ON THE IMAGE SET AND REVERSIBILITY OF SHIFT MORPHISMS OVER DISCRETE ALPHABETS

JORGE CAMPOS, NEPTALÍ ROMERO, AND RAMÓN VIVAS

ABSTRACT. We provide sufficient conditions in order to show that the image set of a continuous and shift-commuting map defined on a shift space over an arbitrary discrete alphabet is also a shift space. Additionally, if such a map is injective, then its inverse is also continuous and shift-commuting.

1. INTRODUCTION

Shift spaces and their morphisms constitute a powerful tool for modeling several phenomena in dynamical systems; this practice is known as symbolic dynamics. They are also an important part of the platform for diverse disciplines like automata, coding, information and system theory. Depending on the context there are two typical ways to establish them: either on bi-infinite sequences of symbols indexed by the set of integer numbers $\mathbb{Z}$, or by using one-sided infinite sequences of symbols indexed by the set of natural numbers $\mathbb{N}$. In this paper we shall deal exclusively with the two-sided setting. Some classic textbooks on the subject are [12] and [13].

The standard construction of shift spaces begins with a finite set of symbols: an *alphabet*. However, in certain environments it is necessary to consider alphabets with infinite symbols; such is the situation when the thermodynamic formalism is developed for the so-called *countable state Markov shifts* (see, for example, [12] and [21]). Shift spaces over alphabets with infinite symbols were also considered by Gromov in his seminal work on endomorphisms of symbolic algebraic varieties and topological invariants of dynamical systems (see [8] and [9]). These important facts justify the attention given to the study of shift spaces over infinite alphabets.

In what follows an alphabet is any nonempty set $\mathcal{A}$ equipped with the discrete topology. Given an alphabet $\mathcal{A}$, the product space $\mathcal{A}^{\mathbb{Z}}$ is considered; this is the set of all bi-infinite sequences over $\mathcal{A}$ endowed with the Cantor metric d, which is

defined for all $x = (x_n)_{n \in \mathbb{Z}}$ and $y = (y_n)_{n \in \mathbb{Z}}$ in $\mathcal{A}^{\mathbb{Z}}$ by

$$d(x, y) = \begin{cases} 0, & \text{if } x_n = y_n \text{ for all } n \in \mathbb{Z}, \\ 2^{-k}, & \text{if } x \neq y \text{ and } k = \min\{|n| : x_n \neq y_n\}. \end{cases}$$

This topological space is called the (two-sided) *full shift over* $\mathcal{A}$. After that, we consider the *shift operator* $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ given by $\sigma(x) = y$, where $y_n = x_{n+1}$ for all $n \in \mathbb{Z}$; it is a homeomorphism. Thus, a *shift space over* $\mathcal{A}$ (or a *subshift over* $\mathcal{A}$) is any nonempty closed subset of X of $\mathcal{A}^{\mathbb{Z}}$ which is shift-invariant, that is, $\sigma(X) = X$; a *shift morphism* (or simply a *morphism*) on the shift space $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is any continuous and *shift-commuting* mapping Φ from X to some full shift $\mathcal{U}^{\mathbb{Z}}$; here the term 'shift-commuting' means $\Phi \circ \sigma = \sigma \circ \Phi$, where σ indistinctly denotes the shift map in both $\mathcal{A}^{\mathbb{Z}}$ and $\mathcal{U}^{\mathbb{Z}}$.

Shift spaces can be equivalently introduced by using a set of special words constructed with the symbols in the alphabet; more precisely, a *word* or a *block* over $\mathcal{A}$ is a finite sequence of symbols in $\mathcal{A}$; the number of such symbols is the *length* of the block. Let $\mathcal{A}^*$ denote the set of blocks over $\mathcal{A}$; it is said that a block $w \in \mathcal{A}^*$ *appears* in $x \in \mathcal{A}^{\mathbb{Z}}$ if there exists an integer closed interval $[i, j] \subset \mathbb{Z}$ ($i \leq j$) such that the restriction $x|_{[i,j]}$ of x to $[i, j]$ is just w; that is, $x|_{[i,j]} = x_i \cdots x_j = w$. It is well known that a nonempty set $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is a shift space over $\mathcal{A}$ if and only if there exists $\mathcal{F} \subset \mathcal{A}^*$ (not necessarily unique) such that $x \in X$ if and only if no block belonging to $\mathcal{F}$ appears in x. Such a set is called a *forbidden set for* X; obviously $\mathcal{F} = \emptyset$ if X is a full shift. There is another set of distinguished words for each shift space. A word $w \in \mathcal{A}^*$ is called *allowed* for the shift space $X \subseteq \mathcal{A}^{\mathbb{Z}}$ if it appears in some point of X. The set $\mathcal{L}(X)$ of the allowed words for the shift space X is called *language* of X; clearly $\mathcal{L}(X) = \{w \in \mathcal{A}^* : w \text{ appears in } x \text{ for some } x \text{ in } X\}$. The language $\mathcal{L}(X)$ is characterized by the *factorial* and *extendable* properties, which are, respectively:

(a) If w is a block in $\mathcal{L}(X)$ and u is a subblock of w, then $u \in \mathcal{L}(X)$.

(b) If $w \in \mathcal{L}(X)$, then there are nonempty blocks $u, v \in \mathcal{L}(X)$ such that the concatenation block uwv is also in $\mathcal{L}(X)$.

Properties (a) and (b) characterize the languages of shift spaces; that is, given a nonempty set $\mathcal{L} \subset \mathcal{A}^*$ satisfying (a) and (b), there is a unique shift space $X \subseteq \mathcal{A}^{\mathbb{Z}}$ such that $\mathcal{L} = \mathcal{L}(X)$. For every integer $n \geq 1$, $\mathcal{L}_n(X)$ denotes the subset of $\mathcal{L}(X)$ whose blocks have length n. It is not hard to show that X is a compact space if and only if $\mathcal{L}_1(X)$ is a finite set, i.e., the alphabet supporting X is a finite set. Notice that the complement of $\mathcal{L}(X)$ is a forbidden set for X; indeed, it is the largest one. Obviously, when X is the full shift, $\mathcal{L}_n(X) = \mathcal{A}^n$.

There is a simple way to obtain morphisms on a shift space $X \subseteq \mathcal{A}^{\mathbb{Z}}$. If $\mathcal{U}$ is an alphabet, m and n are integers with $n - m \geq 0$ and $\varphi : \mathcal{L}_{m+n+1}(X) \to \mathcal{U}$ is an arbitrary function, then the map $\Phi : X \to \mathcal{U}^{\mathbb{Z}}$ defined by

$$\Phi(x)_i = \varphi\left(x|_{[i-m, i+n]}\right), \quad \text{for every } x \text{ in } X \text{ and each } i \in \mathbb{Z}, \qquad (1.1)$$

is a morphism on X. Every map so defined is called a *sliding block code*, the function φ is known as a *local rule* inducing Φ, and the integers m and n are called

respectively *memory* and *anticipation* of Φ. The local rule and the integers m and n provide a sort of window of length $n + m + 1$: it slides through each element of X to determine the value of each component of its image under the corresponding sliding block code. Sometimes it is useful to consider wider windows: take integers M and N with $M \geq m$ and $N \geq n$; the language factorial property implies that $\widehat{\varphi} : \mathcal{L}_{M+N+1}(X) \to \mathcal{U}$ with $\widehat{\varphi}(a_{-M} \cdots a_N) = \varphi(a_{-m} \cdots a_n)$ is well defined for every $a_{-M} \cdots a_N \in \mathcal{L}_{M+N+1}(X)$ and it induces the same sliding block code as φ.

