REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 63, No. 1, 2022, Pages 69–91
Published online: March 28, 2022
https://doi.org/10.33044/revuma.1917

THE $\mathfrak{A}$-PRINCIPAL REAL HYPERSURFACES IN COMPLEX QUADRICS

TEE-HOW LOO

ABSTRACT. A real hypersurface in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ is said to be $\mathfrak{A}$-principal if its unit normal vector field is singular of type $\mathfrak{A}$-principal everywhere. In this paper, we show that a $\mathfrak{A}$-principal Hopf hypersurface in Q^m, $m \geq 3$, is an open part of a tube around a totally geodesic Q^{m+1} in Q^m. We also show that such real hypersurfaces are the only contact real hypersurfaces in Q^m. The classification for complete pseudo-Einstein real hypersurfaces in Q^m, $m \geq 3$, is also obtained.

1. Introduction

A natural research problem that arises in the theory of Riemannian submanifolds, when the ambient spaces are equipped with some additional geometric structures, is to study the interactions between these structures and the submanifold structure on its submanifolds.

For real hypersurfaces in a Hermitian manifold with complex structure J, a geometric condition naturally being considered is to require the line bundle $JT^\perp M$ over M to be invariant under the shape operator S of M, that is, $SJT^\perp M \subset JT^\perp M$. Such real hypersurfaces are known as Hopf hypersurfaces and possess some interesting geometric properties; for instance, Hopf hypersurfaces in a complex projective space $\mathbb{C}P^m$ are curvature adapted and can be realized as tubes around complex submanifolds in $\mathbb{C}P^m$ (cf. [8]).

Similar research has been carried out for real hypersurfaces in quaternionic Kaehler manifolds. Martinez and Perez classified real hypersurfaces M with constant principal curvatures in quaternionic projective spaces $\mathbb{H}P^m$ of which the vector bundle $\mathfrak{J}T^\perp M$ over M is invariant under the shape operator S of M, where $\mathfrak{J}$ is the quaternionic Kaehler structure of $\mathbb{H}P^m$ (cf. [15]). This result has been improved in [1] by removing the constancy assumption on the principal curvatures.

2020 *Mathematics Subject Classification.* Primary 53C40 53B25; Secondary 53C15.
Key words and phrases. Hopf hypersurfaces; Contact structure; pseudo-Einstein real hypersurfaces; Complex quadrics.
This work was supported in part by the FS-UMRG research grant no. RF010B-2018.

The complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ is the unique compact, Kaehler, quaternionic Kaehler manifold with positive scalar curvature. Two natural conditions to be considered are that both $JT^{\perp}M$ and $\mathfrak{J}T^{\perp}M$ are invariant under the shape operator S of real hypersurfaces. Berndt and Suh used these properties to characterize tubes around $G_2(\mathbb{C}^{m+1})$ and tubes around $\mathbb{H}P^{m/2}$ in $G_2(\mathbb{C}^{m+2})$ (cf. [6]). An extension to the non-compact dual of $G_2(\mathbb{C}^{m+2})$ can be found in [5].

In this paper, we study real hypersurfaces, in the m-dimensional complex quadric $Q^m = SO_{m+2}/SO_mSO_2$, $m \geq 2$. The complex quadric Q^m is a Hermitian symmetric space of rank two. It is the only compact non-totally geodesic parallel complex hypersurface in the complex projective space $\mathbb{C}P^{m+1}$ (cf. [17]). This property determines on Q^m, on top of the complex structure J, another distinguished geometric structure $\mathfrak{A}$, which is an S^1-bundle over Q^m generated by conjugations on tangent spaces of Q^m induced by the shape operator of Q^m in $\mathbb{C}P^{m+1}$.

With respect to the structure $\mathfrak{A}$, there are two types of singular tangent vectors for Q^m, namely, $\mathfrak{A}$-principal and $\mathfrak{A}$-isotropic singular tangent vectors, corresponding to the two singular orbits of the isotropy action of Q^m on the unit sphere. A real hypersurface M in Q^m is said to be $\mathfrak{A}$-*principal* (resp. $\mathfrak{A}$-*isotropic*) if the normal bundle of M consists of $\mathfrak{A}$-principal (resp. $\mathfrak{A}$-isotropic) singular tangent vectors in Q^m.

Typical examples of $\mathfrak{A}$-principal (resp. $\mathfrak{A}$-isotropic) real hypersurfaces are the tubes around totally geodesic Q^{m-1} (resp. $\mathbb{C}P^k$, $m = 2k$ is even) in Q^m. These real hypersurfaces have a number of interesting geometric properties; for instance, both of them are Hopf. In addition, tubes around totally geodesic $\mathbb{C}P^k$ in Q^{2k} are the only real hypersurfaces in Q^{2k} with isometric Reeb flow (cf. [7]) while tubes around Q^{m-1} in Q^m appear to be the only known examples of contact real hypersurfaces in Q^m (cf. [2]).

This raises two interesting problems: classifying $\mathfrak{A}$-principal Hopf hypersurfaces and $\mathfrak{A}$-isotropic Hopf hypersurfaces in the complex quadric Q^m. In this paper, we first study the former problem and show that these real hypersurfaces are indeed tubes around Q^{m-1} in Q^m (see Theorem 5.2).

Let M be a real hypersurface in a Kaehler manifold $\hat{M}$. Let $(\phi, \xi, \eta, \langle\,,\rangle)$ be the almost contact metric structure on M induced by the complex structure of $\hat{M}$ (see Section 3 for details). Denote by $\Phi(\cdot,\cdot) := \langle \cdot, \phi\,\cdot\rangle$ the fundamental 2-form. If there exists a non-zero function ρ on M such that $d\eta = \rho\Phi$, then M admits a contact structure. In this case, we call M a *contact real hypersurface* in $\hat{M}$. In [2], Berndt asked whether tubes around totally geodesic Q^{m-1} are the only contact real hypersurfaces in Q^m. We shall give an affirmative answer for this question (see Theorem 6.1).

An almost contact metric manifold M is said to be *pseudo-Einstein* if there exist constants a, b such that its Ricci tensor Ric is given by

$$\operatorname{Ric} X = aX + b\eta(X)\xi.$$

Pseudo-Einstein real hypersurfaces in non-flat complex space forms were studied in [9, 10, 11, 13, 16]. We study pseudo-Einstein real hypersurfaces in Q^m, $m \geq 3$, and show that a complete pseudo-Einstein real hypersurface must be a special kind of tubes around totally geodesic Q^{m-1} (see Theorem 7.6).

This paper is organized as follows: In Section 2, we take a quick revision of geometric structures on complex quadrics Q^m. In Section 3, we fix notations and establish a general framework for understanding the geometry of real hypersurfaces M in Q^m. We derive some general identities for Hopf hypersurfaces in Q^m in Section 4. In particular, we show that the only Hopf hypersurfaces with constant Reeb principal curvature are the $\mathfrak{A}$-principal and $\mathfrak{A}$-isotropic ones (see Lemma 4.5). The main results are proved in the last three sections. In Sections 5 and 6, we show that the following statements are equivalent:

(1) M is an open part of a tube around a totally geodesic Q^{m-1} in Q^m.
(2) M is a $\mathfrak{A}$-principal Hopf hypersurface in Q^m.
(3) M is a contact real hypersurface in Q^m.

We study pseudo-Einstein real hypersurfaces in Section 7. A classification for complete pseudo-Einstein real hypersurfaces in Q^m is obtained.

2. The complex quadrics

We denote by $\mathbb{C}P^{m+1}$ the $(m+1)$-dimensional complex projective space of constant holomorphic sectional curvature 4 with respect to the Fubini–Study metric $\langle\,,\rangle$. Each point $[z]\in\mathbb{C}P^{m+1}$ can be regarded as a complex line in $\mathbb{C}^{m+2}$ spanned by $z\in\mathbb{C}^{m+1}_{\times}$. Up to identification, the tangent space $T_{[z]}\mathbb{C}P^{m+1}$ is given by

$$T_{[z]}\mathbb{C}P^{m+1}=\mathbb{C}^{m+2}\ominus[z]=\{w\in\mathbb{C}^{m+2}:\langle w,z\rangle_{\mathbb{C}}=0\},$$

where $\langle\,,\rangle_{\mathbb{C}}$ is the Hermitian inner product on $\mathbb{C}^{m+2}$.

The m-dimensional complex quadric Q^m is a complex hypersurface characterized by the quadratic equation $z_0^2+z_1^2+\cdots+z_{m+1}^2=0$ in $\mathbb{C}P^{m+1}$, which is isometric to the real Grassmannian of oriented two-planes of $\mathbb{R}^{m+2}$ and is a compact Hermitian symmetric space of rank two.

We denote by J both the complex structure of $\mathbb{C}P^{m+1}$ and that induced on Q^m, and by $\langle\,,\rangle$ as well the induced metric tensor on Q^m. As Q^2 is isometric to $S^2\times S^2$, we will consider $m\geq 3$ in the main part of the paper.

At each $[z]\in Q^m$, up to identification, the normal space $T^{\perp}_{[z]}Q^m=[\bar{z}]$ and tangent space $T_{[z]}Q^m=\mathbb{C}^{m+2}\ominus([z]\oplus[\bar{z}])$. Denote by A_ζ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to a unit vector $\zeta\in T^{\perp}_{[z]}Q^m$. It is known that A_ζ is a self-adjoint involution on $T_{[z]}Q^m$ and satisfies $A_\zeta J+JA_\zeta=0$. In other words, A_ζ is a conjugation on $T_{[z]}Q^m$ with respect to the Hermitian metric $\langle\,,\rangle_{\mathbb{C}}$ given by

$$\langle X,Y\rangle_{\mathbb{C}}=\langle X,Y\rangle+\sqrt{-1}\langle X,JY\rangle$$

for any $X,Y\in T_{[z]}Q^m$.

Let $V(A_\zeta)$ (resp. $JV(A_\zeta)$) be the $(+1)$-eigenspace (resp. the (-1)-eigenspace) of A_ζ. Then we have

$$T_{[z]}Q^m=V(A_\zeta)\oplus JV(A_\zeta),$$

and A_ζ defines a real structure $V(A_\zeta)$ on $T_{[z]}Q^m$. In particular, for $\zeta = \bar{z}$, the shape operator $A_{\bar{z}}w = -\bar{w}$, for each $w \in T_{[z]}Q^m$, $V(A_{\bar{z}}) = \mathbb{R}^{m+2} \cap T_{[z]}Q^m$ and $JV(A_{\bar{z}}) = \sqrt{-1}\mathbb{R}^{m+2} \cap T_{[z]}Q^m$ (cf. [18]).

