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LOCAL SUPERDERIVATIONS ON CARTAN TYPE LIE SUPERALGEBRAS

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Abstract. We characterize the local superderivations on Cartan type Lie superalgebras over the complex field $\mathbb{C}$. Furthermore, we prove that every local superderivation on Cartan type Lie superalgebras is a superderivation. As an application, using the results on local superderivations we characterize the linear 2-local superderivations on Cartan type Lie superalgebras. We prove that every linear 2-local superderivation on Cartan type Lie superalgebras is a superderivation.

1. Introduction

Lie superalgebras, as a generalization of Lie algebras, came from supersymmetry in mathematical physics. They have also promoted the development of Lie algebra, combinatorial mathematics, vertex operator algebra, differential manifold, topology, Lie groups, etc. Since all finite dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero consist of classical Lie superalgebras and Cartan type Lie superalgebras, Cartan type Lie superalgebras have a place in the Lie superalgebras. Cartan type Lie superalgebras over $\mathbb{C}$ are subalgebras of the algebra of all superderivations of the exterior superalgebras. The structural theory of these superalgebras has been playing a key role in the theory of Lie superalgebras.

Local (super)derivations are very important notions in the research of (super)algebras. The concept of local derivation was introduced in 1990 by Kadison [11], Larson and Sourour [13], and these authors studied local derivations of Banach algebras. In 2001 Johnson showed that every local derivation from a

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$\mathbb{C}^*$-algebra A into a Banach A-bimodule is a derivation [9]. Local derivations on the algebra $S(M,\tau)$ were studied deeply in the paper [1]. In recent years, local derivations have aroused the interest of a great many authors; see [6, 8, 20]. In 2016, Ayupov and Kudaybergenov proved that every local derivation on a finite-dimensional semisimple Lie algebra over an algebraically closed field of characteristic zero is a derivation [2]. In 2018, the same authors showed two properties for solvable Lie algebras: one is that local derivation is different from any other derivation, and the second is that there indeed exists a kind of algebras in which each local derivation is a derivation [3]. In 2017, Chen, Wang and Nan mainly studied local superderivations on basic classical Lie superalgebras, and proved that every local superderivation on basic classical Lie superalgebras except for $A(1,1)$ over the complex number field $\mathbb{C}$ is a superderivation [5]. In 2018, Chen and Wang studied local superderivations on Lie superalgebras $\mathfrak{q}(n)$, and proved that every local superderivation on $\mathfrak{q}(n)$, $n>3$, is a superderivation [4].

In this paper, we are interested in determining all local superderivations and linear 2-local superderivations on Cartan type Lie superalgebras over $\mathbb{C}$. Let L be a Cartan type Lie superalgebra over $\mathbb{C}$. The main result in this paper is a complete characterization of the local superderivations on L: we show that every local superderivation is a superderivation for a Cartan type Lie superalgebra.

The paper is organized as follows. In Section 2, we recall some necessary concepts and notations. In Section 3, we establish several lemmas, which will be used to characterize the local superderivations on Cartan type Lie superalgebras. In Section 4, we determine all local superderivations on Cartan type Lie superalgebras. In Section 5, as an application, using the results on local superderivations we determine all linear 2-local superderivations on Cartan type Lie superalgebras.

2. Preliminaries

Throughout, $\mathbb{C}$ is the field of complex numbers, $\mathbb{N}$ the set of nonnegative integers and $\mathbb{Z}_2=\{\bar{0},\bar{1}\}$ the additive group of two elements. For a vector superspace $V=V_{\bar{0}}\oplus V_{\bar{1}}$, we write $|x|$ for the parity of $x\in V_\alpha$, where $\alpha\in\mathbb{Z}_2$. Once the symbol $|x|$ appears in this paper, it will imply that x is a $\mathbb{Z}_2$-homogeneous element. We also adopt the following notation: For the symbols x and y, put $\delta_{x,y}=1$ if $x=y$ and $\delta_{x,y}=0$ otherwise.

2.1. **Lie superalgebras, superderivations.**

Let us recall some definitions relative to Lie superalgebras and superderivations (see [10]).