When X is the full shift $\mathcal{A}^{\mathbb{Z}}$ and $\mathcal{U} = \mathcal{A}$, the concept of sliding block code is just the concrete definition of *cellular automaton* over the alphabet $\mathcal{A}$. Gustav A. Hedlund, in his influential article [10], proved that if $\mathcal{A}$ is finite, then the set of cellular automata in $\mathcal{A}^{\mathbb{Z}}$ matches with the set of continuous and shift-commuting self-mappings of $\mathcal{A}^{\mathbb{Z}}$; so the cellular automata over finite alphabets were characterized. Since Hedlund credited Morton L. Curtis and Roger Lyndon as co-discoverers of this characterization, the result is known as the Curtis–Hedlund–Lyndon theorem. It remains true in the framework of shift spaces over finite alphabets:

Theorem 1.1 (Curtis–Hedlund–Lyndon theorem [13, Theorem 6.2.9])**.** *Let X and Y be shift spaces over finite alphabets. A map $\Psi : X \to Y$ is a sliding block code if and only if it is continuous and shift-commuting.*

This important theorem fails when the shift space is not compact. Take for example the morphism $\Phi : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}$ ($\mathbb{N} = \{0, 1, 2, \dots\}$) given by

$$\Phi(x)_n = \sum_{|j| \leq x_n} x_{j+n}, \quad \text{for all } x \in X \text{ and every } n \in \mathbb{Z}; \tag{1.2}$$

it is continuous and shift-commuting but it cannot be expressed through a local rule as in (1.1). In the study of shift spaces and their morphisms over finite alphabets the compactness has a special role; in particular, it allows us to prove the following two results which are part, together with the Curtis–Hedlund–Lyndon theorem, of the folklore of symbolic dynamics and coding theory:

Theorem 1.2 ([13, Theorem 1.5.13])**.** *If $\Phi : X \to \mathcal{U}^{\mathbb{Z}}$ is a shift morphism, then $\Phi(X)$ is a closed subset of $\mathcal{U}^{\mathbb{Z}}$; therefore, it is a shift space.*

Theorem 1.3 ([13, Theorem 1.5.14])**.** *If $\Phi : X \to \mathcal{U}^{\mathbb{Z}}$ is an injective shift morphism, then the inverse map $\Phi^{-1} : \Phi(X) \to X$ is a sliding block code.*

These two results are known as *closed image property* and *reversibility*, respectively. Like the Curtis–Hedlund–Lyndon theorem, both Theorem 1.2 and Theorem 1.3 are not true in the non-compact context. We refer to [2, Example 1.10.3] and [3, Lemma 5.1], where notable examples of bijective and non-reversible shift morphisms over non-finite alphabets are shown. For its part, the following example shows a non-surjective shift morphism on $\mathbb{N}^{\mathbb{Z}}$ whose image set is not a shift space.

Example 1.4. Let $\Psi : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}$ be the map given by

$$\Psi(x)_j = x_{j-x_j} + x_{j+x_j}, \quad \text{for all } x \in \mathbb{N}^{\mathbb{Z}} \text{ and every } j \in \mathbb{Z}.$$

By direct verification it is shown that this map is a shift morphism. It is also easy to see that the constant sequence $(x_n)_{n \in \mathbb{Z}} = 1^{\mathbb{Z}}$ ($x_n = 1$ for all n) has no preimage under Ψ. Now, for every integer $k \geq 1$, we consider $x^k = (x_j^k)_{j \in \mathbb{Z}} \in \mathbb{N}^{\mathbb{Z}}$, where

$$x_j^k = \begin{cases} 3k + 1 - j, & \text{if } -k \leq j \leq k, \\ 0, & \text{if } j \geq k + 1, \\ 1, & \text{if } j \leq -k - 1. \end{cases}$$

By taking the Ψ-image of each x^k one obtains $\Psi(x^k)|_{[-k,k]} = 1^{2k+1}$ for all $k \geq 1$; therefore, $\Psi(x^k) \to 1^{\mathbb{Z}}$ when $k \to +\infty$. This implies that the image set $\Psi(\mathbb{N}^{\mathbb{Z}})$ is not a shift space; also observe that $(x^k)_{k \geq 1}$ has no convergent subsequences.

Each one of the results in the trilogy —Theorem 1.1, Theorem 1.2 and Theorem 1.3— has been validated in the world of cellular automata by omitting the finiteness of the alphabet. For our aim it is necessary to make some brief comments concerning this trilogy; let us begin by the last two theorems. According to our knowledge of the literature on the subject, the most recent developments have been reported by Ceccherini-Silberstein and Coornaert in [1] and [3]. In these articles the authors consider as alphabet a vector space V and a group G substituting $\mathbb{Z}$; the vector space is endowed with the discrete topology and the Cartesian product V^G is equipped with the product topology. Clearly V^G has the natural vectorial structure induced by that of V. Then the concept of cellular automaton on V^G is introduced analogously to the classical case; see [2, Chapter 1] for details. In this setting the following results are proved:

Theorem 1.5 ([1, Theorem 1.2, Corollary 1.6 and Corollary 1.7])**.** *Let V be a finite-dimensional space and let G be a group.*

   (a) *If $\tau : V^G \to V^G$ is a linear cellular automaton, then $\tau(V^G)$ is closed in V^G; that is, τ has the closed image property.*
   (b) *If $\tau : V^G \to V^G$ is a bijective linear cellular automaton, then τ^{-1} is also a cellular automaton, i.e., τ is reversible.*

New proofs of (a) and (b) are presented in [3] by using the technical tool of Mittag-Leffler's lemma for projective sequences of sets. In that same article it is mentioned that more general results than (a) were obtained by Gromov in [8]. Also in [3] it is proved that when the vector space V is infinite-dimensional, it is possible to construct on $V^{\mathbb{Z}}$ a linear cellular automaton without the closed image property (see [3, Theorem 1.5]) and a non-reversible bijective linear cellular automaton (see [3, Theorem 1.2]).

Related to the first result of the trilogy, we mention that in [20] a generalization of the Curtis–Hedlund–Lyndon theorem for cellular automata over arbitrary discrete alphabets is proved; this is based on the concept of barrier and extended notions of local rule and sliding block code also introduced in [20]. Such generalization is transferred without any difficulty to continuous and shift-commuting mappings between shift spaces over such kind of alphabets. This extended version of the Curtis–Hedlund–Lyndon theorem is part of the foundation on which the discussion in the present paper is developed; see Theorem 2.3 in the next section for

its precise statement and concise commentaries about its proof. There are several references where different versions of the pioneer Curtis–Hedlund–Lyndon theorem are presented, for example [2], [6], [7], [16], [18], [19], [22] and [23]. In some of these versions the topological structure (product topology) or the notions of shift space and sliding block code are modified. We maintain the original relative product topology on the shift spaces and the essence of the notion of local rule in order to preserve the continuity and shift-commuting property as the underlying concepts in the notion of sliding block code.

Having as conceptual basis the characterization of the continuous and shift-commuting maps between shift spaces, i.e., the extended Curtis–Hedlund–Lyndon theorem established in Theorem 2.3 below, the main goal in this paper is to give sufficient conditions in order to guarantee the closed image property and reversibility for such mappings.

Since the sufficient conditions that we will establish for our purposes are somewhat technical, it is necessary to clarify this technicality for a better understanding of the discussion, which even delays the precise establishment of the central results of this work. We have organized the rest of the article as follows. In Section 2 we begin by reviewing the concept of barrier on shift spaces and the extended notions of local rule and sliding block code; these were introduced in [20] as the platform to extend the classic Curtis–Hedlund–Lyndon theorem to cellular automata on arbitrary discrete alphabets (see [20, Theorem 4]), its translation to continuous and shift-commuting maps between shift spaces is Theorem 2.3. Then two special types of barriers are introduced; they are linked to the maps characterized in Theorem 2.3 and determine a wide subset of them on which sufficient conditions will be established for the fulfillment of the closed image property and reversibility. Additionally a crucial lemma is proved. These preliminaries will allow us to establish and prove the main results of the article: Theorem 3.1, Theorem 3.7 and Theorem 3.8; this is precisely the content of the third and final section.

Acknowledgements. The authors would like to thank the referee for helpful comments and suggestions which led to the improvement of the manuscript.

2. The extended environment and technical preliminaries

First we recall that the product topology on the full shift $\mathcal{A}^{\mathbb{Z}}$ has as a basis the set of cylinders $C(h) = \{x \in \mathcal{A}^{\mathbb{Z}} : x|_{\mathrm{dom}(h)} = h\}$, where h is an $\mathcal{A}$-valued function with domain $\mathrm{dom}(h)$, a finite subset of $\mathbb{Z}$, and $x|_{\mathrm{dom}(h)}$ is the restriction of $x : \mathbb{Z} \to \mathcal{A}$ to $\mathrm{dom}(h)$; we denote by $F(\mathcal{A})$ the set of such functions h and by abuse of notation we also say that $\mathrm{dom}(h)$ is the domain of the cylinder $C(h)$.

Definition 2.1. Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a shift space. A *barrier* on X is a partition of X whose elements $C_X(h)$ are of the form $X \cap C(h)$; they are called *cylinders* in X.

This concept was inspired by the notion of barrier introduced by Nash-Williams in his study [17] on well-quasi-ordered sets; for this subject see also [4] and [5].

