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A GENERALIZATION OF PRIMARY IDEALS AND STRONGLY PRIME SUBMODULES

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ABSTRACT. We present $*$-primary submodules, a generalization of the concept of primary submodules of an R-module. We show that every primary submodule of a Noetherian R-module is $*$-primary. Among other things, we show that over a commutative domain R, every torsion free R-module is $*$-primary. Furthermore, we show that in a cyclic R-module, primary and $*$-primary coincide. Moreover, we give a characterization of $*$-primary submodules for some finitely generated free R-modules.

1. Introduction

Throughout this paper, all rings are commutative with identity and all modules are unital. A proper ideal I of a ring R is called a prime (resp. primary) ideal if whenever $ab \in I$, where $a, b \in R$, then either $a \in I$ or $b \in I$ (resp. $a \in I$ or $b^n \in I$ for some positive integer n). The notions of prime and primary ideals have been generalized to modules by various authors ([3], [4] and [6]). Let R be a ring, M an R-module and N a submodule of M. The annihilator of the R-module $\frac{M}{N}$ is denoted by $(N : M)$. A proper submodule N of M is called prime (resp. primary) if whenever $rm \in N$, where $r \in R$, $m \in M$, then either $m \in N$ or $r \in (N : M)$ (resp. $m \in N$ or $r^n \in (N : M)$ for some positive integer n). For more details about prime and primary submodules one can see [2], [5], [7], [11] and [12]. For a proper submodule N of an R-module M and $a \in R$, set $(N :_M a) = \{m \in M \mid am \in N\}$. It is easy to show that $(N :_M a)$ is a submodule of M. Following [1], a proper submodule N of M is said to be classical primary, if $abm \in N$, where $a, b \in R$, $m \in M$, implies that $am \in N$ or $b^n m \in N$ for some $n \in \mathbb{N}$.

Strongly prime submodules have been introduced and studied in [8] and [9]. According to [8], a proper submodule N of an R-module M is said to be strongly prime provided that $(Rx + N : M)y \subseteq N$, for $x, y \in M$, implies that either $x \in N$ or $y \in N$.

In this paper we introduce and investigate $*$-primary submodules, which are a generalization of primary ideals and strongly prime submodules.

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Definition 1.1. A proper submodule N of an R-module M is called *$*$-primary* (resp. *$*$-prime*) if $(Rx+N:M)y \subseteq N$ for $x, y \in M$ implies that either $y \in N$ or $(Rx+N:M)^k \subseteq (N:M)$ for some $k \in \mathbb{N}$ (resp. $y \in N$ or $(Rx+N:M) \subseteq (N:M)$). If the zero submodule of M is $*$-primary, M is called *$*$-primary.* Moreover the R-module M is called *fully $*$-primary* provided that every proper submodule of M is $*$-primary.

We have the following diagram which shows the relationship of strongly prime, prime, primary, $*$-prime and $*$-primary submodules.

$$\begin{array}{ccccc} \text{strongly prime} & \Longrightarrow & \text{prime} & \Longrightarrow & \text{primary} \\ & & \Downarrow & & \\ & & *\text{-prime} & \Longrightarrow & *\text{-primary} \end{array}$$

We give an example which shows that the classical primary submodule and $*$-primary submodules are different. Considering $\mathbb{Q}$ as a $\mathbb{Z}$-module, we observe that $\mathbb{Z}$ is a $*$-primary submodule which is neither a classical primary nor a primary submodule. Moreover, for some prime number p, the submodule $p\mathbb{Z}\oplus 0$ is a classical primary submodule of $\mathbb{Z}\oplus\mathbb{Z}$ which is not a $*$-primary submodule (see Example 1.2).

In the next example we show that the class of $*$-primary submodules is quite different from the class of strongly prime submodules and the class of primary submodules.

Example 1.2. (a) Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module. For every proper submodule N of $\mathbb{Q}$, we have $(N:\mathbb{Q}) = 0$. So all proper submodules of $\mathbb{Q}$ are $*$-prime and hence $*$-primary (one can easily extend this fact to divisible modules over a domain). We know that $\mathbb{Q}$ has no nonzero prime (primary) submodule; therefore it has no nonzero strongly prime submodule.

(b) $*$-primary submodules of the $\mathbb{Z}$-module $\mathbb{Z}_n$ are exactly the primary ideals of the ring $\mathbb{Z}_n$.

