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INTERPOLATION THEORY FOR THE HK-FOURIER TRANSFORM

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Abstract. We use the Henstock–Kurzweil integral and interpolation theory to extend the Fourier cosine transform operator, broadening some classical properties such as the Riemann–Lebesgue lemma. Furthermore, we show that a qualitative difference between the cosine and sine transform is preserved on differentiable functions.

1. Introduction

We shall deal with real Banach spaces denoted by X and with their complexification given by $X + iX$. Also, given two Banach spaces X and Y, we denote by $\mathcal{L}(X, Y)$ the Banach space of all bounded linear operators $T : X \to Y$ with the operator norm given by $\|T\|_{\mathcal{L}(X,Y)} = \sup\{\|T(x)\|_Y : \|x\|_X \le 1\}$. For any $T \in \mathcal{L}(X, Y)$ we define

$$\tilde{T}(x + iy) := T(x) + iT(y) \qquad (x, y \in X).$$

It follows that $\|T\|_{\mathcal{L}(X,Y)} = \|\tilde{T}\|_{\mathcal{L}(X+iX,Y+iY)}$. This procedure has been used by several authors [24, 2, 17].

We recall that for any $p \in [1, \infty)$ and $X \subset \mathbb{R}$, the symbol $\mathcal{L}^1(X)$ denotes the space of all Lebesgue measurable functions $f : X \to \mathbb{R}$ with

$$\|f\|_{\mathcal{L}^p} := \left(\int_X |f(x)|^p \, dx \right)^{1/p} < \infty.$$

Moreover, we denote by $\mathcal{W}_p = \{f : \mathbb{R} \to \mathbb{R} \mid f(x) = 0 \text{ a.e.}\} \equiv$ the subspace of $\mathcal{L}^p(X)$ on which $\|\cdot\|_{\mathcal{L}^p}$ vanishes. It is known that $\|\cdot\|_{\mathcal{L}^p}$ is a seminorm for all $p \in [1, \infty)$ and induces a norm on the quotient space $\mathcal{L}^p(X)/\mathcal{W}_p$, under which it is complete. We will denote this space with respect to its norm by $L^p(X)$, [27]. Similarly, for $p \in [1, \infty)$ we define $\mathcal{L}^p(X, \mathbb{C})$ and $L^p(X, \mathbb{C})$ by considering functions $f : X \to \mathbb{C}$.

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For $p=\infty$ and $f : X \to \mathbb{R}$, we define $\|f\|_\infty$ as the essential supremum of $|f|$, and $\mathcal{L}^\infty(X)$ is the vector space of all Lebesgue measurable functions f for which $\|f\|_\infty < \infty$. Similarly, we define $L^\infty(X)$, $\mathcal{L}^\infty(X,\mathbb{C})$ and $L^\infty(X,\mathbb{C})$. If $A \subsetneq X$ is a Lebesgue measurable set and m denotes the Lebesgue measure, then given a Lebesgue measurable function f defined on A such that $m(X\backslash A)=0$ we will denote by the same symbol f the trivial extension of f to a (measurable) function on X. Furthermore, for a function $f \in \mathcal{L}^p(X)$ or $f \in \mathcal{L}^p(X,\mathbb{C})$, we will call by the same symbol f the (unique) element that defines this function in $L^p(X)$ or in $L^p(X,\mathbb{C})$, respectively. Also, the characteristic function of a set E is given by $\chi_E(x)=1$ if $x\in E$ and zero otherwise.

If f belongs to $L^1(\mathbb{R}) \cap L^p(\mathbb{R})$, the Fourier transform is defined for every real number s as

$$\begin{aligned} \mathcal{F}_p(f)(s) &:= \int_{\mathbb{R}} e^{-isx} f(x)\, dx \\ &= \int_{\mathbb{R}} \cos(sx) f(x)\, dx - i \int_{\mathbb{R}} \sin(sx) f(x)\, dx \\ &= \mathcal{F}_p^c(f)(s) - i\mathcal{F}_p^s(f)(s), \end{aligned} \tag{1.1}$$

where the integral is taken in the Lebesgue sense. $\mathcal{F}_p^c$ and $\mathcal{F}_p^s$ are called Fourier cosine and Fourier sine transforms, respectively. Furthermore, by interpolation theory, the operator $\mathcal{F}_p(f)$ is extended to $L^p(\mathbb{R})$ for $p\in[1,2]$ as a bounded operator

$$\mathcal{F}_p : L^p(\mathbb{R}) \longrightarrow L^q(\mathbb{R})$$

with

$$\|\mathcal{F}_p(f)\|_p \le \gamma_p \|f\|_q,$$

where $1/p+1/q=1$ and

$$\gamma_p = \begin{cases} 1 & \text{if } p=1, \\ (2\pi)^{\frac{1}{q}} \left(\frac{p-1}{p}\right)^{\frac{p-1}{2p}} p^{\frac{1}{2p}} & \text{if } 1<p\le 2. \end{cases}$$

The value of γ_p is given by the Hausdorff–Young inequality [25], the sharp Hausdorff–Young inequality [5, 29], [15, Theorem 5.7] and [3].

For any unbounded subset $X \subset \mathbb{R}$, the space $C_\infty(X)$ denotes the complex valued continuous functions on X vanishing at infinity [25]. We denote the space of bounded variation functions by $BV(\mathbb{R})$ and by $BV_0(\mathbb{R})$ the subspace of functions vanishing at infinity, [12, 4, 31]. Also $BV_0(\mathbb{R},\mathbb{C})$ is the corresponding complexification of $BV_0(\mathbb{R})$.