We present some helpful observations about barriers on a shift space $X \subseteq \mathcal{A}^{\mathbb{Z}}$:

(a) The set of barriers on X is partially ordered by the *finer-than* relation: a barrier $\mathcal{B}'$ is *finer than* the barrier $\mathcal{B}$ if for every $B' \in \mathcal{B}'$ there is $B \in \mathcal{B}$ such that $B' \subseteq B$. When $\mathcal{B}'$ is finer than $\mathcal{B}$ it is also said that $\mathcal{B}'$ is a *refinement* of $\mathcal{B}$ or that $\mathcal{B}$ is coarser than $\mathcal{B}'$. Given a barrier $\mathcal{B}$, it is always possible to get a finer one; to see that, observe that if $C_X(h)$ is a cylinder in X and $\ell \notin \operatorname{dom}(h)$, then $C_X(h)$ is the disjoint union of cylinders $C_X(h_a)$, where h_a is the extension of h to $\operatorname{dom}(h) \cup \{\ell\}$ with $h_a(\ell) = a$ going through the set of symbols such that $C(h_a) \cap X \neq \emptyset$. In particular, if $\mathcal{B}$ is a barrier on X such that for some interval $[i, j] \subset \mathbb{Z}$ the inclusion $\operatorname{dom}(h) \subset [i, j]$ holds for every $C_X(h) \in \mathcal{B}$, then there is a refinement $\mathcal{B}'$ of $\mathcal{B}$ such that $[i, j]$ is the domain of each cylinder in $\mathcal{B}'$.

(b) Take integers m, n, N with $N = m + n + 1 \geq 1$. The set of all allowed N-blocks $\mathcal{L}_N(X)$ can be identified with a barrier whose cylinders have the same domain $[-m, n]$; in fact, every allowed N-block $w = w_0 \cdots w_{N-1}$ in X is identified with the cylinder $C_X(h_w)$, where $h_w : [-m, n] \to \mathcal{A}$ and $h_w(-m + \ell) = w_\ell$ for all $0 \leq \ell \leq N - 1$.

Definition 2.2. Let $\mathcal{A}$ and $\mathcal{U}$ be alphabets. An $\mathcal{U}$-*extended local rule* in the shift space $X \subseteq \mathcal{A}^{\mathbb{Z}}$ (or simply local rule in X) is any $\mathcal{U}$-valued function whose domain is a barrier on X. Given a local rule $\varphi : \mathcal{B} \to \mathcal{U}$ in X, the *map induced by φ* is the transformation $\Psi : X \to \mathcal{U}^{\mathbb{Z}}$ defined by

$$\Psi(x)_n = \varphi(C_X(h_{x,n})), \quad \text{for every } x \in X \text{ and } n \in \mathbb{Z}, \tag{2.1}$$

where $C_X(h_{x,n})$ is the cylinder in $\mathcal{B}$ containing $\sigma^n(x)$, σ being the shift map on $\mathcal{A}^{\mathbb{Z}}$.

We emphasize that if $\mathcal{B}$ is the barrier described in paragraph (b) above, then the action in (2.1) matches that expressed in (1.1). So, the map induced by an extended local rule extends the classical notion of sliding block code. From now on we use the term *extended sliding block code* (ESBC for short) for every transformation Ψ as defined in (2.1). In this extended setting the classical Curtis–Hedlund–Lyndon theorem [13, Theorem 6.2.9] remains valid:

Theorem 2.3 (Curtis–Hedlund–Lyndon theorem)**.** *Let $\mathcal{A}$ and $\mathcal{U}$ be arbitrary discrete alphabets and let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a shift space. A map from X to $\mathcal{U}^{\mathbb{Z}}$ is a shift morphism if and only if it is an ESBC.*

A proof of this theorem is easily obtained by paraphrasing the proof of Theorem 4 in [20], which characterizes the cellular automata on arbitrary discrete alphabets. We only highlight the following facts:

• If $\Psi : X \to \mathcal{U}^{\mathbb{Z}}$ is the ESBC induced by $\varphi : \mathcal{B} \to \mathcal{U}$, then it is a shift morphism. In fact, it is enough to observe that for all $n \in \mathbb{Z}$ the n-coordinate function $\Psi_n : X \to \mathcal{U}$ of Ψ satisfies $\Psi_n = \Psi_0 \circ \sigma^n$, and the 0-coordinate function is constant in each cylinder of $\mathcal{B}$.

• Let $\Psi : X \to \mathcal{U}^{\mathbb{Z}}$ be a morphism. Since in X any nonempty open set is the disjoint union of cylinders in X, it follows that the set of preimages $\Psi_0^{-1}(\{b\})$, with b varying in $\Psi_0(X)$, determines a barrier on X and Ψ_0 defines a local rule inducing Ψ; obviously Ψ_0 is constant in each cylinder of such a barrier.

Example 2.4. To illustrate, we return to the morphism $\Psi : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}$ defined by (1.2). It is clear that the 0-coordinate of Ψ is given by $\Psi_0(x) = \sum_{|j| \leq x_0} x_j$ for all $x = (x_n)_{n \in \mathbb{Z}} \in \mathbb{N}^{\mathbb{Z}}$. Hence, $\Psi_0^{-1}(\{m\}) = \{x \in \mathbb{N}^{\mathbb{Z}} : \sum_{|j| \leq x_0} x_j = m\}$ for all $m \in \mathbb{N}$. In other words, if w is a block with odd length and C_w denotes the cylinder of all $x \in \mathbb{N}^{\mathbb{Z}}$ such that $x|_{[-k,k]} = w$, then the preimage $\Psi_0^{-1}(\{m\})$ is the disjoint union of the cylinders C_w, where $w = w_{-w_0} \cdots w_0 \cdots w_{w_0}$ and $\sum_{|j| \leq w_0} w_j = m$. Therefore the collection $\mathcal{B}^0$ of cylinders C_w, with m varying in $\mathbb{N}$, is a barrier on $\mathbb{N}^{\mathbb{Z}}$ and the local rule $\varphi : \mathcal{B}^0 \to \mathbb{N}$, with $\varphi(C_w) = \sum_{|j| \leq w_0} w_j$, induces Ψ. Note that $\Psi_0^{-1}(\{0\}) = C_0 = \{x \in \mathbb{N}^{\mathbb{Z}} : x_0 = 0\}$ and it is the union of the cylinders $C(h_v)$, where $v \in \mathbb{N}$ and $h_v : \{0, 1\} \to \mathbb{N}$ is given by $h_v(0) = 0$ and $h_v(1) = v$. If in $\mathcal{B}^0$ one substitutes C_0 by the collection of these new cylinders $C(h_v)$, a finer barrier $\mathcal{B}^1$ is obtained and φ acting on $\mathcal{B}^1$ also induces Ψ; clearly $\Psi_0(C(h_v)) = 0$ for all $v \in \mathbb{N}$. We also observe that $\mathcal{B}^0$ is the maximum of all the barriers on $\mathbb{N}^{\mathbb{Z}}$ inducing Ψ; this follows from the following fact: If $C(h)$ is a cylinder on $\mathbb{N}$ such that Ψ_0 is constant on $C(h)$, then there exist a $\mathbb{Z}$-interval $[-N, N]$ and a block $w \in \mathbb{N}^{2N+1}$ such that $[-N, N] \subseteq \operatorname{dom}(h)$ and $h|_{[-N,N]} = w$.

We now introduce two type of barriers and a particular class of sequences related to an ESBC. These notions are fundamental for the rest of the article.

Definition 2.5. Let $\Psi : X \to \mathcal{U}^{\mathbb{Z}}$ be an ESBC.
   (i) A barrier $\mathcal{B}$ on X is said to be *attached to* Ψ if the 0-coordinate function of Ψ is constant on every cylinder of $\mathcal{B}$.
   (ii) If there exists a barrier $\mathcal{B}$ attached to Ψ such that for every $\ell \in \Psi_0(X)$ the number of cylinders in $\mathcal{B}$ with $\Psi_0(B) = \ell$ is finite, then it is said that both Ψ and $\mathcal{B}$ are of *finite degree*.
   (iii) A sequence $(x^k)_{k \in \mathbb{Z}} \subset X$ is called *distinguished* for Ψ if $(\Psi(x^k))_{k \in \mathbb{Z}}$ is convergent. The set of distinguished sequences for Ψ is denoted by $X(\Psi)$.

It is easy to see that a barrier $\mathcal{B}$ on X is attached to an ESBC $\Psi : X \to \mathcal{U}^{\mathbb{Z}}$ if and only if it is induced by the local rule $\varphi : \mathcal{B} \to \mathcal{U}$ defined, for each $C \in \mathcal{B}$, by $\varphi(C) = \Psi_0(C)$. In this way, the barriers attached to an ESBC are the domains of the local rules inducing it.

When the alphabets are finite every sliding block code is of finite degree. Not every barrier attached to a finite degree ESBC is of finite degree; such is the case of the barriers $\mathcal{B}^0$ and $\mathcal{B}^1$ described in Example 2.4. On the other hand, not every ESBC is of finite degree; this is shown in the following example, which also shows that there exist distinguished sequences without convergent subsequences.

