SOME RELATIONS BETWEEN THE SKEW SPECTRUM OF AN ORIENTED GRAPH AND THE SPECTRUM OF CERTAIN CLOSELY ASSOCIATED SIGNED GRAPHS

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ABSTRACT. Let $R_{G'}$ be the vertex-edge incidence matrix of an oriented graph G'. Let $\Lambda(\dot{F})$ be the signed graph whose vertices are identified as the edges of a signed graph $\dot{F}$, with a pair of vertices being adjacent by a positive (resp. negative) edge if and only if the corresponding edges of $\dot{G}$ are adjacent and have the same (resp. different) sign. In this paper, we prove that G' is bipartite if and only if there exists a signed graph $\dot{F}$ such that $R_{G'}^{\mathsf{T}} R_{G'} - 2I$ is the adjacency matrix of $\Lambda(\dot{F})$. It occurs that $\dot{F}$ is fully determined by G'. As an application, in some particular cases we express the skew eigenvalues of G' in terms of the eigenvalues of $\dot{F}$. We also establish some upper bounds for the skew spectral radius of G' in both the bipartite and the non-bipartite case.

1. INTRODUCTION

For a finite simple graph $G = (V, E)$, an *oriented graph* G' is a pair (G, σ'), where σ' is the *orientation* satisfying $\sigma'(ij) \in \{i, j\}$, for every $ij \in E$. If $\sigma'(ij) = j$, we say that the edge ij is oriented from i to j and designate this by $i \to j$ (or $j \leftarrow i$); an oriented edge is also known as an *arc*. Similarly, a *signed graph* $\dot{G}$ is a pair (G, σ), where σ is the *signature* satisfying $\sigma(ij) \in \{1, -1\}$, for every $ij \in E$. The edge set of $\dot{G}$ consists of positive and negative edges, and we interpret a graph as a signed graph with all edges being positive. In both cases we denote $n = |V|$ and say that G is the *underlying graph* (of G' or $\dot{G}$).

The *skew adjacency matrix* (of G') $S_{G'} = (s_{ij})$ is the $n \times n$ matrix defined by: $s_{ij} = 1$ if there is an edge oriented from j to i, $s_{ij} = -1$ if there is an edge oriented from i to j, and $s_{ij} = 0$ otherwise. The eigenvalues of $S_{G'}$ are called the *skew eigenvalues* of G' and they form the *skew spectrum* of G', which consists of purely imaginary numbers. It is easy to verify that non-zero skew eigenvalues come in pairs μ and $-\mu$ (with equal algebraic multiplicity). Consequently, the rank of $S_{G'}$

is even. The largest modulus of skew eigenvalues of G' is called the *(skew) spectral radius* and denoted by $\rho(G')$.

The vertex-edge *incidence matrix* $R_{G'}$ of G' is the matrix whose rows and columns are indexed by $V(G')$ and $E(G')$, respectively, such that its (i, e)-entry is 1 if e is oriented to i, -1 if e is oriented from i, and 0 otherwise.

The *adjacency matrix* (of $\dot{G}$) $A_{\dot{G}} = (a_{ij})$ is obtained from the $(0, 1)$-adjacency matrix of G by reversing the sign of all 1s which correspond to negative edges. The *eigenvalues* and the *spectrum* of $\dot{G}$ are identified as the eigenvalues and the spectrum of $A_{\dot{G}}$, respectively. Since $A_{\dot{G}}$ is symmetric, its eigenvalues are real. The spectral radius is defined as in the case of oriented graphs and denoted by the same symbol.

For a signed graph $\dot{G}$, we denote by $\Lambda(\dot{G})$ the signed graph whose vertices are the edges of $\dot{G}$, such that a pair of vertices is joined by a positive (resp. negative) edge if and only if the corresponding edges of $\dot{G}$ are adjacent and have the same (resp. different) sign. In relation to this definition, we remark that there are several definitions of a signed line graph. A combinatorial one can be found in works of Zaslavsky [8], while the one tailored for the spectral theory can be found in [1, 3, 6]. These definitions differ in sign and the corresponding underlying graph is also the underlying graph of $\Lambda(\dot{G})$. In case of unsigned graphs, the *line graph* $L(G)$ of G is the graph whose vertices are identified with the edges of G, with two vertices being adjacent whenever the corresponding edges are adjacent.