In [30] the Henstock–Kurzweil integral was employed to study the Fourier transform. In [20, 22] it was proved that (1.1) makes sense as a Henstock–Kurzweil integral on $BV_0(\mathbb{R})$. In fact, we have the following statement in [23].

Definition 1.1. The *HK-Fourier transform* exists for every $s \neq 0$, and is defined by

$$\mathcal{F}_{HK} : \mathcal{L}^1(\mathbb{R}) + BV_0(\mathbb{R}) \to C_\infty(\mathbb{R}\backslash\{0\}),$$
$$\mathcal{F}_{HK}(f)(s) := \int_{-\infty}^{\infty} e^{-isx} f(x)\, dx,$$

where the integral is in the Henstock–Kurzweil sense. Analogously, we define the HK-Fourier cosine transform $\mathcal{F}^c_{HK}$ and the HK-sine Fourier transform $\mathcal{F}^s_{HK}$ as in (1.1).

We say "HK-Fourier transform" in order to emphasize the use of the Henstock–Kurzweil integral [30]. Moreover, $\mathcal{F}_{HK}(f)(s)$ is pointwise defined and is continuous except at zero; see example 3(d) in [30]. Note that $\mathcal{F}_{HK}$ is well defined because the Henstock–Kurzweil integral contains the Lebesgue integral, [14, 19]. $\mathcal{F}_1$ can be seen as an extension of the HK-Fourier transform restricted to $BV_0(\mathbb{R})$,

$$\mathcal{F}_{HK} : BV_0(\mathbb{R}) \to C_\infty(\mathbb{R}\backslash\{0\}).$$

Moreover, $\mathcal{F}_p$ is an extension of $\mathcal{F}_1$, so that $\mathcal{F}_p$ is an extension of $\mathcal{F}_{HK}$.

The relation between $\mathcal{F}_p$ and $\mathcal{F}_{HK}$ was first studied in [23], while the operator $\mathcal{F}^c_{HK}$ was studied in [3]. This work builds on these references.

2. Henstock–Kurzweil Fourier transform

The space of Henstock–Kurzweil integrable functions defined on an interval I is denoted by $\mathcal{HK}(I)$. This space is a seminormed space with the Alexiewicz seminorm, defined as

$$\|f\|_{\mathcal{HK}} = \sup\left\{ \left| \int_c^d f(x)\, dx \right| : [c,d] \subset I \right\}.$$

The quotient space $\mathcal{HK}/\mathcal{W}(I)$ will be denoted by $HK(I)$, where $\mathcal{W}(I)$ is the subspace of $HK(I)$ for which the Alexiewicz seminorm vanishes [7]. The completion will be denoted by $\widehat{HK}(I)$ and its complexification will be written as $\widehat{HK}(\mathbb{R},\mathbb{C})$.

We study the HK-Fourier cosine transform defined by

$$\mathcal{F}^c_{HK}(f)(s) = \int_{-\infty}^{\infty} \cos(sx) f(x)\, dx \quad (s \neq 0).$$

Notice that for $s = 0$ and $f \in BV_0(\mathbb{R})$, $\mathcal{F}^c_{HK}(f)(0)$ might not be defined. Also, we have that

$$\mathcal{F}^c_1(f)(s) = \mathcal{F}^c_{HK}(f)(s) \tag{2.1}$$

for all $f \in \mathcal{L}^1(\mathbb{R}) \cap BV_0(\mathbb{R})$ and $s \in \mathbb{R}$. However, a partial result about the question of continuity at $s = 0$ was proved in [3, Theorem 1]. In fact, $\mathcal{F}^c_{HK}$ is bounded while $\mathcal{F}^s_{HK}$ is not. Actually, Theorem 1 and Proposition 3 in [3] imply the following statement.

Theorem 2.1.

(i) The HK-Fourier cosine transform is a bounded linear operator from $BV_0(\mathbb{R})$ into $HK(\mathbb{R})$.
(ii) The Fourier transform is a densely defined closed operator from $L^2(\mathbb{R})$ into $HK(\mathbb{R})$.

We shall show that differences and similitudes between the Fourier cosine and Fourier sine transforms also hold on the classical Sobolev space $W^{1,1}(\mathbb{R})$. It is expected that these transforms are bounded operators with the same domain and codomain for functions with enough smoothness, for example as in the Schwartz space [25]. See also [16].

3. Interpolation theory

We consider a couple (X,Y) of complex Banach spaces such that X and Y are continuously embedded in a Hausdorff topological vector space V, i.e., $X \subset V$ and $Y \subset V$ with continuous inclusion. This couple is called a complex interpolation couple. In this case the intersection $X \cap Y$ is a linear subspace of V, and it is a Banach space under the norm

$$\|v\|_{X\cap Y} = \max\{\|v\|_X, \|v\|_Y\}.$$

The sum $X+Y = \{x+y : x \in X, y \in Y\}$ is a linear subspace of V and it is endowed with the norm

$$\|v\|_{X+Y} = \inf\{\|x\|_X + \|y\|_Y : x \in X,\, y \in Y,\, x+y = v\}.$$

Remark 3.1. It follows from [18] that the space $X+Y$ is isometric to the quotient space $(X\times Y)/D$, where $D = \{(d,-d) \in X\times Y : d \in X\cap Y\}$. Since V is a Hausdorff space, D is closed, so $X+Y$ is a Banach space. Moreover, X and Y are continuously embedded in $X+Y$.