Example 2.6. Let $\Psi : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}$ be the ESBC given by

$$\Psi(x)_j = x_{j-x_j} + x_{j+x_j}, \quad \text{for all } x \in \mathbb{N}^{\mathbb{Z}} \text{ and every } j \in \mathbb{Z}.$$

Let us see that Ψ is not of finite degree. First, take $\ell, m, a, b \in \mathbb{N}$ such that $a+b = \ell$; for this collection of natural numbers define $h : \{-m, 0, m\} \to \mathbb{N}$ by $h(-m) = a$, $h(0) = m$ and $h(m) = b$. It is clear that for every $x \in C(h)$ one has $\Psi_0(x) = \ell$, so the set of cylinders thus defined constitutes a barrier $\mathcal{B}$ attached to Ψ. Also observe

that $\mathcal{B}$ is not of finite degree. Now take any cylinder $C(h')$ such that for some $\ell \in \mathbb{N}$ the equality $\Psi_0(x) = \ell$ holds for each $x \in C(h')$. Suppose that $0 \notin \mathrm{dom}(h')$ and pick $m \notin \mathrm{dom}(h') \cup \{0\}$, so the m-coordinate of $x \in C(h')$ with $x_0 = m$ is restricted to $x_{-m} + x_m = \ell$. This contradicts the fact that the m-coordinate of points in $C(h')$ varies throughout $\mathbb{N}$ even when the 0-coordinate is m. Thus 0 necessarily belongs to $\mathrm{dom}(h')$ and so $x_{-h'(0)} + x_{h'(0)} = \ell$ for all $x \in C(h')$, which forces that $-h'(0), h'(0) \in \mathrm{dom}(h')$. Consequently, $C(h')$ is contained in some cylinder $C(h)$ as described above, and therefore every barrier attached to Ψ is a refinement of $\mathcal{B}$, which clearly implies that Ψ is not of finite degree.

Next, it is not hard to see that $(x_n)_{n \in \mathbb{Z}} = 1^{\mathbb{Z}}$ (i.e., $x_n = 1$ for all $n \in \mathbb{Z}$) has no preimage under Ψ. Now, for every integer $k \geq 1$ consider $x^k = (x_j^k)_{j \in \mathbb{Z}} \in \mathbb{N}^{\mathbb{Z}}$ defined for each $j \in \mathbb{Z}$ as follows:

$$x_j^k = \begin{cases} 3k + 1 - j, & \text{if } -k \leq j \leq k, \\ 0, & \text{if } j \geq k + 1, \\ 1, & \text{if } j \leq -k - 1. \end{cases}$$

After a direct calculation one has $\Psi(x^k)|_{[-k,k]} = 1^{2k+1}$ for all $k \geq 0$, and therefore $\Psi(x^k) \to 1^{\mathbb{Z}}$ when $k \to +\infty$. This shows that the sequence $(x^k)_{k \geq 1}$ is distinguished for Ψ but it has no convergent subsequences.

In what follows, a sequence of elements in a shift space will be called *nice* if it has convergent subsequences. An immediate fact is the following:

If $\Psi : X \to \mathcal{U}^{\mathbb{Z}}$ is an ESBC and every distinguished sequence by Ψ is nice, then Ψ has the closed image property; i.e., $\Psi(X)$ is a shift space.

We stress that the converse of this assertion is false; this is shown in Example 3.3 below. By restricting our attention to ESBCs of finite degree, we will give sufficient conditions, more technical than the totality of distinguished nice sequences, for an ESBC to have the closed image property. Furthermore, these same conditions will guarantee the reversibility of bijective ESBCs. For this purpose the following lemma is crucial. Before continuing, we need to introduce some notation to make sentences more readable. First we recall that $F(\mathcal{A})$ is the set of all $\mathcal{A}$-valued functions whose domains are finite subsets of $\mathbb{Z}$. Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a shift space and let σ be the shift map on $\mathcal{A}^{\mathbb{Z}}$. Observe that for any $n \in \mathbb{Z}$ and every cylinder $C(h)$ in $\mathcal{A}^{\mathbb{Z}}$, $\sigma^{-n}(C(h))$ is also a cylinder; we denote it by $C(h^{[n]})$, where $h^{[n]}$ is the n-translation of h, that is,

$$\mathrm{dom}(h^{[n]}) = \mathrm{dom}(h) + n \quad \text{and} \quad h^{[n]}(j + n) = h(j), \quad \text{for all } j \in \mathrm{dom}(h).$$

On the other hand, it is easy to see that if $\{h_\alpha\}_{\alpha \in \Gamma}$ is a collection in $F(\mathcal{A})$ satisfying $\bigcap_{\alpha \in \Gamma} C(h_\alpha) \neq \emptyset$, then $h_\alpha(m) = h_\beta(m)$ holds for all $\alpha, \beta \in \Gamma$ and every integer m in $\mathrm{dom}(h_\alpha) \cap \mathrm{dom}(h_\beta)$. This last property allows us to define $\bigoplus_{\alpha \in \Gamma} h_\alpha : \bigcup_{\alpha \in \Gamma} \mathrm{dom}(h_\alpha) \to \mathcal{A}$ by

$$\left(\bigoplus_{\alpha \in \Gamma} h_\alpha \right)(m) = h_\alpha(m), \quad \text{whenever } m \in \mathrm{dom}(h_\alpha).$$

Obviously this new function belongs to $F(\mathcal{A})$ if and only if $\bigcup_{\alpha \in \Gamma} \mathrm{dom}(h_\alpha)$ is finite; in this case, $\bigcap_{\alpha \in \Gamma} C(h_\alpha)$ is just the cylinder $C(\bigoplus_{\alpha \in \Gamma} h_\alpha)$. A particular case of this kind of collection is a sequence of functions h_n in $F(\mathcal{A})$ such that h_{n+1} extends h_n. In any case, it is clear that for any collection $\{h_\alpha\}_{\alpha \in \Gamma} \subset F(\mathcal{A})$,

$$\bigcap_{\alpha \in \Gamma} C(h_\alpha) = \{x \in \mathcal{A}^{\mathbb{Z}} : \text{for all } m \in \mathbb{Z}, \; x_m = h_\alpha(m) \text{ if } m \in \mathrm{dom}(h_\alpha)\}.$$

An additional notation: Let $\Psi : X \to \mathcal{U}^{\mathbb{Z}}$ be an ESBC and let $\mathcal{B}$ be a barrier attached to it. For each symbol $\ell \in \Psi_0(X)$, $\mathcal{B}(\ell)$ denotes the set of cylinders B in $\mathcal{B}$ such that $\Psi_0(B) = \ell$. Obviously $\mathcal{B}$ is of finite degree if the cardinal of $\mathcal{B}(\ell)$ is finite for all $\ell \in \Psi_0(X)$.

Lemma 2.7 (Crucial lemma). *If $\Psi : X \to \mathcal{U}^{\mathbb{Z}}$ is an ESBC of finite degree, then for all integers $\ell \geq 0$ and every $(x^k)_{k \geq 1} \in X(\Psi)$ with $\Psi(x^k) \to y$ when $k \to +\infty$, there exist an infinite subset S_ℓ of $\mathbb{N}$ and a cylinder $C_X(h_\ell)$ such that: $S_{\ell+1} \subset S_\ell$, $C_X(h_{\ell+1}) \subset C_X(h_\ell)$, $x^k \in C_X(h_\ell)$ for all $k \in S_\ell$, and $\Psi(z)|_{[-\ell,\ell]} = y|_{[-\ell,\ell]}$ for every $z \in C_X(h_\ell)$.*

Proof. Take a barrier $\mathcal{B}$ attached to Ψ of finite degree. Let $(x^k)_{k \geq 1}$ be a sequence in $X(\Psi)$ with $\Psi(x^k) \to y$ if $k \to +\infty$. From the definition of the Cantor metric it is clear that for each natural number $\ell \geq 0$ there is $N_\ell \geq 1$ such that $x^k \in \Psi_j^{-1}(y_j)$ for all $|j| \leq \ell$ and every $k \geq N_\ell$, where $y = (y_j)_{j \in \mathbb{Z}}$. In particular, for $\ell = 0$ one can choose a cylinder $C_X(h_{y_0,0}) \in \mathcal{B}(y_0)$ and an infinite subset S_0 of $\mathbb{N}$ such that $x^k \in C_X(h_{y_0,0})$ for all $k \in S_0$ and $\Psi_0(z) = y_0$ for each $z \in C_X(h_{y_0,0})$. Further, for $\ell = 1$ there are cylinders $C_X(h_{y_{-1},-1}) \in \mathcal{B}(y_{-1})$ and $C_X(h_{y_1,1}) \in \mathcal{B}(y_1)$ and there exists an infinite set $S_1 \subset S_0$ such that

$$x^k \in C_X(h_{y_1,1}^{[1]}) \cap C_X(h_{y_0,0}) \cap C_X(h_{y_{-1},-1}^{[-1]}), \quad \text{for all } k \in S_1.$$

