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COFINITENESS OF LOCAL COHOMOLOGY MODULES IN THE CLASS OF MODULES IN DIMENSION LESS THAN A FIXED INTEGER

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Abstract. Let n be a non-negative integer, R a commutative Noetherian ring with $\dim(R) \leq n+2$, $\mathfrak{a}$ an ideal of R, and X an arbitrary R-module. In this paper, we first prove that X is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module if X is an $\mathfrak{a}$-torsion R-module such that $\mathrm{Hom}_R\left(\frac{R}{\mathfrak{a}}, X\right)$ and $\mathrm{Ext}^1_R\left(\frac{R}{\mathfrak{a}}, X\right)$ are $\mathrm{FD}_{<n}$ R-modules. Then, we show that $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module and $\{\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) : \dim\left(\frac{R}{\mathfrak{p}}\right) \geq n\}$ is a finite set for all i when $\mathrm{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq n+2$. As a consequence, it follows that $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X))$ is a finite set for all i whenever R is a semi-local ring with $\dim(R) \leq 3$ and X is an $\mathrm{FD}_{<1}$ R-module. Finally, we observe that the category of $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules forms an Abelian subcategory of the category of R-modules.

1. Introduction

We adopt throughout the following notation: let R denote a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ and $\mathfrak{b}$ ideals of R, M a finite (i.e., finitely generated) R-module, X an arbitrary R-module which is not necessarily finite, and n a non-negative integer. We refer the reader to [7, 8, 23] for basic results, notations, and terminology not given in this paper.

Hartshorne, in [14], defined an $\mathfrak{a}$-torsion R-module X to be $\mathfrak{a}$-*cofinite* if the R-module $\mathrm{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ is finite for all i, and asked the following questions.

Question 1.1. Does the category of $\mathfrak{a}$-cofinite R-modules form an Abelian subcategory of the category of R-modules?

Question 1.2. Is $\mathrm{H}^i_{\mathfrak{a}}(M)$ an $\mathfrak{a}$-cofinite R-module for all i?

The following question is also an important problem in local cohomology [16, Problem 4].

Question 1.3. Is $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M))$ a finite set for all i?

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There have been many attempts in the literature to study the above questions. Hartshorne in [14, Proposition 7.6 and Corollary 7.7] showed that the answer to these questions is yes if R is a complete regular local ring and $\mathfrak{a}$ is a prime ideal of R with $\dim\left(\frac{R}{\mathfrak{a}}\right) \leq 1$. Huneke and Koh in [17, Theorem 4.1] and Delfino in [10, Theorem 3] extended Hartshorne's result [14, Corollary 7.7] and provided affirmative answers to Questions 1.2 and 1.3 in more general local rings R and one-dimensional ideals $\mathfrak{a}$. Delfino and Marley in [11, Theorems 1 and 2], Yoshida in [25, Theorem 1.1], Chiriacescu in [9, Theorem 1.4], and Kawasaki in [18, Theorems 1 and 8] showed that the answer to Questions 1.1–1.3 is yes if R is an arbitrary local ring and $\mathfrak{a}$ is an arbitrary ideal of R with $\dim\left(\frac{R}{\mathfrak{a}}\right) \leq 1$. Finally, in [21, Theorems 7.4 and 7.10] and [22, Theorems 2.6 and 2.10], Melkersson provided affirmative answers to these questions for the case that R is an arbitrary ring and either $\dim(R) \leq 2$ or $\mathfrak{a}$ is an arbitrary ideal of R with $\dim\left(\frac{R}{\mathfrak{a}}\right) \leq 1$.

Recall that X is said to be an $\mathrm{FD}_{<n}$ (or *in dimension* $< n$) *R-module* if there is a finite submodule Y of X such that $\dim_R\left(\frac{X}{Y}\right) < n$ [2, 4]. From [26, Theorem 2.3], the class of $\mathrm{FD}_{<n}$ R-modules is closed under taking submodules, quotients, and extensions. We say that X is an $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite R-module* if X is an $\mathfrak{a}$-torsion R-module and $\mathrm{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ is an $\mathrm{FD}_{<n}$ R-module for all i [3, Definition 4.1]. Note that X is an $\mathfrak{a}$-cofinite R-module if and only if X is an $(\mathrm{FD}_{<0}, \mathfrak{a})$-cofinite R-module. Thus, as generalizations of Questions 1.1–1.3, we have the following questions (see [1, Question] and [24, Questions 1.5, 1.6, and 1.8]). Here, the set $\{\mathfrak{p} \in \mathrm{Ass}_R(X) : \dim\left(\frac{R}{\mathfrak{p}}\right) \geq n\}$ is denoted by $\mathrm{Ass}_R(X)_{\geq n}$.

