r + 2k - t$. Therefore, the degrees of H are the same as those of G. In this case, it follows from (3.5) that $N_H(C_3) = s - k$. Note that if $(s, k) = (2, 1)$, then $d_1(H) = 3$. On the other hand, $n_3 = 0$, a contradiction. So, we always have $d_1(H) \geq 4$ or $n_{s+t} = 1$.

5. $d_1(H) = s + t + 1$. From (3.4) we deduce that

$$((s+t+1)^2 - 3(s+t+1) + 2) + 2n_3 = s(s+2t-1) + (t-1)(t-2) - 4k,$$

from which it follows that $n_3 = -t - 2k + 1 < 0$, which is impossible.

6. $d_1(H) = s + t + 2$. Similar to Case 5, we will have a contradiction.

Hence the result. □

Corollary 3.7. *Every graph H L-cospectral with the monster graph G has exactly $s-k$ triangles and as a result all cycles belonging to H are triangles.*

Proof. By Equation (3.5) and Theorem 3.6, $N_H(C_3) = s - k$. On the other hand, H is a $(s-k)$-cyclic graph. Therefore, one can deduce that all cycles belonging to H are triangles. □

Corollary 3.8. *Any graph H L-cospectral with a monster graph is a monster graph.*

Proof. Any connected component is either a tree (and as a result here is a path) or a k-cyclic graph, $k \geq 1$. Since all cycles belonging to H are triangles, if one of its components consists of a triangle then the number of triangles belonging to H is greater than $s-k$, a contradiction. Therefore, all connected components are paths and as a result all triangles have a common vertex. □

Before proving our main result, we state some essential lemmas and notations.

Lemma 3.9 ([7])**.** *Let v be a vertex of a connected graph G and suppose that $v_1, \ldots, v_s$ are pendant vertices of G which are adjacent to v. Let G^* be the graph obtained from G by adding any t $\left(1 \leq t \leq \frac{s(s-1)}{2}\right)$ edges among $v_1, v_2, \ldots, v_s$. Then we have $\mu_1(G) = \mu_1(G^*)$.*

Lemma 3.10 ([5])**.** *No two non-isomorphic starlike trees are L-cospectral.*

Let H be a monster graph and let v be a vertex with maximum degree in H. Now, we remove an edge of any of the triangles, except in edges adjacent to v; then we have a starlike tree, say $S(H)$. Also, $G = G(k, l_1, l_2, \ldots, l_t)$ is a monster graph with k paths which construct $s-k$ triangles and t paths with lengths l_i which meet in the maximum degree G, where $i = 1, 2, \ldots, t$; see Figure 1.

Note that in the proof of Lemma 3.10, it is easy to see that if $S_1 = S(l_1, \ldots, l_t)$ and $S_2 = S(j_1, \ldots, j_t)$ are two non-isomorphic starlike trees, then $\mu_1(S_1) \neq \mu_1(S_2)$, where $l_1 \geq l_2 \geq \cdots \geq l_t \geq 1$ and $j_1 \geq j_2 \geq \cdots \geq j_t \geq 1$.

Corollary 3.11. *Let $G = G(k, l_1, l_2, \ldots, l_t)$ and $H = H(k, j_1, j_2, \ldots, j_t)$ be two monster graphs. If $S(G)$ and $S(H)$ are two non-isomorphic starlike trees corresponding to G and H, respectively, then $\mu_1(S(G)) \neq \mu_1(S(H))$.*

Lemma 3.12. *If G and H are two L-cospectral monster graphs, then $S(H) = S(G)$.*

Proof. By Lemma 3.9, $\mu_1(S(H)) = \mu_1(H)$ and $\mu_1(S(G)) = \mu_1(G)$. If $S(H) \neq S(G)$, then Corollary 3.11 implies that $\mu_1(S(H)) \neq \mu_1(S(G))$ and so $\mu_1(H) \neq \mu_1(G)$, a contradiction. □