To visualize the previous definitions, in Figure 1 we illustrate an underlying graph G, an oriented graph $G' = (G, \sigma')$ (in which the edge orientation is designated by arrows), a signed graph $\dot{G} = (G, \sigma)$ (in which negative edges are dashed), the line graph $L(G)$ and the signed graph $\Lambda(\dot{G})$. Enumerations of the edges in G and $\dot{G}$ correspond to enumerations of the vertices in $L(G)$ and $\Lambda(\dot{G})$. This figure will also help us to visualize some forthcoming results.

Since the skew adjacency matrix is asymmetric with non-real eigenvalues (unless G' has no edges), in our recent work [7] we considered the existence of a related signed graph whose eigenvalues fully determine the skew eigenvalues of G'. It occurs that in such cases the entire theory of real symmetric matrices can be used in the study of skew spectra of oriented graphs. Motivated by these results, in this paper we consider relations between the skew spectrum of an oriented graph G' and the spectrum of a signed graph whose adjacency matrix is given by $R_{G'}^{\mathsf{T}} R_{G'} - 2I$. In particular, we establish some properties of such a signed graph and prove that G' is bipartite if and only if there exists a signed graph $\dot{F}$ such that $R_{G'}^{\mathsf{T}} R_{G'} - 2I$ is the adjacency matrix of $\Lambda(\dot{F})$. It occurs that $\dot{F}$ is fully described by G'. In certain cases we express the characteristic polynomial of $\Lambda(\dot{F})$ in terms of the characteristic polynomial of G'. The paper concludes with some upper bounds for the spectral radius of G' expressed in terms of the spectral radius of $L(G)$ and the spectral radius of $L(\mathrm{bd}(G))$, where $\mathrm{bd}(G)$ denotes the bipartite double of G, defined in the next section.