(c) If $(N:M) = m$ is a maximal ideal of R, then N is a $*$-primary submodule. In particular, if N is a maximal submodule of M then N is a $*$-primary submodule. Moreover, if M is a finitely generated module in which for every proper submodule N of M, $(0:M) = (N:M)$, then N is a $*$-primary submodule.

(d) In the $\mathbb{Z}$-module $\mathbb{Z}\oplus\mathbb{Z}$, submodules of the form $0\oplus m\mathbb{Z}$, $m\mathbb{Z}\oplus 0$ and $p\mathbb{Z}\oplus q\mathbb{Z}$ are not $*$-primary, where p, q are distinct prime numbers and $m \in \mathbb{Z}$. $((\mathbb{Z}(1,q+1)+p\mathbb{Z}\oplus q\mathbb{Z} : \mathbb{Z}\oplus\mathbb{Z})(1,0) \subseteq p\mathbb{Z}\oplus q\mathbb{Z}$, $(1,0) \notin p\mathbb{Z}\oplus q\mathbb{Z}$ and no power of $(\mathbb{Z}(1,q+1)+p\mathbb{Z}\oplus q\mathbb{Z} : \mathbb{Z}\oplus\mathbb{Z})$ is contained in $pq\mathbb{Z}$).

In Section 2, we show that in a Noetherian R-module, every primary submodule is $*$-primary (Proposition 2.8); we also show that torsion free R-modules over commutative domains are $*$-primary (Proposition 2.11). In Section 3, we show that for any cyclic R-module M and submodule N of M, we have that N is strongly prime if and only if N is a $*$-primary submodule with $\frac{R}{(N:M)}$ a reduced ring (Proposition 3.1). Moreover, we show that over a cyclic R-module the $*$-primary submodules are precisely primary submodules (Theorem 3.2). Finally, we investigate $*$-primary submodules of a free R-module M of rank 2 (Proposition 3.7 and Theorem 3.9).

2. $*$-PRIMARY SUBMODULES

In this section we obtain necessary and sufficient conditions under which a submodule N of M is $*$-primary.

For a submodule N of an R-module M set

$$Z_N(M) = \{y \in M \mid (Rx + N : M)y \subseteq N \text{ for some } x \in M \setminus N\}.$$

Proposition 2.1. *Let M be an R-module and N a proper submodule of M. The following are equivalent:*

(1) *N is a $*$-primary submodule of M.*
(2) *For every submodule L of M and $x \in M$, if $(Rx + N : M)L \subseteq N$ then $L \subseteq N$ or $(Rx + N : M)^n \subseteq (N : M)$ for some $n \in \mathbb{N}$.*
(3) *For each $y \in Z_N(M) \setminus N$ and $x \in M \setminus N$ such that $(Rx + N : M)y \subseteq N$, there exists an $n \in \mathbb{N}$ such that $(Rx + N : M)^n(M) \subseteq N$.*
(4) *For each $x \in M$, either $\frac{(Rx+N:M)}{(N:M)}$ is a nilpotent ideal or $(N : (Rx + N : M)) = N$.*
(5) *$\frac{M}{N}$ is a $*$-primary R-module.*

Proof. $(1 \Rightarrow 2)$ Let $(Rx + N : M)L \subseteq N$ and $L \nsubseteq N$. Then there exists $l \in L \setminus N$ such that $(Rx + N : M)l \subseteq N$. Now by (1), we have $(Rx + N : M)^n \subseteq (N : M)$ for some $n \in \mathbb{N}$.

$(2 \Rightarrow 3)$ Assume that $y \in Z_N(M) \setminus N$, $x \in M \setminus N$ and $(Rx + N : M)y \subseteq N$. Set $L = Ry$. By (2) there exists a positive integer $n \in \mathbb{N}$ such that $(Rx + N : M)^n \subseteq (N : M)$ or, equivalently, $(Rx + N : M)^n M \subseteq N$.

$(3 \Rightarrow 4)$ For $x \in N$ it is clear that $\frac{(Rx+N:M)}{(N:M)}$ is a nilpotent ideal. Assume that $x \in M \setminus N$ and $(N :_M (Rx + N : M)) \nsubseteq N$. Then there exists $y \in (N :_M (Rx + N : M)) \setminus N$. Therefore $(Rx + N : M)y \subseteq N$, and hence $y \in Z_N(M) \setminus N$. By (3), $(Rx + N : M)^n \subseteq (N : M)$ for some $n \in \mathbb{N}$, which implies that $\frac{(Rx+N:M)}{(N:M)}$ is a nilpotent ideal.