Throughout this section we shall consider $\mathbb{S} = \{z \in \mathbb{C} : 0 \le \mathrm{Re}(z) \le 1\}$ and we shall use the complex space $X+Y$ and the space $\mathbb{F}(X,Y)$ of functions $f : \mathbb{S} \to X+Y$ holomorphic on the interior of the strip $\mathbb{S}$ and continuous up to its boundary, such that the maps $t \mapsto f(it)$ and $t \mapsto f(1+it)$ are continuous from the real line into X and Y, respectively. Therefore, $\mathbb{F}(X,Y)$ is a Banach space with the norm given by

$$\|f\|_{\mathbb{F}} := \max\left\{\sup_{t\in\mathbb{R}} \|f(it)\|_X, \sup_{t\in\mathbb{R}} \|f(1+it)\|_Y\right\} < \infty.$$

These facts can be consulted in [18, Ch. 2], [6, Ch. 4], [9, Ex. 2.6.6], [28, Ch. 2], [13, Ch. 4] and [10, 1–4].

Definition 3.2. For every $\theta \in (0,1)$, the space $[X,Y]_\theta$ consists of all $a \in X+Y$ such that $a = f(\theta)$ for some $f \in \mathbb{F}(X,Y)$ and the norm on $[X,Y]_\theta$ is

$$\|a\|_{[\theta]} = \inf\{\|f\|_{\mathbb{F}} : f(\theta) = a,\, f \in \mathbb{F}(X,Y)\}.$$

Remark 3.3. The space $X\cap Y$ is dense in $[X,Y]_\theta$ and $[X,Y]_\theta$ is isomorphic to the quotient space $\mathbb{F}(X,Y)/\mathfrak{N}_\theta$, where $\mathfrak{N}_\theta$ is the subset of $\mathbb{F}(X,Y)$ consisting of the functions vanishing at $z=\theta$. Moreover, $\mathfrak{N}_\theta$ is closed (see [6, 18]).

Theorem 3.4. *The space* $[X,Y]_\theta$ *is a Banach space and an intermediate space with respect to* (X,Y)*, i.e.,*

$$X \cap Y \subset [X,Y]_\theta \subset X+Y$$

with continuous inclusion.

Remark 3.5. It follows from [18, Corollary 2.8, Proposition 2.10] that for each $\theta \in (0,1)$,

$$(X,Y)_{\theta,1} \subset [X,Y]_\theta \subset (X,Y)_{\theta,\infty},$$

where the spaces $(X,Y)_{\theta,p}$ are defined by the real method of interpolation. See also [6, Theorem 4.7.1].

Theorem 3.6. *Let* $(X_1,Y_1),(X_2,Y_2)$ *be complex interpolation couples. If* T *belongs to* $\mathcal{L}(X_1,X_2)\cap\mathcal{L}(Y_1,Y_2)$*, then the restriction of* T *to* $[X_1,Y_1]_\theta$ *belongs to* $\mathcal{L}\big([X_1,Y_1]_\theta,[X_2,Y_2]_\theta\big)$ *for every* $\theta\in(0,1)$*. Moreover,*

$$\|T\|_{\mathcal{L}([X_1,Y_1]_\theta,[X_2,Y_2]_\theta)} \le \|T\|^{1-\theta}_{\mathcal{L}(X_1,X_2)}\|T\|^{\theta}_{\mathcal{L}(Y_1,Y_2)}.$$

In order to construct the interpolation space of $L^1(\mathbb{R})$ and $BV_0(\mathbb{R})$ we consider the space $\mathcal{L}^1(\mathbb{R})\cap BV_0(\mathbb{R})$ with given norm $\|\cdot\|_{\mathcal{L}^1(\mathbb{R})\cap BV_0(\mathbb{R})} := \max\{\|\cdot\|_{\mathcal{L}^1},\|\cdot\|_{BV}\}$.

Lemma 3.7. $\mathcal{L}^1(\mathbb{R})\cap BV_0(\mathbb{R})$ *is a Banach space with the given norm.*

Proof. Since $BV_0(\mathbb{R})$ is a Banach space, then given a Cauchy sequence $(f_n)_{n\ge1}$ on $\mathcal{L}^1(\mathbb{R})\cap BV_0(\mathbb{R})$ there is $f\in BV_0(\mathbb{R})$ such that

$$\|f_n - f\|_{BV}\to 0 \quad (n\to\infty).$$

This yields uniform convergence of the sequence to f. Similarly, there exists $[\tilde{f}]\in L^1(\mathbb{R})$ such that

$$\|f_n-\tilde{f}\|_{L^1}\to 0 \quad (n\to\infty).$$

It follows that there exists a subsequence $(f_{n_k})_{k\ge1}$ of $(f_n)_{n\ge1}$ converging pointwise a.e. to f; see [27, 8]. From the fact that $(f_n)_{n\ge1}$ converges uniformly to f, we get that $f(x)=\tilde{f}(x)$ a.e., yielding $f\in\mathcal{L}^1(\mathbb{R})$ and

$$\lim_{n\to\infty}\int_{\mathbb{R}}|f_n(x)-f(x)|\,dx = 0. \qquad \square$$

On the product space $\mathcal{L}^1(\mathbb{R})\times BV_0(\mathbb{R})$ with given norm $\|(f,g)\|_{\mathcal{L}^1\times BV_0} := \|f\|_{\mathcal{L}^1}+\|g\|_{BV}$, we consider the quotient space $\big(\mathcal{L}^1(\mathbb{R})\times BV_0(\mathbb{R})\big)/D$ where $D := \big\{(f,-f)\in\mathcal{L}^1(\mathbb{R})\times BV_0(\mathbb{R}) : f\in\mathcal{L}^1(\mathbb{R})\cap BV_0(\mathbb{R})\big\}$. So, we set

$$\mathcal{L}^1(\mathbb{R})+BV_0(\mathbb{R}) := \big(\mathcal{L}^1(\mathbb{R})\times BV_0(\mathbb{R})\big)/D.$$