The $\mathbb{C}Q$-structure $\mathfrak{A}_{[z]} := \{\lambda A_\zeta : \lambda \in S^1\}$ on $T_{[z]}Q^m$ is independent of the choice of ζ as every pair of unit vectors $\zeta, \zeta' \in T^\perp_{[z]}Q^m$ can be related by $\zeta' = \lambda\zeta$ for some $\lambda \in S^1$, and then $A_{\zeta'} = A_{\lambda\zeta} = \lambda A_\zeta$ holds. It follows that $\mathfrak{A} = \cup_{[z]\in Q^m}\mathfrak{A}_{[z]}$ is an S^1-bundle over Q^m.

To each unit vector field ζ normal to Q^m in $\mathbb{C}P^{m+1}$ we associate a section A_ζ of $\mathfrak{A}$. Denote by $\hat{\nabla}$ and $\nabla^\perp$ the connections corresponding to TQ^m and $T^\perp Q^m$ respectively, induced by the Levi-Civita connection of $\mathbb{C}P^{m+1}$. For all vector fields X and Y tangent to Q^m, we have $\nabla^\perp_X\zeta = q_\zeta(X)J\zeta$ for some 1-form q_ζ on Q^m. Since Q^m is a parallel complex hypersurface in $\mathbb{C}P^{m+1}$ and $A_{J\zeta} = JA_\zeta$, we have

$$0 = (\hat{\nabla}_X A)_\zeta Y = \hat{\nabla}_X A_\zeta Y - A_\zeta \hat{\nabla}_X Y - A_{\nabla^\perp_X \zeta} Y = (\hat{\nabla}_X A_\zeta)Y - q_\zeta(X)JA_\zeta Y.$$

It follows that for each section A of $\mathfrak{A}$, there exists a 1-form q on Q^m such that

$$\hat{\nabla}A = JA \otimes q. \tag{2.1}$$

A non-zero vector $W \in T_{[z]}Q^m$ is said to be *singular* if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for the complex quadric Q^m: $\mathfrak{A}$-principal singular and $\mathfrak{A}$-isotropic singular. A singular tangent vector W is said to be $\mathfrak{A}$-*principal* if there exists a conjugation $A \in \mathfrak{A}_{[z]}$ such that $W \in V(A)$. If $\langle AW, W\rangle = \langle AW, JW\rangle = 0$ for some (and hence for all) $A \in \mathfrak{A}_{[z]}$, then W is called a $\mathfrak{A}$-*isotropic singular vector*.

We have the following characterizations for $\mathfrak{A}$-principal singular tangent vectors.

Lemma 2.1. *Let $W \in T_{[z]}Q^m$ be a unit vector. Then the following are equivalent:*

(a) *W is $\mathfrak{A}$-principal.*
(b) *There exists $A \in \mathfrak{A}_{[z]}$ such that $AW \in \mathbb{C}W$. Furthermore, we have $AW \in \mathbb{C}W$ for each $A \in \mathfrak{A}_{[z]}$.*
(c) *For each $A \in \mathfrak{A}_{[z]}$, $\langle AW, W\rangle^2 + \langle AW, JW\rangle^2 = 1$.*

In general, for each unit tangent vector $W \in T_{[z]}Q^m$ and $A \in \mathfrak{A}_{[z]}$, we can write

$$W = \cos(t)X + \sin(t)JY,$$

where $X, Y \in V(A)$ are orthonormal vectors and $t \in [0, \pi/4]$. The vector W is a $\mathfrak{A}$-principal (resp. $\mathfrak{A}$-isotropic) singular tangent vector when $t = 0$ (resp. $t = \pi/4$).

From the Gauss equation of the complex hypersurface Q^m in $\mathbb{C}P^{m+1}$, the curvature tensor $\hat{R}$ of Q^m is given by

$$\hat{R}(X,Y) = X \wedge Y + JX \wedge JY - 2\langle JX, Y\rangle J + AX \wedge AY + JAX \wedge JAY \tag{2.2}$$

for any X, Y tangent to Q^m and A in $\mathfrak{A}$, where $(U \wedge V)Z = \langle V, Z\rangle U - \langle U, Z\rangle V$.

3. Real hypersurfaces in Q^m

Let M be a connected real hypersurface in Q^m, and let N be a (local) unit vector field normal to M. We define $\xi := -JN$, η the 1-form dual to ξ and $\phi := J_{|TM} - \xi \otimes \eta$. Then (ϕ, ξ, η) is an almost contact structure on M, that is,

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1.$$

Denote by ∇ the Levi-Civita connection, $\langle\, ,\rangle$ the induced Riemannian metric and S the shape operator of M. Then

$$(\nabla_X \phi)Y = \eta(Y)SX - \langle SX, Y\rangle\xi, \quad \nabla_X \xi = \phi SX \tag{3.1}$$

for any X, Y tangent to M.

The real hypersurface M is said to be *Hopf* if the Reeb vector field ξ is principal. It can be verified that M is Hopf if and only if the integral curves of ξ are geodesics in M. The distribution $\mathcal{D} := \ker \eta$ is known as the *maximal holomorphic distribution.*

We call M a $\mathfrak{A}$-*principal* (resp. $\mathfrak{A}$-*isotropic*) real hypersurface if the unit normal vector field N is $\mathfrak{A}$-principal (resp. $\mathfrak{A}$-isotropic) everywhere.

We shall now fix some notations. For any (local) section A in $\mathfrak{A}$ and vector field X tangent to M, we denote by $V := (AN)^T$, the tangential part of AN, $V^\circ = V - \eta(V)\xi$ and

$$BX := AX - \langle V, X\rangle N, \quad AN = V + fN,$$
$$f := \langle AN, N\rangle, \quad g := \langle V, \xi\rangle, \quad k := \|V^\circ\|.$$

We note that the entities V, f, g and k depend on the choice of A. Following these notations, we have the following lemma.

Lemma 3.1.

(a) $B\xi = -f\xi + \phi V$.
(b) $BV = -fV$.
(c) $B\phi V = (k^2 + g^2)\xi + f\phi V - gV$.
(d) $B^2 X = X - \langle X, V\rangle V$.
(e) $f^2 + k^2 + g^2 = 1$.
(f) $\operatorname{Trace} B = -f$.

Proof. It follows from $JA + AJ = 0$ that $0 = (JAN + AJN)^T = \phi V - f\xi - B\xi$. Since $A^2 Z = Z$ for any vector Z tangent to Q^m and $\langle V, V\rangle = k^2 + g^2$, the tangential and normal parts of $A^2 N = N$ give (b) and (e) respectively. For any X tangent to M, $X = A^2 X = B^2 X + \langle X, V\rangle V$. This gives (d). Next, with the help of (a) and (e), we can obtain (c) after putting $X = \xi$ in (d). Finally, (f) can be easily verified as $\operatorname{Trace} B = \operatorname{Trace} A - \langle AN, N\rangle = -f$. □

For any X tangent to M, we define

$$\theta X := JAX - \langle X, B\xi\rangle N.$$

By using the facts $JA + AJ = 0$ and $(JA)^2 Z = Z$ for any Z tangent to Q^m, we can also obtain the following identities.

Lemma 3.2.

(a) $\theta\xi = -V$.
(b) $\theta V = -(k^2+g^2)\xi - f\phi V$.
(c) $\theta\phi V = -fV - gB\xi$.
(d) $\theta^2 X = X - \langle X, B\xi\rangle B\xi$.
(e) $\theta X = \phi BX - \langle X, V\rangle\xi = -B\phi X - \eta(X)V$.
(f) $\operatorname{Trace}\theta = -g$.

Next, we derive some identities from the tangential and normal parts of (2.1).

Lemma 3.3.

(a) $(\nabla_X B)Y = \langle Y, V\rangle SX + \langle SX, Y\rangle V + q(X)\theta Y$.
(b) $\nabla_X V = fSX - BSX + q(X)B\xi$.
(c) $Xf = -2\langle X, SV\rangle + gq(X)$.
(d) $(\nabla_X\theta)Y = \langle Y, B\xi\rangle SX + \langle SX, Y\rangle B\xi - q(X)BY$.
(e) $\nabla_X B\xi = gSX - \theta SX - q(X)V$.
(f) $Xg = -2\langle SB\xi, X\rangle - fq(X)$.
(g) $\nabla_X V^\circ = fSX - g\phi SX - BSX + 2\langle SB\xi, X\rangle\xi + q(X)\phi V$.
(h) $\nabla_X \phi V = gSX + f\phi SX - \phi BSX - \langle SV, X\rangle\xi - q(X)V^\circ$.

Proof. For any X, Y tangent to M, we can obtain (a) and (b) from the tangential and normal parts of $(\hat\nabla_X A)Y = q(X)JAY$ respectively. Next,

$$Xf = -X\langle B\xi, \xi\rangle = -\langle\nabla_X B\xi, \xi\rangle - 2\langle B\xi, \nabla_X\xi\rangle = -2\langle X, SV\rangle + gq(X).$$

We observe that

$$(\hat\nabla_X JA)Y = (\hat\nabla_X J)AY + J(\nabla_X A)Y = -q(X)AY.$$

The tangential and normal parts give (d) and (e) respectively. To obtain (f), we compute

$$Xg = X\langle V, \xi\rangle = \langle\nabla_X V, \xi\rangle + \langle V, \nabla_X\xi\rangle = -2\langle SB\xi, X\rangle - fq(X).$$

Finally, since $V^\circ = V - g\xi$ and $\nabla_X\phi V = (\nabla_X\phi)V + \phi\nabla_X V$, by applying (b), (f) and (3.1) we can derive (g) and (h). ☐

Lemma 3.4.

(a) *If N is $\mathfrak{A}$-principal on a sufficiently small open set $\mathcal{U}\subset M$, then there exists a section A of $\mathfrak{A}$ on $\mathcal{U}$ such that $f = 1$.*
(b) *If N is not $\mathfrak{A}$-principal at $[z]$, then there exist a sufficiently small neighborhood $\mathcal{U}$ of $[z]$ in M and a section A of $\mathfrak{A}$ on $\mathcal{U}$ such that $0 \le f < 1$ and $g = 0$.*
(c) *If N is $\mathfrak{A}$-isotropic on a sufficiently small open set $\mathcal{U}\subset M$, then there exists a section A of $\mathfrak{A}$ on $\mathcal{U}$ such that $k = 1$. Furthermore, we have $k = 1$ for each section A of $\mathfrak{A}$ on $\mathcal{U}$.*

Proof. Statement (a) follows directly from the definition, while the proof of statement (b) can be found in [4]. Statement (c) is just a special case of statement (b). ☐

For each $[z] \in M$, we define a subspace $\mathcal{H}^\perp$ of $T_{[z]}M$ by

$$\mathcal{H}^\perp := \operatorname{Span}\{\xi, V, \phi V\}.$$

Let $\mathcal{H}$ be the orthogonal complement of $\mathcal{H}^\perp$ in $T_{[z]}M$. Then $\dim \mathcal{H} = 2m-2$ when N is $\mathfrak{A}$-principal at $[z]$ and $\dim \mathcal{H} = 2m-4$ otherwise. By virtue of Lemma 3.1, $B\mathcal{H} = \mathcal{H}$ and $B_{|\mathcal{H}}$ has two eigenvalues 1 and -1. For each $\varepsilon \in \{1, -1\}$, denote by $\mathcal{H}(\varepsilon)$ the eigenspace of $B_{|\mathcal{H}}$ corresponding to ε. Then $\dim \mathcal{H}(\varepsilon) = m-1$ (resp. $\dim \mathcal{H}(\varepsilon) = m-2$) when N is $\mathfrak{A}$-principal (resp. N is not $\mathfrak{A}$-principal) at $[z]$. Moreover, we have $\phi\mathcal{H}(\varepsilon) = \mathcal{H}(-\varepsilon)$ by Lemma 3.2 (e).