Definition 2.1. A *Lie superalgebra* is a vector superspace $L=L_{\bar{0}}\oplus L_{\bar{1}}$ with an even bilinear mapping $[\cdot,\cdot]:L\times L\to L$ satisfying the following axioms:

$$[x,y]=-(-1)^{|x||y|}[y,x],$$
$$[x,[y,z]]=[[x,y],z]+(-1)^{|x||y|}[y,[x,z]]$$

for all $x,y,z\in L$.

Definition 2.2. We call a linear map $D : L \to L$ a *superderivation* of a Lie superalgebra L if it satisfies the equation

$$D([x,y]) = [D(x),y] + (-1)^{|D||x|}[x,D(y)]$$

for all $x, y \in L$.

Write $\mathrm{Der}_{\bar{0}}(L)$ (resp. $\mathrm{Der}_{\bar{1}}(L)$) for the set of all superderivations of parity $\bar{0}$ (resp. $\bar{1}$) of L. Denote

$$\mathrm{Der}(L) = \mathrm{Der}_{\bar{0}}(L) \oplus \mathrm{Der}_{\bar{1}}(L).$$

2.2. **Local superderivations and linear 2-local superderivations.** Let us recall some definitions relative to local superderivations and linear 2-local superderivations (see [5, 18]). Let L be a Lie superalgebra.

Definition 2.3. A linear map $\phi : L \to L$ is called a *local superderivation* if for every $x \in L$, there exists a superderivation $D_x \in \mathrm{Der}L$ (depending on x) such that $\phi(x) = D_x(x)$.

Definition 2.4. A linear map $\phi : L \to L$ is called a *linear* 2*-local superderivation* if for any two elements $x, y \in L$, there exists a superderivation $D_{x,y} \in \mathrm{Der}L$ (depending on x, y) such that $\phi(x) = D_{x,y}(x)$ and $\phi(y) = D_{x,y}(y)$.

A local superderivation ϕ of degree α of L is a local superderivation such that $\phi(L_\beta) \subseteq L_{\alpha+\beta}$ for any $\beta \in \mathbb{Z}_2$. Write $\mathrm{LDer}_{\bar{0}}(L)$ (resp. $\mathrm{LDer}_{\bar{1}}(L)$) for the set of all super-biderivations of degree $\bar{0}$ (resp. $\bar{1}$) of L. Denote

$$\mathrm{LDer}(L) = \mathrm{LDer}_{\bar{0}}(L) \oplus \mathrm{LDer}_{\bar{1}}(L).$$

2.3. **Cartan type Lie superalgebras.** Let $n \geq 4$ be an integer and $\Lambda(n)$ be the exterior algebra in n indeterminates $x_1, x_2, \ldots, x_n$ with $\mathbb{Z}_2$-grading structure given by $|x_i| = \bar{1}$. One may define a $\mathbb{Z}$-grading on $\Lambda(n)$ by letting $\deg x_i = 1$, where $1 \leq i \leq n$. Write $n = 2r$ or $n = 2r+1$, where $r \in \mathbb{N}$. Put $\left[\frac{n}{2}\right] = r$. Cartan type Lie superalgebras consist of four series of simple Lie superalgebras contained in the algebra of all superderivations of $\Lambda(n)$:

$$W(n) = \left\{ \sum_{i=1}^{n} f_i \partial_i \mid f_i \in \Lambda(n) \right\},$$

$$S(n) = \left\{ \sum_{i=1}^{n} f_i \partial_i \mid f_i \in \Lambda(n),\ \sum_{i=1}^{n} \partial_i(f_i) = 0 \right\},$$

$$\widetilde{S}(n) = \left\{ (1 - x_1 x_2 \cdots x_n) \sum_{i=1}^{n} f_i \partial_i \mid f_i \in \Lambda(n),\ \sum_{i=1}^{n} \partial_i(f_i) = 0 \right\} \ (n \text{ an even integer}),$$

$$H(n) = \left\{ \mathrm{D_H}(f) \mid f \in \bigoplus_{i=0}^{n-1} \Lambda(n)_i \right\} \ (n > 4),$$

where

$$\mathrm{D_H}(f) = (-1)^{|f|} \sum_{i=1}^{n} \partial_i(f)\partial_{i'},$$

$$i' = \begin{cases} i+r, & \text{if } 1 \leq i \leq \left[\frac{n}{2}\right], \\ i-r, & \text{if } \left[\frac{n}{2}\right] < i \leq 2\left[\frac{n}{2}\right], \\ i, & \text{otherwise.} \end{cases}$$