Some terminology, notation and necessary results are given in Section 2. In particular, one can find some recently established relations between the skew spectrum
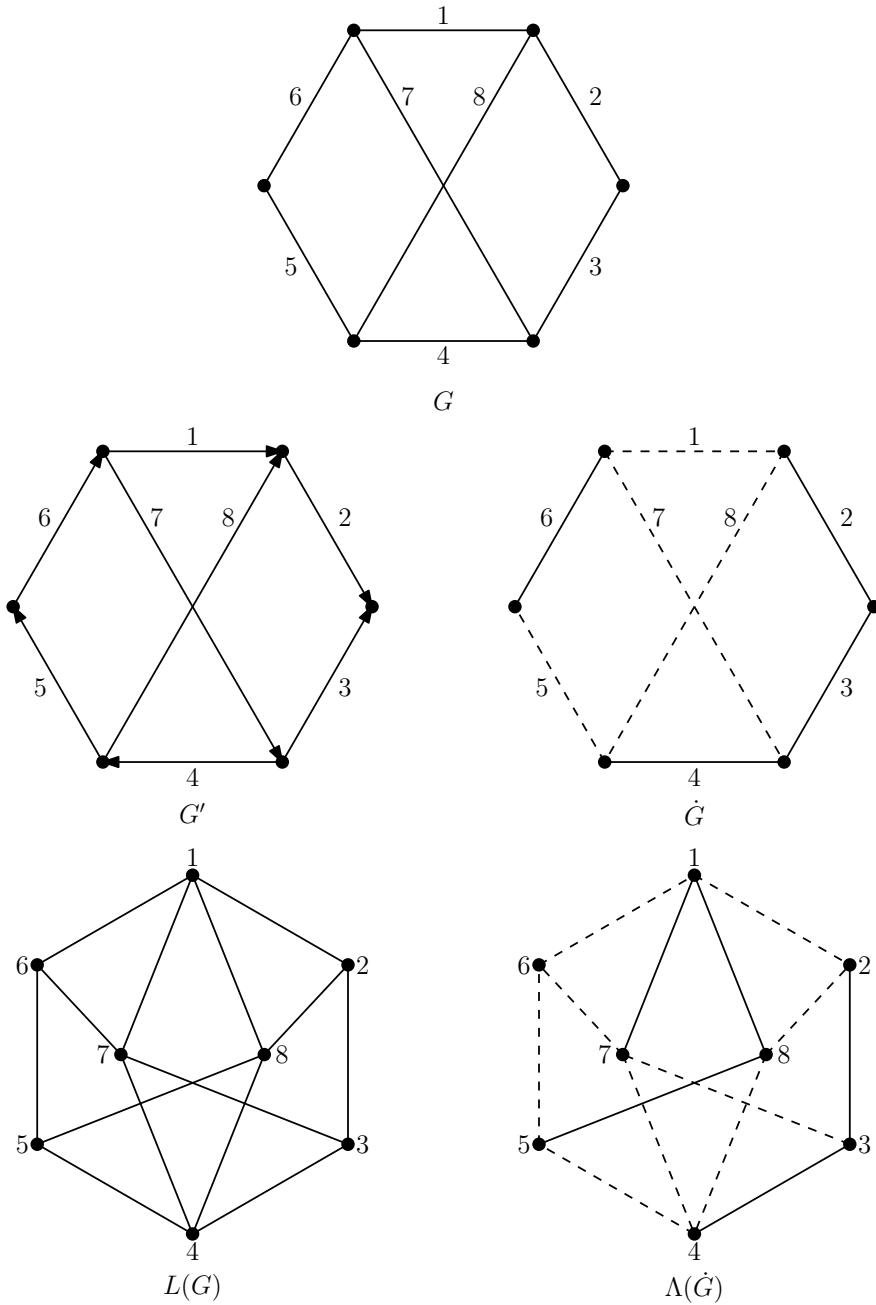
FIGURE 1. A graph G, an oriented graph $G' = (G, \sigma')$, a signed graph $\dot{G} = (G, \sigma)$, the line graph $L(G)$ and the signed graph $\Lambda(\dot{G})$.

of an oriented graph and the spectrum of an associated (for the specific definition of 'associated') signed graph. Results that include similar relations are reported in Section 3.

2. PRELIMINARIES

We first specify some cycles of an oriented and a signed graph. In the forthcoming notation, the subscript denotes the number of vertices of a cycle under consideration. We say that an even oriented cycle C'_{2l}, considered as a subgraph of an oriented graph, is *oriented uniformly* if by traversing the cycle we pass through an odd (resp. even) number of edges oriented in the route direction for l odd (even). A cycle in a signed graph is called *positive* if it contains an even number of negative edges; otherwise, it is said to be *negative*.

For a subset $U \subseteq V(\dot{G})$, let $\dot{G}^U$ be the signed graph obtained from $\dot{G}$ by reversing the sign of every edge between a vertex in U and a vertex in $V(\dot{G}) \setminus U$. We say that $\dot{G}$ and $\dot{G}^U$ are *switching equivalent*. Switching equivalent signed graphs share the same spectrum. We recall from [8] that a signed graph is switching equivalent to its underlying graph if and only if all its cycles are positive. The *negation* $-\dot{G}$ is obtained by reversing the sign of every edge of $\dot{G}$.

For an oriented graph G', the *bipartite double* $\mathrm{bd}(G')$ is the oriented graph whose skew adjacency matrix is defined by the Kronecker product $S_{\mathrm{bd}(G')} = S_{G'} \otimes A_{K_2}$, where A_{K_2} is the adjacency matrix of the complete graph with 2 vertices. Precisely, if the vertices of G' are labelled by $i_1, i_2, \ldots, i_n$, then the vertices of $\mathrm{bd}(G')$ are $i_1 1, i_1 2, i_2 1, i_2 2, \ldots, i_n 1, i_n 2$ and there is an arc $i_u k \to i_v l$ if and only if there is an arc $i_u \to i_v$ and $k \neq l$. The bipartite double of a signed graph $\dot{G}$ is defined analogously, so this is the signed graph determined by $A_{\mathrm{bd}(\dot{G})} = A_{\dot{G}} \otimes A_{K_2}$. Clearly, every bipartite double is bipartite.