Lemma 3.5. *Let M be a real hypersurface in Q^m. Then*

(a) *if M is $\mathfrak{A}$-principal, then $S\mathcal{H}(-1) = 0$;*
(b) *if M is $\mathfrak{A}$-isotropic, then $SV = S\phi V = 0$.*

Proof. Since $f = 1$ when N is $\mathfrak{A}$-principal everywhere, we have $k = g = 0$ and $V = 0$. It follows from Lemma 3.3 (b), (f) that

$$SX - BSX - 2\langle X, S\xi\rangle\xi = 0.$$

By taking the transpose of this equation, we have

$$SX - SBX - 2\langle X, \xi\rangle S\xi = 0$$

for any X tangent to M. In particular, for $X \in \mathcal{H}(-1)$, $2SX = 0$ and so we obtain statement (a).

Suppose that N is $\mathfrak{A}$-isotropic everywhere. Then $f = g = 0$ and $k = 1$. By Lemma 3.3 (c), (f), we have $SV = 0$ and $S\phi V = SB\xi = 0$. ☐

It follows from (2.2), Lemma 3.1 and Lemma 3.2 that the equations of Gauss and Codazzi are given by

$$\begin{aligned} R(X,Y) = X \wedge Y + \phi X \wedge \phi Y - 2\langle \phi X, Y\rangle\phi + BX \wedge BY \\ + \theta X \wedge \theta Y + SX \wedge SY \end{aligned} \tag{3.2}$$

$$\begin{aligned} (\nabla_X S)Y - (\nabla_Y S)X = \eta(X)\phi Y - \eta(Y)\phi X - 2\langle \phi X, Y\rangle\xi \\ + \langle X, V\rangle BY - \langle Y, V\rangle BX + \eta(BX)\theta Y - \eta(BY)\theta X. \end{aligned}$$

Let Ric be the Ricci tensor on M and $h := \operatorname{Trace} S$. Then by (3.2) we have

$$\begin{aligned} \operatorname{Ric} X = (2m-1)X - 3\eta(X)\xi + \langle X, V\rangle V + \langle X, B\xi\rangle B\xi \\ - fBX - g\theta X - (S^2 - hS)X. \end{aligned} \tag{3.3}$$

4. Hopf hypersurfaces in Q^m

In this section, we assume that M is a Hopf hypersurface in Q^m with $\alpha = \langle S\xi, \xi\rangle$.

Lemma 4.1. *Let M be a Hopf hypersurface in Q^m. Then we have*

$$\operatorname{grad}\alpha = (\xi\alpha)\xi - 2(fV^\circ + g\phi V) \tag{4.1}$$

$$(2S\phi S - \alpha(\phi S + S\phi) - 2\phi)X = -2\langle X, V^\circ\rangle\phi V + 2\langle X, \phi V\rangle V^\circ \tag{4.2}$$

for any X tangent to M.

Remark 4.2. This lemma can be obtained by a standard calculation using the Codazzi equation and has been proved in [7]. We will just outline the proof as below.

Proof of Lemma 4.1. For any X, Y tangent to M, we have

$$(\nabla_X S)\xi = (X\alpha)\xi + \alpha\phi SX - S\phi SX.$$

By this equation and the Codazzi equation, we obtain

$$\begin{aligned} 0 &= \langle(\nabla_X S)Y - (\nabla_Y S)X, \xi\rangle + 2\langle\phi X, Y\rangle - 2\langle X, V\rangle\langle Y, B\xi\rangle + 2\langle Y, V\rangle\langle X, B\xi\rangle \\ &= (X\xi)\eta(Y) - (Y\xi)\eta(X) - 2\langle X, V\rangle\langle Y, B\xi\rangle + 2\langle Y, V\rangle\langle X, B\xi\rangle \\ &\quad + \langle\big(2\phi + \alpha(\phi S + S\phi) - 2S\phi S\big)X, Y\rangle. \end{aligned}$$

By substituting $Y = \xi$, we obtain (4.1). By using (4.1) and the above equation, we can get (4.2). $\square$

Acting by ϕ on both sides of (4.2), we obtain

$$(2\phi S\phi S + \alpha S - \alpha\phi S\phi + 2)X - (\alpha^2 + 2)\eta(X)\xi = 2\langle X, V^\circ\rangle V^\circ + 2\langle X, \phi V\rangle\phi V.$$

This implies that $(\phi S\phi)S - S(\phi S\phi) = 0$. Hence there exists a local orthonormal frame $\{X_0 = \xi, X_1, \dots, X_{m-1}, X_m = \phi X_1, \dots, X_{2m-2} = \phi X_{m-1}\}$ such that

$$SX_j = \lambda_j X_j, \quad \phi S\phi X_j = -\mu_j X_j, \qquad j \in \{1, \dots, m-1\}. \tag{4.3}$$

By using (4.2) and (4.3), we get

$$\{-2\lambda_j\mu_j + \alpha(\lambda_j + \mu_j) + 2\}X_j = 2\langle X_j, V^\circ\rangle V^\circ + 2\langle X_j, \phi V\rangle\phi V$$

for $j \in \{1, \dots, m-1\}$. If N is $\mathfrak{A}$-principal, then $\mathcal{D} = \mathcal{H}$ and hence $S\mathcal{H} \subset \mathcal{H}$. On the other hand, $k > 0$ or $V^\circ \neq 0$ when N is not $\mathfrak{A}$-principal. Hence, there is exactly one j, say $j = 1$, such that $-2\lambda_1\mu_1 + \alpha(\lambda_1 + \mu_1) + 2 \neq 0$. This means that $\mathcal{H}$ is spanned by the vectors $X_2, \dots, X_{m-1}, \phi X_2, \dots, \phi X_{m-1}$; so $S\mathcal{H} \subset \mathcal{H}$ and

$$-2\lambda_j\mu_j + \alpha(\lambda_j + \mu_j) + 2 = 0, \qquad j \in \{2, \dots, m-1\}$$

in this case. Furthermore, by selecting an appropriate local section A, we can set $X_1 = (1/k)V^\circ$ and

$$-2\lambda_1\mu_1 + \alpha(\lambda_1 + \mu_1) + 2 - 2k^2 = 0.$$

We have shown the following lemma.

Lemma 4.3. *Let M be a Hopf hypersurface in Q^m, $m \geq 3$. Then $S\mathcal{H} \subset \mathcal{H}$. If E is a vector tangent to $\mathcal{H}$ such that $SE = \lambda E$ and $S\phi E = \mu\phi E$, then*

$$-2\lambda\mu + \alpha(\lambda + \mu) + 2 = 0.$$

Furthermore, if N is not $\mathfrak{A}$-principal, then there exists a section A of $\mathfrak{A}$ such that $SV^\circ = tV^\circ$ and $S\phi V = \omega\phi V$, where t and ω satisfy

$$-2t\omega + \alpha(t + \omega) + 2 - 2k^2 = 0.$$

Lemma 4.4. *Let M be a real hypersurface in Q^m, $m \geq 3$. Then $\phi S + S\phi \neq 0$ on every open set $\mathcal{U} \subset M$.*

Proof. Suppose that $\phi S + S\phi = 0$ on $\mathcal{U}$. It is clear that ξ is principal at each $[z] \in \mathcal{U}$. Since $\dim \mathcal{H} \geq 2m - 4 > 0$, we take a principal vector $X \in \mathcal{H}$ in line with Lemma 4.3. It follows that $\lambda + \mu = 0$ and so $2\lambda^2 + 2 = 0$. This is a contradiction and we obtain the lemma. ☐

Lemma 4.5. *Let M be a Hopf hypersurface in Q^m. Then α is constant if and only if either M is $\mathfrak{A}$-principal or M is $\mathfrak{A}$-isotropic.*

Proof. Suppose that α is a constant. Then by (4.1), we have $fV^\circ + g\phi V = 0$ and so $fk = gk = 0$. Let

$$M_1 = \{[z] \in M : N_{[z]} \text{ is } \mathfrak{A}\text{-principal}\}.$$

If N is not $\mathfrak{A}$-principal everywhere, it follows from Lemma 3.4 that $k \neq 0$ on M_1^c, which implies that $f = g = 0$ on M_1^c and hence N is $\mathfrak{A}$-isotropic on M_1^c.

Now consider the function $F := f^2 + g^2$. We note that F is independent of the choice of $A \in \mathfrak{A}$ and is globally defined on M. Then $F = 1$ on M_1 and $F = 0$ on M_1^c. By the continuity of F, $M = M_1^c$ and so it is $\mathfrak{A}$-isotropic.