Question 1.4. Does the category of $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules form an Abelian subcategory of the category of R-modules?

Question 1.5. Is $\mathrm{H}^i_{\mathfrak{a}}(M)$ an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all i?

Question 1.6. Is $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M))_{\geq n}$ a finite set for all i?

If R is a complete local ring with $\dim\left(\frac{R}{\mathfrak{a}}\right) \leq n+1$, then the answer to Questions 1.5 and 1.6 is yes from [1, Theorems 2.5 and 2.10]. In [24, Corollaries 3.3 and 4.5], the first author and Morsali removed the complete local assumption on R and provided affirmative answers to Questions 1.4–1.6 for the case that $\dim\left(\frac{R}{\mathfrak{a}}\right) \leq n+1$, which are generalizations of Melkersson's results [22, Theorems 2.6 and 2.10]. In this paper, as generalizations of Melkersson's results [21, Theorems 7.4 and 7.10], we show that the answer to Questions 1.4–1.6 is also yes if $\dim(R) \leq n+2$. As a consequence, we provide an affirmative answer to Question 1.3 for the case that R is a semi-local ring with $\dim(R) \leq 3$. This result is a generalization of Marley's result in [19] where he showed that the answer to Question 1.3 is yes if R is a local ring with $\dim(R) \leq 3$ (see [19, Proposition 1.1 and Corollary 2.5]).

In the main result of Section 2, we observe that if $\dim(R) \leq n+2$ and X is an $\mathfrak{a}$-torsion R-module such that $\mathrm{Hom}_R\left(\frac{R}{\mathfrak{a}}, X\right)$ and $\mathrm{Ext}^1_R\left(\frac{R}{\mathfrak{a}}, X\right)$ are $\mathrm{FD}_{<n}$ R-modules, then X is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module. Section 3 is devoted to the study of Questions 1.5 and 1.6. We show that $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module and $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X))_{\geq n}$ is a finite set for all i whenever $\dim(R) \leq n+2$ and $\mathrm{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq n+2$ (e.g., X is an $\mathrm{FD}_{<n}$ R-module).

It follows that if R is a semi-local ring with $\dim(R) \leq 3$ and $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ is an $\mathrm{FD}_{<1}$ R-module for all $i \leq 3$ (e.g., X is an $\mathrm{FD}_{<1}$ R-module), then $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $\mathfrak{a}$-weakly cofinite R-module and $\operatorname{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X))$ is a finite set for all i. Recall that X is said to be an $\mathfrak{a}$-*weakly cofinite R-module* if X is an $\mathfrak{a}$-torsion R-module and the set of associated prime ideals of any quotient module of $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ is finite for all i (see [12, Definition 2.1] and [13, Definition 2.4]). In Section 4, with respect to Question 1.4, we prove that when $\dim(R) \leq n+2$, the category of $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules forms an Abelian subcategory of the category of R-modules.

2. A criterion for cofiniteness

The following two lemmas will be useful in the proof of the main result of this section. Note that when $\mathfrak{b}X = 0$, X is an $\mathrm{FD}_{<n}$ R-module if and only if X is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module.

Lemma 2.1. *Let t be a non-negative integer and let X be an R-module such that $\mathfrak{b}X = 0$ and $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq t$. Then $\operatorname{Ext}^i_{\frac{R}{\mathfrak{b}}}\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module for all $i \leq t$.*

Proof. We prove this by using induction on t. The case $t = 0$ is clear from the isomorphisms

$$\operatorname{Hom}_{\frac{R}{\mathfrak{b}}}\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right) \cong \left(0 :_X \frac{\mathfrak{a}+\mathfrak{b}}{\mathfrak{b}}\right) \cong (0 :_X \mathfrak{a}+\mathfrak{b}) \cong \operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right).$$