We proceed with some theoretical results. For an oriented graph $G' = (G, \sigma')$ and a signed graph $\dot{G} = (G, \sigma)$, we say that the signature σ is *associated* with the orientation σ' if

$$\sigma(ik)\sigma(jk) = s_{ik}s_{jk} \quad \text{holds for every pair of edges } ik \text{ and } jk. \qquad (2.1)$$

Being associated is a symmetric relation.

We know from [7] that, for a graph G and an orientation σ', there exists a signature σ associated with σ' if and only if G is bipartite. The orientation σ' and the signature σ are associated in the following sense: σ' induces two switching equivalent signed graphs (one with signature σ and the other being the negation of the first one) and the signature σ induces two oriented graphs (one with orientation σ' and the other obtained by reversing the orientation of every edge of the first one). Further, if σ' and σ are associated, then we also say that G' and $\dot{G}$ are associated. The reader may observe that the oriented graph G' and the signed graph $\dot{G}$ of Figure 1 are mutually associated.

In the following theorems the exponent stands for the multiplicity of the corresponding eigenvalue.

Theorem 2.1 ([7]). *For a bipartite graph G and an orientation σ', if $\operatorname{rank}(S_{G'}) = 2k$ and σ is associated with σ', then*

$$\pm i\lambda_1, \pm i\lambda_2, \ldots, \pm i\lambda_k, 0^{n-2k}$$

are the skew eigenvalues of $G' = (G, \sigma')$ if and only if

$$\pm\lambda_1, \pm\lambda_2, \ldots, \pm\lambda_k, 0^{n-2k}$$

are the eigenvalues of $\dot{G} = (G, \sigma)$.

Theorem 2.2 ([7]). *Given a graph G and an orientation which determines G' such that $\operatorname{rank}(S_{G'}) = 2k$, let $H' = (H, \sigma')$ denote the bipartite double of G' and $\dot{H} = (H, \sigma)$ denote the signed graph whose signature is associated with σ'. Then*

$$\pm i\lambda_1, \pm i\lambda_2, \ldots, \pm i\lambda_k, 0^{n-2k}$$

are the skew eigenvalues of G' if and only if

$$(\pm\lambda_1)^2, (\pm\lambda_2)^2, \ldots, (\pm\lambda_k)^2, 0^{2(n-2k)}$$

are the eigenvalues of $\dot{H}$.

3. Results

We first consider the matrix $R_{G'}^{\mathsf{T}} R_{G'} - 2I$.

Theorem 3.1. *For the incidence matrix $R_{G'}$ of an oriented graph G', let $A = R_{G'}^{\mathsf{T}} R_{G'} - 2I$. The following statements hold:*

  (i) *A is the adjacency matrix of a signed graph, say $\dot{H}$.*

  (ii) *Every triangle of $\dot{H}$ arises either from a triplet of edges incident with the same vertex of G' or a triangle of G'. Every induced cycle of length ≥ 4 arises from an induced cycle of the same length of G'.*

  (iii) *Every triangle of $\dot{H}$ that arises from a triplet of edges incident with the same vertex is positive. An induced cycle that arises from an induced cycle of G' of an odd (resp. even) length is negative (resp. positive).*

  (iv) *The spectrum of $\dot{H}$ is determined by the spectrum of $R_{G'} R_{G'}^{\mathsf{T}}$.*

Proof. (i): Since every column of $R_{G'}$ contains exactly two non-zero entries, one being 1 and the other -1, we get that every entry of the main diagonal of $R_{G'}^{\mathsf{T}} R_{G'}$ is 2. An off-diagonal (i, j)-entry of $R_{G'}^{\mathsf{T}} R_{G'}$ is equal to $\mathbf{r_i} \cdot \mathbf{r_j}$ (where $\mathbf{r_i}$ is the ith column of $R_{G'}$ and the dot stands for the standard inner product), and thus it belongs to $\{-1, 0, 1\}$. It follows that $R_{G'}^{\mathsf{T}} R_{G'} - 2I$ is a symmetric $(-1, 0, 1)$-matrix with zero diagonal, i.e., it is the adjacency matrix of a signed graph.

