

A CHARACTERIZATION OF THE LORENTZ SPACE $L(p, r)$ IN TERMS OF ORLICZ TYPE CLASSES

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In remembrance of N. M. Rivière (1940–1978), who believed in Lorentz spaces.

ABSTRACT. We describe the Lorentz space $L(p, r)$, $0 < r < p$, $p > 1$, in terms of Orlicz type classes of functions L_Ψ . As a consequence of this result it follows that Stein’s characterization of the real functions on \mathbb{R}^n that are differentiable at almost all the points in \mathbb{R}^n [*Ann. of Math* **113** (1981), no. 2, 383–385], is equivalent to the characterization of those functions given by A. P. Calderón [*Riv. Mat. Univ. Parma* **2** (1951), 203–213].

1. INTRODUCTION

In 1981 E. M. Stein proved that if the gradient ∇F in the distribution sense of a real function F on \mathbb{R}^n belongs to the Lorentz space $L(n, 1)$, then F is differentiable at almost all the points in \mathbb{R}^n , $n > 1$. He further proved that no condition on ∇F weaker than $\|\nabla F\|_{n,1}^* < \infty$ will guarantee the differentiability of F a.e. in \mathbb{R}^n , [8]. E. M. Stein refers to “local” $L(n, 1)$; nevertheless the connection between “local” and “global” should be clear to the reader in this context.

Earlier, in 1951, A. P. Calderón had proved that if ∇F belongs to the Orlicz class

$$L_\Psi = \left\{ f : \int_B \Psi(|f(x)|) dx < \infty \right\}, \quad (1.1)$$

with B a ball in \mathbb{R}^n and Ψ satisfying

$$\int_1^\infty (t/\Psi(t))^{1/(n-1)} dt < \infty, \quad (1.2)$$

$\Psi(t)$ nonnegative, nondecreasing, then F is differentiable at almost all the points of B . A. P. Calderón further showed that no condition on ∇F weaker than (1.1), (1.2) above guarantees the a.e. differentiability on B [1]. Since Calderón’s proof suggests that convexity may not be necessary for Ψ , we will not require it in what follows. We will refer to those classes of functions as Orlicz type classes.

The aim of this paper is to establish the connection between the Lorentz space $L(p, r)$, $0 < r < p$, $p > 1$, and Orlicz type classes that satisfy a condition akin to (1.2) above. The case $p = n$, $r = 1$ is of particular interest as it implies that the differentiability conditions discussed above are equivalent [2].

In a related context, since both Lorentz and Orlicz spaces, as well as the hybrid Lorentz–Orlicz spaces, arise as intermediate spaces of L^p spaces [9], it is also of interest to describe their interconnections. Now, if $I = [0, 1]$ denotes the unit interval in \mathbb{R} , $L^p(I)$ cannot be expressed as the union of the $L^q(I)$ spaces it contains properly; $f(x) = |x|^{-1/p} \ln^{-2/p}(1/|x|)\chi_I \in L^p(I)$ and $f \notin L^q(I)$ for $p < q \leq \infty$. On the other hand, Welland showed that $L^p(I)$, as well as more general Orlicz spaces on I , can be represented as the union of Orlicz spaces that they contain properly [10]. Since the Lorentz spaces are monotone with respect to the second index [4], and since $L(p, r)(I) \subset L(p, p)(I) = L^p(I)$ in the range that is of interest to us, Welland’s result gives that $L(p, r)(I)$ can be described as the union of Orlicz spaces that it contains, but this is insufficient to us. Our result covers $L(p, r)(\mathbb{R}^n)$ and, more to the point, the Orlicz type classes L_Ψ are taken over the family of functions Ψ that satisfy condition (1.3) below.

We will work with classes of measurable functions f defined on \mathbb{R}^n . Let f^* denote the nonincreasing rearrangement of $|f|$, and let $L(p, r)$, [4, 7], denote the Lorentz space of measurable functions f whose nonincreasing rearrangement f^* satisfies

$$\int_0^\infty f^*(t)^r t^{r/p-1} dt < \infty.$$

We will restrict ourselves to the range $0 < r < p$, $p > 1$.

Also consider the Orlicz type class L_Ψ of measurable functions f defined on \mathbb{R}^n , such that the rearrangement f^* of $|f|$ satisfies

$$\int_0^\infty \Psi(f^*(t)) dt < \infty,$$

for a nondecreasing $\Psi \geq 0$ defined on $(0, \infty)$ and satisfying

$$\int_0^\infty \frac{t^{q-1}}{\Psi(t)^{q/p}} dt < \infty, \quad (1.3)$$

where $0 < r < p$, $p > 1$, and $1/p + 1/q = 1/r$.