Suppose that $t > 0$ and that $t-1$ is settled. It is enough to show that $\operatorname{Ext}^t_{\frac{R}{\mathfrak{b}}}\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module, since $\operatorname{Ext}^i_{\frac{R}{\mathfrak{b}}}\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module for all $i \leq t-1$ by the induction hypothesis on $t-1$. From [23, Theorem 11.65], there is a spectral sequence

$$\mathrm{E}_2^{p,q} := \operatorname{Ext}^p_{\frac{R}{\mathfrak{b}}}\left(\operatorname{Tor}^R_q\left(\frac{R}{\mathfrak{b}}, \frac{R}{\mathfrak{a}+\mathfrak{b}}\right), X\right) \underset{p}{\Longrightarrow} \operatorname{Ext}^{p+q}_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right).$$

Let $r \geq 2$ and set $B_r^{t,0} := \operatorname{Im}(\mathrm{E}_r^{t-r,r-1} \longrightarrow \mathrm{E}_r^{t,0})$. Then $B_r^{t,0}$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module because $\mathrm{E}_r^{t-r,r-1}$ is a subquotient of $\mathrm{E}_2^{t-r,r-1}$ that is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module by the induction hypothesis and [15, Proposition 3.4]. Thus, from the short exact sequence

$$0 \longrightarrow B_r^{t,0} \longrightarrow \mathrm{E}_r^{t,0} \longrightarrow \mathrm{E}_{r+1}^{t,0} \longrightarrow 0,$$

$\mathrm{E}_r^{t,0}$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module whenever $\mathrm{E}_{r+1}^{t,0}$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module. There exists a finite filtration

$$0 = \phi^{t+1}H^t \subseteq \phi^t H^t \subseteq \cdots \subseteq \phi^1 H^t \subseteq \phi^0 H^t = \operatorname{Ext}^t_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$$

such that $\mathrm{E}_\infty^{t-i,i} \cong \frac{\phi^{t-i}H^t}{\phi^{t-i+1}H^t}$ for all i, $0 \leq i \leq t$. By assumption, $\operatorname{Ext}^t_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$ is an R-module. Thus, as we noted at the beginning of this section, $\operatorname{Ext}^t_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module and hence $\phi^t H^t$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module. Therefore $\mathrm{E}_\infty^{t,0} \cong \frac{\phi^t H^t}{\phi^{t+1}H^t}$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module and so $\mathrm{E}_{t+2}^{t,0}$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module, because $\mathrm{E}_\infty^{t,0} = \mathrm{E}_{t+2}^{t,0}$ as

$\mathrm{E}_j^{t-j,j-1} = 0 = \mathrm{E}_j^{t+j,1-j}$ for all $j \geq t+2$. Thus $\mathrm{E}_2^{t,0} = \mathrm{Ext}^t_{\frac{R}{\mathfrak{b}}}\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module. □

Lemma 2.2. *Let t be a non-negative integer and let X be an R-module such that $\mathfrak{b}X = 0$ and $\mathrm{Ext}^i_{\frac{R}{\mathfrak{b}}}\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module for all $i \leq t$. Then $\mathrm{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq t$.*

Proof. From [23, Theorem 11.65], there is a spectral sequence

$$\mathrm{E}_2^{p,q} := \mathrm{Ext}^p_{\frac{R}{\mathfrak{b}}}\left(\mathrm{Tor}^R_q\left(\frac{R}{\mathfrak{b}}, \frac{R}{\mathfrak{a}}\right), X\right) \underset{p}{\Longrightarrow} \mathrm{Ext}^{p+q}_R\left(\frac{R}{\mathfrak{a}}, X\right).$$

Let $0 \leq j \leq i \leq t$. By [15, Proposition 3.4], $\mathrm{E}_2^{i-j,j}$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module. Hence $\mathrm{E}_\infty^{i-j,j}$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module as $\mathrm{E}_\infty^{i-j,j} = \mathrm{E}_{i+2}^{i-j,j}$ and $\mathrm{E}_{i+2}^{i-j,j}$ is a subquotient of $\mathrm{E}_2^{i-j,j}$. There exists a finite filtration

$$0 = \phi^{i+1}H^i \subseteq \phi^i H^i \subseteq \cdots \subseteq \phi^1 H^i \subseteq \phi^0 H^i = \mathrm{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$$

such that $\mathrm{E}_\infty^{i-j,j} \cong \frac{\phi^{i-j}H^i}{\phi^{i-j+1}H^i}$ for all j, $0 \leq j \leq i$. Now, from the short exact sequences

$$0 \longrightarrow \phi^{i-j+1}H^i \longrightarrow \phi^{i-j}H^i \longrightarrow \mathrm{E}_\infty^{i-j,j} \longrightarrow 0,$$

for all j, $0 \leq j \leq i$, $\mathrm{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module. Therefore $\mathrm{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ is an $\mathrm{FD}_{<n}$ R-module. □