(ii): If i, j, k are the vertices of a triangle in $\dot{H}$ then we have $\mathbf{r_i} \cdot \mathbf{r_j}, \mathbf{r_j} \cdot \mathbf{r_k}, \mathbf{r_k} \cdot \mathbf{r_i} \in \{-1, 1\}$, which means that the corresponding edges of G' are either incident with the same vertex or form a triangle.

Let $\dot{C}_l$ be an induced cycle of length $l \geq 4$ with vertices $i_1, i_2, \ldots, i_l$ (indexed in the natural order). Then for the corresponding rows of $R_{G'}$, we have $\mathbf{r_i} \cdot \mathbf{r_{i+1}} \in \{1, -1\}$, for $1 \leq i \leq l - 1$ and $\mathbf{r_1} \cdot \mathbf{r_l} \in \{1, -1\}$, which means that they determine the edges of a cycle of G'. If this cycle is not an induced one, then we have $\mathbf{r_j} \cdot \mathbf{r_k} \in \{1, -1\}$ for an additional (non-consecutive) pair of columns, which implies

that the vertices i_j and i_k of $\dot{H}$ are joined by an edge, contradicting the assumption that $\dot{C}_l$ is an induced cycle.

(iii): First, if i and j are adjacent edges of G', then for the (i,j)-entry of A we have

$$a_{ij} = \begin{cases} -1, & \text{if exactly one of } i \text{ and } j \text{ is oriented to the common vertex;} \\ 1, & \text{otherwise.} \end{cases}$$

It follows immediately that every triplet of edges with a common vertex gives rise to a positive triangle of $\dot{H}$. Let C'_l be an induced cycle of length l in G', and assume first that its edges are oriented in the route direction. Then, for l odd (resp. even), the corresponding cycle of $\dot{H}$ is negative (resp. positive). By observing that reversing of the orientation of a single edge of C'_l causes reversing of the sign of exactly two edges in the corresponding cycle of $\dot{H}$, we conclude that, in fact, its signature remains unchanged. Since every orientation of C'_l is obtained by a successive reversing of the orientation of a single edge, we arrive at the desired result.

(iv): This follows since $R_{G'}^{\mathsf{T}} R_{G'}$ and $R_{G'} R_{G'}^{\mathsf{T}}$ share the same non-zero eigenvalues along with their multiplicities. □

Now, we consider $\Lambda(\dot{G})$ in more detail.

Theorem 3.2. *The following statements hold:*
   (i) $\Lambda(\dot{G})$ *is isomorphic to* $\Lambda(-\dot{G})$.
   (ii) $\Lambda(\dot{G})$ *is switching equivalent to its underlying graph.*

Proof. Claim (i) follows by definition of $\Lambda(\dot{G})$, as reversing the sign of two adjacent edges of $\dot{G}$ does not change the sign of the edge between the corresponding vertices of $\Lambda(\dot{G})$.

(ii): We prove that all the cycles in $\Lambda(\dot{G})$ are positive. Similarly to the previous theorem, a triangle in $\Lambda(\dot{G})$ arises either from a triplet of edges with a common vertex or from a triangle (of $\dot{G}$). Other induced cycles arise from induced cycles of the same length. We immediately get that the triangle obtained from a triplet of edges with a common vertex is positive. Let further $\dot{C}_l$ be an induced cycle in $\dot{G}$. If all the edges of $\dot{C}_l$ are negative, then the corresponding cycle of $\Lambda(\dot{G})$ is positive (by definition of $\Lambda(\dot{G})$). For otherwise, observe that every maximal sequence of consecutive negative edges of $\dot{C}_l$ gives rise to exactly 2 negative edges in the corresponding cycle of $\Lambda(\dot{G})$: they are obtained from the first and the last edge of such a sequence and their neighbouring positive edges. Since all the negative edges of $\dot{C}_l$ can be partitioned into the described maximal sequences, we get that the number of negative edges in the corresponding cycle of $\Lambda(\dot{G})$ is even, and so this cycle is positive. Summa summarum, all induced cycles of $\Lambda(\dot{G})$ are positive.