$(4 \Rightarrow 1)$ Suppose that $y, x \in M$ are such that $(Rx + N : M)y \subseteq N$ and $y \notin N$. By hypothesis, $\frac{(Rx+N:M)}{(N:M)}$ is a nilpotent ideal, and hence for some positive integer n, $(Rx + N : M)^n \subseteq (N : M)$.

$(1 \Rightarrow 5)$ Let $\left(\frac{Rx+N}{N} : \frac{M}{N}\right)(y + N) = 0$. Since $\frac{\frac{M}{N}}{\frac{Rx+N}{N}} \cong \frac{M}{Rx+N}$, $(Rx + N : M)(y + N) = 0$. Then $(Rx + N : M)y \subseteq N$. By (1), we get $y \in N$ or $(Rx + N : M)^n \subseteq (N : M)$ for some $n \in \mathbb{N}$. So $y + N = 0$ or $\left(\frac{Rx+N}{N} : \frac{M}{N}\right)^n \subseteq (0 : \frac{M}{N})$.

$(5 \Rightarrow 1)$ Let $(Rx + N : M)y \subseteq N$. Then $\left(\frac{Rx+N}{N} : \frac{M}{N}\right)(y + N) = 0$. By (5) we have $y + N = N$ or $\left(\frac{Rx+N}{N} : \frac{M}{N}\right)^n \subseteq (0 : \frac{M}{N})$. Hence $y \in N$ or $(Rx + N : M)^n \subseteq (N : M)$. □

Proposition 2.2. *Let M be an R-module and N a submodule of M with $Z_N(M) \neq M$. The following statements hold:*

1. *If $(N : M)$ is a primary ideal of a Noetherian ring R, then N is a $*$-primary submodule.*
2. *$Z_N(M) = N$ if and only if $(N : M)$ is a semiprime ideal of R and N is a $*$-primary submodule.*

Proof. (1) Let $(Rx + N : M)y \subseteq N$, $y \notin N$ and $(Rx + N : M)^n \nsubseteq (N : M)$ for every $n \in \mathbb{N}$. We have $(Rx+N : M)(Ry+N : M)M \subseteq (Rx+N : M)(Ry+N) \subseteq N$. Then $(Rx + N : M)(Ry + N : M) \subseteq (N : M)$. Since $(N : M)$ is a primary ideal of R, we have $(Ry + N : M) \subseteq (N : M)$. So $(Ry + N : M)M \subseteq N$ and this is a contradiction.

(2) Let $Z_N(M) = N$ and $(Rx + N : M)y \subseteq N$, where $x, y \in M$. If $x \in N$, we are done. Otherwise, $y \in Z_N(M) = N$. Consequently, suppose that $Z_N(M) \neq N$. There exists $y \in Z_N(M)$ such that $y \notin N$. By definition of $Z_N(M)$ there exists $x \in M \setminus N$ such that $(Rx + N : M)y \subseteq N$. Then $(Rx + N : M)^n \subseteq (N : M)$ and semiprimeness of $(N : M)$ implies that $(Rx + N : M)M \subseteq N$. Therefore $Z_N(M) = M$ and this is a contradiction. □

Proposition 2.3. *Let M be an R-module and m be a maximal ideal of R. Then m^nM is a $*$-primary submodule of M ($m^nM \neq M$, $n \in \mathbb{N}$).*

Proof. Let $(Rx + m^nM : M)y \subseteq m^nM$. If $(Rx + m^nM : M) \subseteq m$, then $(Rx + m^nM : M)^n M \subseteq m^nM$. So $(Rx + m^nM : M)^n \subseteq (m^nM : M)$. If $(Rx + m^nM : M) \nsubseteq m$, then there exists $a \in (Rx + m^nM : M)$ such that $a \notin m$. So $ar + b = 1$, $b \in m$. Hence $1 = 1^n = (ar + b)^n = b^n + sa$ for some $s \in R$. Thus $y = b^n y + say$, and therefore $y \in m^nM$. □

In the next proposition we will consider the basic properties of $*$-primary submodules under module homomorphisms.

