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THE ISOMETRY GROUPS OF LORENTZIAN THREE-DIMENSIONAL UNIMODULAR SIMPLY CONNECTED LIE GROUPS

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ABSTRACT. We determine the isometry groups of all three-dimensional, connected, simply connected and unimodular Lie groups endowed with a left-invariant Lorentzian metric.

1. Introduction

The main purpose of this paper is to consider the problem of computing the isometry groups of three-dimensional, connected, simply connected and unimodular non-abelian Lie groups with respect to the left-invariant Lorentzian metric. The isometry groups of such Lie groups have been given for the Riemannian case in [8]. We recall first that there are, precisely, five such Lie groups: the nilpotent Lie group Nil, the special unitary group $\mathrm{SU}(2)$, the universal covering group $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ of the special linear group, the solvable Lie group Sol and the universal covering group $\widetilde{\mathrm{E}_0}(2)$ of the connected component of the Euclidean group. Left-invariant Lorentzian metrics on these Lie groups were classified in our previous work [1] and are listed in Table 1 and Table 2 below along with their symmetric endomorphisms.

Let (G,g) be a three-dimensional, unimodular, connected and simply connected Lie group endowed with a left-invariant Lorentzian metric. Its group of isometries $\mathrm{Isom}(\mathrm{G},\mathrm{g})$ is a Lie group under the compact open topology and acts on G transitively. Let $\mathrm{Isom}^0(\mathrm{G},\mathrm{g})$ denote its connected component. The isotropy subgroup of $\mathrm{Isom}^0(\mathrm{G},\mathrm{g})$ and the isotropy representation at the identity element e are given by

$$\begin{aligned} \mathrm{Isom}^0_{\mathrm{e}}(\mathrm{G},\mathrm{g}) &:= \left\{\theta \in \mathrm{Isom}^0(\mathrm{G},\mathrm{g}) : \theta(\mathrm{e}) = \mathrm{e}\right\}, \\ \rho : \mathrm{Isom}_{\mathrm{e}}(\mathrm{G},\mathrm{g}) &\to \mathrm{GL}(\mathrm{T_e G}), \quad \theta \mapsto T_{\mathrm{e}}\theta. \end{aligned} \tag{1.1}$$

The representation ρ is faithful since an isometry is determined by its value and its differential at one point. This shows that $\mathrm{Isom}^0_{\mathrm{e}}(\mathrm{G},\mathrm{g})$ is identified with a subgroup of $\mathrm{SO}(2,1)$; more precisely, we will show that $\mathrm{Isom}^0_{\mathrm{e}}(\mathrm{G},\mathrm{g})$ is trivial or exactly conjugated to one of the following subgroups:

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(1) The restricted Lorentz group $\mathrm{SO}^0(2,1)$.
(2) A one-parameter group of rotations around an axis in the three-dimensional Minkowski space $\mathbb{R}^3_1$ spanned by a timelike vector; we will refer to this subgroup as $\mathrm{SO}(2)$.
(3) A one-parameter group of rotations around an axis in $\mathbb{R}^3_1$ spanned by a spacelike vector, which will be referred to as $\mathrm{SO}(1,1)$.
(4) A one-parameter group of rotations around an axis in $\mathbb{R}^3_1$ spanned by an isotropic vector, which will be denoted by $\mathbf{K}$.

To state our main result in this respect, some notation needs to be introduced. For an automorphism $\Phi \in \mathrm{Aut}(\mathrm{G})$, so that for any $\mathrm{a} \in \mathrm{G}$ we have $\Phi \circ \mathrm{L_a} = \mathrm{L}_{\Phi(\mathrm{a})} \circ \Phi$, we say that Φ is an *isometric automorphism with respect to the left-invariant metric* g if $\Phi^*\mathrm{g} = \mathrm{g}$. We write $\mathrm{Aut}(\mathrm{G},\mathrm{g})$ for the subgroup of elements in $\mathrm{Aut}(\mathrm{G})$ that preserve the metric g. Then $\mathrm{Aut}(\mathrm{G},\mathrm{g})$ is a subgroup of $\mathrm{Isom_e}(\mathrm{G},\mathrm{g})$.

We are now ready to give the following theorem.

Theorem 1.1 (Main Theorem)**.** *Let* (G,g) *be a connected, simply connected and unimodular Lie group of dimension 3 with a left-invariant Lorentzian metric* g*.*

(1) *If* (G,g) *is a symmetric space, then* $\mathrm{Isom_e}(\mathrm{G},\mathrm{g}) \neq \mathrm{Aut}(\mathrm{G},\mathrm{g})$ *and* $\mathrm{Isom_e}(\mathrm{G},\mathrm{g}) \cong \mathrm{O}(2,1)$*.*
(2) *If* $\mathrm{G} = \mathrm{Sol}$ *with a left-invariant Lorentzian metric equivalent to* sol7 *(see Table 1), then* $\mathrm{Isom_e}(\mathrm{G},\mathrm{g}) \cong \mathbf{K} \times \mathbb{Z}_2$ *and* $\mathrm{Aut}(\mathrm{G},\mathrm{g}) \cong \mathbb{Z}_2$*.*
(3) *In all other cases,* $\mathrm{Isom_e}(\mathrm{G},\mathrm{g}) = \mathrm{Aut}(\mathrm{G},\mathrm{g})$ *with* $\dim \mathrm{Isom}(\mathrm{G},\mathrm{g}) = 3$ *or* 4*.*

When G is semi-simple, $\mathrm{Aut}^0(\mathrm{G}) = \mathrm{Inn}(\mathrm{G})$, where $\mathrm{Inn}(\mathrm{G})$ is the group of inner automorphisms of G and, as a consequence of Theorem 1.1, we get the following corollary.

Corollary 1.2. *Let* G *be a connected, simply connected and semi-simple Lie group of dimension 3 and* g *be a left-invariant Lorentzian metric on* G*. Then* $\mathrm{Isom}^0_\mathrm{e}(\mathrm{G},\mathrm{g})$ *is a subgroup of* $\mathrm{Inn}(\mathrm{G})$*.*

As a consequence of Theorem 1.1, we recover the result in [6] which showed that $\dim \mathrm{Isom_e}(\mathrm{G},\mathrm{g}) \neq 2$ when $\mathrm{G} = \mathrm{Nil}$, $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ or Sol, and we extend this to all simply connected Lorentzian unimodular Lie groups of dimension 3. The second assertion of Theorem 1.1 corresponds in [6] to the maximal non-Riemannian Lorentz geometries designated by *Lorentz-Sol*.

The isometry groups of left-invariant pseudo-Riemannian metrics have been studied by many authors. We can cite, for example, the works [4, 13, 5, 11]. However, these studies have sought to identify the relationship between the isotropy group $\mathrm{Isom_e}(\mathrm{G},\mathrm{g})$ and the isometric automorphism group $\mathrm{Aut}(\mathrm{G},\mathrm{g})$ or have determined the Lie algebra of the Killing vector fields. Here, we use the same approach as the one used in the Riemannian case in [8]. We note that, roughly speaking, the procedure for calculating the isometry groups involves a connection between the symmetric endomorphism defining the Lie bracket and the Ricci operator of (G,g).

Table 1. Left-invariant Lorentzian metrics on 3D unimodular Lie groups

Lie algebras	$\mathfrak{n}$			$\mathfrak{su}(2)$	$\mathfrak{sol}$						
Natural basis	$[X,Y]=Z$, $[Z,Y]=0$, $[Z,X]=0$.			$[\sigma_x,\sigma_y]=2\sigma_z$, $[\sigma_y,\sigma_z]=2\sigma_x$, $[\sigma_z,\sigma_x]=2\sigma_y$.	$[X_1,X_2]=X_2$, $[X_1,X_3]=-X_3$, $[X_2,X_3]=0$.						
Lie groups	Nil			SU(2)	Sol						
Lorentzian	nil+	nil−	nil0	su	sol1	sol2	sol3	sol4	sol5	sol6	sol7
left-invariant metrics (modulo automorphism)	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&\lambda\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&-\lambda\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}\mu_1&0&0\\0&\mu_2&0\\0&0&-\mu_3\end{pmatrix}$	$\begin{pmatrix}\frac{4}{u^2-v^2}&0&0\\0&1&\frac{u}{v}\\0&\frac{u}{v}&1\end{pmatrix}$	$\begin{pmatrix}\frac{4}{v^2-u^2}&0&0\\0&\frac{u}{v}&-1\\0&-1&\frac{u}{v}\end{pmatrix}$	$\begin{pmatrix}\frac{1}{u+v}&0&0\\0&-\frac{v}{u}&1\\0&1&1\end{pmatrix}$	$\begin{pmatrix}\frac{1}{u}&0&0\\0&-1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&-\frac{2}{u}\\0&1&1\\-\frac{2}{u}&1&1\end{pmatrix}$	$\begin{pmatrix}u^2&0&0\\0&u&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$
Parameters	$\lambda>0$	$\lambda>0$		$\mu_1\geq\mu_2>0$ $\mu_3>0$	$-v<u<v$ $v>0$	$-v<u<v$ $v>0$	$u>0$ $v>0$	$u>0$	$u>0$	$u\neq 0$	
Symmetric	diag	diag	ab2	diag	diag	diag	az$\bar{z}$	az$\bar{z}$	ab2	ab2	a3
endomorphisms **L**	$(\alpha,0,0)$ $\alpha\neq 0$	$(0,0,\gamma)$ $\gamma\neq 0$	$a=0$ $b=0$	(α,β,γ) $\alpha,\beta,\gamma\neq 0$	$(\alpha,\beta,0)$ $\alpha,\beta\neq 0$	$(\alpha,0,\gamma)$ $\alpha,\gamma\neq 0$	$a=0$ $\mathfrak{Re}(z)\neq 0$	$a=0$ $\mathfrak{Re}(z)=0$	$a<0$ $b=0$	$a=0$ $b\neq 0$	$a=0$
Signature of Ricci curvature	$(+,+,+)$	$(+,-,-)$	$(0,0,0)$	$(+,+,+)$ if $\mu_1<\mu_2+\mu_3$ $(+,-,-)$ if $\mu_1>\mu_2+\mu_3$ $(+,0,0)$ if $\mu_1=\mu_2+\mu_3$	$(+,-,-)$ if $u\neq 0$ $(-,0,0)$ if $u=0$	$(+,-,-)$ if $u>0$ $(+,+,+)$ if $u<0$ $(0,0,0)$ if $u=0$	$(+,-,-)$	$(-,0,0)$	$(+,-,-)$	$(-,0,0)$ if $u>0$ $(+,0,0)$ if $u<0$	$(-,0,0)$
Signature of scalar curvature	$\mathfrak{s}=\lambda/2$ $\mathfrak{s}>0$	$\mathfrak{s}=\lambda/2$ $\mathfrak{s}>0$	$\mathfrak{s}=0$	$\mathfrak{s}>0$	$\mathfrak{s}=v^2/2$ $\mathfrak{s}>0$	$\mathfrak{s}=u^2/2$ $\mathfrak{s}>0$ if $u\neq 0$ $\mathfrak{s}=0$ if $u=0$	$\mathfrak{s}=-2v$ $\mathfrak{s}<0$	$\mathfrak{s}=-2u$ $\mathfrak{s}<0$	$\mathfrak{s}=u^2/2$ $\mathfrak{s}>0$	$\mathfrak{s}=0$	$\mathfrak{s}=0$

TABLE 2. Left-invariant Lorentzian metrics on 3D unimodular Lie groups

Lie algebras	$\mathrm{sl}(2,\mathbb{R})$							$\mathrm{e}_0(2)$		
Natural basis	$[X_1,X_2]=2X_3$, $[X_3,X_1]=2X_2$, $[X_3,X_2]=2X_1$.							$[X_1,X_2]=X_3$, $[X_1,X_3]=-X_2$, $[X_3,X_2]=0$.		
Lie groups	$\widetilde{\mathrm{PSL}}(2,\mathbb{R})$							$\widetilde{\mathrm{E}_0}(2)$		
Lorentzian	sll1	sll2	sll3	sll4	sll5	sll6	sll7	ee1	ee2	ee3
left-invariant metrics (modulo	$\begin{pmatrix}-\mu_1&0&0\\0&\mu_2&0\\0&0&\mu_3\end{pmatrix}$	$\begin{pmatrix}\mu_1&0&0\\0&-\mu_2&0\\0&0&\mu_3\end{pmatrix}$	$K\begin{pmatrix}M&\beta&0\\\beta&N&0\\0&0&\frac{a^2\alpha}{N}\end{pmatrix}$	$K\begin{pmatrix}-N&0&\beta\\0&\frac{a^2\alpha}{N}&0\\\beta&0&M\end{pmatrix}$	$K\begin{pmatrix}u&0&v\\0&M&0\\v&0&u\end{pmatrix}$	$K\begin{pmatrix}M&-a&0\\-a&N&0\\0&0&\frac{8a}{b}\end{pmatrix}$	$K\begin{pmatrix}M&N&0\\N&S&R\\0&R&4a^4\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&u&0\\0&0&v\end{pmatrix}$	$\begin{pmatrix}0&-1&0\\-1&u&0\\0&0&v\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&0&u\end{pmatrix}$
automor phism) Parameters	$\mu_1>0$ $\mu_2\geq\mu_3>0$	$\mu_1>0$ $\mu_2>0,\ \mu_3>0.$	$K=\frac{4}{a^2\alpha\sqrt{\alpha^2+\beta^2}}$ $M=\frac{\beta^2-\alpha^2}{\sqrt{\alpha^2+\beta^2}}$ $N=\sqrt{\alpha^2+\beta^2}$ $\alpha>0,\ \beta>0$	$\alpha<0,\ \beta>0$	$K=\frac{16}{v^2-u^2}$ $M=2(u+v)$ $-v<u<v$ $v>0$	$a\neq 0$ $b\neq 0$ $K=\frac{1}{2ab}$ $M=a-8$ $N=a+8$	$K=\frac{2}{a^4(1+2a^2)}$ $M=1-4a^4$ $N=(1+2a^2)^{\frac{3}{2}}$ $S=4a^4+6a^2+1$ $R=2a^3\sqrt{2},\ a>0$	$u\geq v$ $v>0$	$u<0$ $v>0$	$u>0$
Symmetric	diag	diag	$\mathrm{a}^2\mathrm{z}\bar{\mathrm{z}}$	$\mathrm{a}^2\mathrm{z}\bar{\mathrm{z}}$	$\mathrm{a}^2\mathrm{z}\bar{\mathrm{z}}$	ab2	a3	diag	diag	ab2
endomor phisms **L**	(α,β,γ) $\alpha>0,\beta>0$ $\gamma>0$	(α,β,γ) $\alpha<0,\beta>0$ $\gamma<0$	$a\neq 0$ $\mathfrak{Re}(z)>0$	$a\neq 0$ $\mathfrak{Re}(z)<0$	$a\neq 0$ $\mathfrak{Re}(z)=0$	$a\neq 0$ $b\neq 0$	$a\neq 0$	$(\alpha,\beta,0)$ $\alpha>0,\ \beta>0$	$(\alpha,0,\gamma)$ $\alpha>0,\ \gamma<0$	$a>0$ $b=0$
Signature of Ricci curvature	$(+,+,+)$ if $\mu_3<\mu_1-\mu_2$	if $\mu_1<\mu_2-\mu_3$	$(+,-,-)$ $a^2\neq 2\ \mathfrak{Re}(z)$	$(+,-,-)$	$(+,-,-)$	$(+,-,-)$ if $a\neq 2b$	$(+,-,-)$	$(0,0,0)$ if $u=v$	$(+,+,+)$ if $u<-v$	$(+,-,-)$
	$(+,-,-)$ if $\mu_3>\mu_1-\mu_2$	if $\mu_1>\mu_2-\mu_3$	$(-,0,0)$ $a^2=2\ \mathfrak{Re}(z)$			$(-,0,0)$ if $a=2b$		$(+,-,-)$. if $u\neq v$	$(+,-,-)$ if $u>-v$	
	$(+,0,0)$ if $\mu_3=\mu_1-\mu_2$	if $\mu_1=\mu_2-\mu_3$							$(+,0,0)$ if $u=-v$	
	$(-,0,0)$ if $\mu_3=-\mu_1+\mu_2$	if $\mu_1=\mu_2+\mu_3$								
Scalar curvature	$\mathfrak{s}=\frac{2[(\sqrt{\mu_1}+\sqrt{\mu_2})^2+\epsilon\mu_3][(\sqrt{\mu_1}-\sqrt{\mu_2})^2+\epsilon\mu_3]}{\mu_1\mu_2\mu_3}$ $\epsilon=-1$	$\epsilon=1$	$\mathfrak{s}=\frac{1}{2}a^4-2a^2\alpha-2\beta^2$		$\mathfrak{s}=\frac{u}{2}$	$\mathfrak{s}=\frac{1}{2}a(a-4b)$	$\mathfrak{s}=-\frac{3}{2}a^2$ $\mathfrak{s}<0$	$\mathfrak{s}=\frac{(u-v)^2}{2v}$ $\mathfrak{s}\geq 0$	$\mathfrak{s}=\frac{(u-v)^2}{2v}$ $\mathfrak{s}>0$	$\mathfrak{s}=\frac{u}{2}$ $\mathfrak{s}>0$