Assume now that $\Lambda(\dot{G})$ contains a non-induced negative cycle, and let $\dot{C}$ be such a cycle which is shortest in length. Obviously, $\dot{C}$ contains a pair of adjacent vertices such that the edge between them does not belong to $\dot{C}$. Then the edges of $\dot{C}$ and the mentioned additional edge gives 2 cycles such that exactly one of them is negative. The existence of the negative one contradicts either the fact that

all induced cycles are positive or the assumption on the length of $\dot{C}$. Therefore, all cycles of $\Lambda(\dot{G})$ are positive, which by [8] means that this signed graph is switching equivalent to its underlying graph, and we are done. $\qquad\square$

Corollary 3.3. *If G is the underlying graph of $\dot{G}$, then $L(G)$ and $\Lambda(\dot{G})$ share the same spectrum.*

Proof. By definitions of the line graph $L(G)$ and the signed graph $\Lambda(\dot{G})$, we conclude that if G is the underlying graph of $\dot{G}$, then $L(G)$ is the underlying graph of $\Lambda(\dot{G})$, and then the result follows by Theorem 3.2 (ii). $\qquad\square$

To visualize the previous proof, the reader can see Figure 1. We proceed by the question concerning the existence of a signed graph $\dot{F}$ such that $R_{G'}^\mathsf{T} R_{G'} - 2I$ is the adjacency matrix of $\Lambda(\dot{F})$.

Theorem 3.4. *An oriented graph G' is bipartite if and only if there exists a signed graph $\dot{F}$ such that $R_{G'}^\mathsf{T} R_{G'} - 2I$ is the adjacency matrix of $\Lambda(\dot{F})$.*

Proof. Assume that $R_{G'}^\mathsf{T} R_{G'} - 2I$ is the adjacency matrix of $\Lambda(\dot{F})$ for some $\dot{F}$. By Theorem 3.2 (ii), $\Lambda(\dot{F})$ is switching equivalent to its underlying graph, and consequently it does not contain a negative cycle. By Theorem 3.1 (iii), G' does not contain an induced cycle of an odd length, and then the same holds for non-induced ones; hence, G' is bipartite.

Assume now that G' is bipartite. By Theorem 2.1, there exists a signed graph $\dot{G}$ associated with G' in the sense of (2.1). We claim that $\dot{G}$ appears in the role of $\dot{F}$, that is, $R_{G'}^\mathsf{T} R_{G'} - 2I$ is the adjacency matrix of $\Lambda(\dot{G})$. Observe that the signature of $\dot{G}$ is determined by $R_{G'}$ in exactly the same way as the signature of the signed graph determined by $R_{G'}^\mathsf{T} R_{G'} - 2I$. Namely, two adjacent edges of $\dot{G}$ are of the same sign if and only if the corresponding vertices of the signed graph determined by $R_{G'}^\mathsf{T} R_{G'} - 2I$ are joined by a positive edge. Therefore, by definition of $\Lambda(\dot{G})$, $R_{G'}^\mathsf{T} R_{G'} - 2I$ is its adjacency matrix. $\qquad\square$

We record that, by the previous proof, if G' is bipartite and $\dot{G}$ is associated with G', then $R_{G'}^\mathsf{T} R_{G'} - 2I$ is the adjacency matrix of $\Lambda(\dot{G})$. For example, by an appropriate labelling of vertices of G' of Figure 1, one can get that $R_{G'}^\mathsf{T} R_{G'} - 2I$ is the adjacency matrix of $\Lambda(\dot{G})$ of the same figure.