Observe that this intersection is the cylinder $C_X(h_1)$, where $h_1 = \bigoplus_{|j| \leq 1} h_{y_j,j}^{[j]}$ and $h_{y_0,0}^{[0]} = h_{y_0,0}$; note also that $\Psi(z)|_{[-1,1]} = y|_{[-1,1]}$ for all $z \in C_X(h_1)$. Proceeding by recurrence, in each step $\ell \geq 1$ one can select an infinite set $S_\ell \subset S_{\ell-1}$ and cylinders $C_X(h_{y_{-\ell},-\ell}) \in \mathcal{B}(y_{-\ell})$ and $C_X(h_{y_\ell,\ell}) \in \mathcal{B}(y_\ell)$ such that

$$x^k \in \bigcap_{|j| \leq \ell} C_X(h_{y_j,j}^{[j]}), \quad \text{for all } k \in S_\ell.$$

The proof finishes by making $h_\ell = \bigoplus_{|j| \leq \ell} h_{y_j,j}^{[j]}$ and observing that $\Psi(z)|_{[-\ell,\ell]}$ matches $y|_{[-\ell,\ell]}$ for all z in the cylinder $C_X(h_\ell) = \bigcap_{|j| \leq \ell} C_X(h_{y_j,j}^{[j]})$. Notice that the sequence of cylinders $C_X(h_\ell)$ is decreasing because the function $h_{\ell+1}$ extends h_ℓ for every $\ell \geq 0$ and $h_0 = h_{y_0,0}$. $\qquad \square$

Remark 2.8. Let $\Psi : X \to \mathcal{U}^{\mathbb{Z}}$ be an ESBC and let $\mathcal{B}$ be a barrier attached to Ψ of finite degree. With the notation of the preceding lemma we observe that if $(x^k)_{k \geq 1}$ is a sequence in $X(\Psi)$ and ℓ is a natural number, then both the function h_ℓ and the set S_ℓ may be non-unique; they depend on the barrier and the limit of the sequence $(\Psi(x^k))_{k \geq 1}$. Nonetheless, for any chosen sequence $(h_\ell)_{\ell \geq 0}$ one has that $\bigcap_{\ell \geq 0} C(h_\ell) \neq \emptyset$, and so the function $h_\infty := \bigoplus_{\ell \geq 0} h_\ell$ is well defined; obviously

x belongs to $\bigcap_{\ell \geq 0} C(h_\ell)$ if, and only if, $x|_{\mathrm{dom}(h_\infty)} = h_\infty$. It is also clear that although $C_X(h_\ell) \neq \emptyset$ for all $\ell \geq 0$, it may happen that $\bigcap_{\ell \geq 0} C_X(h_\ell) = \emptyset$. On the other hand, if $\bigcap_{\ell \geq 0} C_X(h_\ell) \neq \emptyset$, every point of this set is mapped by Ψ on the limit of the sequence $(\Psi(x^k))_{k \geq 1}$. Clearly in this case

$$\bigcap_{\ell \geq 0} C_X(h_\ell) = \{x \in X : x|_{\mathrm{dom}(h_\infty)} = h_\infty\}.$$

After this discussion, we say that an $\mathcal{A}$-valued function h_∞ is *associated* to a sequence in $X(\Psi)$ and a barrier attached to Ψ of finite degree whenever it is obtained as described above. Although this association may be multivalued (different functions h_∞ may be associated to the same distinguished sequence or barrier), all of these functions are classified according to their domains into five exclusive and exhaustive classes:

C_1: $\mathrm{dom}(h_\infty)$ is bounded.

C_2: $\mathrm{dom}(h_\infty) = \mathbb{Z}$.

C_3: $\mathrm{dom}(h_\infty)$ is bilaterally unbounded, that is, $\mathrm{dom}(h_\infty) \subsetneq \mathbb{Z}$ and for all $N > 0$ there are $a, b \in \mathrm{dom}(h_\infty)$ such that $a < -N$ and $b > N$.

C_4: $\mathrm{dom}(h_\infty)$ is left-unbounded: there is $M \in \mathbb{Z}$ such that $(-\infty, M]$ is the minimal interval containing $\mathrm{dom}(h_\infty)$.

C_5: $\mathrm{dom}(h_\infty)$ is right-unbounded: there is $m \in \mathbb{Z}$ such that $[m, +\infty)$ is the minimal interval containing $\mathrm{dom}(h_\infty)$.

3. Statements and proofs of the main results

Only now are we able to state and demonstrate the main results of this article.

3.1. **The closed image property of ESBCs.** Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a shift space and let $\Psi : X \to \mathcal{U}^{\mathbb{Z}}$ be an ESBC. Since Ψ is shift-commuting, it is clear that the image set $\Psi(X)$ is a shift space if and only if $\Psi(X)$ is a closed subset of $\mathcal{U}^{\mathbb{Z}}$; that is, the limit of the image of every distinguished sequence for Ψ belongs to $\Psi(X)$.

Our first main result is related to the classes C_1 and C_2.

Theorem 3.1. *Let $\Psi : X \to \mathcal{U}^{\mathbb{Z}}$ be an ESBC of finite degree. If a function h_∞ in $C_1 \cup C_2$ is associated to each sequence in $X(\Psi)$, then $\Psi(X)$ is a shift space.*

Proof. Take any point y in the closure of $\Psi(X)$. Let $(y^k)_{k \geq 1}$ be a sequence in $\Psi(X)$ such that $y^k \to y$ when $k \to +\infty$. Pick $(x^k)_{k \geq 1} \subset X$ with $\Psi(x^k) = y^k$ for every $k \geq 1$; obviously $(x^k)_{k \geq 1}$ belongs to $X(\Psi)$. Consider a barrier attached to Ψ of finite degree in such a way that for sequences $(h_\ell)_{\ell \geq 0}$, $(S_\ell)_{\ell \geq 0}$ as in Lemma 2.7, the corresponding function h_∞ is in $C_1 \cup C_2$. First we assume that h_∞ belongs to the class C_2, that is, $\mathrm{dom}(h_\infty) = \mathbb{Z}$; from this same lemma, strictly increasing sequences of integers $(\ell_M)_{M \geq 1}$ and $(k_M)_{M \geq 1}$ can be selected such that $[-M, M] \subset \mathrm{dom}(h_{\ell_M})$, $k_M \in S_{\ell_M}$, and $x^{k_M} \to h_\infty$ when $M \to +\infty$. So, the continuity of Ψ implies that $y \in \Psi(X)$. Now we suppose that h_∞ belongs to the class C_1. As $h_{\ell+1}$ extends h_ℓ for all $\ell \geq 0$ and $\mathrm{dom}(h_\infty) = \bigcup_{\ell \geq 0} \mathrm{dom}(h_\ell)$, there is $L \geq 0$ such that $h_\ell = h_L$ for all $\ell \geq L$. Thus $\bigcap_{\ell \geq 0} C_X(h_\ell)$ is the cylinder $C_X(h_L)$, and again from Lemma 2.7 one deduces that $\Psi(z) = y$ for all $z \in C_X(h_L)$. $\square$

Remark 3.2. From the proof of the preceding theorem it is clear that if $h_\infty \in C_2$, then the sequence $(x^k)_{k \geq 1}$ in $X(\Psi)$ inducing it is nice; however, the same cannot be said if $h_\infty \in C_1$ (see Example 3.3 below); even so, in this case, the limit of $(\Psi(x^k))_{k \geq 1}$ belongs to $\Psi(X)$. We notice that if Ψ is injective and $h_\infty \in C_1$, then $(x^k)_{k \geq 1}$ is nice; indeed, it has a constant subsequence.

We would also like to highlight that if $\Psi : X \to \mathcal{U}^{\mathbb{Z}}$ is an ESBC and the domains of the cylinders in a barrier $\mathcal{B}$ attached to Ψ have a common point, then every function h_∞ associated to any sequence in $X(\Psi)$ belongs to the class C_2. This follows from the next fact: If $m \in \text{dom}(h)$ for all h with $C_X(h) \in \mathcal{B}$, then for all $\ell \geq 0$ the domain of h_ℓ contains the $\mathbb{Z}$-interval $[-\ell + m, m + \ell]$.