TABLE 3. Isometric automorphism groups

Lie groups	**Lorentzian left invariant metrics**		**Symmetric endomorphism**		**Isometric automorphism group**		
G	metric	parameters	kind	parameters	$A = \mathrm{Aut}(\mathrm{G},\mathrm{g})$	infinite	A/A^0 or $\lvert A\rvert$
Nil	nil−		{diag}	$a = b \neq c$	$\mathrm{O}(2)$	✓	$\mathbb{Z}_2$
	nil+		{diag}	$a \neq b = c$	$\mathrm{O}(1,1)$	✓	$\mathbb{Z}_2$
	nil0		{ab2}	a = b	$\mathbf{K} \times \mathbb{Z}_2$	✓	$\mathbb{Z}_2$
SU(2)	su	$\mu_1 \neq \mu_2$	{diag}	$a \neq b \neq c$	D_4		4
	su	$\mu_1 = \mu_2$	{diag}	$a = b \neq c$	$\mathrm{O}(2)$	✓	$\mathbb{Z}_2$
$\widetilde{\mathrm{PSL}}(2,\mathbb{R})$	sll1	$\mu_1 \neq \mu_2 \neq \mu_3$	{diag}	$a \neq b \neq c$	D_4		4
	sll1	$\mu_1 \neq \mu_2 = \mu_3$	{diag}	$a = b \neq c$	$\mathrm{O}(2)$	✓	$\mathbb{Z}_2$
	sll1	$\mu_1 = \mu_2 \neq \mu_3$	{diag}	$a \neq b = c$	$\mathrm{O}(1,1)$	✓	$\mathbb{Z}_2$
	sll1	$\mu_1 = \mu_3 \neq \mu_2$	{diag}	$a = c \neq b$	$\mathrm{O}(1,1)$	✓	$\mathbb{Z}_2$
	sll1	$\mu_1 = \mu_2 = \mu_3$	{diag}	$a = b = c$	$\mathrm{SO}(2,1)$	✓	$\mathbb{Z}_2$
	sll2	$\mu_1 = \mu_2$	{diag}	$a = c \neq b$	$\mathrm{O}(1,1)$	✓	$\mathbb{Z}_2$
	sll2	$\mu_1 \neq \mu_2$	{diag}	$a \neq b \neq c$	D_4		4
	sll6	$a = b$	{ab2}	a = b	$\mathbf{K} \times \mathbb{Z}_2$	✓	$\mathbb{Z}_2$
	sll6	$a \neq b$	{ab2}	a ≠ b	$\mathbb{Z}_2$		2
	sll3, sll4, sll5		{az$\bar{\mathrm{z}}$}	a ≠ 0	$\mathbb{Z}_2$		2
	sll7		{a3}	a ≠ 0	{id}		1
Sol	sol1 sol2	$u \neq 0$ $u \neq 0$	{diag}	$a \neq b \neq c$ $a \neq -b$	D_4		4
	sol1	$u = 0$	{diag}	$a \neq b \neq c$ $a = -b$	D_8		8
	sol2	$u = 0$	{diag}	$a = c \neq b$	$\mathrm{O}(1,1)$	✓	$\mathbb{Z}_2$
	sol4		{az$\bar{\mathrm{z}}$}	$\mathfrak{Re}(z) = 0$	D_4		4
	sol3		{az$\bar{\mathrm{z}}$}	$\mathfrak{Re}(z) \neq 0$	$\mathbb{Z}_2$		2
	sol5, sol6		{ab2}	a ≠ b	$\mathbb{Z}_2$		2
	sol7		{a3}	a = 0	$\mathbb{Z}_2$		2
$\widetilde{\mathrm{E}_0}(2)$	ee1	$u = v$	{diag}	$a = b \neq c$	$\mathrm{O}(2)$	✓	$\mathbb{Z}_2$
	ee1	$u \neq v$	{diag}	$a \neq b \neq c$	D_4		4
	ee2		{diag}	$a \neq b \neq c$	D_4		4
	ee3		{ab2}	a ≠ b	$\mathbb{Z}_2$		2

TABLE 4. Isometry group of Lorentzian 3D unimodular simply connected Lie groups

Lie groups G	**Lorentzian left inv. metrics**: metric, parameters	**Symmetric**	**Symmetric endomorphism**: kind, parameters	**Isometry groups** $\mathrm{Isom}(\mathrm{G},\mathrm{g})$: $\mathrm{Isom}(\mathrm{G},\mathrm{g})$	dim	$\mathrm{Isom}_e(\mathrm{G},\mathrm{g}) \leq \mathrm{Aut}(\mathrm{G})$
Nil	nil−		{diag}, $a = b \neq c$	$\mathrm{Nil} \rtimes \mathrm{O}(2)$	4	✓
	nil+		{diag}, $a \neq b = c$	$\mathrm{Nil} \rtimes \mathrm{O}(1,1)$	4	✓
	nil0	✓	{ab2}, a = b	$\mathrm{Nil} \times \mathrm{O}(2,1)$	6	
$\mathrm{SU}(2)$	su, $\mu_1 \neq \mu_2$		{diag}, $a \neq b \neq c$	$\mathrm{SU}(2) \rtimes \mathrm{D}_4$	3	✓
	su, $\mu_1 = \mu_2$		{diag}, $a = b \neq c$	$\mathrm{SU}(2) \rtimes \mathrm{O}(2)$	4	✓
$\widetilde{\mathrm{PSL}}(2,\mathbb{R})$	sll1, $\mu_1 \neq \mu_2 \neq \mu_3$		{diag}, $a \neq b \neq c$	$\widetilde{\mathrm{PSL}}(2,\mathbb{R}) \rtimes \mathrm{D}_4$	3	✓
	sll1, $\mu_1 \neq \mu_2 = \mu_3$		{diag}, $a = b \neq c$	$\widetilde{\mathrm{PSL}}(2,\mathbb{R}) \rtimes \mathrm{O}(2)$	4	✓
	sll1, $\mu_1 = \mu_2 \neq \mu_3$		{diag}, $a \neq b = c$	$\widetilde{\mathrm{PSL}}(2,\mathbb{R}) \rtimes \mathrm{O}(1,1)$	4	✓
	sll1, $\mu_1 = \mu_3 \neq \mu_2$		{diag}, $a = c \neq b$	$\widetilde{\mathrm{PSL}}(2,\mathbb{R}) \rtimes \mathrm{O}(1,1)$	4	✓
	sll1, $\mu_1 = \mu_2 = \mu_3$	✓	{diag}, $a = b = c$	$\widetilde{\mathrm{PSL}}(2,\mathbb{R}) \times \mathrm{O}(2,1)$	6	
	sll2, $\mu_1 = \mu_2$		{diag}, $a = c \neq b$	$\widetilde{\mathrm{PSL}}(2,\mathbb{R}) \rtimes \mathrm{O}(1,1)$	4	✓
	sll2, $\mu_1 \neq \mu_2$		{diag}, $a \neq b \neq c$	$\widetilde{\mathrm{PSL}}(2,\mathbb{R}) \rtimes \mathrm{D}_4$	3	✓
	sll6, $a = b$		{ab2}, a = b	$\widetilde{\mathrm{PSL}}(2,\mathbb{R}) \rtimes (\mathbf{K} \times \mathbb{Z}_2)$	4	✓
	sll6, $a \neq b$		{ab2}, a ≠ b	$\widetilde{\mathrm{PSL}}(2,\mathbb{R}) \rtimes \mathbb{Z}_2$	3	✓
	sll3, sll4, sll5		{az$\bar{z}$}, a ≠ 0	$\widetilde{\mathrm{PSL}}(2,\mathbb{R}) \rtimes \mathbb{Z}_2$	3	✓
	sll7		{a3}, a ≠ 0	$\widetilde{\mathrm{PSL}}(2,\mathbb{R})$	3	✓
Sol	sol1, $u \neq 0$ sol2, $u \neq 0$		{diag}, $a \neq b \neq c$ $a \neq -b$	$\mathrm{Sol} \rtimes \mathrm{D}_4$	3	✓
	sol1, $u = 0$		{diag}, $a \neq b \neq c$ $a = -b$	$\mathrm{Sol} \rtimes \mathrm{D}_8$	3	✓
	sol2, $u = 0$	✓	{diag}, $a = c \neq b$	$\mathrm{Sol} \times \mathrm{O}(2,1)$	6	
	sol4		{az$\bar{z}$}, $\mathfrak{Re}(z) = 0$	$\mathrm{Sol} \rtimes \mathrm{D}_4$	3	✓
	sol3		{az$\bar{z}$}, $\mathfrak{Re}(z) \neq 0$	$\mathrm{Sol} \rtimes \mathbb{Z}_2$	3	✓
	sol5, sol6		{ab2}, a ≠ b	$\mathrm{Sol} \rtimes \mathbb{Z}_2$	3	✓
	sol7		{a3}, a = 0	$\mathrm{Sol} \times (\mathbf{K} \times \mathbb{Z}_2)$	4	
$\widetilde{\mathrm{E}_0}(2)$	ee1, $u = v$	✓	{diag}, $a = b \neq c$	$\widetilde{\mathrm{E}_0}(2) \times \mathrm{O}(2,1)$	6	
	ee1, $u \neq v$		{diag}, $a \neq b \neq c$	$\widetilde{\mathrm{E}_0}(2) \rtimes \mathrm{D}_4$	3	✓
	ee2		{diag}, $a \neq b \neq c$	$\widetilde{\mathrm{E}_0}(2) \rtimes \mathrm{D}_4$	3	✓
	ee3		{ab2}, a ≠ b	$\widetilde{\mathrm{E}_0}(2) \rtimes \mathbb{Z}_2$	3	✓

The rest of the article is organized as follows. In Section 2 we set up the general notation and we recall some basic facts on the symmetric endomorphism associated with (G, g). We also recall some well-known results on the isometry group of (G, g). In Section 3, we start from the four types of the symmetric endomorphism. This allows us to determine explicitly the isometric automorphisms group $\mathrm{Aut}(\mathrm{G}, \mathrm{g})$ of all possible Lorentzian three-dimensional unimodular Lie groups. Finally, in Section 4, in order to prove the main result, we provide an algorithm to solve, on the Lie algebra level, the problem of finding a complete description of the differential of each isometry at the identity element e. This, in turn, allows us to work in a much simpler setting using the symbolic computation software Maple.