In the remainder of the paper we use Φ to denote the characteristic polynomial of an (oriented or signed) graph under consideration. It coincides with the characteristic polynomial of the corresponding (skew) adjacency matrix. To avoid possible confusion, the graph is indicated in the subscript. Recall that a graph is said to be *bipartite semiregular* if it is bipartite and the vertices belonging to the same part have equal degree.

We start with a simple lemma.

Lemma 3.5. *If a bipartite signed graph $\dot{G}$ is associated with an oriented graph G', then $\dot{G}$ is switching equivalent to its underlying graph if and only if every cycle of G' is oriented uniformly.*

Proof. The result follows since a cycle in $\dot{G}$ is positive if and only if the same cycle in G' is oriented uniformly. □

We now determine the characteristic polynomial of $\Lambda(\dot{G})$ in some particular cases.

Theorem 3.6. *If G' is a bipartite semiregular oriented graph with n_1 vertices of degree r_1 and n_2 ($n_2 \leq n_1$) vertices of degree r_2, such that all its cycles are oriented uniformly, then*

$$\Phi_{\Lambda(\dot{G})}(x) = (x - r_1 + 2)^{n_1 - n_2}(x + 2)^{\beta} \prod_{i=1}^{n_2} ((x - r_1 + 2)(x - r_2 + 2) - \lambda_i^2),$$

where $\dot{G}$ is associated with G', $\beta = n_1 r_1 - n_1 - n_2$ and $\lambda_1, \lambda_2, \ldots, \lambda_{n_2}$ are the first n_2 largest imaginary parts of the skew eigenvalues of G'.

Proof. Since every cycle of G' is oriented uniformly, from Lemma 3.5 we get that $\dot{G}$ is switching equivalent to its underlying graph G, and therefore Φ_G is its characteristic polynomial. By Corollary 3.3, $\Phi_{L(G)}$ is the characteristic polynomial of $\Lambda(\dot{G})$.

Further, we know form [2, Proposition 1.2.18] that the characteristic polynomial of the line graph of a bipartite semiregular graph is given by

$$\Phi_{L(G)}(x) = (x - r_1 + 2)^{n_1 - n_2}(x + 2)^{\beta} \prod_{i=1}^{n_2} ((x - r_1 + 2)(x - r_2 + 2) - \nu_i^2), \quad (3.1)$$

where β is defined in this theorem and $\nu_1, \nu_2, \ldots, \nu_{n_2}$ are the first n_2 largest eigenvalues of G. By the previous part of the proof, we may replace $\Phi_{L(G)}$ with $\Phi_{\Lambda(\dot{G})}$ in (3.1). Finally, since G shares the spectrum with $\dot{G}$, by Theorem 2.1, the first n_2 largest eigenvalues of G coincide with the first n_2 largest imaginary parts of the skew eigenvalues of G', and we are done. □

Here is an immediate corollary concerning bipartite regular oriented graphs.

Corollary 3.7. *If G' is a bipartite regular oriented graph with n vertices and m edges, such that all its cycles are oriented uniformly, then*

$$\Phi_{\Lambda(\dot{G})}(x) = (x + 2)^{m-n} \prod_{i=1}^{n} (x - \lambda_i - r + 2),$$

where $\dot{G}$ is associated with G', r is the vertex degree and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the imaginary parts of the skew eigenvalues of G'.

Proof. The result follows by setting $n_1 = n_2$ ($= \frac{n}{2}$), $r_1 = r_2$ ($= r$) in Theorem 3.6 and taking into account that $2m = rn$. □

In other words, under the assumptions of the previous corollary, if $\pm i\lambda_1, \pm i\lambda_2, \ldots, \pm i\lambda_k, 0^{n-2k}$ are the skew eigenvalues of G', then $\pm \lambda_1 + r - 2, \pm \lambda_2 + r - 2, \ldots, \pm \lambda_k + r - 2, (r-2)^{n-2k}, (-2)^{m-n}$ are the eigenvalues of $\Lambda(\dot{G})$.

In the last two statements we give some upper bounds on $\rho(G')$ expressed in terms of $\rho(L(G))$ (if G' is bipartite) and $\rho(L(\mathrm{bd}(G)))$ (if G' is non-bipartite).