Proposition 2.4. *Let M and M' be R-modules, $K \subseteq N \subseteq M$ and $f : M \to M'$ an epimorphism. The following statements hold:*

1. *If N' is a $*$-primary submodule of M', then $f^{-1}(N')$ is a $*$-primary submodule of M. ($f(M) \nsubseteq N'$).*
2. *If N is a $*$-primary submodule of M with $\ker f \subseteq N$, then $f(N)$ is a $*$-primary submodule of M'.*
3. *N is a $*$-primary submodule of M if and only if $\frac{N}{K}$ is a $*$-primary submodule of $\frac{M}{K}$.*

Proof. (1) Let $(Rx + f^{-1}(N') : M)y \subseteq f^{-1}(N')$, where $x, y \in M$. We claim that $(f(Rx) + N' : M')f(y) \subseteq N'$. Let $r \in (f(Rx) + N' : M')$. Then $f(rM) \subseteq f(Rx)+N'$. So for every $m_0 \in M$ there exists $r_0 \in R$ such that $f(rm_0 - r_0x) \in N'$. Hence $rm_0 \in f^{-1}(N') + Rx$. Therefore $rM \subseteq f^{-1}(N') + Rx$. So $ry \in f^{-1}(N')$, $rf(y) \in N'$, and hence $(f(Rx)+N : M)y \subseteq N$. Since N' is a $*$-primary submodule of M', we have that $f(y) \in N'$ or $(f(Rx) + N' : M')^n \subseteq (N' : M')$ for some $n \in \mathbb{N}$. Hence $y \in f^{-1}(N')$ or $(Rx + f^{-1}(N') : M)^n \subseteq (f^{-1}(N') : M)$.

(2) Let $(Rm'_1 + f(N) : M')m'_2 \subseteq f(N)$, where $m'_1, m'_2 \in M'$. Then there exist $m_1, m_2 \in M$ such that $m'_1 = f(m_1)$, $m'_2 = f(m_2)$. We claim that $(Rm_1 + N : M)m_2 \subseteq N$. Let $r \in (Rm_1 + N : M)$; then $rf(M) \subseteq Rf(m_1) + f(N)$. By assumption, $rf(m_2) \in f(N)$. So $f(rm_2 - n_0) = 0$ for some $n_0 \in \mathbb{N}$ and hence $rm_2 \in N$. Since N is a $*$-primary submodule, we have that $m_2 \in N$ or $(Rm_1 + N : M)^n \subseteq (N : M)$ for some $n \in \mathbb{N}$. So $m'_2 = f(m_2) \in f(N)$ or $(Rf(m_1) + f(N) : M')^n \subseteq (f(N) : f(M))$ for some $n \in \mathbb{N}$.

(3) It is clear by (1) and (2). □

Corollary 2.5. *Let M, N, M_1 and M_2 be R-modules.*

1. *If $f : M \to N$ is an epimorphism and $\mathrm{Rad}(M)$ a $*$-primary submodule, such that $\ker f \subseteq \mathrm{Rad}(M)$ and $\frac{M}{\mathrm{Rad}(M)}$ is a semisimple R-module, then $\mathrm{Rad}(N)$ is a $*$-primary submodule of N.*
2. *Let N_1, N_2 be submodules of M_1, M_2. Then N_1, N_2 are $*$-primary if and only if $N_1 \oplus M_2$ and $M_1 \oplus N_2$ are $*$-primary submodules of $M_1 \oplus M_2$.*

In Proposition 2.4 the surjectivity of f is necessary. For example, consider the homomorphism $\varphi : \mathbb{Z} \to \mathbb{Q}$ via $\varphi(x) = x$. The submodule $6\mathbb{Z}$ is a $*$-primary submodule in $\mathbb{Q}$, but it is not $*$-primary in $\mathbb{Z}$. For the homomorphism $f : \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}$ via $f(x) = (x, 0)$, we have $2\mathbb{Z} \subsetneqq \mathbb{Z}$ is a $*$-primary submodule but $f(2\mathbb{Z}) = 2\mathbb{Z} \oplus 0 \subsetneqq \mathbb{Z} \oplus \mathbb{Z}$ is not a $*$-primary submodule. Also the condition $\ker\varphi \subseteq N$ is necessary. For example, for the surjective homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}_6$, the zero submodule of $\mathbb{Z}$ is $*$-primary but the submodule $\varphi(0) = \overline{0} \subsetneqq \mathbb{Z}_6$ is not $*$-primary.