Therefore, if $a\in\mathcal{L}^1(\mathbb{R})+BV_0(\mathbb{R})$, then it is an equivalence class given by $a=(f,g)+D$. Nevertheless, we shall write $a=f+g$ to simplify notation. Also, we define

$$\|a\|_{\mathcal{L}^1+BV_0} := \inf_{(h,-h)\in D}\|f-h\|_{\mathcal{L}^1}+\|g+h\|_{BV}.$$

This is a norm, by standard arguments. Then we consider the completion of the space $\mathcal{L}^1(\mathbb{R})+BV_0(\mathbb{R})$, denoted by $\mathcal{L}^1\widehat{(\mathbb{R})+B}V_0(\mathbb{R})$. In addition, on the product space $L^1(\mathbb{R}) \times BV_0(\mathbb{R})$ with the usual norm $\|([f],g)\|_{L^1\times BV_0} = \|[f]\|_{L^1} + \|g\|_{BV}$, we make

$$D' = \left\{([f],-f) \in L^1(\mathbb{R}) \times BV_0(\mathbb{R}) : f \in \mathcal{L}^1(\mathbb{R}) \cap BV_0(\mathbb{R})\right\}.$$

Due to $\mathcal{L}^1(\mathbb{R}) \cap BV_0(\mathbb{R})$ being complete, D' is closed in $L^1(\mathbb{R}) \times BV_0(\mathbb{R})$. We define

$$L^1(\mathbb{R}) + BV_0(\mathbb{R}) = \big(L^1(\mathbb{R}) \times BV_0(\mathbb{R})\big)/D'.$$

Thus the sum space $L^1(\mathbb{R}) + BV_0(\mathbb{R})$ is a Banach space with the quotient norm [26]. Its elements are equivalence classes of the form $\bar{a} = \big[f+g\big] = ([f],g) + D'$; however, we will just write $\bar{a} = f+g$. We have the following characterization.

Proposition 3.8. *The space $L^1(\mathbb{R}) + BV_0(\mathbb{R})$ is isometric to $\widehat{\mathcal{L}^1(\mathbb{R}) + BV}_0(\mathbb{R})$.*

Proof. $f \in \mathcal{L}^1(\mathbb{R})$ yields $[f] \in L^1(\mathbb{R})$ with $\|[f]\|_{L^1} = \|f\|_{\mathcal{L}^1}$. Conversely, if $[f]$ belongs to $L^1(\mathbb{R})$ then there is $\tilde{f} \in \mathcal{L}^1(\mathbb{R})$ such that $f = \tilde{f}$ a.e. Then, for each $a = (f,g) + D \in \mathcal{L}^1(\mathbb{R}) + BV_0(\mathbb{R})$, we define $\bar{a} = ([f],g) + D'$ in $L^1(\mathbb{R}) + BV_0(\mathbb{R})$. We get

$$\begin{aligned}\|(f,g)+D\|_{\mathcal{L}^1+BV_0} &= \inf_{(h,-h)\in D} \|f-h\|_{\mathcal{L}^1} + \|g+h\|_{BV} \\ &= \inf_{([h],-h)\in D'} \|[f-h]\|_{L^1} + \|g+h\|_{BV} \\ &= \|([f],g)+D'\|_{L^1+BV_0}.\end{aligned}$$

Therefore, the map $a \mapsto \bar{a}$ from $\mathcal{L}^1(\mathbb{R}) + BV_0(\mathbb{R})$ into $L^1(\mathbb{R}) + BV_0(\mathbb{R})$ has dense range, due to $\mathcal{L}^1(\mathbb{R})$ being dense in $L^1(\mathbb{R})$. The map extends to an isometry from the completion $\widehat{\mathcal{L}^1(\mathbb{R}) + BV}_0(\mathbb{R})$ onto $L^1(\mathbb{R}) + BV_0(\mathbb{R})$, implying the Proposition. ☐

Therefore, we have characterized the real space $L^1(\mathbb{R}) + BV_0(\mathbb{R})$. The complexification space of this space is given by

$$L^1(\mathbb{R},\mathbb{C}) + BV_0(\mathbb{R},\mathbb{C}) := \big(L^1(\mathbb{R}) + BV_0(\mathbb{R})\big) + i\big(L^1(\mathbb{R}) + BV_0(\mathbb{R})\big).$$

Similarly, we define the real space $L^\infty(\mathbb{R}) + \widehat{HK}(\mathbb{R})$ and its complexification $L^\infty(\mathbb{R},\mathbb{C}) + \widehat{HK}(\mathbb{R},\mathbb{C})$. We will consider complex spaces and omit the symbol $(\mathbb{R},\mathbb{C})$ to simplify notation. Furthermore, for the complex interpolation couples

$$\big(L^1, BV_0\big) \quad \text{and} \quad \big(L^\infty, \widehat{HK}\big) \tag{3.1}$$

we say that T is a bounded linear operator from (L^1, BV_0) to $(L^\infty, \widehat{HK})$ if and only if $T \in \mathcal{L}\big(L^1 + BV_0, L^\infty + \widehat{HK}\big)$ such that $T \in \mathcal{L}\big(L^1, L^\infty\big)$ and $T \in \mathcal{L}\big(BV_0, \widehat{HK}\big)$.

We say that the complex spaces $\mathfrak{A}$ and $\mathfrak{B}$ are intermediate spaces between the couples in (3.1) if and only if

$$\mathcal{L}^1 \cap BV_0 \subset \mathfrak{A} \subset L^1 + BV_0 \quad \text{and} \quad L^\infty \cap \widehat{HK} \subset \mathfrak{B} \subset L^\infty + \widehat{HK}.$$

$\mathfrak{A}$ and $\mathfrak{B}$ are called interpolation spaces with respect to the couples in (3.1) if and only if $\mathfrak{A}$ and $\mathfrak{B}$ are intermediate spaces with the following property: $T \in \mathcal{L}(L^1 + BV_0, L^\infty + \widehat{HK})$ implies that the restriction of T to $\mathfrak{A}$ belongs to $\mathcal{L}(\mathfrak{A},\mathfrak{B})$.