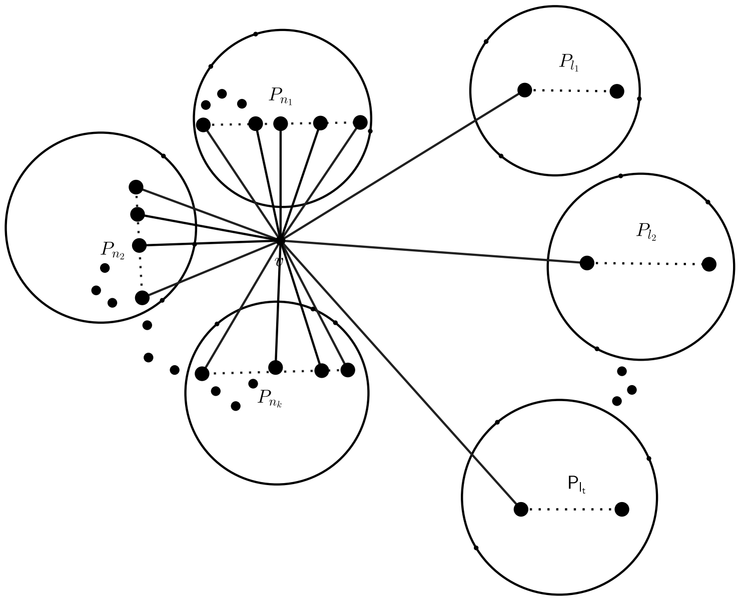

FIGURE 1. The monster graph $M(n, k, s, t)$ and its connected components after removing the vertex v.

By Theorem 3.6, Corollary 3.8 and Lemma 3.12 we have the following theorem.

Theorem 3.13. *All monster graphs are DLS.*

Proof. Let monster graphs G and H be L-cospectral. By Theorem 3.6, G and H have the same degree sequence and by Lemma 3.12, $S(H) = S(G)$. Hence, $(\cup_{l=k+1}^{t} P_{i_k}) = (\cup_{l=k+1}^{t} P_{j_k}) = A$; see Figure 1. On the other hand, by Theorem 3.6 we assume that v and v' denote the vertices with maximum degree in G and H, respectively. Therefore, $G - v = \cup_{l=1}^{k+t} P_{i_l} = (\cup_{l=1}^{k} P_{i_l}) \cup A$ and $H - v' = \cup_{l=1}^{k+t} P_{j_l} = (\cup_{l=1}^{k} P_{j_l}) \cup A$. We claim that $(\cup_{l=1}^{k} P_{j_l}) = (\cup_{l=1}^{k} P_{i_l})$. Our proof is by induction on k, which is the number of paths. Suppose $k = 1$. By the fact that $N_H(C_3) = N_G(C_3)$, Lemma 3.12 and Theorem 3.6, we get $G = H$ and the proof is straightforward.

Let the claim be true for $k - 1$ and let us prove it for k. We show that if two graphs G and H are L-cospectral and they consist of k paths $E = (\cup_{l=1}^{k} P_{j_l})$ and $F = (\cup_{l=1}^{k} P_{i_l})$, respectively, then $E = F$. Since G and H are L-cospectral, by induction hypothesis we deduce that $k - 1$ paths from k paths E and F are equal. Without loss of generality, we can assume that $(\cup_{l=1}^{k-1} P_{j_l}) = (\cup_{l=1}^{k-1} P_{i_l})$. It is sufficient to show that $P_{j_k} = P_{i_k}$. Since $N_H(C_3) = N_G(C_3) = s - k$, $P_{j_k} = P_{i_k}$. Therefore, $(\cup_{l=1}^{k} P_{j_l}) = (\cup_{l=1}^{k} P_{i_l})$ or $E = F$ and so $H = G$. This completes the proof. $\square$

The next corollary follows immediately from Theorems 2.2 and 3.13.

Corollary 3.14. *The complement of a monster graph is DLS.*

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