2. Notation and preliminaries

Let $\mathfrak{g}$ be a three-dimensional unimodular Lie algebra with Lie bracket $[\,,]$ and equipped with a Lorentzian scalar product $\langle\,,\rangle$. Let G be the connected simply connected Lie group with Lie algebra $\mathfrak{g}$ and left-invariant Lorentzian metric g determined by $\langle\,,\rangle$. For left-invariant vector fields X, Y, and Z (or equivalently X, Y, Z in $\mathfrak{g}$), we have that the Levi-Civita connection ∇ of (G, g) satisfies

$$\mathrm{g}(\nabla_{\mathrm{X}}\mathrm{Y},\,\mathrm{Z}) + \mathrm{g}(\mathrm{Y},\,\nabla_{\mathrm{X}}\mathrm{Z}) = 0 \quad \text{and} \quad \nabla_{\mathrm{X}}\mathrm{Y} - \nabla_{\mathrm{Y}}\mathrm{X} - [\mathrm{X}, \mathrm{Y}] = 0,$$

and ∇ is described by the Koszul formula

$$\mathrm{g}(\nabla_{\mathrm{X}}\mathrm{Y}, \mathrm{Z}) = \frac{1}{2}\left\{\mathrm{g}([\mathrm{X},\,\mathrm{Y}]\,, \mathrm{Z}) - \mathrm{g}([\mathrm{Y},\,\mathrm{Z}]\,, \mathrm{X}) + \mathrm{g}([\mathrm{Z},\,\mathrm{X}]\,, \mathrm{Y})\right\}.$$

The curvature tensor denoted by R on (G, g) is defined as the $(1, 3)$-tensor field given by the following expression, for all $\mathrm{X}, \mathrm{Y}, \mathrm{Z} \in \mathfrak{g}$:

$$\mathrm{R}(\mathrm{X}, \mathrm{Y}, \mathrm{Z}) = \nabla_{[\mathrm{X},\mathrm{Y}]}\mathrm{Z} - [\nabla_{\mathrm{X}}, \nabla_{\mathrm{Y}}]\,\mathrm{Z}.$$

Covariant derivatives $\nabla\mathrm{R}$ are defined by the usual product rules, for $\mathrm{W} \in \mathfrak{g}$:

$$\begin{aligned}(\nabla_{\mathrm{W}}\mathrm{R})(\mathrm{X}, \mathrm{Y}, \mathrm{Z}) = \nabla_{\mathrm{W}}\mathrm{R}(\mathrm{X}, \mathrm{Y}, \mathrm{Z}) &- \mathrm{R}(\nabla_{\mathrm{W}}\mathrm{X}, \mathrm{Y}, \mathrm{Z}) - \mathrm{R}(\mathrm{X}, \nabla_{\mathrm{W}}\mathrm{Y}, \mathrm{Z})\\ &- \mathrm{R}(\mathrm{X}, \mathrm{Y}, \nabla_{\mathrm{W}}\mathrm{Z}).\end{aligned}$$

The Ricci curvature ric associated to R is the symmetric bilinear form on $\mathfrak{g}$ given by the trace of the curvature endomorphism on its first and last indexes. More precisely, if $(\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3)$ is a pseudo-orthonormal basis of $\mathfrak{g}$,

$$\mathrm{ric}(\mathrm{X}, \mathrm{Y}) = \mathrm{Tr}(\mathrm{R}(\mathrm{X}, ., \mathrm{Y})) = \sum_{i=1}^{3} \epsilon_i \mathrm{g}(\mathrm{R}(\mathrm{X}, \mathrm{e_i})\mathrm{Y}, \mathrm{e_i}).\ \epsilon_i = \mathrm{g}(\mathrm{e}_i, \mathrm{e}_i) \quad \text{for all } \mathrm{X}, \mathrm{Y} \in \mathfrak{g}.$$

The Ricci operator, which will be denoted by Ric, is the symmetric endomorphism $\mathrm{Ric} : \mathfrak{g} \to \mathfrak{g}$ given by $\mathrm{g}(\mathrm{Ric}(u), v) = \mathrm{ric}(u, v)$, and we denote by $\mathfrak{s}$ the scalar curvature. Note that a Lie group is unimodular if and only if its left-invariant Haar measure is also right-invariant, or equivalently a Lie group is unimodular if and only if the structure constants of the corresponding Lie algebra are trace free, i.e., $\mathrm{Tr}(ad_{\mathrm{X}}) = 0$ for all $\mathrm{X} \in \mathfrak{g}$.

The Bianchi classification provides a list of all real three-dimensional Lie algebras up to isomorphism. It is pointed out in [9] that the five non abelian unimodular

Lie algebras called *Bianchi class A* are Bianchi types II, VI_0, VII_0, VIII, and IX. In addition, up to isomorphism, there exists exactly one connected and simply connected Lie group associated with each type of algebra, which is one of the following:

(1) The Lie algebra $\mathfrak{n}$ with a natural basis $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ satisfying $[\mathrm{X}, \mathrm{Y}] = \mathrm{Z}$. The corresponding simply connected Lie group is the nilpotent Heisenberg group Nil.
(2) The Lie algebra $\mathfrak{su}(2)$ with a natural basis $\sigma_x, \sigma_y, \sigma_z$ satisfying $[\sigma_x, \sigma_y] = 2\sigma_z$, $[\sigma_y, \sigma_z] = 2\sigma_x$, $[\sigma_z, \sigma_x] = 2\sigma_y$. The corresponding simply connected Lie group is $\mathrm{SU}(2)$.
(3) The Lie algebra $\mathrm{sl}(2,\mathbb{R})$ with a natural basis $\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3$ satisfying $[\mathrm{X}_1, \mathrm{X}_2] = 2\mathrm{X}_3$, $[\mathrm{X}_3, \mathrm{X}_1] = 2\mathrm{X}_2$, $[\mathrm{X}_3, \mathrm{X}_2] = 2\mathrm{X}_1$. The corresponding simply connected Lie group is $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$.
(4) The Lie algebra $\mathfrak{sol}$ with a natural basis $\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3$ satisfying $[\mathrm{X}_1, \mathrm{X}_2] = \mathrm{X}_2$, $[\mathrm{X}_1, \mathrm{X}_3] = -\mathrm{X}_3$. The corresponding simply connected Lie group is Sol.
(5) The Lie algebra $\mathrm{e}_0(2)$ with a natural basis $\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3$ satisfying $[\mathrm{X}_1, \mathrm{X}_2] = \mathrm{X}_3$, $[\mathrm{X}_1, \mathrm{X}_3] = -\mathrm{X}_2$. The corresponding simply connected Lie group is $\widetilde{\mathrm{E}_0}(2)$.

It is well known that every automorphism of G induces an automorphism of the Lie algebra $\mathfrak{g} = \mathrm{T_e G}$, but if G is connected and simply connected, every automorphism of $\mathfrak{g}$ can be lifted to a unique automorphism of G and hence $\mathrm{Aut}(\mathrm{G})$ and $\mathrm{Aut}(\mathfrak{g})$ are isomorphic [14]. From now on, we do not make a distinction between automorphisms of $\mathfrak{g}$ and those of G.

Symmetric endomorphism. Recall that a Lorentzian cross product in dimension 3 is determined by a Lorentzian scalar product $\langle\,,\rangle$ and an orientation. Fix an orientation on $\mathfrak{g}$ and let $\times : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ be the Lorentzian cross product satisfying $\langle u, v\times w\rangle = \det([u\,v\,w])$. Since the Lie bracket and the cross product in the Lie algebra $\mathfrak{g}$ are skew-symmetric bilinear forms, they are related by a unique endomorphism $\mathbf{L} : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$\mathbf{L}(\mathrm{X}\times\mathrm{Y}) = [\mathrm{X}, \mathrm{Y}] \quad \text{for all } \mathrm{X}, \mathrm{Y} \in \mathfrak{g}.$$

Furthermore, $\mathfrak{g}$ is unimodular if and only if $\mathbf{L}$ is self-adjoint with respect to $\langle\,,\rangle$. Throughout this paper the self-adjoint map $\mathbf{L}$ is referred to as the symmetric endomorphism. On the other hand, $\mathbf{L}$ mainly behaves like the Ricci operator Ric with the fundamental similarity being that there exists a pseudo-orthonormal basis $\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\}$, with e_3 timelike, so that both Ric and $\mathbf{L}$ take one of the following forms [2]:

$$\text{kind }\{\mathrm{diag}\}(a,b,c) : \begin{pmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{pmatrix}, \qquad \text{kind }\{\mathrm{az\bar{z}}\} : \begin{pmatrix} \mathrm{a} & 0 & 0\\ 0 & \mathrm{b} & -\mathrm{c}\\ 0 & \mathrm{c} & \mathrm{b}\end{pmatrix} \quad \mathrm{c}\neq 0,$$

$$\text{kind } \{\text{ab2}\} : \begin{pmatrix} \mathrm{a} & 0 & 0 \\ 0 & \mathrm{b}+1/2 & -1/2 \\ 0 & 1/2 & \mathrm{b}-1/2 \end{pmatrix}, \qquad \text{kind } \{\text{a3}\} : \begin{pmatrix} \mathrm{a} & \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \mathrm{a} & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \mathrm{a} \end{pmatrix}.$$

A self-adjoint linear map ϕ with respect to $\langle\,,\rangle$ is of kind {a3} if it has three equal eigenvalues associated to a one-dimensional eigenspace; ϕ is of kind {ab2} if it has two eigenvalues a and b, each associated to a one-dimensional eigenspace so that b has multiplicity two; and ϕ is of kind $\{\mathrm{az}\bar{\mathrm{z}}\}$ if it has one real and two complex conjugate eigenvalues z and $\bar{\mathrm{z}}$, where $\mathrm{z} = \mathrm{b} + i\mathrm{c}$.

Isotropy group. A diffeomorphism θ on G is said to be an isometry with respect to the left-invariant Lorentzian metric g if $\theta^*\mathrm{g} = \mathrm{g}$; we denote by $\mathrm{Isom}(\mathrm{G}, \mathrm{g})$ the full group of isometries of (G, g). It is immediate that the left translation group L(G) is a subgroup of $\mathrm{Isom}(\mathrm{G}, \mathrm{g})$ and L(G) acts transitively on G. Furthermore $\mathrm{Isom}(\mathrm{G}, \mathrm{g})$ is generated by the left translation group and the isotropy group, i.e., the subgroup $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G}, \mathrm{g}) = \{\theta \in \mathrm{Isom}(\mathrm{G}, \mathrm{g}) : \theta(\mathrm{e}) = \mathrm{e}\}$ consisting of all isometries which leave the identity element fixed. The isotropy representation ρ, as already mentioned in (1.1), is the linear action of $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G}, \mathrm{g})$ on $\mathrm{T_e G} = \mathfrak{g}$ where $\rho.\theta = T_e\theta$. An isometry is uniquely determined by its 1-jet at e. This is a consequence of the fact that if an isometry θ has a fixed point x such that $T_x\theta$ is the identity map, then θ is the identity. Thus we can identify $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G}, \mathrm{g})$ with its image $\rho.\mathrm{Isom}_{\mathrm{e}}(\mathrm{G}, \mathrm{g})$. We can also consider $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G}, \mathrm{g})$ as a subgroup of $\mathrm{O}(2, 1)$.

3. Isometric automorphism groups

In the present section we will describe $\mathrm{Aut}(\mathfrak{g}, \langle\,,\rangle)$, the isometric automoprhism group (i.e., the group of all automorphisms which leave the metric invariant), as it will serve us as the most important ingredient in the calculation of isometry groups. In this first study we concern ourselves only with the four types of symmetric endomorphism. This is summarized in the following proposition.

Proposition 3.1. *Let $(\mathfrak{g}, \langle\,,\rangle)$ be an oriented Lorentzian unimodular Lie algebra of dimension 3 with corresponding symmetric endomorphism $\mathbf{L}$. Then the following conditions are equivalent:*

(1) $\phi \in \mathrm{Aut}(\mathfrak{g})$.
(2) $\phi \circ \mathbf{L} = \det(\phi)\mathbf{L} \circ (\phi^{-1})^*$.

In particular, $\mathrm{Aut}(\mathfrak{g}, \langle\,,\rangle) = \{\phi \in \mathrm{GL}(\mathfrak{g}) : \phi^*\langle\,,\rangle = \langle\,,\rangle$ *and* $\phi \circ \mathbf{L} = \det(\phi)\mathbf{L} \circ \phi\}$, *where* $\det(\phi) = \pm 1$ *for all* $\phi \in \mathrm{Aut}(\mathfrak{g}, \langle\,,\rangle)$.

Proof. Let $\phi \in \mathrm{Aut}(\mathfrak{g})$. For all $\mathrm{X}, \mathrm{Y} \in \mathfrak{g}$, we have

$$\begin{aligned} \phi([\mathrm{X}, \mathrm{Y}]) = [\phi(\mathrm{X}), \phi(\mathrm{Y})] &\Leftrightarrow \phi \circ \mathbf{L}(\mathrm{X} \times \mathrm{Y}) = \mathbf{L}(\phi(\mathrm{X}) \times \phi(\mathrm{Y})) \\ &\Leftrightarrow \phi \circ \mathbf{L}(\mathrm{X} \times \mathrm{Y}) = \det(\phi)\mathbf{L} \circ (\phi^{-1})^*(\mathrm{X} \times \mathrm{Y}) \\ &\Leftrightarrow \phi \circ \mathbf{L} = \det(\phi)\mathbf{L} \circ (\phi^{-1})^* \end{aligned}$$

On the other hand,

$$\begin{aligned}\phi \in \mathrm{Aut}(\mathfrak{g}, \langle\,,\rangle) &\Leftrightarrow \phi \in \mathrm{Aut}(\mathfrak{g}) \text{ and } \phi^*\langle\,,\rangle = \langle\,,\rangle \\ &\Leftrightarrow \phi \in \mathrm{Aut}(\mathfrak{g}) \text{ and } \phi = (\phi^{-1})^*. \end{aligned}$$ □

By the choice of a positively oriented and pseudo-orthonormal basis, the automorphism group of $\mathfrak{g}$ is identified with a subgroup of $\mathrm{GL}(3,\mathbb{R})$, so we separate the group $\mathrm{Aut}(\mathfrak{g})$ using the determinant as follows: $\mathrm{Aut}(\mathfrak{g}) = \mathrm{Aut}^+(\mathfrak{g}) \cup \mathrm{Aut}^-(\mathfrak{g})$, where $\mathrm{Aut}^+(\mathfrak{g})$ (resp. $\mathrm{Aut}^-(\mathfrak{g})$) is the subgroup of automorphisms with positive determinant (resp. the set of automorphisms with negative determinant).

Having in mind the four types of symmetric endomorphims and the idea of Proposition 3.1, we perform a careful case-by-case study in the following proposition.