One may define a $\mathbb{Z}$-grading on $W(n)$ by letting $\deg x_i = 1 = -\deg \partial_i$, where $1 \leq i \leq n$. Thus $W(n)$ becomes a $\mathbb{Z}$-graded Lie superalgebra of depth 1: $W(n) = \bigoplus_{j=-1}^{\xi_W} W(n)_j$, where $\xi_W = n-1$. Suppose $L = S(n)$ or $H(n)$. Then L is a $\mathbb{Z}$-graded subalgebra of $W(n)$. The $\mathbb{Z}$-grading is defined as follows: $L = \bigoplus_{j=-1}^{\xi_L} L_j$, where $L_j = L \cap W(n)_j$ and

$$\xi_L = \begin{cases} n-2, & \text{if } L = S(n), \\ n-3, & \text{if } H(n). \end{cases}$$

Put

$$\xi_i = x_1 x_2 \cdots x_n \partial_i,$$
$$\widetilde{S}(n)_{-1} = \mathrm{span}_{\mathbb{C}} \{\partial_i - \xi_i \mid 1 \leq i \leq n\},$$
$$\widetilde{S}(n)_i = S(n)_i, \quad \text{for } i > -1.$$

Then $\widetilde{S}(n)$ becomes a $\mathbb{Z}_n$-graded Lie superalgebra: $\widetilde{S}(n) = \bigoplus_{i=-1}^{\xi_{\tilde{S}}} \widetilde{S}(n)_i$, where $\xi_{\tilde{S}} = n-2$. The 0-degree components of these superalgebras are classical Lie algebras:

$$W(n)_0 \cong \mathfrak{gl}(n), \quad S(n)_0 = \widetilde{S}(n)_0 \cong \mathfrak{sl}(n), \quad H(n)_0 \cong \mathfrak{so}(n).$$

Let $L = \bigoplus_{i\in\mathbb{Z}} L_i$ be a $\mathbb{Z}$-graded Lie superalgbra, H_L be the standard Cartan subalgebra of L, $\theta \in H_L^*$ be the zero root, Δ_L be the root system of L. Let us describe the roots of Cartan type Lie superalgebras. If $L = W(n)$, we choose the standard basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$ in H_L^*, and then

$$\Delta_L = \{\varepsilon_{i_1} + \cdots + \varepsilon_{i_k} - \varepsilon_i \mid 1 \leq i_1 < \cdots < i_k \leq n, 1 \leq i \leq n\}.$$

The root systems of $S(n)$ and $\widetilde{S}(n)$ are obtained from the root system of $W(n)$ by removing the roots $\varepsilon_1 + \cdots + \varepsilon_n - \varepsilon_i$, where $1 \leq i \leq n$. Finally if $L = H(n)$, then

$$\Delta_L = \left\{\pm\varepsilon_{i_1} \pm \cdots \pm \varepsilon_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq \left[\frac{n}{2}\right]\right\}.$$

3. General lemmas

Let us now establish several lemmas, which will be used to characterize the local superderivations on Cartan type Lie superalgebras. Put $\mathcal{C} = \sum_{i=1}^{n} x_i \partial_i$ and $\widetilde{H}(n) = \{\mathrm{D_H}(f) \mid f \in \Lambda(n)\}$. By [16], we have the following lemma.