If $N_1 \subsetneqq M_1$ and $N_2 \subsetneqq M_2$ are $*$-primary, then we cannot always say that $N_1 \oplus N_2 \subsetneqq M_1 \oplus M_2$ is a $*$-primary submodule. For example, for every prime number $p \in \mathbb{Z}$, the submodule $p\mathbb{Z} \oplus 0 \subsetneqq \mathbb{Z} \oplus \mathbb{Z}$ is not a $*$-primary submodule.

Fact 2.6. Let M be an R-module and N_1 and N_2 submodules of M.

1. If the intersection of two submodules is a $*$-primary submodule, then not all of them are necessarily $*$-primary; consider for example $M = \mathbb{Z} \oplus \mathbb{Z}$, $N_1 = 0 \oplus 2\mathbb{Z}$, $N_2 = 2\mathbb{Z} \oplus 0$. Also, the intersection of two $*$-primary submodules is not necessarily $*$-primary; for example, take $\mathbb{Z}_6$ as a $\mathbb{Z}$-module. $\langle\overline{2}\rangle$, $\langle\overline{3}\rangle$ are $*$-primary. But $\langle\overline{2}\rangle \cap \langle\overline{3}\rangle = \langle\overline{0}\rangle$ is not a $*$-primary submodule.
2. The property of being $*$-primary in submodules of M is not preserved under isomorphism (for example, $M = \mathbb{Z}$, $N_1 = 6\mathbb{Z}$, $N_2 = 2\mathbb{Z}$).

Proposition 2.7. *Let M be an R-module such that for submodules A, B and C of M we have $A + (B \cap C) = (A + B) \cap (A + C)$; N a $*$-primary submodule of M; and K a submodule of M such that $K \nsubseteq N$ and $(Rx + N : K) = (Rx + N : M)$ for every $x \in M$. Then $K \cap N$ is a $*$-primary submodule of K.*

Proof. Let $(Rx + (K \cap N) : K)y \subseteq K \cap N$, where $x, y \in K$ and $y \notin K \cap N$. Then $y \notin N$. We claim that $(Rx + N : M)y \subseteq N$. Let $r \in (Rx + N : M)$; then $rK \subseteq Rx + N$. So $rK \subseteq (Rx + N) \cap (Rx + K) = Rx + (K \cap N)$. Hence $ry \in N$. Since N is a $*$-primary submodule, $(Rx + N : M)^n \subseteq (N : M)$ for some $n \in \mathbb{N}$. So $(Rx + N : M)^n \subseteq (N \cap K : K)$, and thus $(Rx + (K \cap N) : K)^n \subseteq (N \cap K : K)$. □

In what follows, $*$-primary submodules of a Noetherian ring are investigated.

Proposition 2.8. *Let R be a Noetherian ring and M an R-module. The following statements hold:*

(1) *Primary submodules of M are $*$-primary.*
(2) *For a submodule N of M, if $\sqrt{(N:M)}$ is a maximal ideal of R, then N is a $*$-primary submodule of M.*

Proof. (1) Suppose $x, y \in M$ and $(Rx + N : M)y \subseteq N$. Since N is a primary submodule, $y \in N$ or $\langle x_1, x_2, \ldots, x_n \rangle = (Rx + N : M) \subseteq \sqrt{(N : M)}$. Then there exist $k_1, k_2, \ldots, k_n \in \mathbb{N}$ such that $x_1^{k_1} \in (N : M)$, $\ldots$, $x_n^{k_n} \in (N : M)$. Then $(xR + N : M)^k \subseteq (N : M)$ for some $k \in \mathbb{N}$.

(2) It follows from (1). □

In general, the converse of Proposition 2.8 (1) is not true (see Example 1.2).

Corollary 2.9. *Let M be a Noetherian R-module. Then every primary submodule of M is $*$-primary.*

For an R-module M and a submodule N of M, if $(N : M)$ is a maximal ideal then it is clear that N is $*$-primary. The following example shows that in general if $(N : M)$ is a prime ideal of R, we cannot expect N to be a $*$-primary submodule of M.