Proposition 3.2. *Let $(\mathfrak{g}, \langle\,,\rangle)$ be an oriented Lorentzian unimodular Lie algebra of dimension 3 with corresponding symmetric endomorphism $\mathbf{L}$. Then there exists a pseudo-orthonormal basis $\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\}$, with e_3 timelike, in which $\mathrm{Aut}(\mathfrak{g}, \langle\,,\rangle)$ takes one of the following forms:*

$\boxed{\mathrm{I}}$ *If $\mathbf{L}$ is of kind $\{\mathrm{diag}\}(\alpha,\beta,\beta)$, $\alpha \neq \beta$, then*

$$\mathrm{Aut}(\mathfrak{g}, \langle\,,\rangle) = \left\{ \begin{pmatrix} \det S & 0 \\ 0 & S \end{pmatrix} : S \in \mathrm{O}(1,1) \right\}.$$

$\boxed{\mathrm{II}}$ *If $\mathbf{L}$ is of kind $\{\mathrm{diag}\}(\alpha,\alpha,\beta)$, $\alpha \neq \beta$, then*

$$\mathrm{Aut}(\mathfrak{g}, \langle\,,\rangle) = \left\{ \begin{pmatrix} S & 0 \\ 0 & \det S \end{pmatrix} : S \in \mathrm{O}(2) \right\}.$$

$\boxed{\mathrm{III}}$ *If $\mathbf{L}$ is of kind $\{\mathrm{diag}\}(\alpha,\beta,\alpha)$, $\alpha \neq \beta$, then*

$$\mathrm{Aut}(\mathfrak{g}, \langle\,,\rangle) = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & n & 0 \\ c & 0 & d \end{pmatrix} : S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{O}(1,1) \text{ and } n = \det(S) \right\}.$$

$\boxed{\mathrm{IV}}$ *If $\mathbf{L}$ is of kind $\{\mathrm{diag}\}(\alpha,\alpha,\alpha)$, $\alpha \neq 0$, then $\mathrm{Aut}(\mathfrak{g}, \langle\,,\rangle) = \mathrm{SO}(2,1)$.*

$\boxed{\mathrm{V}}$ *If $\mathbf{L}$ is of kind $\{\mathrm{diag}\}(\alpha,\beta,\gamma)$, $\alpha \neq \beta \neq \gamma$, then*

(a) $$\mathrm{Aut}(\mathfrak{g}, \langle\,,\rangle) = \left\{ \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon\sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix} : \sigma = \pm 1, \varepsilon = \pm 1 \right\} \cup \left\{ \begin{pmatrix} 0 & \varepsilon & 0 \\ \varepsilon\sigma & 0 & 0 \\ 0 & 0 & \sigma \end{pmatrix} : \sigma = \pm 1, \varepsilon = \pm 1 \right\},$$

if $\mathrm{Aut}^-(\mathfrak{g}) \neq \emptyset$, $\alpha = -\beta$ and $\gamma = 0$;

(b) $$\mathrm{Aut}(\mathfrak{g}, \langle\,,\rangle) = \left\{ \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon\sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix} : \sigma = \pm 1, \varepsilon = \pm 1 \right\},$$ *otherwise.*

$\boxed{\text{VI}}$ *If* $\mathbf{L}$ *is of type* $\{\mathrm{ab2}\}$, *then*

(a) $\mathrm{Aut}(\mathfrak{g},\langle\,,\rangle)$
$$= \left\{ \begin{pmatrix} 1 & -\lambda & \lambda \\ \sigma\lambda & \sigma-\sigma(1/2)\lambda^2 & \sigma(1/2)\lambda^2 \\ \sigma\lambda & -\sigma(1/2)\lambda^2 & \sigma+\sigma(1/2)\lambda^2 \end{pmatrix} : \lambda\in\mathbb{R},\ \sigma=\pm1 \right\},\ \textit{if}\ \mathrm{a}=\mathrm{b};$$

(b) $\mathrm{Aut}(\mathfrak{g},\langle\,,\rangle) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix} : \sigma=\pm1 \right\}$, *if* $\mathrm{a}\neq\mathrm{b}$.

$\boxed{\text{VII}}$ *If* $\mathbf{L}$ *is of type* $\{\mathrm{az\bar{z}}\}$, *then*

(a) $\mathrm{Aut}(\mathfrak{g},\langle\,,\rangle) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} : \varepsilon=\pm,\ \sigma=\pm1 \right\}$, *if* $\mathrm{Aut}^-(\mathfrak{g})\neq\emptyset$ *and* $\mathrm{a}=\mathfrak{Re}(\mathrm{z})=0$;

(b) $\mathrm{Aut}(\mathfrak{g},\langle\,,\rangle) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix} : \sigma=\pm1 \right\}$, *otherwise.*

$\boxed{\text{VIII}}$ *If* $\mathbf{L}$ *is of type* $\{\mathrm{a3}\}$, *then*

(a) $\mathrm{Aut}(\mathfrak{g},\langle\,,\rangle) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 1 \end{pmatrix} : \sigma=\pm1 \right\}$, *if* $\mathrm{Aut}^-(\mathfrak{g})\neq\emptyset$ *and* $\mathrm{a}=0$;

(b) $\mathrm{Aut}(\mathfrak{g},\langle\,,\rangle) = \{\mathrm{id}\}$, *otherwise.*

Notation 3.3. To state our next results concisely, it is convenient to introduce the notation
$$\mathbf{K} = \left\{ \begin{pmatrix} 1 & -\lambda & \lambda \\ \lambda & 1-(1/2)\lambda^2 & (1/2)\lambda^2 \\ \lambda & -(1/2)\lambda^2 & 1+(1/2)\lambda^2 \end{pmatrix} : \lambda\in\mathbb{R} \right\}.$$
As already mentioned, $\mathbf{K}$ is called the one-parameter group of rotations in three-dimensional Minkowski space $\mathbb{R}^3_1$ around an axis spanned by an isotropic vector.

Let $D_{2n} = \langle X, Y \mid X^n = Y^2 = 1,\ YXY = X^{-1} \rangle$ be the dihedral group of order $2n$. We set
$$X = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix};$$
then we can easily see that
$$\left\{ \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon\sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix} : \sigma=\pm1,\ \varepsilon=\pm1 \right\} \cup \left\{ \begin{pmatrix} 0 & \varepsilon & 0 \\ \varepsilon\sigma & 0 & 0 \\ 0 & 0 & \sigma \end{pmatrix} : \sigma=\pm1,\ \varepsilon=\pm1 \right\} \cong \mathrm{D}_8.$$
It is clear without calculations that
$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} : \varepsilon=\pm,\ \sigma=\pm1 \right\} \cong \left\{ \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon\sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix} : \sigma=\pm1,\ \varepsilon=\pm1 \right\} \cong \mathrm{D}_4.$$

As simple consequences of Proposition 3.2 we get the following results.

3.1. **Study of** $\mathrm{Aut}(\mathrm{Nil},\mathrm{g})$**.** Note that if $\varphi \in \mathrm{Aut}(\mathfrak{n},\langle\,,\rangle)$, then $\det(\varphi)=1$.

Proposition 3.4. *For any left-invariant Lorentzian metric* g *on* Nil*,*

$$\mathrm{Aut}(\mathrm{Nil},\mathrm{g}) \cong \begin{cases} \mathrm{O}(1,1) & \textit{if } \mathrm{g} \cong \mathsf{nil}+, \\ \mathrm{O}(2) & \textit{if } \mathrm{g} \cong \mathsf{nil}-, \\ \mathbf{K}\times\mathbb{Z}_2 & \textit{if } \mathrm{g} \cong \mathsf{nil0}. \end{cases}$$

3.2. **Study of** $\mathrm{Aut}(\mathrm{SU}(2),\mathrm{g})$**.** Note that if $\varphi \in \mathrm{Aut}(\mathfrak{su}(2),\langle\,,\rangle)$, then $\det(\varphi)=1$.

Proposition 3.5. *For any left-invariant Lorentzian metric* $\mathrm{g} \cong \mathsf{su}$ *on* SU(2)*,*

$$\mathrm{Aut}(\mathrm{SU}(2),\mathrm{g}) \cong \begin{cases} \mathrm{O}(2) & \textit{if } \mu_1=\mu_2, \\ \mathrm{D}_4 & \textit{if } \mu_1\neq\mu_2. \end{cases}$$

3.3. **Study of** $\mathrm{Aut}(\widetilde{\mathrm{PSL}}(2,\mathbb{R}),\mathrm{g})$**.** If $\varphi \in \mathrm{Aut}(\mathrm{sl}(2,\mathbb{R}),\langle\,,\rangle)$, then $\det(\varphi)=1$.

Proposition 3.6. *For any left-invariant Lorentzian metric* $\mathrm{g}\cong\mathsf{sll1}$ *on* $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$*,*

$$\mathrm{Aut}(\widetilde{\mathrm{PSL}}(2,\mathbb{R}),\mathrm{g}) \cong \begin{cases} \mathrm{SO}(2,1) & \textit{if } \mu_1=\mu_2=\mu_3, \\ \mathrm{O}(2) & \textit{if } \mu_1\neq\mu_2=\mu_3, \\ \mathrm{O}(1,1) & \textit{if } \mu_1=\mu_2\neq\mu_3 \;\; \textit{or} \;\; \mu_1=\mu_3\neq\mu_1, \\ \mathrm{D}_4 & \textit{if } \mu_1\neq\mu_2\neq\mu_3. \end{cases}$$

For any left-invariant Lorentzian metric $\mathrm{g}\cong\mathsf{sll2}$ *on* $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$*,*

$$\mathrm{Aut}(\widetilde{\mathrm{PSL}}(2,\mathbb{R}),\mathrm{g}) \cong \begin{cases} \mathrm{O}(1,1) & \textit{if } \mu_1=\mu_2, \\ \mathrm{D}_4 & \textit{if } \mu_1\neq\mu_2. \end{cases}$$

For any left-invariant Lorentzian metric $\mathrm{g}\cong\mathsf{sll3}$, $\mathsf{sll4}$, $\mathsf{sll5}$ *on* $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$*,*

$$\mathrm{Aut}(\widetilde{\mathrm{PSL}}(2,\mathbb{R}),\mathrm{g}) \cong \mathbb{Z}_2.$$

For any left-invariant Lorentzian metric $\mathrm{g}\cong\mathsf{sll7}$ *on* $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$*,*

$$\mathrm{Aut}(\widetilde{\mathrm{PSL}}(2,\mathbb{R}),\mathrm{g}) = \{\mathrm{id}\}.$$

For any left-invariant Lorentzian metric $\mathrm{g}\cong\mathsf{sll6}$ *on* $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$*,*

$$\mathrm{Aut}(\widetilde{\mathrm{PSL}}(2,\mathbb{R}),\mathrm{g}) \cong \begin{cases} \mathbf{K}\times\mathbb{Z}_2 & \textit{if } a=b, \\ \mathbb{Z}_2 & \textit{if } a\neq b. \end{cases}$$

3.4. **Study of** $\mathrm{Aut}(\mathrm{Sol},\mathrm{g})$**.** Note that $\mathrm{Aut}^-(\mathfrak{sol})$ is not empty.

Proposition 3.7. *For any left-invariant Lorentzian metric* $\mathrm{g}\cong\mathsf{sol1}$ *on* Sol*,*

$$\mathrm{Aut}(\mathrm{Sol},\mathrm{g}) \cong \begin{cases} \mathrm{D}_8 & \textit{if } u=0, \\ \mathrm{D}_4 & \textit{if } u\neq 0. \end{cases}$$

For any left-invariant Lorentzian metric $\mathrm{g}\cong\mathsf{sol2}$ *on* Sol*,*

$$\mathrm{Aut}(\mathrm{Sol},\mathrm{g}) \cong \begin{cases} \mathrm{O}(1,1) & \textit{if } u=0, \\ \mathrm{D}_4 & \textit{if } u\neq 0. \end{cases}$$

For any left-invariant Lorentzian metric $\mathrm{g} \cong \mathsf{sol4}$ *on* Sol*,*

$$\mathrm{Aut}(\mathrm{Sol}, \mathrm{g}) \cong \mathrm{D}_4.$$

For any left-invariant Lorentzian metric $\mathrm{g} \cong \mathsf{sol3}, \mathsf{sol5}, \mathsf{sol6}, \mathsf{sol7}$ *on* Sol*,*

$$\mathrm{Aut}(\mathrm{Sol}, \mathrm{g}) \cong \mathbb{Z}_2.$$

3.5. **Study of** $\mathrm{Aut}(\widetilde{\mathrm{E}_0}(2), \mathrm{g})$**.** Note that if $\varphi \in \mathrm{Aut}(\mathfrak{n}, \langle\,,\rangle)$, then $\det(\varphi) = 1$.

Proposition 3.8. *For any left-invariant Lorentzian metric* $\mathrm{g} \cong \mathsf{ee1}$ *on* $\widetilde{\mathrm{E}_0}(2)$*,*

$$\mathrm{Aut}(\widetilde{\mathrm{E}_0}(2), \mathrm{g}) \cong \begin{cases} \mathrm{O}(2) & \text{if } u = v, \\ \mathrm{D}_4 & \text{if } u \neq v. \end{cases}$$

For any left-invariant Lorentzian metric $\mathrm{g} \cong \mathsf{ee2}$ *on* $\widetilde{\mathrm{E}_0}(2)$*,*

$$\mathrm{Aut}(\widetilde{\mathrm{E}_0}(2), \mathrm{g}) \cong \mathrm{D}_4.$$

For any left-invariant Lorentzian metric $\mathrm{g} \cong \mathsf{ee3}$ *on* $\widetilde{\mathrm{E}_0}(2)$*,*

$$\mathrm{Aut}(\widetilde{\mathrm{E}_0}(2), \mathrm{g}) \cong \mathbb{Z}_2.$$

In the next section, we will introduce and complete the solution to the problem in question.

4. Isometry groups

Here we propose a new method of identifying isometry groups that combines two steps: in the first step, we determine all the different kinds of Ricci operators Ric; and in the second, using the covariant derivatives $\nabla\mathrm{R}$, we implement an algorithm that allows us to learn about the relationship between the groups obtained in the previous section and the isotropy groups $\mathrm{Isom_e}(\mathrm{G}, \mathrm{g})$. In what follows, we do some preparations in order to solve the problem in question and, later in this section, we will restrict our attention to the special case when the isotropy group $\mathrm{Isom_e}(\mathrm{G}, \mathrm{g})$ is wider than the isometric automorphism group $\mathrm{Aut}(\mathrm{G}, \mathrm{g})$. Let us begin by quickly reviewing the algebraic ingredients we need to formulate the solution.

4.1. **Useful results for computing the isotropy groups.** We quote a theorem from [3] with some useful preliminary results from [12].

Theorem 4.1 ([3])**.** *Let* (G, g) *be a Lorentzian, three-dimensional, connected, simply connected and unimodular Lie group and let* $\mathbf{L}$ *be its symmetric endomorphism.* (G, g) *is symmetric if and only if one of the following cases occurs:*

1. $\mathrm{G} = \widetilde{\mathrm{PSL}}(2, \mathbb{R})$ *with* $\mathbf{L}$ *of kind* $\{\mathrm{diag}\}(\alpha, \alpha, \alpha)$*,* $\alpha \neq 0$*.*
2. $\mathrm{G} = \mathrm{Sol}$ *with* $\mathbf{L}$ *of kind* $\{\mathrm{diag}\}(\alpha, 0, \alpha)$ *and* g *is flat.*
3. $\mathrm{G} = \widetilde{\mathrm{E}_0}(2)$ *with* $\mathbf{L}$ *of kind* $\{\mathrm{diag}\}(\alpha, \alpha, 0)$ *and* g *is flat.*
4. $\mathrm{G} = \mathrm{Nil}$ *with* $\mathbf{L}$ *of kind* $\{\mathrm{ab2}\}$*,* $\mathrm{a} = \mathrm{b} = 0$ *and* g *is flat.*

A basic tool for computing $\mathrm{Isom_e}(\mathrm{G}, \mathrm{g})$ on the symmetric space is the Ambrose–Hicks–Cartan theorem (see [12], Thm. 17, Ch. 8).

Theorem 4.2 ([12]). *Let* (M, g) *be a complete, connected, simply connected and locally symmetric pseudo-Riemannian manifold. If* $\mathfrak{L} : \mathrm{T_p M} \to \mathrm{T_q M}$ *is a linear isometry that preserves curvature, then there is a unique isometry* $\theta : \mathrm{M} \to \mathrm{M}$ *such that* $T_p\theta = \mathfrak{L}$.