Lemma 3.1. *Let L be a Cartan type Lie superalgebra. Then* $\operatorname{Der} L = \operatorname{ad} L'$, *where*

$$L' \cong \begin{cases} L, & \text{if } L = W(n), \widetilde{S}(n), \\ L \oplus \mathbb{C}\mathcal{C}, & \text{if } L = S(n), \\ \widetilde{H}(n) \oplus \mathbb{C}\mathcal{C}, & \text{if } L = H(n). \end{cases}$$

Let L be a Cartan type Lie superalgebra. By Lemma 3.1 and a simple computation, we have $\Delta_{L'} = \Delta_L$ and the following lemma.

Lemma 3.2. *Let L be a Cartan type Lie superalgebra. Then L' is transitive, that is, if $a \in \bigoplus_{i\geq 0} L'_i$ and $\left[a, L'_{-1}\right] = 0$ then $a = 0$.*

Suppose L is a Cartan type Lie superalgebra. For $i \in \mathbb{Z}$ and $\alpha \in \Delta_{L'}$, we put

$$\begin{aligned} \operatorname{LDer}(L)_{i\times\alpha} = \{\phi \mid \phi \text{ is a linear map from } L \text{ to } L \text{ and there exists } u_x^i \in L'_i \cap L'_\alpha \\ \text{such that } \phi(x) = [u_x^i, x] \text{ for all } x \in L\}. \end{aligned}$$

Then we have the following lemma.

Lemma 3.3. *Let L be a Cartan type Lie superalgebra. Then the following conclusions hold:*

(1) *If $L \neq \widetilde{S}(n)$, then* $\operatorname{LDer}(L) = \bigoplus_{i\in\mathbb{Z},\alpha\in\Delta_{L'}} \operatorname{LDer}(L)_{i\times\alpha}$.

(2) *If $L = \widetilde{S}(n)$, then* $\operatorname{LDer}(L) = \bigoplus_{i\in\mathbb{Z}_n,\alpha\in\Delta_{L'}} \operatorname{LDer}(L)_{i\times\alpha}$.

Proof. (1) By Lemma 3.1, we have the "$\supseteq$" part. Now we verify the "$\subseteq$" part. Let $\phi \in \operatorname{LDer}(L)$. For each $x \in L$, by Lemma 3.1 there exists an element $u_x \in L'$ such that $\phi(x) = [u_x, x]$, where $u_x \in L'$. Since L' has the $\mathbb{Z} \times \Delta_{L'}$-grading, we can write

$$u_x = \sum_{i\in\mathbb{Z},\, \alpha\in\Delta_{L'}} u_x^{i,\alpha},$$

where $u_x^{i,\alpha} \in L'_{i\times\alpha}$. For $i \in \mathbb{Z}$ and $\alpha \in \Delta_{L'}$, we set

$$\phi_{i\times\alpha}(x) = [u_x^{i,\alpha}, x].$$

A direct verification shows that $\phi_{i\times\alpha} \in \operatorname{LDer}(L)_{i\times\alpha}$ and

$$\sum_{i\in\mathbb{Z},\, \alpha\in\Delta_{L'}} \phi_{i\times\alpha}(x) = \sum_{i\in\mathbb{Z},\, \alpha\in\Delta_{L'}} [u_x^{i,\alpha}, x] = [u_x, x] = \phi(x).$$

(2) A similar argument as for $L \neq \widetilde{S}(n)$ works also for $L = \widetilde{S}(n)$. ☐

4. Local superderivations of Cartan type Lie superalgebras

In this section we shall characterize the local superderivations on Cartan type Lie superalgebras. Let L be a Cartan type Lie superalgebra and $\{h_1, \dots, h_l\}$ be the standard basis of H_L. Set $h_0 = \sum_{i=1}^{l} t^i h_i$, where t is a fixed algebraic number from $\mathbb{C}$ of degree bigger than l. Then we have the following propositions.