From Theorem 3.4, we have the interpolation spaces $[L^1, BV_0]_\theta$ and $[L^\infty, \widehat{HK}]_\theta$ with respect to (L^1, BV_0) and $(L^\infty, \widehat{HK})$ for each $\theta \in (0,1)$. We deal with the operators $\mathcal{F}_1^c$ and $\mathcal{F}_{HK}^c$ given in (1.1) and Definition 1.1 and their extensions to the complexification of the spaces. We use the same symbols for the extended operators. Then we define the operator

$$\begin{aligned} &\mathfrak{F}_1^c : L^1(\mathbb{R},\mathbb{C}) + BV_0(\mathbb{R},\mathbb{C}) \longrightarrow L^\infty(\mathbb{R},\mathbb{C}) + \widehat{HK}(\mathbb{R},\mathbb{C}) \\ &\mathfrak{F}_1^c(f+g)(s) := \mathcal{F}_1^c(f)(s) + \mathcal{F}_{HK}^c(g)(s). \end{aligned} \tag{3.2}$$

Formula (3.2) is well defined on $\mathcal{L}^1(\mathbb{R}) + BV_0(\mathbb{R})$. By interpolation theory, $\mathfrak{F}_1^c$ is extended to $L^1(\mathbb{R}) + BV_0(\mathbb{R})$ for each $s \neq 0$. Thus, from Theorem 2.1 and Theorem 3.6 we conclude that

$$\mathfrak{F}_1^c \in \mathcal{L}([L^1, BV_0]_\theta, [L^\infty, \widehat{HK}]_\theta).$$

The following estimate for its norm is valid:

$$\|\mathfrak{F}_1^c\|_{\mathcal{L}([L^1,BV_0]_\theta,[L^\infty,HK]_\theta)} \leq \|\mathcal{F}_1^c\|_{\mathcal{L}(L^1,L^\infty)}^{1-\theta} \|\mathcal{F}_{HK}^c\|_{\mathcal{L}(BV_0,HK)}^{\theta} \leq \mathfrak{C}^\theta,$$

for every $\theta \in (0,1)$, where

$$\mathfrak{C} = 4\pi\,\mathrm{Si}(\pi) \quad \text{and} \quad \mathrm{Si}(x) := \frac{2}{\pi}\int_0^x \frac{\sin(y)}{y}\,dy. \tag{3.3}$$

Furthermore, from Remark 3.5, we have that

$$(L^1, BV_0)_{\theta,1} \subset [L^1, BV_0]_\theta \subset (L^1, BV_0)_{\theta,\infty}$$

holds for each $\theta \in (0,1)$.

Proposition 3.9. *For $f \in [L^1, BV_0]_\theta$, the formula*

$$\mathfrak{F}_1^c(f)(s) = \int_{-\infty}^{\infty} \cos(sx) f(x)\,dx$$

holds true pointwise almost everywhere and the Riemann–Lebesgue lemma is satisfied: $\mathfrak{F}_1^c(f)(s) \to 0$ as $|s| \to \infty$.

Proof. If $f = f_1 + f_0 = \tilde{f}_1 + \tilde{f}_0$ belongs to $L^1(\mathbb{R},\mathbb{C}) + BV_0(\mathbb{R},\mathbb{C})$, then

$$(f_1 - \tilde{f}_1, f_0 - \tilde{f}_0) \in D'.$$

This yields $f_1 - \tilde{f}_1 = \tilde{f}_0 - f_0$ with $f_1 - \tilde{f}_1 \in L^1(\mathbb{R},\mathbb{C})$ and $\tilde{f}_0 - f_0 \in BV_0(\mathbb{R},\mathbb{C})$.

Since $\mathcal{F}_1^c$ and $\mathcal{F}_{HK}^c$ coincide on $L^1(\mathbb{R},\mathbb{C}) \cap BV_0(\mathbb{R},\mathbb{C})$ due to (2.1), we conclude that

$$\mathcal{F}_1^c(f_1) + \mathcal{F}_{HK}^c(f_0) = \mathcal{F}_1^c(\tilde{f}_1) + \mathcal{F}_{HK}^c(\tilde{f}_0).$$

As a consequence, the value of $\mathfrak{F}_1^c(f)(s)$ does not depend on the representation of $f \in [L^1, BV_0]_\theta$, for each $\theta \in (0,1)$. From Theorem 3.6, for every $f \in [L^1, BV_0]_\theta$, there exist $f_1 \in L^1(\mathbb{R},\mathbb{C})$ and $f_0 \in BV_0(\mathbb{R},\mathbb{C})$ such that $f = f_1 + f_0$ and for each

$s \neq 0$,

$$\begin{aligned}
\mathfrak{F}_1^c(f)(s) &= \mathfrak{F}^c(f_1 + f_0)(s) \\
&= \mathcal{F}_1^c(f_1)(s) + \mathcal{F}_{HK}^c(f_0)(s) \\
&= \int_{-\infty}^{\infty} \cos(sx) f_1(x)\, dx + \int_{-\infty}^{\infty} \cos(sx) f_0(x)\, dx \\
&= \int_{-\infty}^{\infty} \cos(sx)\big(f_1(x) + f_0(x)\big)\, dx \\
&= \int_{-\infty}^{\infty} \cos(sx) f(x)\, dx.
\end{aligned} \tag{3.4}$$

Then, (3.4) establishes that the HK-Fourier cosine transform on $[L^1, BV_0]_\theta$ has an integral representation. From the Riemann–Lebesgue lemma [21, Lemma 2] we conclude that $\mathfrak{F}_1^c(f)(s) \to 0$ as $|s| \to \infty$. □

In [1, 8], the Sobolev spaces $W^{1,p}(\mathbb{R})$ are defined and for their complexification $W^{1,p}(\mathbb{R}, \mathbb{C}) := W^{1,p}(\mathbb{R}) + iW^{1,p}(\mathbb{R})$ we have the next statement.