We are now ready to state and prove the main result of this section, which plays an important role in Sections 3 and 4 to study Questions 1.4–1.6.

Theorem 2.3. *Suppose that $\dim(R) \leq n+2$ and X is an $\mathfrak{a}$-torsion R-module such that $\mathrm{Hom}_R\left(\frac{R}{\mathfrak{a}}, X\right)$ and $\mathrm{Ext}^1_R\left(\frac{R}{\mathfrak{a}}, X\right)$ are $\mathrm{FD}_{<n}$ R-modules. Then X is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module.*

Proof. Assume that $\mathfrak{a}$ is nilpotent. Then $\mathfrak{a}^t = 0$ for some integer t. By [15, Proposition 3.4], $\mathrm{Hom}_R\left(\frac{R}{\mathfrak{a}^t}, X\right)$ is an $\mathrm{FD}_{<n}$ R-module and hence $X = (0 :_X \mathfrak{a}^t)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module. Now, assume that $\mathfrak{a}$ is not nilpotent. Since $\Gamma_\mathfrak{a}(R)$ is finite, there is an integer t such that $(0 :_R \mathfrak{a}^t) = \Gamma_\mathfrak{a}(R)$. Set $\mathfrak{b} := (0 :_R \mathfrak{a}^t)$ and $Y := \frac{X}{(0:_X \mathfrak{a}^t)}$. It is easy to see that $\mathfrak{b}Y = 0$, Y is an $(\mathfrak{a}+\mathfrak{b})$-torsion R-module, and $\dim\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}\right) \leq n+1$. Since $(0 :_X \mathfrak{a}^t)$, $\mathrm{Hom}_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$, and $\mathrm{Ext}^1_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$ are $\mathrm{FD}_{<n}$ R-modules from [15, Proposition 3.4], $\mathrm{Hom}_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, Y\right)$ and $\mathrm{Ext}^1_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, Y\right)$ are $\mathrm{FD}_{<n}$ R-modules by the short exact sequence

$$0 \longrightarrow (0 :_X \mathfrak{a}^t) \longrightarrow X \longrightarrow Y \longrightarrow 0.$$

Thus, from [24, Corollary 2.3], $\mathrm{Ext}^i_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, Y\right)$ is an $\mathrm{FD}_{<n}$ R-module for all i. Hence $\mathrm{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, Y\right)$ is an $\mathrm{FD}_{<n}$ R-module for all i by Lemmas 2.1 and 2.2. Therefore X is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module from the above short exact sequence. □

The following corollary is an immediate application of the above theorem.

Corollary 2.4. *Suppose that* $\dim(R) \le n+2$ *and* X *is an arbitrary* R*-module such that* $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *and* $\operatorname{Ext}^1_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *are* $\mathrm{FD}_{<n}$ R*-modules. Then* $\Gamma_{\mathfrak{a}}(X)$ *is an* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite* R*-module.*

Proof. By the short exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(X) \longrightarrow X \longrightarrow \frac{X}{\Gamma_{\mathfrak{a}}(X)} \longrightarrow 0,$$

$\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, \Gamma_{\mathfrak{a}}(X)\right)$ and $\operatorname{Ext}^1_R\left(\frac{R}{\mathfrak{a}}, \Gamma_{\mathfrak{a}}(X)\right)$ are $\mathrm{FD}_{<n}$ R-modules. Thus the assertion follows from Theorem 2.3. □

By putting $n = 0$ in Theorem 2.3 and Corollary 2.4, we have the following results.