Proposition 4.1. *Let L be a Cartan type Lie superalgebra and $\phi \in \operatorname{LDer}(L)$. If $\phi(h_0) = 0$, then $\phi|_{H_L} = 0$.*

Proof. For any h_i $(i = 1, \ldots, l)$, there exists an element

$$u^i = \sum_{1\le k_1<\cdots<k_l\le n} \sum_{j=1}^{n} a^i_{j,k_1,\ldots,k_l} x_{k_1}\cdots x_{k_l}\partial_j \in L',$$

where $a^i_{j,k_1,\ldots,k_l} \in \mathbb{C}$, such that $\phi(h_i) = [u^i, h_i]$. Then

$$\phi(h_i) = -\sum_{1\le k_1<\cdots<k_l\le n} \sum_{j=1}^{n} a^i_{j,k_1,\ldots,k_l}(\varepsilon_{k_1} + \cdots + \varepsilon_{k_l} - \varepsilon_j)(h_i)x_{k_1}\cdots x_{k_l}\partial_j.$$

If $l \neq 1$ or $l = 1$ and $k_1 \neq j$, then there exists $1 \le k \le n$ such that $(\varepsilon_{k_1} + \cdots + \varepsilon_{k_l} - \varepsilon_j)(h_k) \neq 0$. Put

$$h_{ik} = (\varepsilon_{k_1} + \cdots + \varepsilon_{k_l} - \varepsilon_j)(h_k)h_i - (\varepsilon_{k_1} + \cdots + \varepsilon_{k_l} - \varepsilon_j)(h_i)h_k.$$

Then there exists an element

$$u^{ik} = \sum_{1\le k_1<\cdots<k_l\le n} \sum_{j=1}^{n} a^{ik}_{j,k_1,\ldots,k_l} x_{k_1}\cdots x_{k_l}\partial_j \in L',$$

where $a^{ik}_{j,k_1,\ldots,k_l} \in \mathbb{C}$, such that $\phi(h_{ik}) = [u^{ik}, h_{ik}]$. Thus $\phi(h_{ik}) \cap L_{\varepsilon_{k_1}+\cdots+\varepsilon_{k_l}-\varepsilon_j} = 0$. On the other hand,

$$\phi(h_{ik}) = (\varepsilon_{k_1} + \cdots + \varepsilon_{k_l} - \varepsilon_j)(h_k)\phi(h_i) - (\varepsilon_{k_1} + \cdots + \varepsilon_{k_l} - \varepsilon_j)(h_i)\phi(h_k).$$

Hence

$$\begin{aligned}(\varepsilon_{k_1} + \cdots + \varepsilon_{k_l} - \varepsilon_j)(h_k)(\varepsilon_{k_1} + \cdots + \varepsilon_{k_l} - \varepsilon_j)(h_i)a^i_{j,k_1,\ldots,k_l}\\ = (\varepsilon_{k_1} + \cdots + \varepsilon_{k_l} - \varepsilon_j)(h_k)(\varepsilon_{k_1} + \cdots + \varepsilon_{k_l} - \varepsilon_j)(h_i)a^k_{j,k_1,\ldots,k_l},\end{aligned}$$

that is,

$$(\varepsilon_{k_1} + \cdots + \varepsilon_{k_l} - \varepsilon_j)(h_i)a^i_{j,k_1,\ldots,k_l} = (\varepsilon_{k_1} + \cdots + \varepsilon_{k_l} - \varepsilon_j)(h_i)a^k_{j,k_1,\ldots,k_l}.$$

Then

$$\begin{aligned}\phi(h_0) &= \sum_{i=1}^{l} t^i\phi(h_i)\\ &= -\sum_{i=1}^{l}\sum_{1\le k_1<\cdots<k_l\le n} \sum_{j=1}^{n} t^i a^k_{j,k_1,\ldots,k_l}(\varepsilon_{k_1} + \cdots + \varepsilon_{k_l} - \varepsilon_j)(h_i)x_{k_1}\cdots x_{k_l}\partial_j.\end{aligned}$$

Since $\phi(h_0) = 0$, we have $a^k_{j,k_1,\ldots,k_l}\sum_{i=1}^{l} t^i(\varepsilon_{k_1} + \cdots + \varepsilon_{k_l} - \varepsilon_j)(h_i) = 0$. Since t is an algebraic number from $\mathbb{C}$ of degree bigger than l and $(\varepsilon_{k_1} + \cdots + \varepsilon_{k_l} - \varepsilon_j)(h_i)$ are integers, we have $\sum_{i=1}^{l} t^i(\varepsilon_{k_1} + \cdots + \varepsilon_{k_l} - \varepsilon_j)(h_i) \neq 0$. Thus $a^k_{j,k_1,\ldots,k_l} = 0$ and $\phi(h_i) = 0$. The proof is complete. □

To apply Lemma 3.2, we give the following proposition.

