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NEW UNCERTAINTY PRINCIPLES FOR THE (k,a)-GENERALIZED WAVELET TRANSFORM

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Abstract. We present the basic (k,a)-generalized wavelet theory and prove several Heisenberg-type inequalities for this transform. After reviewing Pitt's and Beckner's inequalities for the (k,a)-generalized Fourier transform, we connect both inequalities to show a generalization of uncertainty principles for the (k,a)-generalized wavelet transform. We also present two concentration uncertainty principles, namely the Benedicks–Amrein–Berthier's uncertainty principle and local uncertainty principles. Finally, we connect these inequalities to show a generalization of the uncertainty principle of Heisenberg type and we prove the Faris–Price uncertainty principle for the (k,a)-generalized wavelet transform.

1. Introduction

The classical Fourier transform $\mathscr{F}$, initially defined on $L^1(\mathbb{R}^d)$, extends to an isometry of $L^2(\mathbb{R}^d)$ and commutes with the *rotation group*. Recently, Ben Saïd, Kobayashi and Ørsted gave in [2] a foundation of the deformation theory of the classical situation, by constructing a *generalization* $\mathscr{F}_{k,a}$ *of the Fourier transform*, commuting with *finite Coxeter groups*, and the holomorphic semigroup $\mathscr{I}_{k,a}(z)$ with infinitesimal generator

$$\mathscr{L}_{k,a} := \|x\|^{2-a}\Delta_k - \|x\|^a, \quad a>0,$$

acting on a concrete Hilbert space deforming $L^2(\mathbb{R}^d)$. Here Δ_k is the Dunkl–Laplace operator (see [11]). The deformation parameters consist of a real parameter $a>0$ coming from the interpolation of the minimal unitary representations of two different reductive groups, and a parameter k coming from Dunkl's theory of differential-difference operators [11]. As it turns out, the unitary operator $\mathscr{F}_{k,a}$ includes some known integral transforms as special cases:

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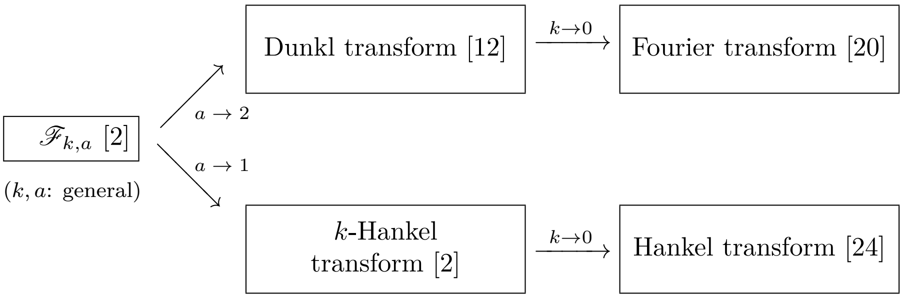
Dedicated to the Emeritus Professor Khalifa Trimèche, the first who introduced the theories of wavelet and uncertainty principles to the Tunisian school of harmonic analysis.



We underline that $\mathscr{F}_{k,a}$ has a rich structure and recently has been gaining a lot of attention (see, e.g., [3, 4, 7, 10, 16, 21, 32, 33, 34, 35]).

One of the aims of the Fourier transform is the study of the wavelet transform. This transform was introduced by Morlet in connection with his study of seismic traces. Thereafter, a more detailed study of the wavelet transform was given by Grossmann and Morlet in [17]. Such paper initiated further study of the topic. Meyer and several other mathematicians became aware of this theory, and they recognized many classical results inside it (see [25, 27, 45]). The wavelet transform has been found to be very useful in many physical and engineering applications, signal processing, seismic recordings, ground vibrations, geophysics, medical imaging, hydrology, gravitational waves, power system analysis, quantum optics and many other areas (see [5, 6]). For more details on the wavelet transform and its basic properties, we refer the reader to [9]. We also refer to [8], where the author extends wavelet theory to the setting of locally compact abelian groups, and to [36] for the wavelet transform on Gelfand pairs. We note also that the notion of the wavelet transform on hypergroups was first introduced by Trimèche in [45].