Corollary 2.5. *Suppose that* $\dim(R) \le 2$ *and* X *is an* $\mathfrak{a}$*-torsion* R*-module such that* $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *and* $\operatorname{Ext}^1_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *are finite* R*-modules. Then* X *is an* $\mathfrak{a}$*-cofinite* R*-module.*

Corollary 2.6. *Suppose that* $\dim(R) \le 2$ *and* X *is an arbitrary* R*-module such that* $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *and* $\operatorname{Ext}^1_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *are finite* R*-modules. Then* $\Gamma_{\mathfrak{a}}(X)$ *is an* $\mathfrak{a}$*-cofinite* R*-module.*

3. Cofiniteness and associated primes of local cohomology modules

The following is the main result of this section; it shows that the answer to Questions 1.5 and 1.6 is yes if $\dim(R) \le n+2$.

Theorem 3.1. *Suppose that* $\dim(R) \le n+2$ *and* X *is an arbitrary* R*-module. Then the following statements are equivalent:*

(i) $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is an* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite* R*-module for all* i*;*
(ii) $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *is an* $\mathrm{FD}_{<n}$ R*-module for all* i*;*
(iii) $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *is an* $\mathrm{FD}_{<n}$ R*-module for all* $i \le n+2$.

Proof. (i)⇒(ii). This follows by [3, Theorem 2.1].

(iii)⇒(i). We first show that if t is a non-negative integer such that $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \le t+1$, then $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all $i \le t$. We prove this by using induction on t. The case $t = 0$ follows from Corollary 2.4. Suppose that $t > 0$ and that $t-1$ is settled. It is enough to show that $\mathrm{H}^t_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module, because $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all $i \le t-1$ from the induction hypothesis on $t-1$. By [3, Theorem 2.3], $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, \mathrm{H}^t_{\mathfrak{a}}(X)\right)$ and $\operatorname{Ext}^1_R\left(\frac{R}{\mathfrak{a}}, \mathrm{H}^t_{\mathfrak{a}}(X)\right)$ are $\mathrm{FD}_{<n}$ R-modules. Therefore $\mathrm{H}^t_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module from Theorem 2.3. This terminates the induction argument. Thus $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all $i \ne n+2$ from [7, Theorem 6.1.2]. By [3, Theorem 2.3], $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, \mathrm{H}^{n+2}_{\mathfrak{a}}(X)\right)$ is an $\mathrm{FD}_{<n}$ R-module. Also, from [7, Exercise 7.1.7], $\operatorname{Supp}_R(\mathrm{H}^{n+2}_{\mathfrak{a}}(X)) \subseteq \operatorname{Max}(R)$, because each R-module can be viewed as the direct limit of its finite submodules. Thus $\mathrm{H}^{n+2}_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module by [24, Lemma 2.1]. □

Corollary 3.2. *Suppose that* $\dim(R) \leq n+2$*, X is an arbitrary R-module, and t is a non-negative integer such that* $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *is an* $\mathrm{FD}_{<n}$ *R-module for all $i \leq t+1$ (resp. for all $i \leq n+2$). Then* $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is an* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite R-module for all $i \leq t$ (resp. for all i). In particular,* $\operatorname{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X))_{\geq n}$ *is a finite set for all $i \leq t$ (resp. for all i).*

Proof. The first assertion follows from the proof of Theorem 3.1. The last assertion follows by the first one and [8, Exercise 1.2.28]. □

We have the following corollaries by taking $n = 0$ in Theorem 3.1 and Corollary 3.2.

Corollary 3.3 (see [21, Theorem 7.10])**.** *Suppose that* $\dim(R) \leq 2$ *and X is an arbitrary R-module. Then the following statements are equivalent:*

(i) $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is an $\mathfrak{a}$-cofinite R-module for all i;*
(ii) $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *is a finite R-module for all i;*
(iii) $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *is a finite R-module for all $i \leq 2$.*

Corollary 3.4. *Suppose that* $\dim(R) \leq 2$ *and X is an arbitrary R-module such that* $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *is a finite R-module for all $i \leq 2$. Then* $\operatorname{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X))$ *is a finite set for all i.*

If R is a local ring with $\dim\left(\frac{R}{\mathfrak{a}}\right) \leq 2$, then the answer to Question 1.3 is yes by Bahmanpour–Naghipour's result [6, Theorem 3.1] (see also [20, Theorem 3.3(c)]). In [24, Corollary 5.6], the first author and Morsali generalized this result to arbitrary semi-local rings. In the next result, by putting $n = 1$ in Corollary 3.2, we provide an affirmative answer to Question 1.3 for the case that R is a semi-local ring with $\dim(R) \leq 3$. Note that our result is a generalization of Marley's result in [19], where he showed that if R is a local ring with $\dim(R) \leq 3$ and M is a finite R-module, then $\operatorname{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M))$ is a finite set for all i (see [19, Proposition 1.1 and Corollary 2.5]). Note also that, if R is a semi-local ring and X is an $(\mathrm{FD}_{<1}, \mathfrak{a})$-cofinite R-module, then X is an $\mathfrak{a}$-weakly cofinite R-module by [5, Theorem 3.3].