Proposition 4.2. *Let L be a Cartan type Lie superalgebra, $\phi \in \mathrm{LDer}(L)$ and $\phi|_{L_{-1}\oplus H_L} = 0$. Then the following conclusions hold:*

(1) *If* $L \neq \widetilde{S}(n)$ *and* $\phi \in \mathrm{LDer}(L)_{i\times\alpha}$, *where* $i \in \mathbb{Z}$ *and* $\alpha \in \Delta_{L'}$, *then* ϕ *is zero.*
(2) *If* $L = \widetilde{S}(n)$ *and* $\phi \in \mathrm{LDer}(L)_{i\times\alpha}$, *where* $i \in \mathbb{Z}_n$ *and* $\alpha \in \Delta_{L'}$, *then* ϕ *is zero.*

Proof. For any $x \in \bigoplus_{j\geq 0} L_j$, there exists an element $u_x \in L'_i \cap L'_\alpha$ such that

$$\begin{aligned}\phi(x) &= \phi(x + \partial_1 + \cdots + \partial_n - \delta_{L,\tilde{S}}(\xi_1 + \cdots + \xi_n))\\ &= [u_x, x + \partial_1 + \cdots + \partial_n - \delta_{L,\tilde{S}}(\xi_1 + \cdots + \xi_n)] \in \bigoplus_{j\geq 0} L_{i+j}.\end{aligned}$$

Since

$$[u_x, \partial_1 + \cdots + \partial_n - \delta_{L,\tilde{S}}(\xi_1 + \cdots + \xi_n)] \in L_{i-1}$$

and $[u_x, x] \in \bigoplus_{j\geq 0} L_{i+j}$, we have

$$[u_x, \partial_1 + \cdots + \partial_n - \delta_{L,\tilde{S}}(\xi_1 + \cdots + \xi_n)] = 0.$$

Because $\partial_1 - \delta_{L,\tilde{S}}\xi_1, \ldots, \partial_n - \delta_{L,\tilde{S}}\xi_n$ belong to different root spaces, we have that $[u_x, \partial_j - \delta_{L,\tilde{S}}\xi_j] = 0$ for all $1 \leq j \leq n$. By Lemma 3.2, we have $u_x \in L'_{-1}$. Note that every root space of L'_{-1} is one-dimensional. Then there is $1 \leq k \leq l$ and $a_x \in \mathbb{C}$ such that $u_x = a_x(\partial_k - \delta_{L,\tilde{S}}\xi_k)$. Take $h \in H_L$ satisfying $[(\partial_k - \delta_{L,\tilde{S}}\xi_k), h] \neq 0$. By the equation

$$[a_x(\partial_k - \delta_{L,\tilde{S}}\xi_k), x] = \phi(x) = \phi(h + x) = [a_{h+x}(\partial_k - \delta_{L,\tilde{S}}\xi_k), h + x],$$

we have $a_{h+x}[(\partial_k - \delta_{L,\tilde{S}}\xi_k), h] = 0$ for all $x \in \bigoplus_{i\geq 1} L_i$ or $x \in L_0 \cap L_\beta$, where $\theta \neq \beta \in \Delta_L$. Then $a_{h+x} = 0$, that is, $\phi(x) = 0$ for all $x \in \bigoplus_{i\geq 1} L_i$ or $x \in L_0 \cap L_\beta$, where $\theta \neq \beta \in \Delta_L$. This, together with $\phi|_{L_{-1}\oplus H_L} = 0$, implies that $\phi = 0$. The proof is complete. □

By Proposition 4.2, we have the following local superderivations vanishing propositions.