Motivated by the previous works, in [34], with the aid of the harmonic analysis associated to the (k, a)-generalized Fourier transform, we define and study the (k, a)-generalized wavelet transform $\Phi_h^{k,a}$ (see Definition 2.18).

The main objective of this paper is the study of some quantitative uncertainty principles associated with the (k, a)-generalized wavelet transform $\Phi_h^{k,a}$ on $\mathbb{R}^d$. In the classical setting, the notion of the quantitative uncertainty principles for the wavelet transform was first introduced by Wilczok [48]. Next, this subject has been extended for the generalized wavelet transforms (see [18, 28, 31, 38]).

Roughly speaking, the uncertainty principle states that a non-zero integrable function f and its Fourier transform $\mathscr{F}(f)$, cannot both be sharply localized. To make such a principle concrete, many classical qualitative uncertainty principles (Hardy, Cowling–Price, Morgan, Beurling and Miyachi, etc.), state that f and $\mathscr{F}(f)$ cannot both have arbitrarily rapid Gaussian decay, unless f is identically zero.

It is worth mentioning that "quantitative uncertainty principles" is just another name for some special inequalities. These inequalities give us information about how a function and its Fourier transform relate. They are called "quantitative uncertainty principles" since they are similar to the classical Heisenberg uncertainty principle, which has had a big part to play in the development and understanding of quantum physics. We refer the reader to the survey [13], the book [19] and the references [1, 40, 14, 22, 26, 29, 30, 41, 42, 46] for numerous versions of uncertainty principles for the Fourier transform in different settings.

In this article we investigate the previous kinds of uncertainty principles for the (k,a)-generalized wavelet transforms $\Phi_h^{k,a}$, where h is a (k,a)-generalized wavelet (see Definition 2.15). Indeed, we prove for this transform in particular the following Heisenberg-type uncertainty inequality:

Theorem A (see Theorem 3.8). *For every $p,q>0$, there exists a positive constant $M_{p,q}(k,a)$ such that for every $f\in L^2_{k,a}(\mathbb{R}^d):=L^2(\mathbb{R}^d,d\nu_{k,a}(x))$, we have*

$$\left(\int_{\mathbb{R}_+^{d+1}}\|y\|^p|\Phi_h^{k,a}(f)(b,y)|^2\,d\mu_{k,a}(b,y)\right)^{\frac{q}{p+q}} \times\left(\int_{\mathbb{R}_+^{d+1}}b^{-q}|\Phi_h^{k,a}(f)(b,y)|^2\,d\mu_{k,a}(b,y)\right)^{\frac{p}{p+q}}\geq M_{p,q}(k,a)C_h\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}.$$

Here $d\nu_{k,a}(x):=\omega_{k,a}(x)dx$, where $\omega_{k,a}$ is a weight function (see (2.2)*), the measure $d\mu_{k,a}$ is given by* (2.13)*, and C_h is the constant defined by* (2.14)*.*

Next, two weighted uncertainty principles for the transform $\Phi_h^{k,a}$ are also studied. In particular, we obtain the following Pitt's uncertainty:

Theorem B (see Theorem 4.1). *There exists a positive constant $C_{k,a}(\lambda)$ such that for any arbitrary $f\in\mathscr{S}(\mathbb{R}^d)$ and for any $0\leq\lambda<\frac{2\gamma+d+a-2}{2}$, we have*

$$C_h\int_{\mathbb{R}^d}\|\xi\|^{-2\lambda}\big|\mathscr{F}_{k,a}(f)(\xi)\big|^2\,d\nu_{k,a}(\xi) \leq C_{k,a}(\lambda)\int_{\mathbb{R}_+^{d+1}}\|y\|^{2\lambda}\big|\Phi_h^{k,a}(f)(b,y)\big|^2\,d\mu_{k,a}(b,y),$$

where $\mathscr{S}(\mathbb{R}^d)$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$ and γ is the index of the multiplicity function k (see (2.1)*).*

Using this theorem and the harmonic analysis associated with the (k,a)-generalized wavelet transform we derive the following Beckner-type uncertainty inequality for $\Phi_h^{k,a}$.