Corollary 3.5. *Suppose that R is a semi-local ring with* $\dim(R) \leq 3$*, X is an arbitrary R-module, and t is a non-negative integer such that* $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *is an* $\mathrm{FD}_{<1}$ *R-module for all $i \leq t+1$ (resp. for all $i \leq 3$). Then* $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is an $\mathfrak{a}$-weakly cofinite R-module for all $i \leq t$ (resp. for all i). In particular,* $\operatorname{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X))$ *is a finite set for all $i \leq t$ (resp. for all i).*

4. Abelianness of the category of cofinite modules

The following theorem is the main result of this section; it shows that the answer to Question 1.4 is also yes if $\dim(R) \leq n+2$.

Theorem 4.1. *If* $\dim(R) \leq n+2$*, then the category of* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite R-modules forms an Abelian subcategory of the category of R-modules.*

Proof. The proof is similar to that of [24, Theorem 3.1]. We bring it here for the sake of completeness. Assume that X and Y are $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules and $f : X \to Y$ is an R-homomorphism. We show that $\ker f$, $\operatorname{im} f$, and $\operatorname{coker} f$ are $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules. From the short exact sequence

$$0 \longrightarrow \operatorname{im} f \longrightarrow Y \longrightarrow \operatorname{coker} f \longrightarrow 0,$$

$\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, \operatorname{im} f\right)$ is an $\mathrm{FD}_{<n}$ R-module. Hence $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, \ker f\right)$ and $\operatorname{Ext}^1_R\left(\frac{R}{\mathfrak{a}}, \ker f\right)$ are $\mathrm{FD}_{<n}$ R-modules by the short exact sequence

$$0 \longrightarrow \ker f \longrightarrow X \longrightarrow \operatorname{im} f \longrightarrow 0.$$

Therefore $\ker f$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module by Theorem 2.3. Thus $\operatorname{im} f$ and $\operatorname{coker} f$ are $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules from the above short exact sequences. ☐

As an immediate application of the above theorem, we have the following corollary.

Corollary 4.2. *Suppose that* $\dim(R) \leq n + 2$*,* N *is a finite* R*-module, and* X *is an* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite* R*-module. Then* $\operatorname{Ext}^j_R(N, X)$ *and* $\operatorname{Tor}^R_j(N, X)$ *are* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite* R*-modules for all* j*.*

We have the following results by taking $n = 0$ in Theorem 4.1 and Corollary 4.2.

Corollary 4.3 (see [21, Theorem 7.4])**.** *If* $\dim(R) \leq 2$*, then the category of* $\mathfrak{a}$*-cofinite* R*-modules forms an Abelian subcategory of the category of* R*-modules.*

Corollary 4.4. *Suppose that* $\dim(R) \leq 2$*,* N *is a finite* R*-module, and* X *is an* $\mathfrak{a}$*-cofinite* R*-module. Then* $\operatorname{Ext}^j_R(N, X)$ *and* $\operatorname{Tor}^R_j(N, X)$ *are* $\mathfrak{a}$*-cofinite* R*-modules for all* j*.*

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References

[1] N. Abazari and K. Bahmanpour, Extension functors of local cohomology modules and Serre categories of modules, *Taiwanese J. Math.* **19** (2015), no. 1, 211–220. MR 3313413.

[2] M. Aghapournahr and K. Bahmanpour, Cofiniteness of weakly Laskerian local cohomology modules, *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* **57(105)** (2014), no. 4, 347–356. MR 3288929.

[3] M. Aghapournahr, A. J. Taherizadeh and A. Vahidi, Extension functors of local cohomology modules, *Bull. Iranian Math. Soc.* **37** (2011), no. 3, 117–134. MR 2901589.