Conversely, we have two cases: M is $\mathfrak{A}$-principal and M is $\mathfrak{A}$-isotropic. If M is $\mathfrak{A}$-principal, then $f = 1$, $g = 0$ and $V = 0$. On the other hand, we have $f = g = 0$ when M is $\mathfrak{A}$-isotropic. By using (4.1), we deduce that $\operatorname{grad} \alpha = (\xi\alpha)\xi$ in both cases. It follows that

$$(XY - \nabla_X Y)\alpha = (X\xi\alpha)\eta(Y) + (\xi\alpha)\langle Y, \phi SX\rangle.$$

Hence

$$0 = (X\xi\alpha)\eta(Y) - (Y\xi\alpha)\eta(X) + (\xi\alpha)\langle Y, (\phi S + S\phi)X\rangle.$$

Substituting $Y = \xi$ gives $X\xi\alpha = (\xi\xi\alpha)\eta(X)$. Hence $(\xi\alpha)(\phi S + S\phi) = 0$. It follows from Lemma 4.4 that $\phi S + S\phi \neq 0$ on a dense open subset of M. Hence $\xi\alpha = 0$ by its continuity and so $\operatorname{grad} \alpha = 0$. Accordingly, α is a constant. ☐

Lemma 4.6. *Assuming the notation and hypotheses in Lemma 4.3, if M is neither $\mathfrak{A}$-principal nor $\mathfrak{A}$-isotropic, then*

$$(\xi\alpha)(\lambda + \mu) = -2g(\lambda - \mu)\langle BE, E\rangle - 2f(\lambda - \mu)\langle BE, \phi E\rangle \tag{4.4}$$

$$(\xi\alpha)(t + \omega) = 2fg(t - \omega) \tag{4.5}$$

$$\operatorname{grad}(\xi\alpha) = (\xi\alpha)\xi + 2f(\omega - \alpha)\phi V - 2g(t - \alpha)V^\circ. \tag{4.6}$$

Proof. By using (4.1), we have

$$
\begin{aligned}
(XY-\nabla_X Y)\alpha = (X\xi\alpha)\eta(Y) &- 2(Xf)\langle V^\circ, Y\rangle - 2(Xg)\langle \phi V, Y\rangle \\
&+ (\xi\alpha)\langle \nabla_X\xi, Y\rangle - 2f\langle \nabla_X V^\circ, Y\rangle - 2g\langle \nabla_X\phi V, Y\rangle.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
0 &= (X\xi\alpha)\eta(Y) - 2(Xf)\langle V^\circ, Y\rangle - 2(Xg)\langle \phi V, Y\rangle \\
&\quad - (Y\xi\alpha)\eta(X) + 2(Yf)\langle V^\circ, X\rangle + 2(Yg)\langle \phi V, X\rangle \\
&\quad + (\xi\alpha)\langle \nabla_X\xi, Y\rangle - 2f\langle \nabla_X V^\circ, Y\rangle - 2g\langle \nabla_X\phi V, Y\rangle \\
&\quad - (\xi\alpha)\langle \nabla_Y\xi, X\rangle + 2f\langle \nabla_Y V^\circ, X\rangle + 2g\langle \nabla_Y\phi V, X\rangle \\
&= \{X(\xi\alpha) + 2g(t-2\alpha)\langle V^\circ, X\rangle - 4f(\omega-\alpha)\langle \phi V, X\rangle\}\eta(Y) \\
&\quad - \{Y(\xi\alpha) + 2g(t-2\alpha)\langle V^\circ, Y\rangle - 4f(\omega-\alpha)\langle \phi V, Y\rangle\}\eta(X) \\
&\quad + \langle(\xi\alpha)(\phi S + S\phi)X + 2g(\phi BS + SB\phi)X + 2f\langle(SB - BS)X, Y\rangle
\end{aligned}
$$

for any $X, Y \in TM$. In particular, if $X = E$ and $Y = \phi E$, then we get (4.4). On the other hand, (4.5) can be obtained by putting $X = V^\circ$ and $Y = \phi V$ in the preceding equation. Finally, letting $X = \xi$ gives (4.6). ☐

5. Tubes around Q^{m-1} in Q^m

The totally geodesic complex hypersurface Q^{m-1} in Q^m is determined by the equations

$$z_0^2 + \cdots + z_m^2 = 0, \quad z_{m+1} = 0.$$

The complex hypersurface Q^{m-1} is a singular orbit of the cohomogeneity one action $SO_{m+1} \subset SO_{m+2}$ on Q^m. The other singular orbit is a totally geodesic totally real m-dimensional sphere $S^m = SO_{m+1}/SO_m$.

The distance between the two singular orbits of the SO_{m+1}-action is $\pi/2\sqrt{2}$ and each principal orbit of the action is a tube of radius $r \in]0, \pi/2\sqrt{2}[$ around the totally geodesic $Q^{m-1} \subset Q^m$. A principal orbit of the action is a homogeneous space of the form SO_{m+1}/S_{m-1} which is an S^1-bundle over Q^{m-1}, and an S^{m-1}-bundle over S^m.

From the construction of $\mathfrak{A}$ it is clear that $T_{[z]}Q^{m-1}$ and $T^\perp_{[z]}Q^{m-1}$ are A-invariant for each $A \in \mathfrak{A}_{[z]}$. Moreover, since the real codimension of Q^{m-1} in Q^m is 2, for each unit vector $\zeta \in T^\perp_{[z]}Q^{m-1}$, $[z] \in Q^{m-1}$, there exists $A \in \mathfrak{A}_{[z]}$ such that $A\zeta = \zeta$ and so $AJ\zeta = -J\zeta$. Hence

$$T_{[z]}Q^{m-1} = (V(A) \ominus \mathbb{R}\zeta) \oplus J(V(A) \ominus \mathbb{R}\zeta).$$

It follows that the Jacobi operator $\hat{R}_\zeta := \hat{R}(\cdot, \zeta)\zeta$ is given by

$$\hat{R}_\zeta Y = Y + AY - 2\langle Y, \zeta\rangle\zeta + 2\langle Y, J\zeta\rangle J\zeta.$$

It has two constant eigenvalues, 0 and 2, with corresponding eigenspaces $J(V(A) \ominus \mathbb{R}\zeta) \oplus \mathbb{R}\zeta$ and $(V(A) \ominus \mathbb{R}\zeta) \oplus \mathbb{R}J\zeta$.

We will use the standard Jacobi field method to determine the principal curvatures and their corresponding eigenspaces of a tube around a totally geodesic Q^{m-1} in Q^m.

Fix $r \in]0, \pi/2\sqrt{2}[$. For each $[z] \in Q^{m-1}$ and unit vector $\zeta \in T^{\perp}_{[z]}Q^{m-1}$, denote by $\gamma_\zeta(s)$ the unit speed geodesic in Q^m that passes through $[z]$ at $s = 0$ with initial velocity ζ.

Let $\mathcal{Y}$ be the Jacobi field along γ_ζ with initial values $\mathcal{Y}(0) \in T_{[z]}Q^{m-1}$ and $\dot{\mathcal{Y}}(0) + S_\zeta \mathcal{Y}(0) = \dot{\mathcal{Y}}(0) \in T^{\perp}_{[z]}Q^{m-1}$, where S_ζ denotes the shape operator of Q^{m-1} with respect to ζ. Then $\dot{\gamma}_\zeta(r)$ is a unit vector normal to the tube M_r of radius r around Q^{m-1} at $\gamma_\zeta(r)$ and the tangent space of M_r at $\gamma_\zeta(r)$ is spanned by $\mathcal{Y}(r)$. Moreover, the shape operator S of M_r with respect to $N = -\dot{\gamma}_\zeta(r)$ can be determined by the equation (cf. [3, p. 225])

$$S\mathcal{Y}(r) = \dot{\mathcal{Y}}(r).$$

To determine the principal curvatures of M_r and their corresponding eigenspaces, we consider the following Jacobi field:

$$\mathcal{Y}_X(t) = \begin{cases} (1/\sqrt{2}t)\sin(\sqrt{2}t)\mathcal{E}_X(t), & X = J\zeta; \\ (1/\sqrt{2}t)\cos(\sqrt{2}t)\mathcal{E}_X(t), & X \in V(A) \ominus \mathbb{R}\zeta; \\ \mathcal{E}_X(t), & X \in J(V(A) \ominus \mathbb{R}\zeta), \end{cases}$$

where $\mathcal{E}_X$ is the parallel vector field along γ_ζ with $\mathcal{E}_X(0) = X$. It follows that M_r has three constant principal curvatures, $\sqrt{2}\cot(\sqrt{2}r)$, $-\sqrt{2}\tan(\sqrt{2}r)$ and 0, with eigenspaces $\mathbb{R}J\zeta$, $V(A) \ominus \mathbb{R}\zeta$ and $J(V(A) \ominus \mathbb{R}\zeta)$ respectively, of which we have identified the subspaces obtained by parallel translation along γ_ζ from $[z]$ to $\gamma_\zeta(r)$.

We can see that the unit vector N for M_r is $\mathfrak{A}$-principal and the shape operator S satisfies $\phi S + S\phi = -\sqrt{2}\tan(\sqrt{2}r)\phi$. We summarize these observations in the following theorem.

Theorem 5.1 ([2])**.** *Let M be the tube of radius $r \in]0, \pi/2\sqrt{2}[$ around the totally geodesic Q^{m-1} in Q^m. Then the normal bundle of M consists of $\mathfrak{A}$-principal singular tangent vectors of Q^m, and M has three constant principal curvatures:*

$$\alpha = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0.$$

The corresponding eigenspaces are

$$T_\alpha = \mathbb{R}JN, \quad T_\lambda = V(A) \ominus \mathbb{R}N, \quad T_\mu = J(V(A) \ominus \mathbb{R}N),$$

and the corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\lambda) = m - 1 = m(\mu),$$

where A is a conjugation such that $AN = N$ and N is a unit vector normal to M. Further, the shape operator S satisfies $\phi S + S\phi = -\sqrt{2}\tan(\sqrt{2}r)\phi$.

Theorem 5.1 tells us that a tube around a totally geodesic Q^{m-1} in Q^m is Hopf and $\mathfrak{A}$-principal. We shall show that the converse is also true.

Theorem 5.2. *Let M be a Hopf hypersurface of the complex quadric Q^m, $m \geq 3$. Then M is $\mathfrak{A}$-principal if and only if M is an open part of a tube around a totally geodesic Q^{m-1} in Q^m.*

Proof. Suppose that M is Hopf and $\mathfrak{A}$-principal. For each $[z] \in M$, since $S\mathcal{H}(-1) = 0$ and $\phi\mathcal{H}(-1) = \mathcal{H}(1)$, after putting $X \in \mathcal{H}(-1)$ in (4.2), we have $\alpha S\phi X = -2\phi X$, which implies that $\alpha \neq 0$ and $S\phi X = -(2/\alpha)\phi X$. By Lemma 4.5, α is a constant; without loss of generality, we put $\alpha = \sqrt{2}\cot(\sqrt{2}r)$ with $0 < r < \pi/2\sqrt{2}$. Hence we see that M has three constant principal curvatures:

$$\alpha = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\frac{2}{\alpha} = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0.$$

The corresponding principal curvature spaces are

$$\mathcal{T}_\alpha = \mathbb{R}\xi, \quad \mathcal{T}_\lambda = \mathcal{H}(1), \quad \mathcal{T}_\mu = \mathcal{H}(-1)$$

and the corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\lambda) = m - 1 = m(\mu).$$

We will use the Jacobi field method again to determine the focal submanifold of M. As before, denote by $\gamma_N(s)$ is the unit speed geodesic in Q^m that passes through $[z] \in M$ at $s = 0$ with initial velocity $N_{[z]}$. Since M is a real hypersurface in Q^m, we may identify the unit normal bundle $B(M)$ as M, and the focal map $\Phi_r([z]) = \gamma_N(r)$.

Let $\mathcal{Y}_X$ be the Jacobi field along γ_N with initial values $\mathcal{Y}_X(0) = X \in T_xM$ and $\dot{\mathcal{Y}}_X(0) = -SX$. Then

$$d\Phi_r(\sigma)X = \mathcal{Y}_X(r).$$

As N is $\mathfrak{A}$-principal, by using (2.2), the normal Jacobi operator $R_N := \hat{R}(\cdot, N)N$ is given by

$$R_NY = Y + BY + 2\eta(X)\xi.$$

It follows that R_N has two constant eigenvalues, 0 and 2, with corresponding eigenspaces $\mathcal{T}_\mu$ and $\mathcal{T}_\lambda \oplus T_\alpha$, respectively.