Theorem C (see Theorem 4.3). *For any function $f \in \mathscr{S}(\mathbb{R}^d)$, we have the inequality*

$$\int_{\mathbb{R}_+^{d+1}} \log\|y\|\,|\Phi_h^{k,a}(f)(b,y)|^2\,d\mu_{k,a}(b,y) + C_h \int_{\mathbb{R}^d} \log\|\xi\|\,|\mathscr{F}_{k,a}(f)(\xi)|^2\,d\nu_{k,a}(\xi) \geq \frac{2}{a}\left[\frac{\Gamma'\left(\frac{2\gamma+d+a-2}{2a}\right)}{\Gamma\left(\frac{2\gamma+d+a-2}{2a}\right)} + \log a\right] C_h \|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}.$$

(Here Γ denotes the well-known Euler's gamma function.)

Finally, the concentration uncertainty principles for the (k,a)-generalized wavelet transforms are also investigated. In fact, firstly we present the Benedicks–Amrein–Berthier uncertainty principle for the (k,a)-generalized wavelet transforms, which shows that it is impossible for a non-trivial function and its (k,a)-generalized wavelet transform to be both supported on sets of finite measure. More precisely, we prove:

Theorem D (see Theorem 5.1). *Let E_1 and E_2 be two subsets of $\mathbb{R}^d$ with finite measure. There exists a positive constant $C_{k,a}(E_1,E_2)$ such that for any arbitrary function $f \in L^2_{k,a}(\mathbb{R}^d)$, we have*

$$\int_0^\infty \int_{\mathbb{R}^d\setminus E_1} \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y) + C_h \int_{\mathbb{R}^d\setminus E_2} |\mathscr{F}_{k,a}(f)(\xi)|^2\,d\nu_{k,a}(\xi) \geq \frac{C_h\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}}{C_{k,a}(E_1,E_2)}.$$

As a corollary we derive the following general form of Heisenberg-type uncertainty inequality for the (k,a)-generalized wavelet transforms:

Corollary A (see Corollary 5.2). *Let $s,t>0$. Then there exists a positive constant $\mathfrak{C}_{k,a}(s,t)$ such that for any arbitrary function $f \in L^2_{k,a}(\mathbb{R}^d)$, we have*

$$\left(\int_{\mathbb{R}_+^{d+1}} \|y\|^{2s}\left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y)\right)^{\frac{t}{2}} \left(\int_{\mathbb{R}^d} \|\xi\|^{2t}|\mathscr{F}_{k,a}(f)(\xi)|^2\,d\nu_{k,a}(\xi)\right)^{\frac{s}{2}} \geq \mathfrak{C}_{k,a}(s,t)(C_h)^{\frac{q}{2}}\|f\|^{s+t}_{L^2_{k,a}(\mathbb{R}^d)}.$$

In our second result of the concentration uncertainty principles for the (k,a)-generalized wavelet transforms, we will be studying the local uncertainty inequalities comparing the concentration around a single point and in a subset of finite measure. More precisely, we prove:

Theorem E (see Theorem 5.4). *Let $E \subset \mathbb{R}^d$ with finite measure. Then there exists a positive constant $\mathfrak{C}(k,a,s)$ such that for any arbitrary $f \in L^2_{k,a}(\mathbb{R}^d)$ and for any $s \in \left(0, \frac{2\gamma+d+a-2}{2}\right)$, we have*

$$\int_E |\mathscr{F}_{k,a}(f)(\xi)|^2\,d\nu_{k,a}(\xi) \leq \frac{\mathfrak{C}(k,a,s)(\nu_{k,a}(E))^{\frac{2s}{2\gamma+d+a-2}}}{C_h}\left\|\|y\|^s\,\Phi_h^{k,a}(f)\right\|^2_{L^2_{\mu_{k,a}}(\mathbb{R}_+^{d+1})}.$$

Next, we also derive a general form of Heisenberg-type uncertainty inequality for the (k,a)-generalized wavelet transforms. Finally we prove the following Faris–Price's uncertainty principle for $\Phi_h^{k,a}$:

Theorem F (see Theorem 5.10). *Let η, p be two real numbers such that $p \geq 1$ and $0 < \eta < 2\gamma + d + a - 2$. Then there is a positive constant $C_{k,a}(\eta,p)$ such that for every function f in $L^2_{k,a}(\mathbb{R}^d)$ and for every measurable subset $T \subset \mathbb{R}^{d+1}_+$ such that $0 < \mu_{k,a}(T) := \int_T d\mu_{k,a}(b,y) < \infty$, we have*

$$\left(\int_T \left|\Phi_h^{k,a}(f)(b,y)\right|^p d\mu_{k,a}(b,y)\right)^{\frac{1}{p}} \leq C_{k,a}(\eta,p)\big(\mu_{k,a}(T)\big)^{\frac{1}{p(p+1)}} \left\| \, \|(\tfrac{1}{b},y)\|^\eta \Phi_h^{k,a}(f)\right\|_{L^2_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)}^{\frac{4\gamma+2d+2a-4}{(2\gamma+d+a-2+\eta)(p+1)}} \times \left(\|f\|_{L^2_{k,a}(\mathbb{R}^d)}\|h\|_{L^2_{k,a}(\mathbb{R}^d)}\right)^{\frac{(2\gamma+d+a-2+\eta)(p+1)-(4\gamma+2d+2a-4)}{(2\gamma+d+a-2+\eta)(p+1)}}.$$

The rest of the paper is organized as follows. In Section 2 we recall the main results of the harmonic analysis associated with the operator $\mathscr{L}_{k,a}$ and we present an overview of the (k,a)-generalized wavelet theory. Section 3 deals with deriving many variants of Heisenberg's inequalities for the previous transform. In Section 4 we present two weighted uncertainty principles for the (k,a)-generalized wavelet transform. Finally, the last section is devoted to studying two concentration uncertainty principles for the (k,a)-generalized wavelet transform, namely the Benedicks–Amrein–Berthier's uncertainty principle and local uncertainty principles.

2. Preliminaries

This section gives an introduction to the theory of the (k,a)-Laguerre semigroup, the (k,a)-generalized Fourier transform and the (k,a)-generalized wavelet transform. Main references are [2, 7, 16, 21, 34].

2.1. **The (k,a)-Laguerre semigroup.** We consider $\mathbb{R}^d$ with the Euclidean inner product $\langle\, , \rangle$ for which the basis $\{e_i,\ i = 1,\dots,d\}$ is orthogonal and $\|x\| = \sqrt{\langle x,x\rangle}$. For α in $\mathbb{R}^d\backslash\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α, i.e.,

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{\|\alpha\|^2}\alpha.$$

A finite set $R \subset \mathbb{R}^d\backslash\{0\}$ is called a *root system* if $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha(R) = R$ for all $\alpha \in R$. For a given root system R the reflections σ_α, $\alpha \in R$, generate a finite group $W \subset O(d)$, called the *reflection group* associated with R.

We fix a positive root system $R_+ = \{\alpha \in R : \langle \alpha, \beta\rangle > 0\}$ for some $\beta \in \mathbb{R}^d\backslash \bigcup_{\alpha\in R} H_\alpha$. We will assume that $\langle\alpha,\alpha\rangle = 2$ for all $\alpha \in R_+$. A function

$k : \mathscr{R} \to \mathbb{C}$ is called a *multiplicity function* if it is invariant under the action of the associated reflection group W. For brevity, we introduce the index

$$\gamma = \gamma(k) = \sum_{\alpha \in R_+} k(\alpha). \tag{2.1}$$

Moreover, let $\omega_{k,a}$ denote the weight function

$$\omega_{k,a}(x) = \|x\|^{a-2} \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)}, \tag{2.2}$$

which is W-invariant and homogeneous of degree $2\gamma + a - 2$. We also introduce the constant d_k given by

$$d_k := \int_{S^{d-1}} \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)} \, d\sigma(x),$$

where $d\sigma$ denotes the Lebesgue surface measure on the unit sphere S^{d-1}. The normalization constant

$$c_{k,a} = \int_{\mathbb{R}^d} e^{-\frac{\|x\|^a}{a}} \omega_{k,a}(x) \, dx$$

is needed for later use.

In the following we use this notation:

- $C^p(\mathbb{R}^d)$, the space of functions of class C^p on $\mathbb{R}^d$;
- $D(\mathbb{R}^d)$, the space of C^∞-functions on $\mathbb{R}^d$ with compact support;
- $\mathscr{S}(\mathbb{R}^d)$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$.