[4] D. Asadollahi and R. Naghipour, Faltings' local-global principle for the finiteness of local cohomology modules, *Comm. Algebra* **43** (2015), no. 3, 953–958. MR 3298115.

[5] K. Bahmanpour, On the category of weakly Laskerian cofinite modules, *Math. Scand.* **115** (2014), no. 1, 62–68. MR 3250048.

[6] K. Bahmanpour and R. Naghipour, Cofiniteness of local cohomology modules for ideals of small dimension, *J. Algebra* **321** (2009), no. 7, 1997–2011. MR 2494753.

[7] M. P. Brodmann and R. Y. Sharp, *Local Cohomology: An Algebraic Introduction with Geometric Applications*, Cambridge Studies in Advanced Mathematics, 60, Cambridge University Press, Cambridge, 1998. MR 1613627.

[8] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, 39, Cambridge University Press, Cambridge, 1993. MR 1251956.

[9] G. Chiriacescu, Cofiniteness of local cohomology modules over regular local rings, *Bull. London Math. Soc.* **32** (2000), no. 1, 1–7. MR 1718769.

[10] D. Delfino, On the cofiniteness of local cohomology modules, *Math. Proc. Cambridge Philos. Soc.* **115** (1994), no. 1, 79–84. MR 1253283.

[11] D. Delfino and T. Marley, Cofinite modules and local cohomology, *J. Pure Appl. Algebra* **121** (1997), no. 1, 45–52. MR 1471123.

[12] K. Divaani-Aazar and A. Mafi, Associated primes of local cohomology modules, *Proc. Amer. Math. Soc.* **133** (2005), no. 3, 655–660. MR 2113911.

[13] K. Divaani-Aazar and A. Mafi, Associated primes of local cohomology modules of weakly Laskerian modules, *Comm. Algebra* **34** (2006), no. 2, 681–690. MR 2211948.

[14] R. Hartshorne, Affine duality and cofiniteness, *Invent. Math.* **9** (1969/70), 145–164. MR 0257096.

[15] S. H. Hassanzadeh and A. Vahidi, On vanishing and cofiniteness of generalized local cohomology modules, *Comm. Algebra* **37** (2009), no. 7, 2290–2299. MR 2536919.

[16] C. Huneke, Problems on local cohomology, in *Free resolutions in commutative algebra and algebraic geometry (Sundance, UT, 1990)*, 93–108, Res. Notes Math., 2, Jones and Bartlett, Boston, MA. MR 1165320.

[17] C. Huneke and J. Koh, Cofiniteness and vanishing of local cohomology modules, *Math. Proc. Cambridge Philos. Soc.* **110** (1991), no. 3, 421–429. MR 1120477.

[18] K. Kawasaki, On a category of cofinite modules which is Abelian, *Math. Z.* **269** (2011), no. 1-2, 587–608. MR 2836085.

[19] T. Marley, The associated primes of local cohomology modules over rings of small dimension, *Manuscripta Math.* **104** (2001), no. 4, 519–525. MR 1836111.

[20] T. Marley and J. C. Vassilev, Cofiniteness and associated primes of local cohomology modules, *J. Algebra* **256** (2002), no. 1, 180–193. MR 1936885.

[21] L. Melkersson, Modules cofinite with respect to an ideal, *J. Algebra* **285** (2005), no. 2, 649–668. MR 2125457.

[22] L. Melkersson, Cofiniteness with respect to ideals of dimension one, *J. Algebra* **372** (2012), 459–462. MR 2990020.

[23] J. J. Rotman, *An Introduction to Homological Algebra*, Pure and Applied Mathematics, 85, Academic Press, New York, 1979. MR 0538169.

[24] A. Vahidi and S. Morsali, Cofiniteness with respect to the class of modules in dimension less than a fixed integer, *Taiwanese J. Math.* **24** (2020), no. 4, 825–840. MR 4124548.

[25] K.-I. Yoshida, Cofiniteness of local cohomology modules for ideals of dimension one, *Nagoya Math. J.* **147** (1997), 179–191. MR 1475172.

[26] T. Yoshizawa, Subcategories of extension modules by Serre subcategories, *Proc. Amer. Math. Soc.* **140** (2012), no. 7, 2293–2305. MR 2898693.

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