The aim of this paper is to prove that for $0 < r < p$, $p > 1$,

$$L(p, r) = \bigcup_{\Psi} L_\Psi, \quad 1/p + 1/q = 1/r. \quad (1.4)$$

Or, in other words, $f \in L(p, r)$ if and only if

$$\int_0^\infty \Psi(f^*(t)) dt < \infty$$

for some Ψ that satisfies (1.3) above.

The proof is accomplished in two parts, each dealing with an inclusion in (1.4). We only point out that the constants c that appear below may vary from occurrence to occurrence, and are independent of f .

2. EMBEDDING OF ORLICZ TYPE CLASSES INTO LORENTZ SPACES

We begin by showing that the Orlicz type classes corresponding to functions Ψ that satisfy (1.3) above are continuously included in an appropriate Lorentz space. More precisely, we have

Proposition 2.1. *Let f be a nonnegative, nonincreasing function defined on $(0, \infty)$ such that*

$$\int_0^\infty \Psi(f(t)) dt < \infty,$$

where $\Psi(t)$ satisfies (1.3) above.

Then, we have

$$\int_0^\infty f(t)^r t^{r/p-1} dt \leq c \left(\int_0^\infty \frac{t^{q-1}}{\Psi(t)^{q/p}} dt \right)^{r/q} \left(\int_0^\infty \Psi(f(t)) dt \right)^{r/p}. \tag{2.1}$$

Proof. Let

$$J = \int_0^\infty f(t)^r t^{r/p-1} dt,$$

and consider the interval I_k where $2^k < f \leq 2^{k+1}$, $-\infty < k < \infty$. Clearly

$$J \leq \sum_k 2^{(k+1)r} \int_{I_k} t^{r/p-1} dt \leq (p/r) \sum_k 2^{(k+1)r} |I_k|^{r/p}. \tag{2.2}$$

Now, multiplying and dividing by $\Psi(2^k)^{r/p}$, by Hölder’s inequality with conjugate indices $(p/r, q/r)$, it readily follows that the sum in the right-hand side of (2.2) is dominated by

$$\begin{aligned} \sum_k 2^{(k+1)r} \Psi(2^k)^{-r/p} |I_k|^{r/p} \Psi(2^k)^{r/p} \\ \leq \left(\sum_k \frac{2^{(k+1)q}}{\Psi(2^k)^{q/p}} \right)^{r/q} \left(\sum_k |I_k| \Psi(2^k) \right)^{r/p}. \end{aligned} \tag{2.3}$$

Consider the sum in the first factor in (2.3) above. Each summand there can be estimated by

$$\frac{2^{(k+1)q}}{\Psi(2^k)^{q/p}} \leq c \int_{2^{k-1}}^{2^k} \frac{t^{q-1}}{\Psi(t)^{q/p}} dt,$$

and, consequently, the sum does not exceed

$$c \sum_k \int_{2^{k-1}}^{2^k} \frac{t^{q-1}}{\Psi(t)^{q/p}} dt = c \int_0^\infty \frac{t^{q-1}}{\Psi(t)^{q/p}} dt. \tag{2.4}$$

As for the second sum, since

$$|I_k| \Psi(2^k) \leq \int_{I_k} \Psi(f(t)) dt,$$

it readily follows that

$$\sum_k |I_k| \Psi(2^k) \leq \int_0^\infty \Psi(f(t)) dt. \tag{2.5}$$

Thus, combining (2.2), (2.3), (2.4), and (2.5) above, the estimate (2.1) holds, and the proof is finished. \square

3. EMBEDDING OF LORENTZ SPACES INTO ORLICZ TYPE CLASSES

We complete the proof by showing that if $f \in L(p, r)$ for an appropriate range of values of p, r , then f is in an Orlicz type class L_Ψ , where Ψ depends on f . More precisely, we have

Proposition 3.1. *Let $0 < r < p, p > 1$. Let f be a nonnegative, nonincreasing function defined on $(0, \infty)$ such that*

$$\int_0^\infty f(t)^r t^{r/p-1} dt < \infty.$$

Then, with $1/p + 1/q = 1/r$, there exists a nonnegative, nondecreasing function $\Psi(t)$ defined on $(0, \infty)$ satisfying (1.3) above for which

$$\int_0^\infty \Psi(f(t)) dt < \infty. \tag{3.1}$$

Proof. Let $f_0(t)$ be a strictly positive, strictly decreasing function on $(0, \infty)$ such that

$$\lim_{t \rightarrow 0^+} f_0(t) = \infty, \quad \lim_{t \rightarrow \infty} f_0(t) = 0,$$

and

$$\int_0^\infty f_0(t)^r t^{r/p-1} dt < \int_0^\infty f(t)^r t^{r/p-1} dt.$$

Let now $g_0 = f + f_0$. Then,

$$f(t) < g_0(t), \quad \text{for all } t, \tag{3.2}$$

and

$$\int_0^\infty g_0(t)^r t^{r/p-1} dt \leq \max(2, 2^r) \int_0^\infty f(t)^r t^{r/p-1} dt < \infty. \tag{3.3}$$