In this paper we assume that k is a non-negative multiplicity function satisfying

$$2\gamma + d + a - 2 > 0.$$

The Dunkl operators T_j, $j = 1, \dots, d$, on $\mathbb{R}^d$ associated with the finite reflection group W and multiplicity function k are given, for f in $C^1(\mathbb{R}^d)$ and $x \in \mathbb{R}^d_{\text{reg}} = \mathbb{R}^d \setminus \cup_{\alpha \in \mathscr{R}} H_\alpha$, by

$$T_j f(x) := \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} k(\alpha) \alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle},$$

where $\alpha_j = \langle \alpha, e_j \rangle$. The Dunkl operators form a commutative system of differential-difference operators.

We define the *Dunkl–Laplace operator* Δ_k on $\mathbb{R}^d$ for f in $C^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d_{\text{reg}}$, by

$$\Delta_k f(x) := \sum_{j=1}^d T_j^2 f(x) = \Delta f(x) + 2 \sum_{\alpha \in R^+} k(\alpha) \left(\frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle^2} \right),$$

where Δ and ∇ are, respectively, the usual Euclidean Laplacian and gradient operators on $\mathbb{R}^d$.

Notation.

- For $p \in [1,\infty]$, p' denotes the conjugate exponent of p.
- $d\nu_{k,a}(x) := \omega_{k,a}(x)dx$.
- $L^p_{k,a}(\mathbb{R}^d)$, $1 \le p \le \infty$, is the space of measurable functions on $\mathbb{R}^d$ such that

$$\|f\|_{L^p_{k,a}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |f(x)|^p \, d\nu_{k,a}(x) \right)^{\frac{1}{p}} < \infty, \quad \text{if } 1 \le p < \infty,$$
$$\|f\|_{L^\infty_{k,a}(\mathbb{R}^d)} := \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} |f(x)| < \infty.$$

For $p = 2$, we provide this space with the inner product

$$\langle f, g\rangle_{L^2_{k,a}(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x)\overline{g(x)} \, d\nu_{k,a}(x).$$

Consider the operator

$$\mathscr{L}_{k,a} := \|x\|^{2-a}\Delta_k - \|x\|^a, \quad a > 0.$$

In the following proposition we recall some spectral properties of the operator $\mathscr{L}_{k,a}$.

Proposition 2.1. *Let $a > 0$ and k be as above. Then:*

(1) *The differential-difference operator $\mathscr{L}_{k,a}$ is an essentially self-adjoint operator on $L^2_{k,a}(\mathbb{R}^d)$.*
(2) *There is no continuous spectrum of $\mathscr{L}_{k,a}$.*
(3) *The discrete spectrum of $-\mathscr{L}_{k,a}$ is given by*

$$\begin{cases} \{2na + 2m + 2\gamma + d + a - 2 : n, m \in \mathbb{N}\} & \text{if } d \ge 2, \\ \{2na + 2\gamma + a \pm 1 : n \in \mathbb{N}\} & \text{if } d = 1. \end{cases}$$

(4) *The map $\mathbb{C}^+ \times L^2_{k,a}(\mathbb{R}^d) \to L^2_{k,a}(\mathbb{R}^d)$, $(z, f) \mapsto e^{-z\mathscr{L}_{k,a}} f$, is continuous (here $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$).*
(5) *The operator norm of $e^{-z\mathscr{L}_{k,a}}$ equals $\exp\big(- \frac{1}{a}(2\gamma + d + a - 2)\,\mathrm{Re}(z)\big)$.*
(6) *If $\mathrm{Re}(z) > 0$, then $e^{-z\mathscr{L}_{k,a}}$ is a Hilbert–Schmidt operator.*
(7) *If $\mathrm{Re}(z) = 0$, then $e^{-z\mathscr{L}_{k,a}}$ is a unitary operator.*

Remark 2.2. (i) For $z \in \mathbb{C}^+$, the operator $e^{-z\mathscr{L}_{k,a}}$ is called (k, a)-*Laguerre semigroup* [2]. In the $(k, a) \equiv (0, 1)$ case, $e^{-z\mathscr{L}_{0,1}}$ is the Laguerre semigroup studied by Kobayashi and Mano [23]. In the $(k, a) \equiv (0, 2)$ case, $e^{-z\mathscr{L}_{0,2}}$ is the Hermite semigroup studied by Howe [20].