We remark that the curvature tensor R of a three-dimensional Lorentzian manifold (M, g) is completely determined by the Ricci curvature tensor. This means that, given any $\mathfrak{L} \in \mathrm{GL}(\mathfrak{g})$, we have that $\mathfrak{L}$ preserves the curvature R if and only if $\mathfrak{L}$ commutes with the Ricci operator Ric, i.e., $\mathfrak{L}^*\mathrm{R} = \mathrm{R} \Leftrightarrow [\mathfrak{L}, \mathrm{Ric}] = 0$.

It is well known that any isometry $\theta \in \mathrm{Isom}(\mathrm{G}, \mathrm{g})$ preserves the metric g and the curvature tensor R; for that reason, we introduce a novel group, called *symmetrical group* and denoted by $\mathrm{Sym}(\mathrm{G}, \mathrm{g})$. The symmetrical group can be expressed as

$$\mathrm{Sym}(\mathrm{G}, \mathrm{g}) := \{\mathfrak{L} \in \mathrm{GL}(\mathfrak{g}) : \mathfrak{L}^*\langle\,,\rangle = \langle\,,\rangle \text{ and } [\mathfrak{L}, \mathrm{Ric}] = 0\},$$

where $\langle\,,\rangle$ is induced by the metric g.

Remark 4.3. Given that the isotropy group $\mathrm{Isom_e}(\mathrm{G}, \mathrm{g})$ is isometric as a Lie group to a subgroup of $O(\mathfrak{g}) \cong \mathrm{O}(2,1)$, we can write

$$\mathrm{Aut}(\mathrm{G}, \mathrm{g}) \subset \mathrm{Isom_e}(\mathrm{G}, \mathrm{g}) \subset \mathrm{Sym}(\mathrm{G}, \mathrm{g}).$$

An application of Theorem 4.2 and Theorem 4.1 is the following.

Corollary 4.4. *If* (G, g) *a is a symmetric space, then* $\mathrm{Isom_e}(\mathrm{G}, \mathrm{g}) \cong \mathrm{O}(2,1)$.

Proof. By Theorem 4.2, for any linear map $\phi \in \mathrm{GL}(\mathfrak{g})$ preserving the metric g and the curvature tensor R at the identity element, ϕ can be lifted to an unique isometry $\theta \in \mathrm{Isom_e}(\mathrm{G}, \mathrm{g})$ such that $T_e\theta = \phi$. Consequently, for any $\phi \in \mathrm{Sym}(\mathrm{G}, \mathrm{g})$ we have that $\phi \in \mathrm{Isom_e}(\mathrm{G}, \mathrm{g})$, and hence $\mathrm{Isom_e}(\mathrm{G}, \mathrm{g}) \cong \mathrm{Sym}(\mathrm{G}, \mathrm{g})$. From Theorem 4.1 we immediately obtain that a symmetric space is a flat space or a negative constant sectional curvature space. The corollary follows. □

Using the same procedure as in Section 3, we now give a list of all such symmetrical groups $\mathrm{Sym}(\mathrm{G}, \mathrm{g})$ associated to each kind of the curvature operator Ric. A pseudo-orthonormal basis $\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\}$ with timelike e_3 is considered for the sake of simplicity.

Kind $\{\mathrm{diag}\}(a, b, c)$:

(1) if $a \neq b \neq c$, then $\mathrm{Sym}(\mathrm{G}, \mathrm{g})$ is a finite group;
(2) if $a = b \neq c$, then $\mathrm{Sym}(\mathrm{G}, \mathrm{g}) = \begin{pmatrix} \mathrm{O}(2) & 0 \\ 0 & \pm 1 \end{pmatrix}$;
(3) if $a \neq b = c$, then $\mathrm{Sym}(\mathrm{G}, \mathrm{g}) = \begin{pmatrix} \pm 1 & 0 \\ 0 & \mathrm{O}(1,1) \end{pmatrix}$;
(4) if $a = c \neq b$, then $\mathrm{Sym}(\mathrm{G}, \mathrm{g}) = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_2 \\ 0 & \pm 1 & 0 \\ \lambda_3 & 0 & \lambda_4 \end{pmatrix} : \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} \in \mathrm{O}(1,1) \right\}$;
(5) if $a = b = c$, then $\mathrm{Sym}(\mathrm{G}, \mathrm{g}) = O(\mathfrak{g}) \cong \mathrm{O}(2,1)$.

Kind $\{\mathrm{az\bar{z}}\}$: then $\mathrm{Sym}(\mathrm{G}, \mathrm{g})$ is a finite group.

Kind $\{\mathrm{ab2}\}$:

(1) if $\mathrm{a} \neq \mathrm{b}$, then $\mathrm{Sym}(\mathrm{G}, \mathrm{g})$ is a finite group;
(2) if $\mathrm{a} = \mathrm{b}$, then

$$\mathrm{Sym}(\mathrm{G}, \mathrm{g}) = \left\{ \begin{pmatrix} \varepsilon & -\lambda & \lambda \\ \sigma\lambda & \sigma\varepsilon(1-(1/2)\lambda^2) & (1/2)\sigma\varepsilon\lambda^2 \\ \sigma\lambda & -(1/2)\sigma\varepsilon\lambda^2 & \sigma\varepsilon(1+(1/2)\lambda^2) \end{pmatrix} : \lambda \in \mathbb{R},\, \varepsilon = \pm 1,\, \sigma = \pm 1 \right\}.$$

Kind $\{\mathrm{a3}\}$: then $\mathrm{Sym}(\mathrm{G}, \mathrm{g})$ is a finite group.

We restrict ourselves to the kinds of the curvature operator Ric and symmetric endomorphism $\mathbf{L}$ giving infinite groups, namely the kinds $\{\mathrm{diag}\}(a, b, c)$, $a = b = c$, $a = b \neq c$, $a = c \neq b$, $a \neq b = c$ and $\{\mathrm{ab2}\}$ with $\mathrm{a} = \mathrm{b}$. Let us first introduce the following terminology.

Definition 4.5. The curvature operator (resp. the symmetric endomorphism) is said to be of *infinite kind* if it has an infinite symmetrical group (resp. infinite isometric automorphism group); otherwise it is said to be of *finite kind*.

Remark 4.6. In view of Remark 4.3, if $\mathbf{L}$ is of infinite kind, then Ric is necessarily of infinite kind.

This motivates the following definition.

Definition 4.7. Let $(V, \langle\, , \rangle)$ denote a Lorentzian vector space. We say that two self-adjoint endomorphisms ϕ_1 and ϕ_2 on V have the same *infinite kind* if there exists a pseudo-orthonormal basis B for V with respect to which the matrix of ϕ_1 (resp. ϕ_2) is

i. $\{\mathrm{diag}\}(a, a, a)$ (resp. $\{\mathrm{diag}\}(a', a', a')$) for some $a, a' \in \mathbb{R}$;
ii. $\{\mathrm{diag}\}(a, a, c), a \neq c$ (resp. $\{\mathrm{diag}\}(a', a', c'), a' \neq c'$) for some $a, c, a', c' \in \mathbb{R}$;
iii. $\{\mathrm{diag}\}(a, b, a), a \neq b$ (resp. $\{\mathrm{diag}\}(a', b', a'), a' \neq b'$) for some $a, b, a', b' \in \mathbb{R}$;
iv. $\{\mathrm{diag}\}(a, b, b), a \neq b$ (resp. $\{\mathrm{diag}\}(a', b', b'), a' \neq b'$) for some $a, b, a', b' \in \mathbb{R}$;
v. $\{\mathrm{ab2}\}, \mathrm{a} = \mathrm{b}$ (resp. $\{\mathrm{a'b'2}\}, \mathrm{a'} = \mathrm{b'}$) for some $\mathrm{a}, \mathrm{b}, \mathrm{a'}, \mathrm{b'} \in \mathbb{R}$.

For use in the proof of Proposition 4.9, we record the following obvious statement.

Proposition 4.8. *Let* $(\mathrm{G}_1, \mathrm{g}_1)$ *and* $(\mathrm{G}_2, \mathrm{g}_2)$ *be two Lorentzian three-dimensional Lie groups with Ricci operator respectively* Ric_1 *and* Ric_2. *If* Ric_1 *and* Ric_2 *have the same infinite kind, then* $\mathrm{Sym}(\mathrm{G}_1, \mathrm{g}_1) = \mathrm{Sym}(\mathrm{G}_2, \mathrm{g}_2)$.

Proof. $\mathrm{Sym}(\mathrm{G}, \mathrm{g})$ depends only on the five infinite kinds mentioned above of the Ricci operator Ric. □

Proposition 4.9. *Let* (G, g) *be a Lorentzian Lie group with Lie algebra* $\mathfrak{g} = T_eG$ *such that its symmetric endomorphism* $\mathbf{L}$ *and its Ricci operator* Ric *have the same infinite kind. Then*

$$\mathrm{Isom_e}(\mathrm{G}, \mathrm{g}) = \mathrm{Aut}(\mathrm{G}, \mathrm{g}) \quad or \quad \mathrm{Isom_e}(\mathrm{G}, \mathrm{g}) = \mathrm{Aut}(\mathrm{G}, \mathrm{g}) \times \{\pm \mathrm{id}\}. \tag{4.1}$$

In particular, if (G,g) *is a symmetric space, then* $\mathrm{Isom_e}(\mathrm{G},\mathrm{g}) = \mathrm{Aut}(\mathrm{G},\mathrm{g}) \times \{\pm\mathrm{id}\}$*; otherwise,* $\mathrm{Isom_e}(\mathrm{G},\mathrm{g}) = \mathrm{Aut}(\mathrm{G},\mathrm{g})$.

Proof. It is enough to prove that the set $\mathrm{Aut}^-(\mathrm{G},\mathrm{g})$ is empty. Let $\theta \in \mathrm{Aut}(\mathrm{G},\mathrm{g})$ be given. We have the equalities $\phi \circ \mathbf{L} = \det(\phi)\mathbf{L} \circ \phi$ and $\phi \circ \mathrm{Ric} = \mathrm{Ric} \circ \phi$, where $\phi = T_e\theta$. If we additionally assume that $\phi \in \mathrm{Aut}^-(\mathfrak{g})$, then ϕ satisfies $\phi \circ \mathbf{L} = -\mathbf{L} \circ \phi$ and $\phi \circ \mathrm{Ric} = \mathrm{Ric} \circ \phi$. It follows from the hypothesis that $\mathrm{Sym}(\mathrm{G},\mathrm{g})$ can be written by substituting $\mathbf{L}$ in place of Ric, since according to Proposition 4.8, $\mathrm{Sym}(\mathrm{G},\mathrm{g})$ is uniquely determined by the corresponding kind of Ric. Also, we can write $\phi \circ \mathbf{L} = -\mathbf{L} \circ \phi$ and $\phi \circ \mathbf{L} = \mathbf{L} \circ \phi$. This, in turn, implies that the set $\mathrm{Aut}^-(\mathrm{G},\mathrm{g})$ is empty and hence $\mathrm{Aut}(\mathrm{G},\mathrm{g}) \subset \mathrm{Isom_e}(\mathrm{G},\mathrm{g}) \subset \mathrm{Aut}(\mathrm{G},\mathrm{g}) \times \{\pm\mathrm{id}\}$. Therefore $\mathrm{Aut}(\mathrm{G},\mathrm{g})$ is a subgroup of index at most 2 in $\mathrm{Isom_e}(\mathrm{G},\mathrm{g})$, and the property (4.1) follows. The rest of the proposition is proved by the fact that (G,g) is a symmetric space if and only if there exists an isometry $\theta \in \mathrm{Isom_e}(\mathrm{G},\mathrm{g})$ such that $T_e\theta = -\mathrm{id}$. □

From considering the above discussion, it is natural to wonder when a Ricci operator behaves as a symmetric endomorphism. This means that we only have to find all possible kinds of Ric as a function of the kind of $\mathbf{L}$.

We now state a lemma that leads us to the intuition behind the design of Algorithm 1.

Lemma 4.10. *Let* (G,g) *be a Lorentzian, connected and simply connected Lie group,* $\mathbf{L}$ *its symmetric endomorphism and* Ric *its Ricci operator.*

- [I] *If* $G = \mathrm{Nil}$*, then we have three possibilities for kinds of* $\mathbf{L}$*:*
  - [1] $\mathbf{L}$ *is of kind* $\{\mathrm{diag}\}(\alpha,0,0)$ *with* $\alpha \neq 0$*, then* Ric *is of kind* $\{\mathrm{diag}\}(a,b,b)$*,* $a \neq b$ *(same kinds).*
  - [2] $\mathbf{L}$ *is of kind* $\{\mathrm{diag}\}(0,0,\gamma)$ *with* $\gamma \neq 0$*, then* Ric *is of kind* $\{\mathrm{diag}\}(a,a,c)$*,* $a \neq c$ *(same kinds).*
  - [3] $\mathbf{L}$ *is of kind* $\{\mathrm{ab2}\}$ *with* $\mathrm{a} = \mathrm{b}$*, then* $\mathrm{Ric} = 0$ *(different kinds and symmetric space).*
- [II] *If* $G = \mathrm{SU}(2)$*, then we have four possibilities for kinds of* $\mathbf{L}$*:*
  - [1] $\mathbf{L}$ *is of kind* $\{\mathrm{diag}\}(\alpha,\alpha,\gamma)$ *with* $\alpha \neq \gamma$*, then* Ric *is of kind* $\{\mathrm{diag}\}(a,a,c)$*,* $a \neq c$ *(same kinds).*
  - [2] $\mathbf{L}$ *is of kind* $\{\mathrm{diag}\}(\alpha,\beta,\gamma)$ *with* $\alpha \neq \beta \neq \gamma$*:*
    - (i) *if* $\gamma = \alpha - \beta$*, then* Ric *is of kind* $\{\mathrm{diag}\}(a,b,b)$*,* $a \neq b$ *(different kinds),*
    - (ii) *if* $\gamma = -(\alpha - \beta)$*, then* Ric *is of kind* $\{\mathrm{diag}\}(a,b,a)$*,* $a \neq b$ *(different kinds),*
    - (iii) *if* $\gamma \neq \pm(\alpha - \beta)$*, then* Ric *is of kind* $\{\mathrm{diag}\}(a,b,c)$*,* $a \neq b \neq c$ *(of finite kinds).*
- [III] *If* $G = \widetilde{\mathrm{PSL}}(2,\mathbb{R})$*, then we have fourteen possibilities for kinds of* $\mathbf{L}$*:*
  - [1] $\mathbf{L}$ *is of kind* $\{\mathrm{diag}\}(\alpha,\beta,\gamma)$ *with* $\alpha \neq \beta \neq \gamma$*,*
    - (i) *if* $\gamma = \alpha - \beta$*, then* Ric *is of kind* $\{\mathrm{diag}\}(a,b,b)$*,* $a \neq b$ *(different kinds).*

(ii) *if* $\alpha = \gamma - \beta$, *then* Ric *is of kind* $\{\text{diag}\}(a, a, c)$, $a \neq c$ *(different kinds).*