To compute $d\Phi_r([z])X$, $X \in T_{[z]}M$, we select the Jacobi field

$$\mathcal{Y}_X(t) = \begin{cases} \left(\cos(\sqrt{2}t) - (\alpha/\sqrt{2})\sin(\sqrt{2}t)\right)\mathcal{E}_X(t), & X = \xi; \\ \left(\cos(\sqrt{2}t) - (\lambda/\sqrt{2})\sin(\sqrt{2}t)\right)\mathcal{E}_X(t), & X \in T_\lambda; \\ \mathcal{E}_X(t), & X \in T_\beta, \end{cases} \tag{5.1}$$

where $\mathcal{E}_X$ is the parallel vector field along $\gamma_{[z]}$ with $\mathcal{E}_X(0) = X$. Then we have $d\Phi_r([z])X = \mathcal{Y}_X(r) = 0$ if and only if $X = \xi$, and we conclude that Φ_r has constant rank $2m - 2$. It follows that Φ_r is locally a submersion onto a submanifold $\tilde{M}$ in Q^m of real dimension $2m - 2$.

Note that $T_\lambda \oplus T_\mu = \mathcal{D}_{[z]}$ is invariant under J, J is invariant under parallel translation along geodesics, and the tangent space $T_{\Phi_r([z])}\tilde{M}$ of $\tilde{M}$ at $\Phi_r([z])$ is obtained by parallel translation of $T_\lambda \oplus T_\beta$ along the geodesic $\gamma_{[z]}$. Hence $\tilde{M}$ is a complex $(m-1)$-dimensional complex submanifold in Q^m, that is, a complex hypersurface.

Now we claim that $\tilde{M}$ is totally geodesic. To prove this claim, we note that the vector $\zeta = \dot{\gamma}_N(r)$ is a unit normal vector of $\tilde{M}$ at $\Phi_r([z])$ and the shape operator $\tilde{S}_\zeta$ of $\tilde{M}$ in Q^m with respect to ζ can be determined by $\tilde{S}_\zeta X = -\dot{\mathcal{Y}}_X(r)$, where $X \in T_\lambda \oplus T_\mu$ and $\mathcal{Y}_X$ is the Jacobi field given by (5.1). First, it is clear that $\dot{\mathcal{Y}}_X(r) = 0$ for $X \in T_\beta$. Next, as $\lambda = -\sqrt{2}\tan(\sqrt{2}r)$ we see that $\dot{\mathcal{Y}}_X(r) = 0$ for $X \in T_\lambda$. Hence, $\tilde{M}$ is a totally geodesic complex hypersurface in Q^m.

By the rigidity of totally geodesic submanifolds, M is an open part of a tube of radius r around a connected, complete, totally geodesic complex hypersurface $\tilde{M}$ of Q^m. According to the classification of totally geodesic submanifolds in Q^m (cf. [12]), $\tilde{M}$ is the totally geodesic complex hypersurface Q^{m-1} in Q^m. This implies that M is locally congruent to a tube around Q^{m-1} in Q^m. ☐

6. Contact real hypersurfaces in Q^m

Let M be a real hypersurface in a Kaehler manifold $\hat{M}$. Denote by $\Phi(\cdot,\cdot) := \langle\cdot, \phi\ \cdot\rangle$ the fundamental 2-form. If there exists a non-zero function ρ on M such that $d\eta = \rho\Phi$, then M admits a contact structure. In this case, we call M a *contact real hypersurface* in $\hat{M}$. Since $d\eta(X,Y) = \langle(\phi S + S\phi)X, Y\rangle$, a real hypersurface M in $\hat{M}$ is contact if and only if

$$\phi S + S\phi = \rho\phi \tag{6.1}$$

for some non-zero function ρ on M.

Theorem 6.1. *Let M be a real hypersurface in Q^m. Then M is contact if and only if M is an open part of a tube around a totally geodesic Q^{m-1} in Q^m.*

Proof. Suppose that M is a contact real hypersurface, thas is, it satisfies (6.1). Then it is clear that M is Hopf. Furthermore, ρ must be a non-zero constant (cf. [4]). We first consider the case where M is neither 𝔄-isotropic nor 𝔄-principal. Then there exists an open subset $\mathcal{U} \subset M$ on which $0 < k < 1$. Without loss of generality, we assume $\mathcal{U} = M$.

Let λ, μ, t, ω and E be as stated in Lemma 4.3. We can assume that E is a unit vector. By the Codazzi equation, we have

$$\begin{aligned}
0 &= \langle(\nabla_E S)\phi E - (\nabla_{\phi E} S)E, V^\circ\rangle \\
&= \langle(t\mathbb{I} - S)\nabla_E V^\circ, \phi E\rangle - \langle(t\mathbb{I} - S)\nabla_{\phi E} V^\circ, E\rangle \\
&= -tg(\lambda+\mu) - t(\lambda-\mu)\langle BE, \phi E\rangle + 2g\lambda\mu.
\end{aligned} \tag{6.2}$$

Similarly, we compute

$$\begin{aligned}
0 &= \langle(\nabla_E S)\phi E - (\nabla_{\phi E} S)E, \phi V\rangle \\
&= \langle(t\mathbb{I} - S)\nabla_E V^\circ, \phi E\rangle - \langle(t\mathbb{I} - S)\nabla_{\phi E} V^\circ, E\rangle \\
&= \omega f(\lambda+\mu) - \omega(\lambda-\mu)\langle BE, E\rangle - 2f\lambda\mu.
\end{aligned} \tag{6.3}$$

It follows from (4.4)–(4.5) and (6.2)–(6.3) that $(\xi\alpha)\{(\lambda+\mu)t\omega - (t+\omega)\lambda\mu\} = 0$. By applying Lemma 4.3 and the fact that $\lambda+\mu = \omega+t = \rho$, we obtain $(\xi\alpha)\rho k^2 = 0$ and hence $\xi\alpha = 0$.

Since V° and ϕV are orthogonal, we obtain $g(t-\alpha) = f(\omega-\alpha) = 0$ by (4.6). If $fg \neq 0$, then $t = \omega = \alpha$. But these imply that $\alpha = (t+\omega)/2 = \rho/2$ is a constant, a contradiction to Lemma 4.5. Hence we have either $f = 0$ or $g = 0$. Without loss of generality, we assume $g = 0$, hence $f \neq 0$ and $\alpha = \omega = \rho - t$. By substituting these into the second equation in Lemma 4.3, we get $-2\alpha^2 + \rho\alpha - 2 + 2k^2 = 0$. By applying Lemma 3.3 (c) and (4.1), we see that

$$0 = (-4\alpha + \rho)\,\mathrm{grad}\,\alpha + 2\,\mathrm{grad}(k^2) = -2f(-4\alpha + \rho)V - 2\,\mathrm{grad}(f^2) = -3\rho.$$

This is a contradiction. Consequently, M is either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal. It is clear that N is not $\mathfrak{A}$-isotropic everywhere (for otherwise we have $2\rho\phi V = (\phi S + S\phi)V = 0$ by virtue of Lemma 3.5, which is impossible). Hence M is $\mathfrak{A}$-principal. According to Theorem 5.2, M is an open part of a tube around a totally geodesic Q^{m-1} in Q^m.

Conversely, as shown in Theorem 5.1, the shape operator of a tube of radius r around a totally geodesic Q^{m-1} in Q^m satisfies $\phi S + S\phi = -\sqrt{2}\tan(\sqrt{2}r)\phi$. Hence it is contact, and this completes the proof. □

Remark 6.2. Contact real hypersurfaces in Kaehler manifolds with constant mean curvature were studied in [4].

Next, we study real hypersurfaces M in Q^m under a weaker version of (6.1), i.e.,

$$\phi(\phi S + S\phi - \rho\phi)\phi = 0, \tag{6.4}$$

for some function ρ on M. We shall first derive some identities from the condition (6.4). Note that (6.4) is equivalent to

$$\langle(\phi S + S\phi - \rho\phi)Y, Z\rangle = 0$$

for any vector fields Y and Z in $\mathcal{D}$. Differentiating this equation covariantly in the direction of X in $\mathcal{D}$ we get

$$\begin{aligned}\langle\phi SY, \nabla_X Z\rangle &+ \langle(\nabla_X\phi)SY + \phi(\nabla_X S)Y + \phi S\nabla_X Y, Z\rangle\\ &+ \langle S\phi Y, \nabla_X Z\rangle + \langle(\nabla_X S)\phi Y + S(\nabla_X\phi)Y + S\phi\nabla_X Y, Z\rangle\\ &- d\rho(X)\langle\phi Y, Z\rangle - \rho\langle\phi Y, \nabla_X Z\rangle - \rho\langle(\nabla_X\phi)Y + \phi\nabla_X Y, Z\rangle = 0.\end{aligned}$$

By using (3.1) and (6.4), this equation can be rewritten as

$$\begin{aligned}-\langle Z, \phi S\xi\rangle\langle\phi SX, Y\rangle + \langle Y, \phi S\xi\rangle\langle\phi SX, Z\rangle - \langle(\nabla_X S)Y, \phi Z\rangle + \langle(\nabla_X S)Z, \phi Y\rangle\\ + \eta(SY)\langle SX, Z\rangle - \eta(SZ)\langle SX, Y\rangle - d\rho(X)\langle\phi Y, Z\rangle = 0.\end{aligned} \tag{6.5}$$

Now by replacing X, Y and Z cyclically in (6.5) and then summing these equations, with the help of the Codazzi equation, Lemma 3.2 (e) and (6.4), we obtain

$$\mathfrak{S}(\rho\langle X, \phi S\xi\rangle + d\rho(X))\langle\phi Y, Z\rangle = 0,$$

where $\mathfrak{S}$ denotes the cyclic sum over X, Y and Z. Let X be an arbitrary vector in $\mathcal{D}$. Since $m \geq 3$, we may take $Y \perp X, \phi X$ and $Z = \phi Y$ in the above equation to obtain

$$\rho\langle X, \phi S\xi\rangle + d\rho(X) = 0$$

for any X in $\mathcal{D}$.

In a special case where ρ is a non-zero constant, the above equation implies that $\phi S\xi = 0$, which means that ξ is principal and so $(\phi S+S\phi-\rho\phi)\xi = 0$. Consequently, we have $\phi S + S\phi - \rho\phi = 0$, for some non-zero constant ρ, and hence it follows from Theorem 6.1 that we have proved the following result.

Theorem 6.3. *Let M be a real hypersurface in Q^m, $m \geq 3$. Then M satisfies*

$$\phi(\phi S + S\phi - \epsilon\phi)\phi = 0$$

for some constant $\epsilon \neq 0$ if and only if M is an open part of a tube around a totally geodesic Q^{m-1} in Q^m.