(ii) Suppose that $a > 0$ and the multiplicity function k satisfies

$$2\gamma + d > \max(1, 2 - a).$$

For $\mathrm{Re}(z) \ge 0$, the operator $e^{-z\mathscr{L}_{k,a}}$ has a distribution kernel $\Lambda_{k,a}(x, y, z)$ such that

$$e^{-z\mathscr{L}_{k,a}} f(x) = \frac{1}{c_{k,a}} \int_{\mathbb{R}^d} \Lambda_{k,a}(x, y, z) f(y) \, d\nu_{k,a}(y) \tag{2.3}$$

for $f \in L^2_{k,a}(\mathbb{R}^d)$.

2.2. **The (k,a)-generalized Fourier transform.** The object of this subsection is to recall the main properties of the (k,a)-generalized Fourier transform $\mathscr{F}_{k,a}$ given by

$$\mathscr{F}_{k,a} := e^{\frac{i\pi}{2a}(2\gamma+d+a-2)} \exp\left(\frac{i\pi}{2a}\mathscr{L}_{k,a}\right). \tag{2.4}$$

This is a unitary operator on the space $L^2_{k,a}(\mathbb{R}^d)$. Some notable special cases include:

- $a=2$, $k\equiv 0$: $\mathscr{F}_{k,a}$ is the Euclidean Fourier transform (see [20]).
- $a=2$, $k>0$: we recover the Dunkl transform (see [12]).
- $a=1$, $k\equiv 0$: $\mathscr{F}_{k,a}$ is the Hankel transform and it appears in [24] as the unitary inversion operator of the Schrödinger model of the minimal representation of the group $O(d+1,2)$.
- $a=1$, $k>0$: we recover the k-Hankel transform (see [3, 4]). In fact, the unitary operator $\mathscr{H}_k := \mathscr{F}_{k,1}$ may be regarded as the Dunkl analogue of the Hankel-type transform $\mathscr{F}_{0,1}$. This unitary operator $\mathscr{H}_k$ can be written by means of the Dunkl intertwining operator V_k and the classical Bessel functions. Recently in [3, 4] the authors have studied the harmonic analysis associated with this transform.

Let us collect the main properties of the (k,a)-generalized Fourier transform (cf. [2]).

Proposition 2.3. *Let $a>0$ and k be as above.*

(i) Plancherel's theorem for $\mathscr{F}_{k,a}$. *The (k,a)-generalized Fourier transform $\mathscr{F}_{k,a}$ is an isometric isomorphism on $L^2_{k,a}(\mathbb{R}^d)$ and we have*

$$\int_{\mathbb{R}^d} |f(x)|^2 \, d\nu_{k,a}(x) = \int_{\mathbb{R}^d} |\mathscr{F}_{k,a}(f)(\lambda)|^2 \, d\nu_{k,a}(\lambda). \tag{2.5}$$

(ii) Parseval's formula for $\mathscr{F}_{k,a}$. *For all f,g in $L^2_{k,a}(\mathbb{R}^d)$ we have*

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)} \, d\nu_{k,a}(x) = \int_{\mathbb{R}^d} \mathscr{F}_{k,a}(f)(\lambda)\overline{\mathscr{F}_{k,a}(g)(\lambda)} \, d\nu_{k,a}(\lambda).$$

(iii) Inversion formula. *The (k,a)-generalized Fourier transform $\mathscr{F}_{k,a}$ is of finite order if and only if $a\in\mathbb{Q}$. If $a\in\mathbb{Q}$ is of the form $a=\frac{s}{t}$, with s,t positive, then $\mathscr{F}^{2s}_{k,a}=\mathrm{Id}$. In particular,*

$$\mathscr{F}^{-1}_{k,a} = \mathscr{F}^{2s-1}_{k,a}.$$

By the Schwartz kernel theorem there exists a distribution kernel $B_{k,a}(\lambda,x)$ such that

$$\mathscr{F}_{k,a}(f)(\lambda) = \frac{1}{c_{k,a}} \int_{\mathbb{R}^d} f(x) B_{k,a}(x,\lambda) \, d\nu_{k,a}(x), \quad \text{for all } \lambda\in\mathbb{R}^d. \tag{2.6}$$