Example 2.10. Let $M = \mathbb{Z} \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module and $N = 2\mathbb{Z} \oplus 0$. The ideal $(N : M) = 0$ is a prime ideal of $\mathbb{Z}$ but N is not a $*$-primary submodule. For $(\mathbb{Z}(2,2)+2\mathbb{Z}\oplus 0 : \mathbb{Z}\oplus\mathbb{Z})(1,0) \subseteq 2\mathbb{Z}\oplus 0$, $(1,0) \notin 2\mathbb{Z}\oplus 0$ and powers of $(\mathbb{Z}(2,2)+2\mathbb{Z}\oplus 0 : \mathbb{Z} \oplus \mathbb{Z})$ are nonzero.

A nonzero module is a compressible module if it can be embedded in each of its nonzero submodules.

Theorem 2.11. *Let R be an integral domain, M an R-module and N a proper submodule of M. If $\frac{M}{N}$ is a torsion free or compressible module, then N is a $*$-prime ($*$-primary) submodule. Moreover, $T(M) \subsetneqq M$ is a $*$-primary submodule.*

Proof. Let L be a submodule of M and $x \in M$ such that $(Rx + N : M)L \subseteq N$ and $L \nsubseteq N$. Then $\left(\frac{Rx+N}{N} : \frac{M}{N}\right)\left(\frac{L}{N}\right) = N$. If $\frac{M}{N}$ is torsion free, then $\left(\frac{Rx+N}{N} : \frac{M}{N}\right) = 0$. Hence $(Rx + N : M) = 0 \subseteq (N : M)$. If $\frac{M}{N}$ is compressible, there exists a monomorphism $f : \frac{M}{N} \to \frac{L}{N}$. Now $(Rx + N : M)f\left(\frac{M}{N}\right) \subseteq N$ implies that $(Rx + N : M)M \subseteq N$. Thus $(Rx + N : M) \subseteq (N : M)$. □

Corollary 2.12. *Let R be an integral domain and M a torsion free R-module.*

(1) *Every direct summand of M is $*$-primary.*
(2) *For every maximal ideal m of R, the R-module M_m is $*$-primary as an R_m-module.*

3. $*$-PRIMARY SUBMODULES IN SOME FINITELY GENERATED MODULES

In this section we characterize $*$-primary submodules in cyclic modules. Also we investigate $*$-primary submodules in the free R-module $R \oplus R$.

Proposition 3.1. *Let M be a cyclic R-module and N a submodule of M. The following are equivalent:*

(1) *N is a $*$-primary submodule and $\frac{R}{(N:M)}$ is a reduced ring.*
(2) *N is a $*$-primary submodule and $(N : M)$ is a semiprime ideal of R.*
(3) *N is a strongly prime submodule of M.*

Proof. ($3 \Rightarrow 1$) Let $M = Rx$ and let N be a strongly prime submodule of M. Let $a + (N : M) \neq (N : M)$ and $n \in \mathbb{N}$ the smallest natural number such that $(a + (N : M))^n \subseteq (N : M)$. Then there exists $r_0 \in R$ such that $ar_0x \notin N$. We have $(Ra^{n-1}x + N : M)ar_0x \subseteq N$. Since N is strongly prime, $a^{n-1}x \in N$. So $\langle a^{n-1}x\rangle \subseteq N$ and this is a contradiction.

($1 \Rightarrow 2$) and ($2 \Rightarrow 3$) are clear. □

Theorem 3.2. *Let M be a cyclic R-module and N a proper submodule of M. The following are equivalent:*

(1) *N is a primary submodule.*
(2) *N is a $*$-primary submodule.*
(3) *$(N : M)$ is a primary ideal.*

Proof. ($2 \Rightarrow 3$) Let $M = Rx$, $ab \in (N : M)$, $aM \nsubseteq N$ and $b^nM \nsubseteq N$ for every $n \in \mathbb{N}$. Then there exist $r_0, r_1 \in R$ such that $ar_0x \notin N$, $b^nr_1x \notin N$. We have $(Rbm + N : M)ar_0x \subseteq N$ for every $m \in M$. Since N is a $*$-primary submodule and $ar_0x \notin N$, $(Rbm + N : M)^k \subseteq (N : M)$ for some $k \in \mathbb{N}$. So $(Rbx + N : M)^k bx \subseteq N$. Therefore $b^kbx \in N$ and this is a contradiction.