Remark 6.4. Theorem 6.3 was proved in [14] for real hypersurfaces in non-flat complex space forms.

7. Pseudo-Einstein real hypersurfaces in Q^m

Suppose that M is pseudo-Einstein, that is,

$$\operatorname{Ric} X = aX + b\eta(X)\xi, \tag{7.1}$$

where a, b are constants. By (3.3), we see that M is pseudo-Einstein if and only if

$$PX = (2m-a-1)X - (3+b)\eta(X)\xi + \langle X, V\rangle V + \langle X, B\xi\rangle B\xi - fBX - g\theta X, \tag{7.2}$$

where $P := S^2 - hS$.

Lemma 7.1. *Let M be a pseudo-Einstein real hypersurface in Q^m, $m \geq 3$. If $b \neq 0$, then M is Hopf.*

Proof. It follows from the hypothesis (7.1) that

$$(\nabla_X \operatorname{Ric})Y = b\{\langle \phi SX, Y\rangle\xi + \eta(Y)\phi SX\}.$$

Take an orthonormal basis $\{e_1, \ldots, e_{2m-1}\}$ on $T_{[z]}M$. Then

$$X(\operatorname{Trace}\operatorname{Ric}) = \sum_{j=1}^{2m-1} \langle (\nabla_X \operatorname{Ric})e_j, e_j\rangle = 0,$$

$$\operatorname{div}\operatorname{Ric}(X) = \sum_{j=1}^{2m-1} \langle (\nabla_{e_j} \operatorname{Ric})X, e_j\rangle = b\langle \phi S\xi, X\rangle.$$

By the well-known formula $d(\operatorname{Trace}\operatorname{Ric}) = 2\operatorname{div}\operatorname{Ric}$, we obtain $b\phi S\xi = 0$. Hence we conclude that M is Hopf if $b \neq 0$. $\square$

Theorem 7.2. *Let M be a pseudo-Einstein real hypersurface in Q^m, $m \geq 3$. Then M is either $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.*

Proof. Suppose that M is neither $\mathfrak{A}$-principal nor $\mathfrak{A}$-isotropic. Then there exists an open subset $\mathcal{U} \subset M$ on which $0 < f < 1$ and $g = 0$. Without loss of generality, we assume $\mathcal{U} = M$. It follows from (7.2) that $P\xi = (2m-a-2-2k^2-b)\xi - 2f\phi V$. If $b \neq 0$, then M is Hopf by Lemma 7.1 and so $(\alpha^2 - h\alpha)\xi = (2m - a - 2 - 2k^2 -$

$b)\xi - 2f\phi V$, which implies that $f = 0$, a contradiction. Hence we have $b = 0$. It follows that P has at most five distinct eigenvalues:

$$\sigma_0 = 2m - a, \quad \sigma_1 = 2m - a - 1 - f, \quad \sigma_2 = 2m - a - 1 + f,$$
$$\sigma_3 = 2m - a - 2 - 2k, \quad \sigma_4 = 2m - a - 2 + 2k,$$

with eigenspaces

$$\mathcal{T}_0 = \mathbb{R}V, \quad \mathcal{T}_1 = \mathcal{H}(1), \quad \mathcal{T}_2 = \mathcal{H}(-1), \quad \mathcal{T}_3 = \mathbb{R}W_3, \quad \mathcal{T}_4 = \mathbb{R}W_4,$$

where

$$W_3 = r\xi + \frac{s}{k}\phi V, \quad W_4 = -s\xi + \frac{r}{k}\phi V, \quad r = \sqrt{1+k}, \quad s = \sqrt{1-k}.$$

Since $f, k > 0$ and $f^2 + k^2 = 1$, we can easily verify the following:

$$\left.\begin{array}{l} \sigma_0 \notin \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\} \\ \sigma_1 \notin \{\sigma_0, \sigma_2, \sigma_3, \sigma_4\} \\ \sigma_3 \notin \{\sigma_0, \sigma_1, \sigma_2, \sigma_4\} \end{array}\right\}.$$

Since $PS = SP$, we conclude that

$$S\mathcal{H}(1) \subset \mathcal{H}(1), \quad SV = tV, \quad SW_3 = \kappa W_3,$$

where t and κ are functions satisfying

$$t^2 - ht = \sigma_0, \quad \kappa^2 - h\kappa = \sigma_3. \tag{7.3}$$

Let

$$U := SW_4 - \tau W_4 \in \mathcal{H}(-1), \quad \tau = \frac{1}{2}\langle SW_4, W_4\rangle.$$

A straightforward calculation gives

$$\left.\begin{array}{ll} BW_3 = W_4, & \\ \operatorname{grad} r = stk^{-1}V, & \operatorname{grad} s = -rtk^{-1}V \\ \operatorname{grad}(sk^{-1}) = -r(2-k)tk^{-3}V, & \operatorname{grad}(rk^{-1}) = -s(2+k)tk^{-3}V \\ (\phi S + S\phi)W_3 = -s(\kappa + t)k^{-1}V, & (\phi S + S\phi)W_4 = -r(\tau + t)k^{-1}V + \phi U \\ S\phi SW_3 = -s\kappa tk^{-1}V, & S\phi SW_4 = -r\tau tk^{-1}V + S\phi U \\ \phi BSW_3 = -r\kappa k^{-1}V, & \phi BSW_4 = -s\tau k^{-1}V - \phi U \\ SB\phi W_3 = sftk^{-1}V, & SB\phi W_4 = rftk^{-1}V. \end{array}\right\} \tag{7.4}$$

By applying (7.4), we compute

$$\begin{aligned}(\nabla_X S)W_3 &= (X\kappa)W_3 + (\kappa\mathbb{I} - S)\nabla_X W_3 \\ &= (X\kappa)W_3 - \frac{r(2-k)t}{k^3}\langle X, V\rangle(\kappa\mathbb{I} - S)\phi V + \frac{2(\kappa - t)}{rk}\langle X, SB\xi\rangle V \\ &\quad + \frac{r}{k}(\kappa\mathbb{I} - S)\phi SX - \frac{s}{k}(\kappa\mathbb{I} - S)\phi BSX.\end{aligned}$$

By the Codazzi equation, we have

$$
\begin{aligned}
0 &= \langle(\nabla_X S)Y - (\nabla_Y S)X, W_3\rangle + 2r\left\{\langle\phi X, Y\rangle - \frac{\langle X, V\rangle\langle Y, B\xi\rangle - \langle Y, V\rangle\langle X, B\xi\rangle}{k}\right\}\\
&= (X\kappa)\langle Y, W_3\rangle - \frac{r(2-k)t}{k^3}\langle X, V\rangle\langle Y, (\kappa\mathbb{I} - S)\phi V\rangle\\
&\quad - (Y\kappa)\langle X, W_3\rangle + \frac{r(2-k)t}{k^3}\langle Y, V\rangle\langle X, (\kappa\mathbb{I} - S)\phi V\rangle\\
&\quad - \frac{2(\kappa - t)}{rk}\langle X, V\rangle\langle Y, SB\xi\rangle - \frac{2r}{k}\langle X, V\rangle\langle Y, B\xi\rangle + \frac{st}{k}\langle X, V\rangle\langle Y, S\xi\rangle\\
&\quad + \frac{2(\kappa - t)}{rk}\langle Y, V\rangle\langle X, SB\xi\rangle + \frac{2r}{k}\langle Y, V\rangle\langle X, B\xi\rangle - \frac{st}{k}\langle Y, V\rangle\langle X, S\xi\rangle\\
&\quad + 2r\langle\phi X, Y\rangle + \frac{r}{k}\langle\kappa(\phi S + S\phi)X - 2S\phi SX, Y\rangle - \frac{s\kappa}{k}\langle(\phi BS + SB\phi)X, Y\rangle.
\end{aligned}
\tag{7.5}
$$

Next we claim that $U = 0$. For otherwise, we have $\sigma_2 = \sigma_4$ or $1 + f = 2k$. It follows that $f = 3/5$. Hence $t = 0$ and so $2m - a = 0$ by Lemma 3.3 and (7.3). By putting $X = W_4$ and $Y \in \mathcal{H}(1)$ in (7.5), we obtain $r\langle\kappa\phi U - 2S\phi U, Y\rangle + s\kappa\langle\phi U, Y\rangle = 0$. Since $S\mathcal{H}(1) \subset \mathcal{H}(1)$, we have $S\phi U = \mu\phi U$, where

$$\mu = \frac{r+s}{2r}\kappa = \frac{2}{3}\kappa.$$

Since $\mu^2 - h\mu = \sigma_1 = -8/5$, we have

$$\frac{4}{9}\kappa^2 - \frac{2}{3}h\kappa = -\frac{8}{5}.$$

Comparing with (7.3), we obtain $\kappa^2 = -18/5$, a contradiction. Hence we conclude that $U = 0$ or $SW_4 = \tau W_4$ and so

$$\tau^2 - h\tau = \sigma_4. \tag{7.6}$$

Moreover, we have $S\mathcal{H}(\varepsilon) \subset \mathcal{H}(\varepsilon)$, $\phi S\phi\mathcal{H}(\varepsilon) \subset \mathcal{H}(\varepsilon)$ and $(SB - BS)\mathcal{H}(\varepsilon) = 0$ for any $\varepsilon \in \{1, -1\}$.