Example 3.3. Let X denote the shift space of all sequences $x = (x_n)_{n \in \mathbb{Z}}$ in $\mathbb{N}^{\mathbb{Z}}$ such that 0 appears in x and $x_n \neq x_m$ for all $n \neq m$. For every $x \in X$, let $0(x)$ be the integer where 0 appears in x. Define $\Psi : X \to \mathbb{Z}^{\mathbb{Z}}$ by $\Psi(x)_n = 0(x) - n$ for all $n \in \mathbb{Z}$ and every $x \in X$. It is easy to check that Ψ is a shift morphism. Observe that if $C_X(f_n)$ is the cylinder of all $x \in X$ such that $0(x) = n$, then $\mathcal{B} = \{C_X(f_n) : n \in \mathbb{Z}\}$ is a barrier attached to Ψ of finite degree; indeed, for every $n \in \mathbb{Z}$ one has $\Psi_0^{-1}(n) = C_X(f_n)$ and $\Psi(C_X(f_n))$ is the singleton $\{y^n\}$, where $y^n = (y_m^n)_{m \in \mathbb{Z}}$ with $y_m^n = n - m$. In this way, the image set $\Psi(X)$ is the discrete set $\{y^n : n \in \mathbb{Z}\}$, which is clearly a shift space. In addition, for every $(x^k)_{k \geq 1} \in X(\Psi)$, the sequence $(\Psi(x^k))_{k \geq 1}$ is eventually constant. Therefore, if one follows the indications in Lemma 2.7 and Remark 2.8 to construct h_∞ from the barrier $\mathcal{B}$, one obtains that for every distinguished sequence for Ψ there exists $n \in \mathbb{Z}$ such that the function h_∞ has domain $\{n\}$ and $h_\infty(n) = 0$. Obviously in this case the function h_∞ is unique, it belongs to the class C_1 and there are infinitely many non-nice distinguished sequences for Ψ.

Some extra information about this shift space X: $\mathcal{L}_1(X) = \mathbb{N}$ and for all $a, b \in \mathbb{N}$ with $a \neq b$, both ab and ba are allowed blocks in X.

In our search for sufficient conditions to guarantee the closed image property for finite degree ESBCs, it remains to examine the cases when the function h_∞ is in some of the classes C_3, C_4 or C_5. For this end, we only deal with shift spaces having certain finiteness properties on the allowed symbols.

Definition 3.4. A shift space $X \subset \mathcal{A}^{\mathbb{Z}}$ is said to be:

(a) *right-finite*, if for all $a \in \mathcal{L}_1(X)$, the set $\{b \in \mathcal{A} : ab \in \mathcal{L}(X)\}$ is finite;
(b) *left-finite*, if for all $a \in \mathcal{L}_1(X)$, the set $\{b \in \mathcal{A} : ba \in \mathcal{L}(X)\}$ is finite;
(c) *bilaterally-finite*, if it is both right-finite and left-finite.

In [18] the term "row-finite shift" is introduced with the same meaning as our "right-finite"; see also [7], where the notion of column-finite shift is introduced.

For the next lemma we assume that $\Psi : X \to \mathcal{U}^{\mathbb{Z}}$ is an ESBC, $\mathcal{B}$ is a barrier attached to Ψ of finite degree, $(x^k)_{k \geq 1}$ is a sequence in $X(\Psi)$ and h_∞ is a function associated to $(x^k)_{k \geq 1}$ and $\mathcal{B}$.

Lemma 3.5. *For the sequence $(x^k)_{k\geq 1}$ to be nice it is sufficient that one of the following conditions holds:*

   (a) $h_\infty \in C_3$ and X is right-finite or left-finite.
   (b) $h_\infty \in C_4$ and X is right-finite.
   (c) $h_\infty \in C_5$ and X is left-finite.

Proof. (a) We assume X to be right-finite; the left-finite case is treated analogously. Let $(h_\ell)_{\ell\geq 0}$ and $(S_\ell)_{\ell\geq 0}$ be sequences as in Lemma 2.7 such that the associated function h_∞ is in C_3. For each $\ell \geq 0$ we denote by $I_\ell = [a_\ell, b_\ell]$ the minimal interval in $\mathbb{Z}$ such that $\mathrm{dom}(h_\ell) \subset I_\ell$; clearly $I_\ell \subset I_{\ell+1}$ for all $\ell \geq 0$ and $\bigcup_{\ell\geq 0} I_\ell = \mathbb{Z}$. Hence, there exists a first integer ℓ_1 such that h_{ℓ_1} has gaps; that is, $\mathrm{dom}(h_\ell) = I_\ell$ for all $0 \leq \ell < \ell_1$ and there are integers $k_1 \geq 1$ and $a_{\ell_1} \leq a_{\ell_1}^1 < b_{\ell_1}^1 < \cdots < a_{\ell_1}^{k_1} < b_{\ell_1}^{k_1} \leq b_{\ell_1}$ such that $\mathrm{dom}(h_{\ell_1}) = I_{\ell_1} \setminus \bigcup_{i=1}^{k_1} (a_{\ell_1}^i, b_{\ell_1}^i)$; the open intervals $(a_{\ell_1}^i, b_{\ell_1}^i)$, $i = 1, \ldots, k_1$, are the gaps in $\mathrm{dom}(h_{\ell_1})$. On the other hand, since the block $w_1 = x^k|_{[a_{\ell_1}, a_{\ell_1}^1]}$ is the same for all $k \in S_{\ell_1}$ and X is right-finite, there exist a $(b_{\ell_1}^1 - a_{\ell_1}^1 - 1)$-block w and an infinite subset S'_{ℓ_1} of S_{ℓ_1} such that $x^k|_{[a_{\ell_1}, b_{\ell_1}^1]} = w_1 w$ for all $k \in S'_{\ell_1}$. Repeating this procedure in each gap of $\mathrm{dom}(h_{\ell_1})$, both the function h_{ℓ_1} and the set S_{ℓ_1} are upgraded so that $\mathrm{dom}(h_{\ell_1}) = [a_{\ell_1}, b_{\ell_1}]$, $S_{\ell_1} \subset S_{\ell_1 - 1}$ and $x^k \in C_X(h_{\ell_1})$ for all $k \in S_{\ell_1}$; for simplicity we have used the same notation for such an update. Thus, with recursive arguments, two new sequences $(h_\ell)_{\ell\geq 0}$ and $(S_\ell)_{\ell\geq 0}$ are constructed so that for all $\ell \geq 0$ the following assertions hold:

   • $\mathrm{dom}(h_\ell) = [a_\ell, b_\ell]$, $h_{\ell+1}$ extends h_ℓ and $\bigcup_{\ell\geq 0}[a_\ell, b_\ell] = \mathbb{Z}$.
   • S_ℓ is an infinite subset of $\mathbb{N}$ with $S_{\ell+1} \subset S_\ell$ and $x^k \in C_X(h_\ell)$ for all $k \in S_\ell$.

The proof of part (a) ends by noting that the new function $h_\infty = \bigoplus_{\ell\geq 0} h_\ell$ has domain $\mathbb{Z}$ and therefore the sequence $(x^k)_{k\geq 1}$ is nice; see the proof of Theorem 3.1.

The proofs for (b) and (c) essentially follow the same scheme of the previous one. For instance, in the case (b) take $(h_\ell)_{\ell\geq 0}$, $(S_\ell)_{\ell\geq 0}$ as in Lemma 2.7 and consider the corresponding funtion h_∞; if $[a_\ell, b_\ell]$ and $(-\infty, M]$ are the minimal intervals containing respectively $\mathrm{dom}(h_\ell)$ and $\mathrm{dom}(h_\infty)$, then one can assume that $M \in \mathrm{dom}(h_0)$ and h_0 has gaps, say $(a_0^0, b_0^0), \ldots, (a_0^{k_0}, b_0^{k_0})$. Next observe that the blocks $x^k|_{[a_0, a_0^0]}, x^k|_{[b_0^j, a_0^{j+1}]}$ with $0 \leq j < k_0$ and $x^k|_{[b_0^{k_0}, M]}$ do not change when $k \in S_0$; so the right-finiteness property allows us to select an $(M - a_0)$-block w and an infinite set S_0 (same nomenclature for short) such that $x^k|_{[a_0, M+1]} = w$ for all $k \in S_0$. Therefore, both the sequence of functions $(h_\ell)_{\ell\geq 0}$ and the nested sequence of infinite subsets $(S_\ell)_{\ell\geq 0}$ can be upgraded in such a way that for all $\ell \geq 0$ one has $\mathrm{dom}(h_\ell) = [a_\ell, M + 1 + \ell]$ and $x^k \in C_X(h_\ell)$ for every $k \in S_\ell$. This leads to a new function h_∞ whose domain is $\mathbb{Z}$ and then $(x^k)_{k\geq 1}$ is a nice sequence. $\square$

Remark 3.6. Observe that when X is bilaterally-finite, every sequence $(x^k)_{k\geq 1}$ in $X(\Psi)$ is nice. In fact, if there exists a barrier attached to Ψ of finite degree such that $h_\infty \in C_1$, then the sequences $(h_\ell)_{\ell\geq 0}$ and $(S_\ell)_{\ell\geq 0}$ can be upgraded as above in such a way that the new function h_∞ has $\mathbb{Z}$ as its domain.