($3 \Rightarrow 2$) Let $(Ry + N : M)z \subseteq N$ and $z \notin N$. Then there exist $r, s \in R$ such that $(Rrx + N : Rx)sx \subseteq N$. Hence$(Rrx + N : Rx)s \subseteq (N : x) = (N : Rx) = (N : M)$. Since $(N : M)$ is primary, $(Rrx + N : Rx) \subseteq \sqrt{(N : M)}$ and $r^k \in (N : M)$ for some $k \in \mathbb{N}$. Let $t_1t_2 \cdots t_k \in (Rrx + N : Rx)^k$. Then $t_1t_2 \cdots t_{k-1}t_kRx \subseteq t_1t_2 \cdots t_{k-1}(Rrx+N) \subseteq rt_1t_2 \cdots t_{k-1}(Rx+N) \subseteq r^2t_1t_2 \cdots t_{k-2}(Rrx+N) \subseteq \cdots \subseteq r^kRrx + N \subseteq N$.

($3 \Leftrightarrow 1$) Let N be a submodule of $M = Rm$. Then for every $n \in N$, $n = rm$, we have $r \in (N : M)$ and hence $N = (N : M)M$, which implies that every cyclic module is multiplication, and we are done. □

Corollary 3.3. *Let M be a cyclic R-module. Then M is $*$-primary if and only if $\operatorname{ann}(M)$ is a primary ideal of R.*

Fact 3.4. Let M be a finitely generated $\mathbb{Z}$-module. Then $M \cong \mathbb{Z}_{p_1^{\alpha_1}} \oplus \mathbb{Z}_{p_2^{\alpha_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{\alpha_k}} \oplus \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n \text{ times}}$, where $p_1, \ldots, p_k$ are prime numbers and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers. Obviously $\underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n \text{ times}}$ is $*$-primary and $\mathbb{Z}_{p_i^{\alpha_i}}$ are fully $*$-primary for each i.

By means of the following proposition, which has an essential role in the remainder of this section, we can determine some $*$-primary submodules of a finitely generated free R-module F. We give its proof for the sake of completeness.

Proposition 3.5 ([10, Proposition 2.3])**.** *Let R be a domain and let $\{a_i\}_{i=1}^n \subseteq R$ be such that $R = Ra_1 + Ra_2 + \cdots + Ra_n$. Then $R(a_1, \ldots, a_n)$ is a direct summand of the free R-module $F = R^n$.*

Proof. $R = Ra_1 + Ra_2 + \cdots + Ra_n$. Then there exist $s_1, s_2, \ldots, s_n \in R$ such that $1 = s_1a_1 + s_2a_2 + \cdots + s_na_n$. Let $N = \{(x_1, x_2, \ldots, x_n) \in F \mid s_1x_1 + s_2x_2 + \cdots + s_nx_n = 0\}$. Consider the functions $f : R \to R^n$ defined by $f(r) = r(a_1, \ldots, a_n)$ and $g : R^n \to R$ defined by $g(r_1, \ldots, r_n) = s_1r_1 + s_2r_2 + \cdots + s_nr_n$. The homomorphism $(g \circ f)$ is the identity. Then $R^{(n)} = \operatorname{Im} f \oplus \ker g = R(a_1, \ldots, a_n) \oplus N$. □

In the following, we will study $*$-primary submodules in some finitely generated free modules; we get the same results obtained by Pusat-Yılmaz for prime submodules in [10].

Proposition 3.6. *Let R be a commutative ring and $F = R^{(n)}$ a finitely generated free R-module. The following statements hold:*

(1) *If R is a domain and $\{a_i\}_{i=1}^n \subseteq R$ such that $R = Ra_1 + Ra_2 + \cdots + Ra_n$, then $R(a_1, \ldots, a_n)$ is a $*$-primary submodule of F.*

(2) *Let $\{c_1, c_2, \ldots, c_n\} \subseteq F$ and $A = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$. If $R\det(A)$ is a maximal ideal of R, then $N = Rc_1 + Rc_2 + \cdots + Rc_n$ is a $*$-primary submodule of F.*

Proof. (1) By Proposition 3.5, we have that $\frac{F}{R(a_1,\ldots,a_n)} \cong N$ is a torsion free module; then by Theorem 2.11 the submodule $R(a_1, \ldots, a_n)$ is $*$-primary.