Let $X \in \mathcal{H}$; replacing X by ϕX in (7.5) gives

$$-2rkX + r\kappa(\phi S\phi X - SX) - 2rS\phi S\phi X - s\kappa(\phi BS\phi X - SBX) = 0.$$

By taking the transpose of this equation, we obtain

$$-2rkX + r\kappa(\phi S\phi X - SX) - 2r\phi S\phi SX - s\kappa(\phi SB\phi X - BSX) = 0.$$

It follows that

$$2r(\phi S\phi S - S\phi S\phi)X = s\kappa\phi(SB - BS)\phi X - s\kappa(BS - SB)X = 0$$

for any $X \in \mathcal{H}$. This implies that $S_{|\mathcal{H}(-1)}$ and $\phi S\phi_{|\mathcal{H}(-1)}$ are simultaneously diagonalized by orthonormal vectors $X_1, \ldots, X_{m-2}$ in $\mathcal{H}(-1)$, say

$$SX_j = \lambda_j X_j, \quad \phi S\phi X_j = -\mu_j X_j, \qquad j \in \{1, \ldots, m-2\}.$$

It follows that

$$\lambda_j^2 - h\lambda_j = \sigma_2. \tag{7.7}$$

Moreover, since each $\phi X_j \in \mathcal{H}(1)$ and $S\phi X_j = \mu_j \phi X_j$, we also have

$$\mu_j^2 - h\mu_j = \sigma_1. \tag{7.8}$$

Letting $X = X_j$ and $Y = \phi X_j$ in (7.5) gives

$$2rk + r\kappa(\lambda_j + \mu_j) - 2r\lambda_j\mu_j + s\kappa(\lambda_j - \mu_j) = 0,$$

which can be rewritten as

$$\{(1-k+f)\lambda_j - (1-k-f)\mu_j\}\,\kappa = 2f(\lambda_j\mu_j - k). \tag{7.9}$$

By a similar calculation, we have

$$\begin{aligned}(\nabla_X S)W_4 = (X\tau)W_4 &- \frac{s(2+k)t}{k^3}\langle X, V\rangle(\tau\mathbb{I} - S)\phi V + \frac{2(\tau - t)}{sk}\langle X, SB\xi\rangle V \\ &+ \frac{s}{k}(\tau\mathbb{I} - S)\phi SX - \frac{r}{k}(\tau\mathbb{I} - S)\phi BSX\end{aligned}$$

and

$$\begin{aligned}0 = {} & (X\tau)\langle Y, W_4\rangle - \frac{s(2+k)t}{k^3}\langle X, V\rangle\langle Y, (\tau\mathbb{I} - S)\phi V\rangle \\ & - (Y\tau)\langle X, W_4\rangle + \frac{s(2+k)t}{k^3}\langle Y, V\rangle\langle X, (\tau\mathbb{I} - S)\phi V\rangle \\ & - \frac{2(\tau - t)}{sk}\langle X, V\rangle\langle Y, SB\xi\rangle - \frac{2s}{k}\langle X, V\rangle\langle Y, B\xi\rangle + \frac{rt}{k}\langle X, V\rangle\langle Y, S\xi\rangle \\ & + \frac{2(\tau - t)}{sk}\langle Y, V\rangle\langle X, SB\xi\rangle + \frac{2s}{k}\langle Y, V\rangle\langle X, B\xi\rangle - \frac{rt}{k}\langle Y, V\rangle\langle X, S\xi\rangle \\ & - 2s\langle \phi X, Y\rangle + \frac{s}{k}\langle \tau(\phi S + S\phi)X - 2S\phi SX, Y\rangle - \frac{r\tau}{k}\langle(\phi BS + SB\phi)X, Y\rangle.\end{aligned} \tag{7.10}$$

Similarly, putting $X = X_j$ and $Y = \phi X_j$ in (7.10) gives

$$\{(1+k+f)\lambda_j - (1+k-f)\mu_j\}\,\tau = 2f(\lambda_j\mu_j + k). \tag{7.11}$$

In the following calculation, we replace λ_j and μ_j by λ and μ respectively for simplicity. First, eliminating the variable h in (7.3) and (7.6)–(7.8) gives

$$\mu\lambda^2 = (\mu^2 + 1 + f - \sigma_0)\lambda + (\sigma_0 - 1 + f)\mu, \tag{7.12}$$

$$\mu\kappa_\epsilon^2 = (\mu^2 + 1 + f - \sigma_0)\kappa_\epsilon + (\sigma_0 - 2 - 2\epsilon k)\mu, \tag{7.13}$$

where $\epsilon \in \{1, -1\}$ and we have put $\kappa_1 = \kappa$ and $\kappa_{-1} = \tau$. Using this unified notation, (7.9) and (7.11) can be expressed as

$$\{(1-\epsilon k+f)\lambda - (1-\epsilon k-f)\mu\}\,\kappa_\epsilon = 2f(\lambda\mu - \epsilon k). \tag{7.14}$$

It follows from (7.13)–(7.14) that

$$\begin{aligned}2\mu f^2(\lambda\mu - \epsilon k)^2 = {} & (\mu^2 + 1 + f - \sigma_0)f(\lambda\mu - \epsilon k)\{(1-\epsilon k+f)\lambda - (1-\epsilon k-f)\mu\} \\ & + \frac{\mu}{2}(\sigma_0 - 2 - 2\epsilon k)\{(1-\epsilon k+f)\lambda - (1-\epsilon k-f)\mu\}^2.\end{aligned}$$

By applying (7.12), we can eliminate the variable λ^2 in the preceding equation and obtain

$$k^2\{(\mu^2C_1+C_2)\lambda+(-\mu^2C_1+C_3)\mu\}+\epsilon k\{(\mu^2C_4+C_5)\lambda+(\mu^2C_6+C_7)\mu\}=0, \tag{7.15}$$

where

$$\begin{aligned}
C_1 &= 2f-\sigma_0\\
C_2 &= 2f(1+f)+(1-f)\sigma_0-\sigma_0^2\\
C_3 &= -2f(1+f)-(1-3f)\sigma_0+\sigma_0^2\\
C_4 &= -2f(1+f)+\sigma_0\\
C_5 &= -2f(1+f)^2+(-1+f+2f^2)\sigma_0+\sigma_0^2\\
C_6 &= 2f(1-f)-\sigma_0\\
C_7 &= 2f(1-f^2)+(1-3f)\sigma_0-\sigma_0^2.
\end{aligned}$$

After substituting $\epsilon=\pm1$ in (7.15), we have

$$(\mu^2C_1+C_2)\lambda+(-\mu^2C_1+C_3)\mu=(\mu^2C_4+C_5)\lambda+(\mu^2C_6+C_7)\mu=0. \tag{7.16}$$

It follows that

$$-2C_1\mu^4+D_1\mu^2+D_2=0, \tag{7.17}$$

where

$$\begin{aligned}
D_1 &= \frac{C_1(C_5+C_7)+C_2C_6-C_3C_4}{2f^2}=-8f(1+f)+8f\sigma_0+\sigma_0^2,\\
D_2 &= \frac{C_2C_7-C_3C_5}{2f^2}=-4f(1+f)^2+(-2+4f+6f^2)\sigma_0+(3-f)\sigma_0^2-\sigma_0^3.
\end{aligned}$$

On the other hand, by using (7.12) and the first equation of (7.16), we obtain

$$C_1\mu^4+D_3\mu^2+D_4=0, \tag{7.18}$$

where

$$\begin{aligned}
D_3 &= \frac{C_1\{3C_2-C_3+2\sigma_0(\sigma_0-1)\}-2C_3\sigma_0}{4f^2}=4f(1+f)-4f\sigma_0-\sigma_0^2,\\
D_4 &= \frac{C_2\{C_2-C_3+2\sigma_0(\sigma_0-1)\}}{4f^2}=2f(1+f)^2+(1-2f-3f^3)\sigma_0-2\sigma_0^2+\sigma_0^3.
\end{aligned}$$

It follows from (7.17)–(7.18) that $\sigma_0^2(\mu^2+1+f-\sigma_0)=0$. If $\sigma_0=0$, then (7.18) reduces to $(\mu^2+1+f)^2=0$, which implies that $f<0$, a contradiction. Hence we have $\mu^2+1+f-\sigma_0=0$. After substituting this back into (7.18), we obtain $1+f-\sigma_0=0$. Since σ_0 is a constant, f is also a constant. By virtue of Lemma 3.3, we have $t=0$ and so (7.3) implies that $\sigma_0=0$, a contradiction. Consequently, this case does not exist. $\square$

Theorem 7.3. *Let M be a real hypersurface of the complex quadric Q^m, $m\geq3$. Then M is pseudo-Einstein, that is, it satisfies* (7.1)*, if and only if one of the following holds:*

(a) *M is an open part of a tube of radius r around a totally geodesic Q^{m-1} in Q^m where $a = -b = 2m$ and $2\cot^2(\sqrt{2}r) = m-2$.*
(b) *$m = 3$, $a = 6$, $b = -4$ and M is a $\mathfrak{A}$-isotropic Hopf hypersurface with principal curvatures 0, λ and $1/\lambda$. The corresponding principal curvature space for 0 is $\mathcal{H}^\perp$. Moreover, $\lambda^2 \neq 1$ on an open dense subset of M.*

Proof. Suppose that M is pseudo-Einstein. According to Theorem 7.2, we have two cases: M is $\mathfrak{A}$-principal and M is $\mathfrak{A}$-isotropic.

Case I. M is $\mathfrak{A}$-principal.

In this case, we have $f = 1$, $g = 0$ and $V = 0$. Hence, (7.2) is descended to

$$PX = (2m - a - 1)X - (2 + b)\eta(X)\xi - BX.$$

Since $S\mathcal{H}(-1) = 0$, we obtain $2m - a = 0$ and hence

$$PX = -X - (2 + b)\eta(X)\xi - BX.$$

We claim that M is Hopf. Suppose that M is not Hopf. Then we have $b = 0$ by Lemma 7.1. It follows that $PX = -2$ for any $X \perp \mathcal{H}(-1)$. Furthermore, M has three distinct principal curvatures (for otherwise M must be Hopf): 0, λ and μ, with multiplicities $m-1$, m_1 and $m-m_1$ respectively, where λ and μ are solutions for

$$z^2 - hz + 2 = 0.$$

Hence, we have $\lambda + \mu = h$ and $\lambda\mu = 2$ so that

$$0 = m_1\lambda + (m - m_1)\mu - h = \frac{(m_1 - 1)\lambda^2 + 2(m - m_1 - 1)}{\lambda}.$$

This contradicts the fact $m \geq 3$. Hence the claim is proved.

By Theorem 5.1 and Theorem 5.2, we conclude that M is an open part of a tube of radius $r \in \,]0, \pi/2\sqrt{2}[$ around the totally geodesic Q^{m-1} in Q^m, and M has three constant principal curvatures,

$$\alpha = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0,$$

with multiplicities 1, $m-1$, $m-1$ respectively. It follows that

$$h = \alpha + (m-1)\lambda.$$

Moreover, α and λ satisfy

$$\alpha^2 - h\alpha + 2 + b = 0, \quad \lambda^2 - h\lambda + 2 = 0.$$

By using these equations, we obtain $\cot^2(\sqrt{2}r) = (m-2)/2$ and $b = -2m$. This gives statement (a) in the theorem.

Case II. M is $\mathfrak{A}$-isotropic.