Lemma 3.10. *For each $\theta \in (0, 1)$,*

$$W^{1,1}(\mathbb{R}, \mathbb{C}) \subset [L^1(\mathbb{R}, \mathbb{C}), BV_0(\mathbb{R}, \mathbb{C})]_\theta$$

with continuous inclusion.

Proof. First we recall that $W^{1,1}(\mathbb{R}) \subset \mathcal{L}^1(\mathbb{R}) \cap BV_0(\mathbb{R})$. From [4, Theorem 7.5] we have

$$\begin{aligned}
\|u\|_{L^1 \cap BV_0} &:= \max\big\{\|u\|_{L^1}, \|u\|_{BV}\big\} \\
&\leq \|u\|_{L^1} + \|u\|_{BV} \\
&= \|u\|_{L^1} + \|u'\|_{L^1} \\
&= \|u\|_{W^{1,1}}.
\end{aligned}$$

If u belongs to $W^{1,1}(\mathbb{R}, \mathbb{C})$, from [18, Proposition 2.4] we get

$$\|u\|_{[\theta]} \leq \max\{\|u\|_{L^1}, \|u\|_{BV}\} \leq \|u\|_{W^{1,1}(\mathbb{R},\mathbb{C})},$$

for every $\theta \in (0, 1)$. □

Corollary 3.11. *For $u \in W^{1,1}(\mathbb{R}, \mathbb{C})$, $\mathfrak{F}_1^c(u)$ belongs to $HK(\mathbb{R}, \mathbb{C})$.*

The proof of Corollary 3.11 follows from the fact that $W^{1,1}(\mathbb{R}) \subset BV_0(\mathbb{R})$, and then by Theorem 2.1,

$$\mathcal{F}_{HK}^c\big(W^{1,1}(\mathbb{R})\big) \subset HK(\mathbb{R}).$$

Therefore, the range of the Sobolev space $W^{1,1}(\mathbb{R}, \mathbb{C})$ under the HK-Fourier cosine transform is contained in $HK(\mathbb{R}, \mathbb{C})$. Explicitly,

$$\mathfrak{F}_1^c\big(W^{1,1}(\mathbb{R}, \mathbb{C})\big) \subset HK(\mathbb{R}, \mathbb{C}). \tag{3.5}$$

The Fourier cosine and sine transforms are continuous operators on $L^2(\mathbb{R}, \mathbb{C})$, while their qualitative differences appear even on the space of functions $W^{1,1}(\mathbb{R}, \mathbb{C})$ that have a degree of regularity. In the following example we show this difference.

Example 3.12. Let us define

$$h(x) := \begin{cases} \dfrac{1}{2-\log(x)} & \text{if } x \in (0,1], \\ 0 & \text{if } x > 1. \end{cases}$$

For each $x \in (0,1)$, we have

$$h'(x) = \frac{1}{x[2-\log(x)]^2}.$$

We extend h over $\mathbb{R}$ as an odd map. Also, we consider an even function $\varphi \in C_c^\infty(\mathbb{R})$ such that $0 \le \varphi(x) \le 1$, with $\varphi(x) = 1$ for $|x| \le 1/2$ and vanishing for $|x| \ge 1$. We define $f(x) := h(x)\varphi(x)$, for all $x \in \mathbb{R}$. Thus, f is an odd map belonging to $W^{1,1}(\mathbb{R}) \subset L^1(\mathbb{R}) \cap BV_0(\mathbb{R})$ and

$$\mathcal{F}^s_{HK}(f)(s) = 2\int_0^\infty \sin(sx) f(x)\, dx, \quad \text{for all } s \ge 0.$$

We analyze the convergence of the Henstock–Kurzweil integral:

$$\int_0^\infty \mathcal{F}^s_{HK}(f)(s)\, ds. \tag{3.6}$$

For any fixed $s > 0$, the map $x \mapsto \sin(sx)$ belongs to $HK[0,M]$ for $0 < M < \infty$, and we have

$$\|\sin(s\cdot)\|_{HK[0,M]} = \sup_{0<u<v<M} \left| \int_u^v \sin(st)\, dt \right| \le \frac{2}{s}.$$

Thus, for $0 < b < \infty$, we get from Lebesgue's dominated convergence theorem, Fubini's theorem and Hake's theorem [4]:

$$\int_0^b \int_0^\infty \sin(sx) f(x)\, dx\, ds = \lim_{M\to\infty} \int_0^M \frac{1-\cos(bx)}{x} f(x)\, dx.$$

In fact,

$$\int_0^M \frac{1-\cos(bx)}{x} f(x)\, dx = \int_0^b \frac{1-\cos(y)}{y} f(y/b)\, dy.$$

Now, for $\delta = 1/4$,

$$\int_0^b \frac{1-\cos(y)}{y} f(y/b)\, dy = \int_0^\delta \frac{1-\cos(y)}{y} f(y/b)\, dy + \int_\delta^b \frac{1-\cos(y)}{y} f(y/b)\, dy. \tag{3.7}$$