The following result complements Theorem 3.1 in order to establish sufficient conditions to guarantee the closed image property for ESBCs of finite degree.

Theorem 3.7. *Let $\Psi : X \to \mathcal{U}^{\mathbb{Z}}$ be an ESBC of finite degree. If one of the following conditions is fulfilled, then $\Psi(X)$ is a shift space:*

(a) *X is bilaterally-finite.*
(b) *X is right-finite and a function $h_\infty \in C_1 \cup C_2 \cup C_3 \cup C_4$ is associated to each sequence in $X(\Psi)$.*
(c) *X is left-finite and a function $h_\infty \in C_1 \cup C_2 \cup C_3 \cup C_5$ is associated to each sequence in $X(\Psi)$.*

Proof. It is a straightforward combination of the arguments developed in Theorem 3.1 (see Remark 3.2) and Lemma 3.5. □

3.2. **The reversibility property of ESBCs.** We begin this last part of the article by recalling that an ESBC $\Psi : X \to Y$ is said to be *reversible* if it is bijective and its inverse is also an ESBC.

We have already mentioned that in the case of finite alphabets, every bijective shift morphism is reversible ([13, Theorem 1.5.14]), and this is not true when those symbol sets are not finite (see [2, Example 1.10.3] and [3, Lemma 5.1]). It is also well known that in the classical cellular automata context, reversibility is equivalent to injectivity; this was proved independently by Hedlund [10] and Richardson [19]. We consider it relevant to highlight that the notion of reversibility has captured the interest of researchers in several scientific and technological scenarios; see, for example, [11], [14] and [15].

Our third and last main contribution is the following theorem.

Theorem 3.8. *A bijective ESBC of finite degree is reversible if one of the conditions established in Theorems* 3.1 *or* 3.7 *is satisfied.*

Proof. Let $X \subset \mathcal{A}^{\mathbb{Z}}$ and $Y \subset \mathcal{U}^{\mathbb{Z}}$ be shift spaces. Take a bijective ESBC $\Psi : X \to Y$ of finite degree. We denote by $\Phi : Y \to X$ its inverse; clearly it is shift-commuting. So, from Theorem 2.3 it follows that Φ is an ESBC whenever it is continuous; in other words, Ψ is reversible if the 0-coordinate function $\Phi_0 : Y \to \mathcal{A}$ of Φ is locally constant. Recall that the alphabet $\mathcal{U}$ is endowed with the discrete topology and $\Phi_n = \Phi_0 \circ \sigma^n$ for every $n \in \mathbb{Z}$, where σ denotes the shift map on $\mathcal{U}^{\mathbb{Z}}$.

Suppose that Φ_0 is not locally constant; then for some $y \in Y$ there exists a sequence $(y^k)_{k \geq 1} \subset Y$ such that $y^k \to y$ when $k \to +\infty$ and $\Phi_0(y^k) \neq \Phi_0(y)$. Now, let $x = (x_n)_{n \in \mathbb{Z}}$ and $x^k = (x_n^k)_{n \in \mathbb{Z}}$ be the unique elements in X such that $\Psi(x) = y$ and $\Psi(x^k) = y^k$ for all $k \geq 1$; obviously $(x^k)_{k \geq 1}$ is a distinguished sequence for Ψ and $x_0^k \neq x_0$ for all $k \geq 1$. Take a barrier $\mathcal{B}$ attached to Ψ of finite degree, let $(h_\ell)_{\ell \geq 0}$ and $(S_\ell)_{\ell \geq 0}$ be sequences related to $(x^k)_{k \geq 1}$ and $\mathcal{B}$ as in Lemma 2.7, and let h_∞ be an associated function to $(x^k)_{k \geq 1}$ and $\mathcal{B}$. Notice that if $(x^k)_{k \geq 1}$ is nice, then the injectivity of Ψ leads to a contradiction; in fact, if $z \in X$ is a limit point of $(x^k)_{k \geq 1}$, then $\Psi(z) = \Psi(x)$; however, $z \neq x$ because $x_0^k \neq x_0$ for all k. The proof ends by noting that if either $h_\infty \in C_1 \cup C_2$, or $h_\infty \notin C_1 \cup C_2$ and X has the appropriate finiteness property (right-finite or left-finite), then the sequence $(x^k)_{k \geq 1}$ is always nice (see Remark 3.2 and proof of Lemma 3.5). □

The following corollary is obvious.

Corollary 3.9. *If $\Psi : X \to \mathcal{U}^{\mathbb{Z}}$ is an injective ESBC of finite degree and one of the conditions established in Theorems* 3.1 *and* 3.7 *is satisfied, then $\Psi(X)$ is a shift space and $\Psi : X \to \Psi(X)$ is reversible.*

Our final example shows a reversible finite degree sliding block code whose inverse is neither of finite degree nor a sliding block code.

Example 3.10. Let $\Psi : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}$ be the sliding block code of memory 0 and anticipation 1 defined for each $x = (x_n)_{n \in \mathbb{Z}} \in \mathbb{N}^{\mathbb{Z}}$ by

$$\Psi(x)_n = x_n + 2x_{n+1}, \quad \text{for all } n \in \mathbb{Z}.$$

Clearly Ψ is induced by the local rule $\psi : \mathbb{N}^2 \to \mathbb{N}$ given by $\psi(uv) = u + 2v$. It is not difficult to verify that Ψ is injective but not onto; on the other hand, since for each natural number y there are only a finite number of 2-blocks uv in $\mathbb{N}^2$ solving $u + 2v = y$, the sliding block code Ψ is of finite degree. From the preceding discussions it follows that the image set $Y = \Psi(\mathbb{N}^{\mathbb{Z}})$ is a shift space and its inverse $\Phi : Y \to \mathbb{N}^{\mathbb{Z}}$ is an ESBC. Let us see that Φ is not of finite degree. Suppose the opposite; then there exists a barrier $\mathcal{B}$ on Y attached to Φ such that for each $a \in \mathbb{N}$, the number of cylinders in $\mathcal{B}$ with Φ_0-image equal to a is finite. Let $C_Y(h_1), \ldots, C_Y(h_m)$ be the cylinders in $\mathcal{B}$ satisfying $\Phi_0(C_Y(h_i)) = 0$, $i = 1, \ldots, m$. Consider, for each $j \in \mathbb{N}$, $x^j = (x_k^j)_{k \in \mathbb{Z}}$ with $x_k^j = 0$ for $k \neq 1$ and $x_1^j = j$; clearly the Ψ-image $y^j = (y_k^j)_{k \in \mathbb{Z}}$ of x^j is given by $y_0^j = 2j$, $y_1^j = j$ and $y_k^j = 0$ if $k \notin \{0, 1\}$. Since j is arbitrary, one has that $\{0, 1\} \cap \mathrm{dom}(h_i) = \emptyset$ and $h_i(\ell) = 0$ for all $\ell \in \mathrm{dom}(h_i)$ and every $i = 1, \ldots, m$. On the other hand, if one takes any index $1 \leq i \leq m$ and any $\ell \in \mathrm{dom}(h_i)$, then $y = (y_k)_{k \in \mathbb{Z}}$, with $y_{\ell-1} = 2$, $y_\ell = 1$ and $y_k = 0$ otherwise, belongs to Y and $\Phi_0(y) = 0$; however, $y \notin C_Y(h_i)$ whatever the index $i = 1, \ldots, m$, which is a contradiction.

Now we will show that Φ is not a sliding block code. Assume, on the contrary, and without loss of generality, that Φ is induced by a local rule with the same memory and anticipation, say $L \geq 1$; that is, there exists $\varphi : \mathcal{L}_{2L+1}(Y) \to \mathbb{N}$ such that $\Phi(y)_k = \varphi\left(y|_{[-L+k,k+L]}\right)$ for all $y \in Y$ and every $k \in \mathbb{Z}$; we note in particular that $\varphi\left(y|_{[-L,L]}\right) = x_0^y$, where x^y is the unique element of $\mathbb{N}^{\mathbb{Z}}$ such that $\Phi(y) = x^y$. However, this assumption is negated by the following fact. For the elements $x = (x_k)_{k \in \mathbb{Z}}$ and $x' = (x'_k)_{k \in \mathbb{Z}}$ in $\mathbb{N}^{\mathbb{Z}}$ given by

$$x_k = \begin{cases} 2^{L+1-k}, & \text{if } -L \leq k \leq L+1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$x'_k = \begin{cases} 2^{2\ell+1}, & \text{if } k = L - 2\ell + 1 \text{ and } 0 \leq \ell \leq L, \\ 0, & \text{otherwise} \end{cases}$$

one obtains, after a direct calculation, that $y = \Psi(x)$ and $z = \Psi(x')$ have the same central block $y|_{[-L,L]} = z|_{[-L,L]} = 2^{2L+2}2^{2L+1} \cdots 2^3 2^2$, but $x_0 \neq x'_0$.