(iii) *if* $\alpha = -\gamma + \beta$, *then* Ric *is of kind* $\{\text{diag}\}(a, b, a)$, $a \neq b$ *(different kinds).*

(iv) *if* $\alpha \neq \pm(\gamma - \beta)$ *and* $\alpha \neq \gamma + \beta$, *then* Ric *is of kind* $\{\text{diag}\}(a, b, c)$, $a \neq b \neq c$ *(of finite kinds).*

[2] **L** *is of kind* $\{\text{diag}\}(\alpha, \alpha, \gamma)$ *with* $\alpha \neq \gamma$, *then* Ric *is of kind* $\{\text{diag}\}(a, a, c)$, $a \neq c$ *(same kinds).*

[3] **L** *is of kind* $\{\text{diag}\}(\alpha, \beta, \alpha)$ *with* $\alpha \neq \beta$, *then* Ric *is of kind* $\{\text{diag}\}(a, b, a)$, $a \neq b$ *(same kinds).*

[4] **L** *is of kind* $\{\text{diag}\}(\alpha, \beta, \beta)$ *with* $\alpha \neq \beta$, *then* Ric *is of kind* $\{\text{diag}\}(a, b, b)$, $a \neq b$ *(same kinds).*

[5] **L** *is of kind* $\{\text{diag}\}(\alpha, \alpha, \alpha)$, *then* Ric *is of kind* $\{\text{diag}\}(a, a, a)$, $a \neq 0$ *(same kinds and symmetric space).*

[6] **L** *is of type* $\{\mathrm{az\bar{z}}\}$,

(i) *if* $\mathrm{a} \neq 2\,\mathfrak{Re}(\mathrm{z})$, *then* Ric *is of type* $\{\mathrm{a'z'\bar{z'}}\}$ *(of finite kinds).*

(ii) *if* $\mathrm{a} = 2\,\mathfrak{Re}(\mathrm{z})$, *then* Ric *is of kind* $\{\text{diag}\}(a', b', b')$, $a' \neq b'$ *(different kinds).*

[7] **L** *is of type* $\{\mathrm{a3}\}$, *then* Ric *is of type* $\{\mathrm{a'3}\}$ *(of finite kinds).*

[8] **L** *is of kind* $\{\mathrm{ab2}\}$ *with* $\mathrm{a} = \mathrm{b} \neq 0$, *then* $\dfrac{-1}{\mathrm{a}}$.Ric *is of kind* $\{\mathrm{a'b'2}\}$, $\mathrm{a'} = \mathrm{b'}$ *(same kinds).*

[9] **L** *is of kind* $\{\mathrm{ab2}\}$ *with* $\mathrm{a} \neq \mathrm{b} \neq 0$,

(i) *if* $\mathrm{a} \neq 2\mathrm{b}$ *then* Ric *is of kind* $\{\mathrm{a'b'2}\}$, $\mathrm{a'} \neq \mathrm{b'}$ *(of finite kinds).*

(ii) *if* $\mathrm{a} = 2\mathrm{b}$, *then* Ric *is of kind* $\{\text{diag}\}(a', b', b')$, $a' \neq b'$ *(different kinds).*

[IV] *If* $G = \mathrm{Sol}$, *then we have nine possibilities for kinds of* **L***:*

[1] **L** *is of kind* $\{\text{diag}\}(\alpha, \beta, 0)$ *with* $\alpha > 0$, $\beta < 0$ *and* $\alpha = -\beta$, *then* Ric *is of kind* $\{\text{diag}\}(a, a, c)$, $a \neq c$ *(different kinds).*

[2] **L** *is of kind* $\{\text{diag}\}(\alpha, \beta, 0)$ *with* $\alpha > 0$, $\beta < 0$ *and* $\alpha \neq -\beta$, *then* Ric *is of kind* $\{\text{diag}\}(a, b, c)$, $a \neq b \neq c$ *(of finite kinds).*

[3] **L** *is of kind* $\{\text{diag}\}(\alpha, 0, \gamma)$ *with* $\alpha \neq \gamma$, *then* Ric *is of kind* $\{\text{diag}\}(a, b, c)$, $a \neq b \neq c$ *(of finite kinds).*

[4] **L** *is of kind* $\{\text{diag}\}(\alpha, 0, \alpha)$ *with* $\alpha \neq 0$, *then* $\mathrm{Ric} = 0$ *(different kinds and symmetric space).*

[5] **L** *is of type* $\{\mathrm{az\bar{z}}\}$ *with* $\mathfrak{Re}(z) \neq 0$, *then* Ric *is of type* $\{\mathrm{a'z'\bar{z'}}\}$ *(of finite kinds).*

[6] **L** *is of type* $\{\mathrm{az\bar{z}}\}$ *with* $\mathfrak{Re}(z) = 0$, *then* Ric *is of kind* $\{\text{diag}\}(a', b', b')$, $a' \neq b'$ *(different kinds).*

[7] **L** *is of kind* $\{\mathrm{ab2}\}$ *with* $\mathrm{a} \neq \mathrm{b}$,

(i) *if* $\mathrm{a} = 0$ *then* Ric *is of kind* $\{\mathrm{a'b'2}\}$, $\mathrm{a'} = \mathrm{b'}$ *(different kinds).*

(ii) *if* $\mathrm{b} = 0$ *then* Ric *is of kind* $\{\mathrm{a'b'2}\}$, $\mathrm{a'} \neq \mathrm{b'}$ *(of finite kinds).*

8 **L** *is of kind* $\{\mathrm{a3}\}$ *with* $\mathrm{a}=0$, *then* Ric *is of kind* $\{\mathrm{a'b'2}\}$, $\mathrm{a'}=\mathrm{b'}=0$ *(different kinds).*

V *If* $G=\widetilde{\mathrm{E}_0}(2)$, *then we have five possibilities for kinds of* **L***:*

1 **L** *is of kind* $\{\mathrm{diag}\}(\alpha,0,\gamma)$ *with* $\alpha>0$, $\gamma<0$,
  (i) *if* $\alpha=-\gamma$, *then* Ric *is of kind* $\{\mathrm{diag}\}(a,b,a)$, $a\neq b$ *(different kinds).*
  (ii) *if* $\alpha\neq-\gamma$, *then* Ric *is of kind* $\{\mathrm{diag}\}(a,b,c)$, $a\neq b\neq c$ *(of finite kinds).*

2 **L** *is of kind* $\{\mathrm{diag}\}(\alpha,\alpha,0)$ *with* $\alpha\neq 0$, *then* $\mathrm{Ric}=0$ *(different kinds and symmetric space).*

3 **L** *is of kind* $\{\mathrm{diag}\}(\alpha,\beta,0)$ *with* $\alpha\neq\beta$, *then* Ric *is of kind* $\{\mathrm{diag}\}(a,b,c)$, $a\neq b\neq c$ *(of finite kinds).*

4 **L** *is of kind* $\{\mathrm{ab2}\}$ *with* $\mathrm{a}\neq 0$, $\mathrm{b}=0$, *then* Ric *is of kind* $\{\mathrm{a'b'2}\}$, $\mathrm{a'}\neq\mathrm{b'}$ *(of finite kinds).*

We require the following lemma, which is essentially [8, Corollary 2.8], and a useful result from [12].

Lemma 4.11 ([8])**.** *If* $\mathrm{Isom_e(G,g)}$ *is a finite group, then* $\mathrm{Isom_e(G,g)}=\mathrm{Aut(G,g)}$.

Remark 4.12. Notice that if Ric is of finite kind, then $\mathrm{Isom_e(G,g)}=\mathrm{Aut(G,g)}$.

Any isometry $\theta\in\mathrm{Isom_e(G,g)}$ preserves the metric g, the curvature tensor R, and its covariant derivative $\nabla\mathrm{R}$. More precisely, we have the following result.

Proposition 4.13 ([12])**.** *If* $\theta\in\mathrm{Isom_e(G,g)}$ *then* $\theta^*\nabla\mathrm{R}=\nabla\mathrm{R}$, *i.e., for any* $U,V,W,Z\in\mathfrak{g}=\mathrm{T_eG}$, *if* $\psi=T_e\theta$ *then*

$$\psi(\nabla\mathrm{R}(U,V,W,Z))=\nabla\mathrm{R}(\psi(U),\psi(V),\psi(W),\psi(Z)).$$

These are all the ingredients that we need. So, we are now in a position to offer an efficient algorithm.

We only care about the case where Algorithm 1 produces a empty set, so the best way to proceed in this case is as follows. We introduce a modified symmetrical group, denoted by $\mathrm{Sym'(G,g)}$, defined by

$$\mathrm{Sym'(G,g)}=\{\psi\in\mathrm{GL}(\mathfrak{g}):\ \psi^*\langle\,,\rangle=\langle\,,\rangle,\ \psi^*\mathrm{R}=\mathrm{R},\ \psi^*\nabla\mathrm{R}=\nabla\mathrm{R}\},$$

where $\langle\,,\rangle$ is induced by the metric g.

Remark 4.14. Since from Remark 4.3 it follows that $\mathrm{Isom_e(G,g)}\subset O(\mathfrak{g})$, we further deduce that

$$\mathrm{Aut(G,g)}\subset\mathrm{Isom_e(G,g)}\subset\mathrm{Sym'(G,g)}\subset\mathrm{Sym(G,g)}.$$

By computing the covariant derivative $\nabla\mathrm{R}$ for each case that we have called *different kinds* in Lemma 4.10, we find that all the modified symmetrical groups $\mathrm{Sym'(G,g)}$ are finite groups except in a single case presented in Item 8 on the solvable Lie group Sol, where $\mathrm{Sym'(G,g)}$ becomes one-dimensional. This case will be explained in more detail in the next paragraph.

Algorithm 1: Isometric automorphism groups and isotropy groups

Inputs : • $[G_i]_{i=1}^5 := \{\mathrm{Nil},\ \mathrm{SU}(2),\ \widetilde{\mathrm{PSL}}(2,\mathbb{R}),\ \mathrm{Sol},\ \widetilde{\mathrm{E}_0}(2)\}$,
• integers $K_1 \cdots K_5$,
• $\left[[\mathbf{L}_{i,j}]_{j=1}^{K_i}, i = 1 \cdots 5\right]$:= a list of symmetric endomorphisms associated to each G_i, $i = 1 \cdots 5$.

Outputs: • a list of isometric automorphism groups $\mathrm{Aut(G, g)}$,
• a list of isotropy groups $\mathrm{Isom_e(G, g)}$.

```
for i = 1 to 5 do
    foreach j ∈ {1, 2, ..., K_i} do
        kind L ← the kind of L_{i,j}          // Part 1: Isometric automorphism groups
        A[i, j] ← Aut(G_i, g) associated to kind L
        [e2, e1] := L_{i,j} · e3,  [e2, e3] := L_{i,j} · e1,  [e3, e1] := L_{i,j} · e2   // will be
          used to compute Ric
            /* B = {e1, e2, e3} is a pseudo-orthonormal basis with e3 timelike */
        Ric ←Ricci operator                   // with respect to this basis B
        kind Ric ← the kind of Ric            // Part 2: Isotropy groups
        if kind Ric = finite then
            I[i, j] ← A[i, j]                 /* if Ric is of finite kind */
        else
            if kind Ric = symmetric then
                I[i, j] ← O(2, 1)             /* if (G, g) is a symmetric space */
            else
                if kind Ric= kind L then
                    I[i, j] ← A[i, j]         /* if Ric and L have the same infinite kind
                      */
                else
                    I[i, j] ← ∅               /* to proceed in this case, we must use the
                      covariant derivative */
                end
            end
        end
    end
end
return (A, I)
```

Special case. We deal with the case when the symmetric endomorphism $\mathbf{L}$ is of kind $\{\mathrm{a}3\}$, $\mathrm{a} = 0$ on the solvable Lie group Sol which can be identified with $\mathbb{R} \ltimes_\phi \mathbb{R}^2$ equipped with the following group operation $\star$:

$$\left(\mathrm{x}, \begin{bmatrix} \mathrm{y} \\ \mathrm{z} \end{bmatrix}\right) \star \left(m, \begin{bmatrix} n \\ p \end{bmatrix}\right) = \left(\mathrm{x} + m, \begin{bmatrix} \mathrm{y} \\ \mathrm{z} \end{bmatrix} + \phi(\mathrm{x}) \begin{bmatrix} n \\ p \end{bmatrix}\right),$$

where

$$\phi(\mathrm{x}) = \begin{pmatrix} e^{\mathrm{x}} & 0 \\ 0 & e^{-\mathrm{x}} \end{pmatrix}$$

and the covering projection π is

$$\left(\mathrm{x}, \begin{bmatrix} \mathrm{y} \\ \mathrm{z} \end{bmatrix}\right) \mapsto \left(\phi(\mathrm{x}), \begin{bmatrix} \mathrm{y} \\ \mathrm{z} \end{bmatrix}\right).$$

If we denote by $\partial_{\mathrm{x}}, \partial_{\mathrm{y}}, \partial_{\mathrm{z}}$ the usual coordinate vector fields on $\mathbb{R}^3$, we can define the following left-invariant vector fields on Sol:

$$\mathrm{E}_1 = \partial_{\mathrm{x}}, \quad \mathrm{E}_2 = \exp(\mathrm{x})\partial_{\mathrm{y}}, \quad \mathrm{E}_3 = \exp(-\mathrm{x})\partial_{\mathrm{z}}, \tag{4.2}$$

so that the set $\{\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3\}$ is a basis of the Lie algebra $\mathfrak{sol}$ with non-vanishing commutation relations $[E_1, E_2] = E_2$, $[E_1, E_3] = -E_3$, and $E_1^* = d\mathrm{x}$, $E_2^* = \exp(-\mathrm{x})d\mathrm{y}$, $\mathrm{E}_3^* = \exp(\mathrm{x})\mathrm{dz}$, where E_i^* denotes the dual form to E_i.

The left-invariant Lorentzian metric g on Sol admitting $\mathbf{L}$ of kind $\{\mathrm{a3}\}$, $\mathrm{a} = 0$ as symmetric endomorphism is equivalent to the metric $\mathsf{sol7}$ (see Table 1):

$$\mathrm{g} = \exp(\mathrm{x})\mathrm{dxdz} + \exp(-2\mathrm{x})\mathrm{dy}^2. \tag{4.3}$$

The isotropy group of this metric is one-dimensional.