Proposition 4.3. *Let* L *be a Cartan type Lie superalgebra,* $\phi \in \mathrm{LDer}(L)$ *and* $\phi|_{H_L} = 0$. *Then the following conclusions hold:*

(1) *If* $L \neq \widetilde{S}(n)$ *and* $\phi \in \mathrm{LDer}(L)_{i\times\alpha}$, *where* $i \in \mathbb{Z}$ *and* $\theta \neq \alpha \in \Delta_{L'}$, *then* ϕ *is zero.*
(2) *If* $L = \widetilde{S}(n)$ *and* $\phi \in \mathrm{LDer}(L)_{i\times\alpha}$, *where* $i \in \mathbb{Z}_n$ *and* $\theta \neq \alpha \in \Delta_{L'}$, *then* ϕ *is zero.*

Proof. Since $\alpha \neq \theta$, there exists an element $h \in H_L$ such that $\alpha(h) \neq 0$. Let $1 \leq k \leq n$. By Lemma 3.1, we know that there are $u_k, v_k \in L'_\alpha$ such that

$$\phi(\partial_k - \delta_{L,\tilde{S}}\xi_k) = [u_k, \partial_k - \delta_{L,\tilde{S}}\xi_k],$$

$$\phi(h + \partial_k - \delta_{L,\tilde{S}}\xi_k) = [v_k, h + \partial_k - \delta_{L,\tilde{S}}\xi_k].$$

Since $\phi(H_L) = 0$,

$$\begin{aligned}[u_k, \partial_k - \delta_{L,\tilde{S}}\xi_k] &= \phi(\partial_k - \delta_{L,\tilde{S}}\xi_k) = \phi(h + \partial_k - \delta_{L,\tilde{S}}\xi_k)\\ &= [v_k, h + \partial_k - \delta_{L,\tilde{S}}\xi_k] = -\alpha(h)v_k + [v_k, \partial_k - \delta_{L,\tilde{S}}\xi_k]\end{aligned}$$

for all $h \in H_L$. Then

$$\alpha(h)v_k = [v_k, \partial_k - \delta_{L,\tilde{S}}\xi_k] - [u_k, \partial_k - \delta_{L,\tilde{S}}\xi_k] \in L_\alpha \cap L_{\alpha-\varepsilon_k} = 0$$

for all $h \in H_L$. Since $\alpha \neq \theta$, there exists $h \in H_L$ such that $\alpha(h) = 1$. Therefore, we have $v_k = 0$. Then

$$\phi(\partial_k - \delta_{L,\tilde{S}}\xi_k) = \phi(h + \partial_k - \delta_{L,\tilde{S}}\xi_k) = [v_k, h + \partial_k - \delta_{L,\tilde{S}}\xi_k] = 0,$$

that is, $\phi|_{L_{-1}} = 0$. By Proposition 4.2, we have $\phi = 0$. □

Proposition 4.4. *Let L be a Cartan type Lie superalgebra and $\phi \in \mathrm{LDer}(L)$. Then the following conclusions hold:*

(1) *If $L \neq \widetilde{S}(n)$ and $\phi \in \mathrm{LDer}(L)_{i\times\theta}$, where $i \in \mathbb{Z}$, then ϕ is a superderivation.*
(2) *If $L = \widetilde{S}(n)$ and $\phi \in \mathrm{LDer}(L)_{i\times\theta}$, where $i \in \mathbb{Z}_n$, then ϕ is a superderivation.*

Proof. Let $1 \leq k \leq n$. For $\partial_k - \delta_{L,\tilde{S}}\xi_k$, there exists an element $u_k \in L'_\theta$ such that

$$\phi(\partial_k - \delta_{L,\tilde{S}}\xi_k) = [u_k, \partial_k - \delta_{L,\tilde{S}}\xi_k].$$