Since $f(y/b) \to 0$ as $b \to \infty$, we have that

$$\lim_{b\to\infty} \int_0^\delta \frac{1-\cos(y)}{y} f(y/b)\, dy = 0.$$

For the second integral on the right side of (3.7) we have

$$\int_\delta^b \frac{1-\cos(y)}{y} f(y/b)\, dy = \int_\delta^b \frac{f(y/b)}{y}\, dy + \int_\delta^b \frac{-\cos(y)}{y} f(y/b)\, dy = I_1 + I_2.$$

Integrating by parts,

$$\lim_{b\to\infty} I_2 = \lim_{b\to\infty} -\int_\delta^b -\sin(y)\left(\frac{b^{-1}yf'(y/b)-f(y/b)}{y^2}\right)dy$$
$$= \lim_{b\to\infty}\int_{\delta/b}^{1}\frac{\sin(bt)}{bt}f'(t)\,dt - \int_{\delta}^{b}\frac{\sin(y)f(y/b)}{y^2}\,dy.$$

By Lebesgue's dominated convergence theorem we conclude that

$$\lim_{b\to\infty}\int_\delta^b \frac{\sin(y)f'(y/b)}{by}\,dy = \lim_{b\to\infty}\int_\delta^b -\frac{\sin(y)}{y^2}f(y/b)\,dy = 0.$$

Therefore the limit of I_2 is zero. Writing explicitly the integrand of the integral I_1 we get

$$I_1 = \int_{\delta/b}^{1/2}\frac{1}{u[2-\log(u)]} + \int_{1/2}^{1}\frac{\varphi(u)}{u[2-\log(u)]}\,du$$

and

$$\lim_{b\to\infty}\int_{\delta/b}^{1/2}\frac{1}{u[2-\log(u)]}\,du = \infty.$$

Therefore, the integral in (3.6) does not exist and $\mathcal{F}^s_{HK}(f)$ does not belong to $HK(\mathbb{R})$. In conclusion,

$$\mathcal{F}^s_2\big(W^{1,1}\big) = \mathcal{F}^s_1\big(W^{1,1}\big) = \mathcal{F}^s_{HK}\big(W^{1,1}(\mathbb{R})\big) \not\subseteq HK(\mathbb{R}). \tag{3.8}$$

Then, the HK-Fourier sine transform remains unbounded on $W^{1,1}(\mathbb{R})$, in contrast with relation (3.5). Also, in [3, Example 1] it was established that

$$\mathcal{F}^s_{HK}\big(BV_0(\mathbb{R})\backslash W^{1,1}(\mathbb{R})\big) \not\subseteq HK(\mathbb{R}).$$

The function given in Example 3.12 is a slight variation of one considered in [11].

We can proceed in the same way to define the space $L^2(\mathbb{R},\mathbb{C}) + BV_0(\mathbb{R},\mathbb{C})$. Therefore, we have the continuous inclusions

$$L^2(\mathbb{R},\mathbb{C})\cap BV_0(\mathbb{R},\mathbb{C}) \subset [L^2(\mathbb{R},\mathbb{C}), BV_0(\mathbb{R},\mathbb{C})]_\theta \subset L^2(\mathbb{R},\mathbb{C}) + BV_0(\mathbb{R},\mathbb{C}),$$

for $0<\theta<1$. Now for the extended operators $\mathcal{F}^c_2$ and $\mathcal{F}^c_{HK}$ on $L^2(\mathbb{R},\mathbb{C})$ and on $BV_0(\mathbb{R},\mathbb{C})$ respectively, we define the map

$$\mathfrak{F}^c_2 : L^2(\mathbb{R},\mathbb{C}) + BV_0(\mathbb{R},\mathbb{C}) \longrightarrow L^2(\mathbb{R},\mathbb{C}) + \widehat{HK}(\mathbb{R},\mathbb{C})$$
$$\mathfrak{F}^c_2(f+g) := \mathcal{F}^c_2(f) + \mathcal{F}^c_{HK}(g).$$

So, by Theorem 3.6, we have

$$\mathfrak{F}^c_2 \in \mathcal{L}([L^2, BV_0]_\theta, [L^2, \widehat{HK}]_\theta),$$

with the following estimate for its norm:

$$\|\mathfrak{F}^c_2\|_{\mathcal{L}([L^2,BV_0]_\theta,[L^2,\widehat{HK}]_\theta)} \le \|\mathcal{F}^c_2\|^{1-\theta}_{\mathcal{L}(L^2,L^2)}\|\mathcal{F}^c_{HK}\|^{\theta}_{\mathcal{L}(BV_0,\widehat{HK})} \le (2\pi)^{\frac{1-\theta}{2}}\mathfrak{C}^\theta,$$

for every $\theta \in (0,1)$ and $\mathfrak{C}$ given by (3.3).

Similarly, for the operators $\mathcal{F}_p^c$ and $\mathcal{F}_{HK}^c$ on $L^p(\mathbb{R},\mathbb{C})$ and on $BV_0(\mathbb{R},\mathbb{C})$ respectively, we define

$$\begin{aligned}&\mathfrak{F}_p^c : L^p(\mathbb{R},\mathbb{C}) + BV_0(\mathbb{R},\mathbb{C}) \longrightarrow L^q(\mathbb{R},\mathbb{C}) + \widehat{HK}(\mathbb{R},\mathbb{C})\\ &\mathfrak{F}_p^c(f) := \mathcal{F}_p^c(f_p) + \mathcal{F}_{HK}^c(g),\end{aligned}$$

where $f = f_p + g$ with $1/p + 1/q = 1$. This operator is a generalization of the map considered in [3, Corollary 1]. For the couples given by $X_1 = L^p(\mathbb{R},\mathbb{C})$ and $X_2 = L^q(\mathbb{R},\mathbb{C})$, with $1 \le p \le 2$, and $Y_1 = BV_0(\mathbb{R},\mathbb{C})$ and $Y_2 = \widehat{HK}(\mathbb{R},\mathbb{C})$ we have from Theorem 3.6 that