Finally, we define the function $g(t)$. Let J_k be the interval where $2^k < g_0(t) \leq 2^{k+1}$, and $[a_k, b_k]$ its closure. Then $g(t)$ is defined by

$$g(a_k) = 2^{k+1}, \quad g(b_k) = 2^k$$

and extended linearly on $[a_k, b_k]$. It follows that $g(t)$ is strictly decreasing, continuous on $(0, \infty)$, absolutely continuous, invertible, and

$$g_0(t)/2 < g(t) < 2g_0(t). \tag{3.4}$$

Furthermore, since $g(t)$ is decreasing and $r < p$ it follows that

$$g(\varepsilon)^r \varepsilon^{r/p} \leq \int_0^\varepsilon g(t)^r t^{r/p-1} dt, \quad \varepsilon > 0,$$

and, consequently,

$$\lim_{\varepsilon \rightarrow 0^+} g(\varepsilon)^r \varepsilon^{r/p} = 0. \tag{3.5}$$

Likewise, for large N , we have

$$g(N)^r N^{r/p} \leq c \int_{N/2}^N g(t)^r t^{r/p-1} dt,$$

and, consequently, since $\int_0^\infty g(t)^r t^{r/p-1} dt < \infty$,

$$\lim_{N \rightarrow \infty} g(N)^r N^{r/p} = 0. \tag{3.6}$$

Let now $\Psi(t)$ be defined by the equation

$$\Psi(g(t)) = g(t)^r t^{r/p-1}, \tag{3.7}$$

and let $\varphi(t)$ be given by

$$\Psi(t) = t^r \varphi(t). \tag{3.8}$$

From (3.7) and (3.8) it follows that

$$\varphi(g(t)) = t^{r/p-1}. \tag{3.9}$$

This gives that $\varphi(t)$ increases, and, consequently, $\Psi(t)$, and $\Psi(t)/t^r$, are increasing.

Next we verify that Ψ satisfies (1.3). Since $\Psi(t) = t^r \varphi(t)$ and $r/p + r/q = 1$, by (3.9) it follows that

$$\int_0^\infty \frac{t^{q-1}}{\Psi(t)^{q/p}} dt = \int_0^\infty \frac{t^{q-1} t^{r-q}}{\varphi(t)^{q/p}} dt = \int_0^\infty t^{r-1} \varphi(t)^{1-q/r} dt. \tag{3.10}$$

By the substitution $t = g(u)$, since $r/p - 1 = -r/q$ and $1 - q/r = -q/p$, the right-hand side of (3.10) becomes

$$- \int_0^\infty g(u)^{r-1} (u^{r/p-1})^{1-q/r} g'(u) du = - \int_0^\infty g(u)^{r-1} g'(u) u^{r/p} du. \tag{3.11}$$

Now, on account of (3.5) and (3.6), integration by parts gives that (3.11) evaluates to

$$c \int_0^\infty g(u)^r u^{r/p-1} du,$$

which by (3.3) and (3.4) is finite, and (1.3) holds.

Moreover, by (3.7) it follows that

$$\int_0^\infty \Psi(g(u)) du < \infty,$$

and, consequently, by (2.5) and (3.2),

$$\int_0^\infty \Psi(f(u)/2) du \leq \int_0^\infty \Psi(g_0(u)/2) du \leq \int_0^\infty \Psi(g(u)) du < \infty. \tag{3.12}$$

Repeating the above argument with $2f$ replacing f above, (3.12) becomes

$$\int_0^\infty \Psi(2f(u)/2) du = \int_0^\infty \Psi(f(u)) du < \infty,$$

(3.1) holds, and the proof is finished. □

Since as noted in the proof $\Psi(t)/t$ increases when $r = 1$, $\Psi(t)$ can be regularized to a convex function $\Psi_0(t)$ such that $\Psi_0(t) \leq \Psi(t) \leq \Psi_0(2t)$, and, therefore, in this case the Orlicz type class L_Ψ is essentially equivalent to an Orlicz space [5, 9].

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