For $\partial_1 + \cdots + \partial_n - \delta_{L,\tilde{S}}(\xi_1 + \cdots + \xi_n)$, there exists an element $u \in L'_\theta$ such that

$$\phi(\partial_1 + \cdots + \partial_n - \delta_{L,\tilde{S}}(\xi_1 + \cdots + \xi_n)) = [u, \partial_1 + \cdots + \partial_n - \delta_{L,\tilde{S}}(\xi_1 + \cdots + \xi_n)].$$

Since ϕ is a linear map,

$$[u, \partial_1 + \cdots + \partial_n - \delta_{L,\tilde{S}}(\xi_1 + \cdots + \xi_n)] = [u_1, \partial_1 - \delta_{L,\tilde{S}}\xi_1] + \cdots + [u_n, \partial_n - \delta_{L,\tilde{S}}\xi_n].$$

Then

$$[u, \partial_k - \delta_{L,\tilde{S}}\xi_k] = [u_i, \partial_k - \delta_{L,\tilde{S}}\xi_k]$$

for all $1 \leq k \leq n$. Put $\phi' = \phi - \operatorname{ad} u$. Then $\phi'(\partial_k - \delta_{L,\tilde{S}}\xi_k) = 0$ for all $1 \leq k \leq n$, that is, $\phi'|_{L_{-1}} = 0$. Since $\phi \in \mathrm{LDer}(L)_{i\times\theta}$, we have $\phi'|_{H_L} = 0$. By Proposition 4.2, we have $\phi' = 0$. Then $\phi = \operatorname{ad} u$ is a superderivation. The proof is complete. □

By Propositions 4.1, 4.3 and 4.4, we have the following theorem.

Theorem 4.5. *Let L be a Cartan type Lie superalgebra. Then*

$$\mathrm{LDer}(L) = \mathrm{Der}(L).$$

Proof. Only the "$\subseteq$" part needs a verification. Let $\phi \in \mathrm{LDer}(L)$. By Lemma 3.1, there exists an element $u \in L'$ such that $\phi(h_0) = [u, h_0]$. Put $\phi' = \phi - \operatorname{ad} u$. Then $\phi'(h_0) = 0$. By Proposition 4.1, we have $\phi'|_{H_L} = 0$. Propositions 4.3 and 4.4 together with Lemma 3.3 imply that $\phi' \in \mathrm{Der}(L)$. Therefore, $\phi \in \mathrm{Der}(L)$. □

5. Applications

Let us characterize the linear 2-local superderivation of Cartan type Lie superalgebras. In [17] P. Šemrl introduced the concept of 2-local derivations. Moreover, the author proved that every 2-local derivation on $\mathcal{B}(H)$ is a derivation. Similarly, some authors started to describe 2-local derivations. In [12] S. Kim and J. Kim give a short proof of the fact that every 2-local derivation on the algebra $M_n(\mathbb{C})$ is a derivation. A similar description for the finite-dimensional case appeared later in [15]. In the paper [14], 2-local derivations and automorphisms have been described on matrix algebras over finite-dimensional division rings. Later J. Zhang and H. Li [19] extended the above result for arbitrary symmetric digraph matrix algebras and constructed an example of 2-local derivation which is not a derivation on the algebra of all upper triangular complex 2×2 matrices. In [7] Fošner and Fošner introduced the concept of 2-local superderivations on the associative superalgebra and proved that every 2-local superderivation on the superalgebra $M_n(\mathbb{C})$ is a superderivation. In 2017, Chen, Wang and Nan [18] mainly studied 2-local superderivations on basic classical Lie superalgebras, and proved that every 2-local superderivation on basic classical Lie superalgebras except for $A(1,1)$ over the complex number field $\mathbb{C}$ is a superderivation.

Using the above results on local superderivations we have the following theorem.

Theorem 5.1. *Let L be a Cartan type Lie superalgebra. Then every linear 2-local superderivation of L is a superderivation.*

Proof. By the definition of linear 2-local superderivation, we know that every linear 2-local superderivation of L is a local superderivation. Let ϕ be a linear 2-local superderivation of L. Then $\phi \in \mathrm{LDer}(L)$. By Theorem 4.5, we have $\phi \in \mathrm{Der}(L)$. □

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