In this case, we have $f = 0$ and $SV = S\phi V = 0$. Hence, $2m - a = 0$ and (7.2) is descended to

$$PX = -X - (3 + b)\eta(X)\xi + \langle X, V\rangle V + \langle X, \phi V\rangle \phi V.$$

It follows that P has at most three distinct eigenvalues,

$$\sigma_0 = 0, \quad \sigma_1 = -1, \quad \sigma_2 = -4 - b,$$

with eigenspaces

$$\mathcal{T}_0 = \mathbb{R}V \oplus \mathbb{R}\phi V, \quad \mathcal{T}_1 = \mathcal{H}, \quad \mathcal{T}_2 = \mathbb{R}\xi.$$

If $\sigma_2 \notin \{\sigma_0, \sigma_1\}$, then M is Hopf as $\dim \mathcal{T}_2 = 1$. On the other hand, if $\sigma_2 \in \{\sigma_0, \sigma_1\}$, then $b \neq 0$ and so M is also Hopf by Lemma 7.1. Hence, we conclude that M is Hopf in this case. We take an orthonormal basis $\{X_1, \ldots, X_{m-2}, \phi X_1, \ldots, \phi X_{m-2}\}$ in $\mathcal{H}$ such that

$$SX_j = \lambda_j X_j, \quad S\phi X_j = \mu_j \phi X_j, \qquad j \in \{1, \ldots, m-2\}.$$

By (4.2), we have

$$2\lambda_j\mu_j - \alpha(\lambda_j + \mu_j) - 2 = 0. \tag{7.19}$$

Moreover, all λ_j and μ_j must be solutions of

$$z^2 - hz + 1 = 0. \tag{7.20}$$

We consider

$$\mathcal{E}_j = \{[z] : \lambda_j = \mu_j\}; \quad \mathcal{E} = \bigcap_{j=1}^{m-2} \mathcal{E}_j.$$

If $\operatorname{Int}\mathcal{E} \neq \emptyset$, then we have $\phi S - S\phi = 0$ on $\operatorname{Int}\mathcal{E}$ and by a result in [7], there are four principal curvatures: $\alpha = 2\cot 2r$, $\lambda_1 = \cot r$, $\lambda_2 = -\tan r$, $\beta = 0$. The corresponding principal curvature spaces are

$$\mathcal{T}_\alpha = \mathbb{R}\xi, \quad \mathcal{T}_{\lambda_1}, \quad \mathcal{T}_{\lambda_2}, \quad \mathcal{T}_\beta = \mathbb{R}V \oplus \mathbb{R}\phi V,$$

where $\phi\mathcal{T}_{\lambda_1} = \mathcal{T}_{\lambda_1}$, $\phi\mathcal{T}_{\lambda_2} = \mathcal{T}_{\lambda_2}$ and $\mathcal{H} = \mathcal{T}_{\lambda_1} \oplus \mathcal{T}_{\lambda_2}$. Since λ_1, λ_2 are solutions of (7.20), we have $\lambda_1\lambda_2 = 1$. This is a contradiction and so $\operatorname{Int}\mathcal{E} = \emptyset$.

Without loss of generality, we assume that $\mathcal{E}_1^c \neq \emptyset$. It follows that λ_1, μ_1 are distinct solutions of (7.20) on $\mathcal{E}_1^c$. Hence, $\lambda_1 + \mu_1 = h$ and $\lambda_1\mu_1 = 1$. Substituting these into (7.19) gives $h\alpha = 0$. Hence, $\alpha = 0$ in view of (7.20) and so $4 + b = -\sigma_2 = 0$.

Suppose that $m \geq 4$. If there exists $i \in \{2, \ldots, m-2\}$ such that $\mathcal{E}_1^c \cap \operatorname{Int}\mathcal{E}_i \neq \emptyset$, then we have $\lambda_i = \mu_i \in \{\lambda_1, \mu_1\}$, say $\lambda_i = \lambda_1$, as there are only three principal curvatures in this case. It follows from (7.19) that $\lambda_i^2 = 1 = \lambda_1\mu_1$. This contradicts the fact that $\lambda_1 \neq \mu_1$. Hence we conclude that $\mathcal{E}_1^c \supset \cap_{j=1}^{m-2}\mathcal{E}_j^c \neq \emptyset$. It follows that

$$h = (m-2)(\lambda_1 + \mu_1) = (m-2)h.$$

This is a contradiction. Hence $m = 3$ and $\mathcal{E}^c = \mathcal{E}_1^c$, which is open and dense in M. This gives statement (b). The converse is trivial. □

Remark 7.4. Pseudo-Einstein Hopf hypersurfaces in Q^m were studied in [19]. However, the classification was inaccurate as the real hypersurfaces listed in [19, Main Theorem 2 (ii)] are not pseudo-Einstein. As we have verified in the proof of Theorem 7.3, if such a real hypersurface is pseudo-Einstein, two of its principal curvatures $\lambda_1 = \cot r$ and $\lambda_2 = -\tan r$ must be solutions for (7.20), which is clearly impossible. Unfortunately, this argument was overlooked by [19, Remark 6.2].

Remark 7.5. The author does not know any example of the real hypersurfaces stated in Theorem 7.3 (b). However, even if it exists, this example is local in the sense that it is not extendible to a complete real hypersurface on the basis of Theorem 7.6 below.

Theorem 7.6. *Let M be a complete real hypersurface of the complex quadric Q^m, $m \geq 3$. Then M is pseudo-Einstein, that is, it satisfies* (7.1)*, if and only if it is congruent to a tube of radius r around a totally geodesic Q^{m-1} in Q^m where $a = -b = 2m$ and $2\cot^2(\sqrt{2}r) = m - 2$.*

Proof. Suppose that M is complete pseudo-Einstein and satisfies the properties in Theorem 7.3 (b). Then we have $f = g = 0$ and $k = 1$. Consider a unit vector field X tangent to $\mathcal{H}$ with $SX = \lambda X$ and $S\phi X = (1/\lambda)\phi X$. Furthermore, taking the reciprocal if necessary, we have $\lambda^2 \geq 1$. Note that $\langle S, S\rangle = \lambda^2 + (1/\lambda)^2 \geq 2$, with equality holding if and only if $\lambda^2 = 1$. Since M is compact, $\langle S, S\rangle$ is bounded. Suppose that the maximum for $\langle S, S\rangle$ is attained at a point $[z] \in M$. Then $\lambda^2 > 1$ and so λ is differentiable at $[z]$ by Theorem 7.3 (b).

By the Codazzi equation, we have

$$\langle(\nabla_X S)V - (\nabla_V S)X, X\rangle = -\langle BX, X\rangle.$$

At the points on which λ is differentiable, by applying Lemma 3.3 and Lemma 3.5 to the preceding equation, we have

$$V\lambda = (1 + \lambda^2)\langle BX, X\rangle. \tag{7.21}$$

Similarly, we have

$$\begin{aligned} \phi V\lambda &= \langle(\nabla_{\phi V} S)X, X\rangle = \langle(\nabla_X S)\phi V, X\rangle - \langle BX, \phi X\rangle \\ &= -(1 + \lambda^2)\langle BX, \phi X\rangle. \end{aligned} \tag{7.22}$$

Since $[z]$ is a critical point, it follows from (7.21)–(7.22) that $\langle BX, X\rangle = \langle BX, \phi X\rangle = 0$ at the point $[z]$. Since $\dim \mathcal{H} = 2$ and $B\mathcal{H} \subset \mathcal{H}$, we get $BX = \langle BX, X\rangle X + \langle BX, \phi X\rangle\phi X = 0$. This is a contradiction and the proof is complete. $\square$

Acknowledgement

The authors would like to thank the referee for reading the manuscript in great detail and for providing valuable comments which led to the improvement of the present paper.

References

[1] J. Berndt, Real hypersurfaces in quaternionic space forms, *J. Reine Angew. Math.* **419** (1991), 9–26. MR 1116915.

[2] J. Berndt and Y. J. Suh, On the geometry of homogeneous real hypersurfaces in the complex quadric, in *Proceedings of the 16th International Workshop on Differential Geometry and the 5th KNUGRG-OCAMI Differential Geometry Workshop [Volume 16]*, 1–9, Natl. Inst. Math. Sci. (NIMS), Taejŏn, 2012. MR 3014814.

[3] J. Berndt, S. Console and C. Olmos, *Submanifolds and Holonomy*, Chapman & Hall/CRC Research Notes in Mathematics, 434, Chapman & Hall/CRC, Boca Raton, FL, 2003. MR 1990032.

[4] J. Berndt and Y. J. Suh, Contact hypersurfaces in Kähler manifolds, *Proc. Amer. Math. Soc.* **143** (2015), no. 6, 2637–2649. MR 3326043.

[5] J. Berndt and Y. J. Suh, Real hypersurfaces in complex two-plane Grassmannians, *Monatsh. Math.* **127** (1999), no. 1, 1–14. MR 1666307.

[6] J. Berndt and Y. J. Suh, Hypersurfaces in noncompact complex Grassmannians of rank two, *Internat. J. Math.* **23** (2012), no. 10, 1250103, 35 pp. MR 2999048.

[7] J. Berndt and Y. J. Suh, Real hypersurfaces with isometric Reeb flow in complex quadrics, *Internat. J. Math.* **24** (2013), no. 7, 1350050, 18 pp. MR 3084731.

[8] A. A. Borisenko, On the global structure of Hopf hypersurfaces in a complex space form, *Illinois J. Math.* **45** (2001), no. 1, 265–277. MR 1849998.

[9] T. E. Cecil and P. J. Ryan, Focal sets and real hypersurfaces in complex projective space, *Trans. Amer. Math. Soc.* **269** (1982), no. 2, 481–499. MR 0637703.

[10] T. A. Ivey and P. J. Ryan, Hopf hypersurfaces of small Hopf principal curvature in $\mathbb{C}H^2$, *Geom. Dedicata* **141** (2009), 147–161. MR 2520069.

[11] H. S. Kim and P. J. Ryan, A classification of pseudo-Einstein hypersurfaces in $\mathbf{C}P^2$, *Differential Geom. Appl.* **26** (2008), no. 1, 106–112. MR 2393977.

[12] S. Klein, Totally geodesic submanifolds of the complex quadric, *Differential Geom. Appl.* **26** (2008), no. 1, 79–96. MR 2393975.

[13] M. Kon, Pseudo-Einstein real hypersurfaces in complex space forms, *J. Differential Geometry* **14** (1979), no. 3, 339–354 (1980). MR 0594705.

[14] S. H. Kon and T.-H. Loo, On characterizations of real hypersurfaces in a complex space form with η-parallel shape operator, *Canad. Math. Bull.* **55** (2012), no. 1, 114–126. MR 2932991.

[15] A. Martínez and J. D. Pérez, Real hypersurfaces in quaternionic projective space, *Ann. Mat. Pura Appl.* (4) **145** (1986), 355–384. MR 0886718.

[16] S. Montiel, Real hypersurfaces of a complex hyperbolic space, *J. Math. Soc. Japan* **37** (1985), no. 3, 515–535. MR 0792990.

[17] H. Nakagawa and R. Takagi, On locally symmetric Kaehler submanifolds in a complex projective space, *J. Math. Soc. Japan* **28** (1976), no. 4, 638–667. MR 0417463.

[18] H. Reckziegel, On the geometry of the complex quadric, in *Geometry and Topology of Submanifolds, VIII (Brussels, 1995/Nordfjordeid, 1995)*, 302–315, World Sci. Publ., River Edge, NJ, 1996. MR 1434581.

[19] Y. J. Suh, Pseudo-Einstein real hypersurfaces in the complex quadric, *Math. Nachr.* **290** (2017), no. 11-12, 1884–1904. MR 3683465.

T. H. Loo
Institute of Mathematical Sciences, University of Malaya, 50603 Kuala Lumpur, Malaysia
looth@um.edu.my

Received: February 6, 2020
Accepted: August 26, 2020