Theorem 3.8. *For the spectral radius $\rho(G')$ of a connected bipartite oriented graph G' with underlying graph G, we have*

(i) $\rho(G') \leq \frac{1}{2}(\rho(L(G)) + 2)$, *with equality if and only if G' is regular and all its cycles are oriented uniformly, and*

(ii) *if G' is not a path, $\rho(G') \leq \rho(L(G))$, with equality if and only if G' is a uniformly oriented cycle.*

Proof. By Theorem 2.1, we have $\rho(G') = \rho(\dot{G})$, where as before $\dot{G}$ is associated with G'. Observe that the largest eigenvalue of $\dot{G}$ does not exceed the largest eigenvalue of G, along with equality if and only if $\dot{G}$ and G are switching equivalent; this fact is explicitly proved in [5]. Now, since $\dot{G}$ is bipartite, its spectrum is symmetric about the origin, and so we have that $\rho(\dot{G}) \leq \rho(G)$, with equality if and only if they are switching equivalent. By the result of Shi [4], we have $\rho(G) \leq \frac{1}{2}(\rho(L(G)) + 2)$ (with equality if and only if G is regular) and, if G is not a path, $\rho(G) \leq \rho(L(G))$ (with equality if and only if G is a cycle). Thus, we get the inequalities of (i) and (ii). Equality cases follow by $\rho(G') = \rho(G)$ if and only if $\dot{G}$ is switching equivalent to G, that is, by Lemma 3.5, if and only if the cycles of G' are oriented uniformly. $\qquad\square$

The upper bound of (i) gives a better estimate whenever $\rho(L(G)) > 2$. Now, the non-bipartite case.

Theorem 3.9. *For the spectral radius $\rho(G')$ of a non-bipartite oriented graph G' with underlying graph G, we have $\rho(G') < \frac{1}{2}(\rho(L(\mathrm{bd}(G))) + 2)$ and $\rho(G') < \rho(L(\mathrm{bd}(G)))$.*

Proof. Following the proof of the previous theorem, we get

$$\rho(G') = \rho(\mathrm{bd}(\dot{G})) \leq \rho(\mathrm{bd}(G)), \tag{3.2}$$

and then we get $\rho(G') \leq \frac{1}{2}(\rho(L(\mathrm{bd}(G))) + 2)$ and $\rho(G') \leq \rho(L(\mathrm{bd}(G)))$.

It remains to show that, in these upper bounds, equality cannot occur. Assuming the contrary, we get that the inequality of (3.2) must reduce to equality, i.e., we get $\rho(\mathrm{bd}(\dot{G})) = \rho(\mathrm{bd}(G))$, which implies that $\mathrm{bd}(\dot{G})$ and $\mathrm{bd}(G)$ are switching equivalent. Let C'_l be an odd cycle (of G') with vertices $i_1, i_2, \ldots, i_l$ (indexed in the natural order), and let C'_{2l} be the corresponding cycle of $\mathrm{bd}(G')$. Then every vertex i_u of C'_l produces its two copies in C'_{2l}, say i_u1 and i_u2, and (according to the definition of a bipartite double) the vertices of C'_{2l} are ordered as $i_11, i_22, i_31, \ldots, i_l1, i_12, i_21, i_32, \ldots, i_l2$. It follows also by definition that an edge $i_u i_v$ of C'_l gives its two copies in C'_{2l}, i_u1i_v2 and i_u2i_v1, whose orientation is determined by the orientation of $i_u i_v$ in the sense that if $i_u \to i_v$, then $i_u1 \to i_v2$ and $i_u2 \to i_v1$. This, together with the vertex ordering of C'_{2l}, implies that C'_{2l} contains an even number of edges oriented in the route direction. In other words, C'_{2l} is not oriented uniformly, and thus the corresponding cycle of $\mathrm{bd}(\dot{G})$ is negative. Hence, by Lemma 3.5, $\mathrm{bd}(\dot{G})$ is not switching equivalent to its underlying graph, which contradicts our assumption. $\qquad\square$

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