(2) By [10, Proposition 3.7] $R\det(A) \subseteq (N : F) \subseteq \sqrt{R\det(A)}$. Since $R\det(A)$ is a maximal ideal of R, $(N : M)$ is too, and therefore N is $*$-primary. □

Proposition 3.7. *Let R be a commutative ring and let $a_i, b_i \in R$ $(i = 1, 2)$ be such that $R = Rb_1 + Rb_2$. Then $R(a_1, a_2) + R(b_1, b_2)$ is a $*$-primary submodule of $F = R \oplus R$ if and only if $R(a_1b_2 - a_2b_1)$ is a primary ideal of R.*

Proof. There exist elements $s_1, s_2 \in R$ such that $1 = s_1b_1 + s_2b_2$. Then by Proposition 3.5, $F = L \oplus L'$, where $L = R(b_1, b_2)$ and $L' = \{(x, y) \in F \mid s_1x + s_2y = 0\}$. We have $R(-s_2, s_1) \subseteq L'$. Now $F = L + R(-s_2, s_1)$. It follows that $L' = (L \cap L') + R(-s_2, s_1) = R(-s_2, s_1)$. Now $F = L \oplus L'$ and $N = L \oplus (N \cap L')$ give that $\frac{F}{N} \cong \frac{L'}{N \cap L'} = \frac{R(-s_2,s_1)}{Rd(-s_2,s_1)} \cong \frac{R}{Rd}$. Thus by Proposition 2.4, N is a $*$-primary submodule of F if and only if Rd is a primary ideal of R. □

In the above proposition the condition that $\{b_1, b_2\}$ is a spanning set of R is a necessary condition. For, set $F = \mathbb{Z}\oplus\mathbb{Z}$, $N = \mathbb{Z}(6,6)+\mathbb{Z}(15,15)$; then $(\mathbb{Z}(1,2)+N : \mathbb{Z}\oplus\mathbb{Z})(1,1) \subseteq N$, $(1,1) \notin N$ and $3^n \neq 0$ for every $n \in \mathbb{N}$.

Corollary 3.8. *In the $\mathbb{Z}$-module $\mathbb{Z}\oplus\mathbb{Z}$, the submodule $\mathbb{Z}(a_1, a_2)$ such that a_1, a_2 are coprime is a $*$-primary submodule.*

The converse of Proposition 3.6 (2) is generally not true. For consider $F = \mathbb{Z}\oplus\mathbb{Z}$ and $N = \mathbb{Z}(2,6)+\mathbb{Z}(1,3)$. By Proposition 3.7, the submodule N of F is a $*$-primary submodule but the zero ideal is not maximal.

Theorem 3.9. *Let R be a Noetherian domain and $R(\det A)$ a nonzero primary ideal of R, where $A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$. Then $N = R(a_1, a_2) + R(b_1, b_2)$ is a $*$-primary submodule of $R \oplus R$.*

Proof. Let $(R(x_1, x_2) + N : R \oplus R)(y_1, y_2) \subseteq N$ and $(R(x_1, x_2) + N : R \oplus R)^n \nsubseteq (N : M)$ for every $n \in \mathbb{N}$. Since R is a Noetherian ring, there exists $r \in (R(x_1, x_2)+N : R\oplus R)$ such that $r^k \notin (N : M)$ for every $k \in \mathbb{N}$ and $r(y_1, y_2) \in N$. Then $r(y_1, y_2) = s_1(a_1, a_2) + s_2(b_1, b_2)$ for some elements $s_i \in R$ $(i = 1, 2)$. In matrix notation, we have $r(y_1, y_2) = (s_1, s_2)\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = (s_1, s_2)A$. Then $r(y_1, y_2) \operatorname{adj} A = (s_1, s_2)A \operatorname{adj} A = (s_1, s_2) \det A$. Let $\operatorname{adj} A = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. Then $r(y_1, y_2)\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = (s_1, s_2)D$, where $D = \det A$. So $r(y_1 b_{1j} + y_2 b_{2j}) = Ds_j \in \langle D\rangle$ for $j = 1, 2$. Since $\langle D\rangle$ is a primary ideal of R, $(y_1, y_2)B = D(t_1, t_2)$. Therefore $(y_1, y_2) = t_1(a_1, a_2) + t_2(b_1, b_2) \in N$. ☐

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