Finally, we construct a barrier on Y attached to Φ, and define on it an extended local rule inducing Φ. For each $N \geq 1$ and $y = (y_n)_{n \in \mathbb{Z}} \in Y$, consider the set

$$S_N(y) = \{w_{-N} \cdots w_{N+1} \in \mathbb{N}^{2N+2} : w_i + 2w_{i+1} = y_i, \ i = -N, \ldots, N\}.$$

The next properties are immediate:

(i) The block $x^y|_{[-N, N+1]}$ is always in $S_N(y)$.

(ii) If $N \geq 2$, a block $w_{-N} \cdots w_{N+1} \in S_N(y)$ iff $w_{-N+1} \cdots w_N$ belongs to $S_{N-1}(y)$, $w_{-N} + 2w_{-N+1} = y_{-N}$ and $w_N + 2w_{N+1} = y_N$.

(iii) If $w_{-N} \cdots w_{N+1}$ and $w'_{-N} \cdots w'_{N+1}$ are blocks in $S_N(y)$ with $w_i = w'_i$ for some i, then those blocks match.

Additionally, since for each $w_{-1} w_0 w_1 w_2 \in S_1(y)$ one has $w_1 = \frac{1}{2}(y_0 - w_0)$, the cardinal $\#S_1(y)$ of $S_1(y)$ is bounded above by $\lfloor \frac{y_0}{2} \rfloor + 1$, where $\lfloor x \rfloor$ means the integer part of x. From property (ii) above, $\#S_2(y) \leq \lfloor \lfloor \frac{y_0}{2} \rfloor / 2 \rfloor + 1 = \lfloor \frac{y_0}{4} \rfloor + 1$. Thus, by a recursive argument one concludes that $1 \leq \#S_N(y) \leq \lfloor \frac{y_0}{2^N} \rfloor + 1$, and therefore there exists a first integer $r(y) \geq 1$ such that $\#S_N(y) = 1$ for all $N \geq r(y)$. Obviously, if $1 \leq N < r(y)$, the system of Diophantine equations $w_i + 2w_{i+1} = y_i$, with $|i| \leq N$, has at least two solutions in $\mathbb{N}^{2N+2}$ and $x^y_{-N} \cdots x^y_{N+1}$ is the unique solution of such a system when $N \geq r(y)$.

For each $y \in Y$ we consider the function $h^y : \{-r(y), \ldots, r(y)\} \to \mathbb{N}$ defined by $h^y(i) = y_i$ for all $0 \leq i \leq r(y)$. It is straightforward to check that the collection $\mathcal{B}$ of the cylinders $C_Y(h^y)$ is a barrier attached on Y to Φ and $\varphi : \mathcal{B} \to \mathbb{N}$ with $\varphi(C_Y(h^y)) = x^y_0$ induces Φ. Note that $r(z) = r(y)$ for all $z \in C_Y(h^y)$; indeed, either $C_Y(h^y) \cap C_Y(h^z) = \emptyset$ if $z \notin C_Y(h^y)$ or $C_Y(h^y) = C_Y(h^z)$ for all $z \in C_Y(h^y)$.

References

[1] T. Ceccherini-Silberstein and M. Coornaert, Induction and restriction of cellular automata, *Ergodic Theory Dynam. Systems* **29** (2009), no. 2, 371–380. MR 2486775.

[2] T. Ceccherini-Silberstein and M. Coornaert, *Cellular Automata and Groups*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2010. MR 2683112.

[3] T. Ceccherini-Silberstein and M. Coornaert, On the reversibility and the closed image property of linear cellular automata, *Theoret. Comput. Sci.* **412** (2011), no. 4-5, 300–306. MR 2778463.

[4] C. A. Di Prisco and S. Todorcevic, Canonical forms of shift-invariant maps on $[\mathbb{N}]^\infty$, *Discrete Math.* **306** (2006), no. 16, 1862–1870. MR 2251567.

[5] R. Fraïssé, *Theory of Relations*, Studies in Logic and the Foundations of Mathematics, 118, North-Holland, Amsterdam, 1986. MR 832435.

[6] D. Gonçalves, M. Sobottka and C. Starling, Sliding block codes between shift spaces over infinite alphabets, *Math. Nachr.* **289** (2016), no. 17-18, 2178–2191. MR 3583264.

[7] D. Gonçalves, M. Sobottka and C. Starling, Two-sided shift spaces over infinite alphabets, *J. Aust. Math. Soc.* **103** (2017), no. 3, 357–386. MR 3725178.

[8] M. Gromov, Endomorphisms of symbolic algebraic varieties, *J. Eur. Math. Soc. (JEMS)* **1** (1999), no. 2, 109–197. MR 1694588.

[9] M. Gromov, Topological invariants of dynamical systems and spaces of holomorphic maps. I, *Math. Phys. Anal. Geom.* **2** (1999), no. 4, 323–415. MR 1742309.

[10] G. A. Hedlund, Endomorphisms and automorphisms of the shift dynamical system, *Math. Systems Theory* **3** (1969), 320–375. MR 0259881.

[11] J. Kari, Reversible cellular automata: from fundamental classical results to recent developments, *New Gener. Comput.* **36** (2018), 145–172.

[12] B. P. Kitchens, *Symbolic Dynamics*, Universitext, Springer-Verlag, Berlin, 1998. MR 1484730.

[13] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, Cambridge, 1995. MR 1369092.

[14] K. Morita, Reversible computing and cellular automata—a survey, *Theoret. Comput. Sci.* **395** (2008), no. 1, 101–131. MR 2401003.

[15] K. Morita, Reversible cellular automata, in *Computational Complexity. Vols. 1–6*, 2668–2684, Springer, New York, 2012. MR 3074639.

[16] J. Müller and C. Spandl, A Curtis–Hedlund–Lyndon theorem for Besicovitch and Weyl spaces, *Theoret. Comput. Sci.* **410** (2009), no. 38-40, 3606–3615. MR 2553315.

[17] C. St. J. A. Nash-Williams, On well-quasi-ordering transfinite sequences, *Proc. Cambridge Philos. Soc.* **61** (1965), 33–39. MR 0173640.

[18] W. Ott, M. Tomforde and P. N. Willis, *One-sided shift spaces over infinite alphabets*, NYJM Monographs, 5, State University of New York, University at Albany, Albany, NY, 2014. MR 3207861.

[19] D. Richardson, Tessellations with local transformations, *J. Comput. System Sci.* **6** (1972), 373–388. MR 0319678.

[20] N. Romero, A. Rovella and F. Vilamajó, Remark on cellular automata and shift preserving maps, *Appl. Math. Lett.* **19** (2006), no. 6, 576–580. MR 2221516.

[21] O. M. Sarig, Thermodynamic formalism for countable Markov shifts, *Ergodic Theory Dynam. Systems* **19** (1999), no. 6, 1565–1593. MR 1738951.

[22] M. Sobottka and D. Gonçalves, A note on the definition of sliding block codes and the Curtis–Hedlund–Lyndon theorem, *J. Cell. Autom.* **12** (2017), no. 3-4, 209–215. MR 3620337.

[23] S. Wacker, Cellular automata on group sets and the uniform Curtis–Hedlund–Lyndon theorem, in *Cellular Automata and Discrete Complex Systems*, 185–198, Lecture Notes in Comput. Sci., 9664, Springer, 2016. MR 3533912.

J. Campos
Universidad Nacional Experimental Politécnica Antonio José de Sucre, Departamento de Estudios Básicos, Sección de Matemática, Barquisimeto, Venezuela
jorgkmpos@gmail.com

N. Romero✉
Universidad Centroccidental Lisandro Alvarado, Departamento de Matemática, Decanato de Ciencias y Tecnología, Apartado Postal 400, Barquisimeto, Venezuela
nromero@ucla.edu.ve

R. Vivas
Universidad Nacional Experimental Politécnica Antonio José de Sucre, Departamento de Estudios Básicos, Sección de Matemática, Barquisimeto, Venezuela
ramon.alberto.vivas@gmail.com