Indeed, we consider a one-parameter group $\xi_{\mathrm{t}} : \mathbb{R} \ltimes_\phi \mathbb{R}^2 \to \mathbb{R} \ltimes_\phi \mathbb{R}^2$ of diffeomorphisms defined by

$$\xi_{\mathrm{t}}\left(\mathrm{x}, \begin{bmatrix} \mathrm{y} \\ \mathrm{z} \end{bmatrix}\right) = \left(\mathrm{x}, \begin{bmatrix} -\frac{1}{3}\,\mathrm{t}\,\exp(3\mathrm{x}) + \mathrm{y} + \frac{1}{3}\,\mathrm{t} \\ -\frac{1}{6}\,\exp(3\mathrm{x})\,\mathrm{t}^2 + \mathrm{t}\,\mathrm{y} + \mathrm{z} + \frac{1}{6}\,\mathrm{t}^2 \end{bmatrix}\right).$$

We can easily verify that ξ_t leaves the identity element invariant and the differential of ξ_{t} with respect to the basis (4.2) has the matrix

$$\xi_{\mathrm{t}*} = \begin{pmatrix} 1 & 0 & 0 \\ -\mathrm{t}\,\exp(3\mathrm{x}) & 1 & 0 \\ \frac{-\mathrm{t}^2}{2}\,\exp(3\mathrm{x}) & \mathrm{t} & 1 \end{pmatrix}.$$

By inspection we see that ξ_{t} is itself an isometry with respect to the metric g given in (4.3). It follows that $\mathrm{Isom}_{\mathrm{e}}(\mathrm{Sol}, \mathrm{g}) = \mathbf{K} \times \mathbb{Z}_2$, since the other two connected components of $\mathrm{Sym}(\mathrm{Sol}, \mathrm{g})$ do not preserve the covariant derivative $\nabla \mathrm{R}$ and hence $\mathrm{Isom}_{\mathrm{e}}(\mathrm{Sol}, \mathrm{g}) \neq \mathrm{Aut}(\mathrm{Sol}, \mathrm{g})$.

4.2. **Results.** Let (G, g) be a Lorentzian and simply connected Lie group with $\mathbf{L}$ its symmetric endomorphism and Ric its Ricci operator. The following calculations are performed for each case listed in Lemma 4.10.

[I] Lie group: Nil

- [1] $\mathbf{L}$ is of kind $\{\mathrm{diag}\}(\alpha, 0, 0)$ (non-symmetric space).
  - ☑ Ric and $\mathbf{L}$ have the same infinite kind, then $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G}, \mathrm{g}) = \mathrm{Aut}(\mathrm{G}, \mathrm{g})$.

Result 1: $\mathrm{Aut}(\mathrm{G}, \mathrm{g}) \cong \mathrm{O}(1,1)$ and $\mathrm{Isom}(\mathrm{G}, \mathrm{g}) \cong \mathrm{Nil} \rtimes \mathrm{O}(1,1)$.

[2] $\mathbf{L}$ is of kind $\{\text{diag}\}(0,0,\gamma)$ (non-symmetric space).

☑ Ric and $\mathbf{L}$ have the same infinite kind, then $\text{Isom}_\text{e}(\text{G},\text{g}) = \text{Aut}(\text{G},\text{g})$.

Result 2: $\text{Aut}(\text{G},\text{g}) \cong \text{O}(2)$ and $\text{Isom}(\text{G},\text{g}) \cong \text{Nil} \rtimes \text{O}(2)$.

[3] $\mathbf{L}$ is of kind $\{\text{ab2}\}$ with $\text{a} = \text{b} = 0$.

☒ g is flat and (Nil,g) is a symmetric space.

Result 3: $\text{Aut}(\text{G},\text{g}) \cong \mathbf{K} \times \mathbb{Z}_2$ and $\text{Isom}(\text{G},\text{g}) \cong \text{Nil} \times \text{O}(2,1)$.

[II] Lie group: SU(2)

[1] $\mathbf{L}$ is of kind $\{\text{diag}\}(\alpha,\alpha,\gamma)$, $\alpha \neq \gamma$ (non-symmetric space).

☑ Ric and $\mathbf{L}$ have the same infinite kind, then $\text{Isom}_\text{e}(\text{G},\text{g}) = \text{Aut}(\text{G},\text{g})$.

Result 1: $\text{Aut}(\text{G},\text{g}) \cong \text{O}(2)$ and $\text{Isom}(\text{G},\text{g}) \cong \text{SU}(2) \rtimes \text{O}(2)$.

[2] $\mathbf{L}$ is of kind $\{\text{diag}\}(\alpha,\beta,\gamma)$, $\alpha \neq \beta \neq \gamma$.

(i) If $\gamma = \alpha - \beta$, then Ric is of kind $\{\text{diag}\}(a,b,b)$, $a \neq b$ (of infinite kind). Let $\Theta \in \text{Isom}^0_\text{e}(\text{G},\text{g})$ and put $T_e\Theta = \psi$; then there exists $t \in \mathbb{R}$ such that

$$M(\psi,\text{B}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(t) & \sinh(t) \\ 0 & \sinh(t) & \cosh(t) \end{pmatrix}.$$

Since $\nabla\text{R}(\psi(\text{e}_2),\psi(\text{e}_3),\psi(\text{e}_2),\psi(\text{e}_1)) - \psi(\nabla\text{R}(\text{e}_2,\text{e}_3,\text{e}_2,\text{e}_1)) = 0$, we have $2\sinh(t)(\alpha-\beta)(\alpha-2\beta) = 0$, i.e., $\sinh(t) = 0$.

☑ $\text{Isom}^0_\text{e}(\text{G},\text{g})$ is trivial. Hence $\text{Isom}_\text{e}(\text{G},\text{g}) = \text{Aut}(\text{G},\text{g})$.

(ii) If $\gamma = -(\alpha-\beta)$, then Ric is of kind $\{\text{diag}\}(a,b,a)$, $a \neq b$ (of infinite kind). Let $\Theta \in \text{Isom}^0_\text{e}(\text{G},\text{g})$ and put $T_e\Theta = \psi$; then there exists $t \in \mathbb{R}$ such that

$$M(\psi,\text{B}) = \begin{pmatrix} \cosh(t) & 0 & \sinh(t) \\ 0 & 1 & 0 \\ \sinh(t) & 0 & \cosh(t) \end{pmatrix}.$$

Since $\nabla\text{R}(\psi(\text{e}_2),\psi(\text{e}_3),\psi(\text{e}_2),\psi(\text{e}_1)) - \psi(\nabla\text{R}(\text{e}_2,\text{e}_3,\text{e}_2,\text{e}_1)) = 0$, we have $2\sinh(t)2\alpha^2(\alpha-\beta) = 0$, i.e., $\sinh(t) = 0$.

☑ $\text{Isom}^0_\text{e}(\text{G},\text{g})$ is trivial. Hence $\text{Isom}_\text{e}(\text{G},\text{g}) = \text{Aut}(\text{G},\text{g})$.

(iii) If $\gamma \neq \pm(\alpha-\beta)$,

☑ Ric is of finite kind, then $\text{Isom}_\text{e}(\text{G},\text{g}) = \text{Aut}(\text{G},\text{g})$.

Result 2: $\text{Aut}(\text{G},\text{g}) \cong \text{D}_4$ and $\text{Isom}(\text{G},\text{g}) \cong \text{SU}(2) \rtimes \text{D}_4$.

[III] Lie group: $\widetilde{\text{PSL}}(2,\mathbb{R})$

[1] $\mathbf{L}$ is of kind $\{\text{diag}\}(\alpha,\beta,\gamma)$, $\alpha \neq \beta \neq \gamma$.

(i) If $\gamma = \alpha - \beta$, then Ric is of kind $\{\text{diag}\}(a,b,b)$, $a \neq b$ (of infinite kind). Let $\Theta \in \text{Isom}^0_\text{e}(\text{G},\text{g})$ and put $T_e\Theta = \psi$; then there exists $t \in \mathbb{R}$ such that

$$M(\psi,\text{B}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(t) & \sinh(t) \\ 0 & \sinh(t) & \cosh(t) \end{pmatrix}.$$

Since $\nabla\mathrm{R}(\psi(\mathrm{e}_2),\psi(\mathrm{e}_3),\psi(\mathrm{e}_2),\psi(\mathrm{e}_1))-\psi(\nabla\mathrm{R}(\mathrm{e}_2,\mathrm{e}_3,\mathrm{e}_2,\mathrm{e}_1))=0$, we have $2\sinh(t)(\alpha-\beta)(\alpha-2\beta)=0$, i.e., $\sinh(t)=0$.

☑ $\mathrm{Isom}^0_{\mathrm{e}}(\mathrm{G},\mathrm{g})$ is trivial. Hence $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G},\mathrm{g})=\mathrm{Aut}(\mathrm{G},\mathrm{g})$.

(ii) If $\alpha=\gamma-\beta$, then Ric is of kind $\{\mathrm{diag}\}(a,a,c)$, $a\neq c$ (of infinite kind). Let $\Theta\in\mathrm{Isom}^0_{\mathrm{e}}(\mathrm{G},\mathrm{g})$ and put $T_e\Theta=\psi$; then there exists $t\in\mathbb{R}$ such that

$$M(\psi,\mathrm{B})=\begin{pmatrix}\cos(t)&\sin(t)&0\\-\sin(t)&\cos(t)&0\\0&0&1\end{pmatrix}.$$

Since $\nabla\mathrm{R}(\psi(\mathrm{e}_2),\psi(\mathrm{e}_3),\psi(\mathrm{e}_2),\psi(\mathrm{e}_1))-\psi(\nabla\mathrm{R}(\mathrm{e}_2,\mathrm{e}_3,\mathrm{e}_2,\mathrm{e}_1))=0$, we have $2\sin^3(t)\beta(\gamma-\beta)(\gamma-2\beta)=0$, i.e., $\sin(t)=0$.

☑ $\mathrm{Isom}^0_{\mathrm{e}}(\mathrm{G},\mathrm{g})$ is trivial. Hence $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G},\mathrm{g})=\mathrm{Aut}(\mathrm{G},\mathrm{g})$.

(iii) If $\gamma=-(\alpha-\beta)$, then Ric is of kind $\{\mathrm{diag}\}(a,b,a)$, $a\neq b$ (of infinite kind). Let $\Theta\in\mathrm{Isom}^0_{\mathrm{e}}(\mathrm{G},\mathrm{g})$ and put $T_e\Theta=\psi$; then there exists $t\in\mathbb{R}$ such that

$$M(\psi,\mathrm{B})=\begin{pmatrix}\cosh(t)&0&\sinh(t)\\0&1&0\\\sinh(t)&0&\cosh(t)\end{pmatrix}.$$

Since $\nabla\mathrm{R}(\psi(\mathrm{e}_2),\psi(\mathrm{e}_3),\psi(\mathrm{e}_2),\psi(\mathrm{e}_1))-\psi(\nabla\mathrm{R}(\mathrm{e}_2,\mathrm{e}_3,\mathrm{e}_2,\mathrm{e}_1))=0$, we have $2\sinh(t)2\alpha^2(\alpha-\beta)=0$, i.e., $\sinh(t)=0$.

☑ $\mathrm{Isom}^0_{\mathrm{e}}(\mathrm{G},\mathrm{g})$ is trivial. Hence $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G},\mathrm{g})=\mathrm{Aut}(\mathrm{G},\mathrm{g})$.

(iv) If $\gamma\neq\pm(\alpha-\beta)$ and $\alpha\neq\gamma-\beta$

☑ Ric is of finite kind, then $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G},\mathrm{g})=\mathrm{Aut}(\mathrm{G},\mathrm{g})$.

Result 1: $\mathrm{Aut}(\mathrm{G},\mathrm{g})\cong\mathrm{D}_4$ and $\mathrm{Isom}(\mathrm{G},\mathrm{g})\cong\widetilde{\mathrm{PSL}}(2,\mathbb{R})\rtimes\mathrm{D}_4$.

[2] **L** is of kind $\{\mathrm{diag}\}(\alpha,\alpha,\gamma)$, $\alpha\neq\gamma$ (non-symmetric space).

☑ Ric and **L** have the same infinite kind, then $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G},\mathrm{g})=\mathrm{Aut}(\mathrm{G},\mathrm{g})$.

Result 2: $\mathrm{Aut}(\mathrm{G},\mathrm{g})\cong\mathrm{O}(2)$ and $\mathrm{Isom}(\mathrm{G},\mathrm{g})\cong\widetilde{\mathrm{PSL}}(2,\mathbb{R})\rtimes\mathrm{O}(2)$.

[3] **L** is of kind $\{\mathrm{diag}\}(\alpha,\beta,\alpha)$, $\alpha\neq\beta$ (non-symmetric space).

☑ Ric and **L** have the same infinite kind, then $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G},\mathrm{g})=\mathrm{Aut}(\mathrm{G},\mathrm{g})$.

Result 3: $\mathrm{Aut}(\mathrm{G},\mathrm{g})\cong\mathrm{O}(1,1)$ and $\mathrm{Isom}(\mathrm{G},\mathrm{g})\cong\widetilde{\mathrm{PSL}}(2,\mathbb{R})\rtimes\mathrm{O}(1,1)$.

[4] **L** is of kind $\{\mathrm{diag}\}(\alpha,\beta,\beta)$, $\alpha\neq\beta$ (non-symmetric space).

☑ Ric and **L** have the same infinite kind, then $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G},\mathrm{g})=\mathrm{Aut}(\mathrm{G},\mathrm{g})$.

Result 4: $\mathrm{Aut}(\mathrm{G},\mathrm{g})\cong\mathrm{O}(1,1)$ and $\mathrm{Isom}(\mathrm{G},\mathrm{g})\cong\widetilde{\mathrm{PSL}}(2,\mathbb{R})\rtimes\mathrm{O}(1,1)$.

[5] **L** is of kind $\{\mathrm{diag}\}(\alpha,\alpha,\alpha)$, $\alpha\neq 0$ (symmetric space).

☒ Ric and **L** have the same infinite kind, then $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G},\mathrm{g})=\mathrm{Aut}(\mathrm{G},\mathrm{g})\times\{\pm\mathrm{id}\}$.

Result 5: $\mathrm{Aut}(\mathrm{G},\mathrm{g})\cong\mathrm{SO}(2,1)$ and $\mathrm{Isom}(\mathrm{G},\mathrm{g})\cong\widetilde{\mathrm{PSL}}(2,\mathbb{R})\times\mathrm{O}(2,1)$.

6 **L** is of type $\{az\bar{z}\}$.

(i) If $a \neq 2\,\mathfrak{Re}(z)$, one gets that

☑ Ric is of finite kind, then $\mathrm{Isom_e}(G, g) = \mathrm{Aut}(G, g)$.

(ii) If $a = 2\,\mathfrak{Re}(z)$, Ric is of kind $\{\mathrm{diag}\}(a', b', b')$ (of infinite kind). Let $\Theta \in \mathrm{Isom}^0_e(G, g)$ and put $T_e\Theta = \psi$; then there exists $t \in \mathbb{R}$ such that

$$M(\psi, B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(t) & \sinh(t) \\ 0 & \sinh(t) & \cosh(t) \end{pmatrix}.$$

Since $\nabla R(\psi(e_2), \psi(e_3), \psi(e_2), \psi(e_1)) - \psi(\nabla R(e_2, e_3, e_2, e_1)) = 0$, we have $4\sinh(t)\,\mathfrak{Im}(z)|z|^2 = 0$, i.e., $\sinh(t) = 0$.