$$\mathfrak{F}_p^c \in \mathcal{L}\big([L^p, BV_0]_\theta, [L^q, \widehat{HK}]_\theta\big)$$

for every $\theta \in (0,1)$, and the following estimate for the norm:

$$\|\mathfrak{F}_p^c\|_{\mathcal{L}([L^p,BV_0]_\theta,[L^q,\widehat{HK}]_\theta)} \le \|\mathcal{F}_p^c\|_{\mathcal{L}(L^p,L^q)}^{1-\theta} \|\mathcal{F}_{HK}^c\|_{\mathcal{L}(BV_0,\widehat{HK})}^{\theta} \le \gamma_p^{1-\theta}\mathfrak{C}^\theta,$$

where $\mathfrak{C}$ is given in (3.3).

For $1 < p < 2$, the relation between $\mathfrak{F}_p^c$, $\mathfrak{F}_1^c$ and $\mathfrak{F}_2^c$ is given by the decomposition of $L^p(\mathbb{R})$ in [23], which implies that for each $f_p + g \in L^p(\mathbb{R}) + BV_0(\mathbb{R})$ there exist $f_1 \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ and $f_2 \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ such that

$$f_p + g = (f_1 + g) + f_2 = f_1 + (f_2 + g).$$

Corollary 3.13. *For $1 < p < 2$, $\mathfrak{F}_p^c(f_p + g) = \mathfrak{F}_1^c(f_1 + g) + \mathfrak{F}_2^c(f_2)$.*

Proof. This follows by taking a sequence $(f_n)_{n\ge1}$ on $L^p(\mathbb{R})$ such that $f_n \to f$ with $f = f_1 + f_2 \in L^p(\mathbb{R})$, and using that the sequence $f_n = f_{1,n} + f_{2,n}$ has the property $f_{i,n} \to f_i$ in the norm of $L^i(\mathbb{R})$, $i = 1, 2$; see [23]. ☐

Proposition 3.14. *For $f \in W^{1,p}(\mathbb{R})$ with $1 < p \le 2$, the formula*

$$\mathcal{F}_p(f')(s) = is\mathcal{F}_p(f)(s)$$

holds true pointwise almost everywhere.

Proof. If $f \in W^{1,p}(\mathbb{R})$ then f, f' belong to $L^p(\mathbb{R})$ with $f(x) \to 0$ as $|x| \to \infty$; see [8, Corollary 8.9]. Next, for each $n \ge 1$, we let $\varphi_n(x) := \chi_{[-n,n]}(x)f(x)$ and $\gamma_n(x) := \chi_{[-n,n]}f'(x)$. So $\|\varphi_n - f\|_{L^p} \to 0$ as $n \to \infty$, and there exists a subsequence $(\varphi_{n_k})_{k\ge1}$ such that $\mathcal{F}_p(\varphi_{n_k})(s) \to \mathcal{F}_p(f)(s)$ a.e. as $k \to \infty$. Therefore,

$$\mathcal{F}_p(f)(s) = \lim_{k\to\infty} \mathcal{F}_p(\varphi_{n_k})(s) = \lim_{k\to\infty} \int_{-n_k}^{n_k} e^{-isx} f(x)\,dx$$

almost everywhere. Integrating by parts [8, Corollary 8.10], for each $k \ge 1$,

$$\int_{-n_k}^{n_k} e^{-isx} f'(x)\,dx = e^{-isx} f(x)\Big|_{-n_k}^{n_k} - \int_{-n_k}^{n_k} -ise^{-isx} f(x)\,dx.$$

Thus,

$$\begin{aligned}\mathcal{F}_p(f')(s) &= \lim_{k\to\infty} \mathcal{F}_p(\gamma_{n_k})(s) = \lim_{k\to\infty} \int_{-n_k}^{n_k} e^{-isx} f'(x)\,dx \\ &= is \lim_{k\to\infty} \int_{-n_k}^{n_k} e^{-isx} f(x)\,dx = is\,\mathcal{F}_p(f)(s)\end{aligned}$$

almost everywhere. □

Proposition 3.15. *If $f \in W^{1,p}(\mathbb{R})$ with $1 < p \le 2$, then $\mathcal{F}_p(f) \in L^1(\mathbb{R})$.*

Proof. If $f \in W^{1,p}(\mathbb{R})$ then $\mathcal{F}_p(f)$ belongs to $L^q(\mathbb{R})$ with $1/p + 1/q = 1$ and for $A = \{s \in \mathbb{R} : |s| \ge 1\}$ we get, by Proposition 3.14 and Hölder's inequality,

$$\int_A |\mathcal{F}_p(f)(s)|\,ds = \int_A \left| \frac{1}{s}\,\mathcal{F}_p(f')(s)\,ds \right| \le \|1/(\cdot)\|_{L^p(A)} \|\mathcal{F}_p(f')\|_{L^q(A)} < \infty,$$

with $1/p + 1/q = 1$. Therefore, $\mathcal{F}_p(f) \in L^1(\mathbb{R})$. □

As a consequence of Proposition 3.15, the range of $W^{1,p}(\mathbb{R})$, for $1 < p \le 2$, under the action of the L^p-Fourier transform operator is contained in $L^1(\mathbb{R})$. Explicitly,

$$\mathcal{F}_p\big(W^{1,p}(\mathbb{R})\big) \subset L^1(\mathbb{R}) \subsetneq HK(\mathbb{R}).$$

This relation contrasts with (3.8).

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