☑ $\mathrm{Isom}^0_e(G, g)$ is trivial. Hence $\mathrm{Isom_e}(G, g) = \mathrm{Aut}(G, g)$.

Result 6: $\mathrm{Aut}(G, g) \cong \mathbb{Z}_2$ and $\mathrm{Isom}(G, g) \cong \widetilde{\mathrm{PSL}}(2, \mathbb{R}) \rtimes \mathbb{Z}_2$.

7 **L** is of kind $\{a3\}$, with $a \neq 0$.

☑ Ric is of finite kind, this implies that $\mathrm{Isom_e}(G, g) = \mathrm{Aut}(G, g)$.

Result 7: $\mathrm{Aut}(G, g) = \{\mathrm{id}\}$ and $\mathrm{Isom}(G, g) = \widetilde{\mathrm{PSL}}(2, \mathbb{R})$.

8 **L** is of kind $\{ab2\}$, with $a = b$ (non-symmetric space)

☑ Ric and **L** have same infinite kind, then $\mathrm{Isom_e}(G, g) = \mathrm{Aut}(G, g)$.

Result 8: $\mathrm{Aut}(G, g) \cong \mathbf{K} \times \mathbb{Z}_2$ and $\mathrm{Isom}(G, g) \cong \widetilde{\mathrm{PSL}}(2, \mathbb{R}) \rtimes (\mathbf{K} \times \mathbb{Z}_2)$.

9 **L** is of kind $\{ab2\}$, with $a \neq b \neq 0$.

(i) If $a \neq 2b$, one gets that

☑ Ric is of finite kind, then $\mathrm{Isom_e}(G, g) = \mathrm{Aut}(G, g)$.

(ii) If $a = 2b$, one gets that Ric is of kind $\{\mathrm{diag}\}(a', b', b')$ (of infinite kind). Let $\Theta \in \mathrm{Isom}^0_e(G, g)$ and put $T_e\Theta = \psi$; then there exists $t \in \mathbb{R}$ such that

$$M(\psi, B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(t) & \sinh(t) \\ 0 & \sinh(t) & \cosh(t) \end{pmatrix}.$$

Since $\nabla R(\psi(e_2), \psi(e_3), \psi(e_2), \psi(e_1)) - \psi(\nabla R(e_2, e_3, e_2, e_1)) = 0$, we have $2b\sinh(t) = 0$, i.e., $\sinh(t) = 0$.

☑ $\mathrm{Isom}^0_e(G, g)$ is trivial; this implies that $\mathrm{Isom_e}(G, g) = \mathrm{Aut}(G, g)$.

Result 9: $\mathrm{Aut}(G, g) \cong \mathbb{Z}_2$ and $\mathrm{Isom}(G, g) \cong \widetilde{\mathrm{PSL}}(2, \mathbb{R}) \rtimes \mathbb{Z}_2$.

IV Lie group: Sol

1 **L** is of kind $\{\mathrm{diag}\}(\alpha, \beta, 0)$, $\alpha \neq \beta \neq 0$ and $\alpha = -\beta$, then Ric is of kind $\{\mathrm{diag}\}(a, a, c)$, $a \neq c$ (of infinite kind). Let $\Theta \in \mathrm{Isom}^0_e(G, g)$ and put $T_e\Theta = \psi$; let $t \in \mathbb{R}$ be such that

$$M(\psi, B) = \begin{pmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\nabla R(\psi(e_2), \psi(e_3), \psi(e_2), \psi(e_1)) - \psi(\nabla R(e_2, e_3, e_2, e_1)) = 0$, we have $4\sin^3(t)\beta^3 = 0$, i.e., $\sin(t) = 0$.

☑ $\mathrm{Isom}^0_{\mathrm{e}}(\mathrm{G},\mathrm{g})$ is trivial. This implies that $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G},\mathrm{g}) = \mathrm{Aut}(\mathrm{G},\mathrm{g})$.

Result 1: $\mathrm{Aut}(\mathrm{G},\mathrm{g}) \cong \mathrm{D}_8$ and $\mathrm{Isom}(\mathrm{G},\mathrm{g}) \cong \mathrm{Sol} \rtimes \mathrm{D}_8$.

[2] **L** is of kind $\{\mathrm{diag}\}(\alpha,\beta,0)$, $\alpha \neq \beta \neq 0$ and $\alpha \neq -\beta$, then Ric is of kind $\{\mathrm{diag}\}(a,b,c)$, $a \neq b \neq c$.

☑ Ric is of finite kind. This implies that $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G},\mathrm{g}) = \mathrm{Aut}(\mathrm{G},\mathrm{g})$.

Result 2: $\mathrm{Aut}(\mathrm{G},\mathrm{g}) \cong \mathrm{D}_4$ and $\mathrm{Isom}(\mathrm{G},\mathrm{g}) \cong \mathrm{Sol} \rtimes \mathrm{D}_4$.

[3] **L** is of kind $\{\mathrm{diag}\}(\alpha,0,\gamma)$, $\alpha \neq \gamma \neq 0$.

☑ Ric is of finite kind. This implies that $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G},\mathrm{g}) = \mathrm{Aut}(\mathrm{G},\mathrm{g})$.

Result 3: $\mathrm{Aut}(\mathrm{G},\mathrm{g}) \cong \mathrm{D}_4$ and $\mathrm{Isom}(\mathrm{G},\mathrm{g}) \cong \mathrm{Sol} \rtimes \mathrm{D}_4$.

[4] **L** is of kind $\{\mathrm{diag}\}(\alpha,0,\alpha)$, $\alpha \neq 0$.

☒ g is flat and $(\mathrm{Sol},\mathrm{g})$ is a symmetric space.

Result 4: $\mathrm{Aut}(\mathrm{G},\mathrm{g}) \cong \mathrm{O}(1,1)$ and $\mathrm{Isom}(\mathrm{G},\mathrm{g}) \cong \mathrm{Sol} \times \mathrm{O}(2,1)$.

[5] **L** is of type $\{\mathrm{az}\bar{\mathrm{z}}\}$ with $\mathfrak{Re}(z) \neq 0$.

☑ Ric is of finite kind, then $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G},\mathrm{g}) = \mathrm{Aut}(\mathrm{G},\mathrm{g})$.

Result 5: $\mathrm{Aut}(\mathrm{G},\mathrm{g}) \cong \mathbb{Z}_2$ and $\mathrm{Isom}(\mathrm{G},\mathrm{g}) \cong \mathrm{Sol} \rtimes \mathbb{Z}_2$.

[6] **L** is of type $\{\mathrm{az}\bar{\mathrm{z}}\}$ with $\mathfrak{Re}(z) = 0$, one gets that Ric is of kind $\{\mathrm{diag}\}(a',b',b')$ (of infinite kind). Let $\Theta \in \mathrm{Isom}^0_{\mathrm{e}}(\mathrm{G},\mathrm{g})$ and put $T_e\Theta = \psi$; then there exists $t \in \mathbb{R}$ such that

$$M(\psi,\mathrm{B}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(t) & \sinh(t) \\ 0 & \sinh(t) & \cosh(t) \end{pmatrix}.$$

Since $\nabla\mathrm{R}(\psi(\mathrm{e}_2),\psi(\mathrm{e}_3),\psi(\mathrm{e}_2),\psi(\mathrm{e}_1)) - \psi(\nabla\mathrm{R}(\mathrm{e}_2,\mathrm{e}_3,\mathrm{e}_2,\mathrm{e}_1)) = 0$, we have $4\sinh(t)(\mathfrak{Im}(z))^3 = 0$, i.e., $\sinh(t) = 0$.

☑ $\mathrm{Isom}^0_{\mathrm{e}}(\mathrm{G},\mathrm{g})$ is trivial. This implies that $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G},\mathrm{g}) = \mathrm{Aut}(\mathrm{G},\mathrm{g})$.

Result 6: $\mathrm{Aut}(\mathrm{G},\mathrm{g}) \cong \mathrm{D}_4$ and $\mathrm{Isom}(\mathrm{G},\mathrm{g}) \cong \mathrm{Sol} \rtimes \mathrm{D}_4$.

[7] **L** is of kind $\{\mathrm{ab2}\}$, with $\mathrm{a} \neq \mathrm{b}$.

(i) If $\mathrm{a} = 0$, one gets that Ric is of kind $\{\mathrm{a'b'2}\}$ with $\mathrm{a'} = \mathrm{b'}$ (of infinite kind). Let $\Theta \in \mathrm{Isom}^0_{\mathrm{e}}(\mathrm{G},\mathrm{g})$ and put $T_e\Theta = \psi$; then there exists $t \in \mathbb{R}$ such that

$$M(\psi,\mathrm{B}) = \begin{pmatrix} 1 & -t & t \\ t & 1-(1/2)t^2 & (1/2)t^2 \\ t & -(1/2)t^2 & 1+(1/2)t^2 \end{pmatrix}.$$

Since $\nabla\mathrm{R}(\psi(\mathrm{e}_2),\psi(\mathrm{e}_2),\psi(\mathrm{e}_1),\psi(\mathrm{e}_1)) - \psi(\nabla\mathrm{R}(\mathrm{e}_2,\mathrm{e}_2,\mathrm{e}_1,\mathrm{e}_1)) = 0$, we have $2\,t\,b^2 = 0$, i.e., $t = 0$.

☑ $\mathrm{Isom}^0_{\mathrm{e}}(\mathrm{G},\mathrm{g})$ is trivial. Hence $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G},\mathrm{g}) = \mathrm{Aut}(\mathrm{G},\mathrm{g})$.

(ii) If $b = 0$, one gets that

☑ Ric is of finite kind, then $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G},\mathrm{g}) = \mathrm{Aut}(\mathrm{G},\mathrm{g})$.

Result 7: $\mathrm{Aut}(\mathrm{G},\mathrm{g}) \cong \mathbb{Z}_2$ and $\mathrm{Isom}(\mathrm{G},\mathrm{g}) \cong \mathrm{Sol} \rtimes \mathbb{Z}_2$.

[8] **L** is of kind $\{\mathrm{a3}\}$, with $\mathrm{a} = 0$, one gets that Ric is of kind $\{\mathrm{a'b'2}\}$, $\mathrm{a'} = \mathrm{b'}$ (of infinite kind); see Special case.

☒ $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G},\mathrm{g}) \neq \mathrm{Aut}(\mathrm{G},\mathrm{g})$.

Result 8: $\mathrm{Aut}(\mathrm{G},\mathrm{g}) \cong \mathbb{Z}_2$ and $\mathrm{Isom}(\mathrm{G},\mathrm{g}) \cong \mathrm{Sol} \times (\mathbf{K} \times \mathbb{Z}_2)$.

V Lie group: $\widetilde{\mathrm{E}_0}(2)$

1 **L** is of kind $\{\mathrm{diag}\}(\alpha,0,\gamma)$, $\alpha\neq\gamma\neq 0$.

(i) If $\alpha=-\gamma$, then Ric is of kind $\{\mathrm{diag}\}(a,b,a)$, $a\neq b$ (of infinite kind). Let $\Theta\in\mathrm{Isom}^0_{\mathrm{e}}(\mathrm{G},\mathrm{g})$ and put $T_e\Theta=\psi$; then there exists $t\in\mathbb{R}$ such that

$$M(\psi,\mathrm{B})=\begin{pmatrix}\cosh(t) & 0 & \sinh(t)\\ 0 & 1 & 0\\ \sinh(t) & 0 & \cosh(t)\end{pmatrix}.$$

Since $\nabla\mathrm{R}(\psi(\mathrm{e}_1),\psi(\mathrm{e}_2),\psi(\mathrm{e}_1),\psi(\mathrm{e}_1))-\psi(\nabla\mathrm{R}(\mathrm{e}_1,\mathrm{e}_2,\mathrm{e}_1,\mathrm{e}_1))=0$, we have $4\gamma^3\sinh^3(t)$, i.e., $\sinh(t)=0$.

☑ $\mathrm{Isom}^0_{\mathrm{e}}(\mathrm{G},\mathrm{g})$ is trivial, one gets $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G},\mathrm{g})=\mathrm{Aut}(\mathrm{G},\mathrm{g})$.

(ii) If $\alpha\neq-\gamma$, then Ric is of kind $\{\mathrm{diag}\}(a,b,c)$, $a\neq b\neq c$ (of finite kind),

☑ Ric is of finite kind. This implies that $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G},\mathrm{g})=\mathrm{Aut}(\mathrm{G},\mathrm{g})$.

Result 1: $\mathrm{Aut}(\mathrm{G},\mathrm{g})\cong\mathrm{D}_4$ and $\mathrm{Isom}(\mathrm{G},\mathrm{g})\cong\widetilde{\mathrm{E}_0}(2)\rtimes\mathrm{D}_4$.

2 **L** is of kind $\{\mathrm{diag}\}(\alpha,\alpha,0)$, $\alpha\neq 0$ (symmetric space)

☒ g is flat and $(\widetilde{\mathrm{E}_0}(2),\mathrm{g})$ is a symmetric space.

Result 2: $\mathrm{Aut}(\mathrm{G},\mathrm{g})\cong\mathrm{O}(2)$ and $\mathrm{Isom}(\mathrm{G},\mathrm{g})\cong\widetilde{\mathrm{E}_0}(2)\times\mathrm{O}(2,1)$.

3 **L** is of kind $\{\mathrm{diag}\}(\alpha,\beta,0)$, $\alpha\neq\beta\neq 0$.

☑ Ric is of finite kind, then $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G},\mathrm{g})=\mathrm{Aut}(\mathrm{G},\mathrm{g})$.

Result 3: $\mathrm{Aut}(\mathrm{G},\mathrm{g})\cong\mathrm{D}_4$ and $\mathrm{Isom}(\mathrm{G},\mathrm{g})\cong\widetilde{\mathrm{E}_0}(2)\rtimes\mathrm{D}_4$.

4 **L** is of kind $\{\mathrm{ab2}\}$, with $\mathrm{a}\neq\mathrm{b}$.

☑ Ric is of finite kind, then $\mathrm{Isom}_{\mathrm{e}}(\mathrm{G},\mathrm{g})=\mathrm{Aut}(\mathrm{G},\mathrm{g})$.

Result 4: $\mathrm{Aut}(\mathrm{G},\mathrm{g})\cong\mathbb{Z}_2$ and $\mathrm{Isom}(\mathrm{G},\mathrm{g})\cong\widetilde{\mathrm{E}_0}(2)\rtimes\mathbb{Z}_2$.

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