Using the relations (2.3), (2.4) and (2.6), it is easy to see that

$$B_{k,a}(x,y) = e^{\frac{i\pi}{2a}(2\gamma+d+a-2)} \Lambda_{k,a}\left(x,y,\frac{i\pi}{2}\right).$$

In general no closed expression for $B_{k,a}(x,y)$ is available. The paper [2] lists explicit formulae whenever $d=1$ and $a>0$ is arbitrary, or whenever $d\geq 2$ is arbitrary and $a\in\{1,2\}$. The explicit expression of the kernel $B_{k,a}(x,y)$ in the previous special cases is given by the following proposition.

Proposition 2.4. (i) *Suppose that $d=1$ and $a>0$. For $d=1$, there is but a single choice of root system, $\mathscr{R}=\{\pm 1\}$ (up to scaling), and $W=\mathbb{Z}_2$, as well as $k>\frac{1}{2}(1-a)$. In this case we have*

$$B_{k,a}(x,y)=\Gamma\left(\frac{2k+a-1}{a}\right)\left(\widetilde{J}_{\frac{2k-1}{a}}\left(\frac{2}{a}|xy|^{\frac{a}{2}}\right)+\frac{xy}{(ia)^{\frac{2}{a}}}\widetilde{J}_{\frac{2k+1}{a}}\left(\frac{2}{a}|xy|^{\frac{a}{2}}\right)\right),$$

where $\widetilde{J}_\nu$ is the normalized Bessel function given by

$$\widetilde{J}_\nu(t)=\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{2^{2n}n!\Gamma(n+\nu+1)}. \tag{2.7}$$

(ii) *Suppose that $d\geq 2$. In the polar coordinates $x=r\omega$ and $y=s\eta$, the kernel $B_{k,a}(x,y)$ is given by*

$$B_{k,a}(x,y)=\begin{cases}\Gamma\left(\gamma+\frac{d-1}{2}\right)\left(\widetilde{V_{k,a}}\left(\widetilde{J}_{\gamma+\frac{d-3}{2}}\left(\sqrt{2rs(1+\cdot)}\right)\right)\right)(\omega,\eta) & \text{if } a=1,\\ \left(\widetilde{V_{k,a}}(e^{-irs\cdot})\right)(\omega,\eta) & \text{if } a=2.\end{cases}$$

Here $\widetilde{V_{k,a}}$ is defined as follows, if h is a continuous function of one variable:

$$\widetilde{V_{k,a}}(h)(x,y)=\int_{\mathbb{R}^d}h(\langle z,y\rangle)\,d\zeta_x^k(z),$$

where $d\zeta_x^k$ is a positive probability measure on $\mathbb{R}^d$, with support in the closed ball $\overline{\mathscr{B}}_d(0,\|x\|)$ of center 0 and radius $\|x\|$ (see [39]).

We continue by stating basic properties of the kernel $B_{k,a}$ of the (k,a)-generalized Fourier transform $\mathscr{F}_{k,a}$.

Proposition 2.5. (i) *The distribution $B_{k,a}(\cdot,\cdot)$ solves the following differential-difference equations on $\mathbb{R}^d\times\mathbb{R}^d$:*

$$\begin{cases}\|\lambda\|^{2-a}\Delta_k^\lambda B_{k,a}(\lambda,x)=-\|x\|^a B_{k,a}(\lambda,x),\\ \|x\|^{2-a}\Delta_k^x B_{k,a}(\lambda,x)=-\|\lambda\|^a B_k(\lambda,x).\end{cases}$$

Here, the superscript in Δ_k^x, etc. indicates the relevant variable.

(ii) *For $z,t\in\mathbb{C}^d$, we have $B_{k,a}(z,t)=B_{k,a}(t,z)$, $B_{k,a}(z,0)=1$, and, for all $\lambda>0$, $B_{k,a}(\lambda z,t)=B_{k,a}(z,\lambda t)$.*

Remark 2.6. When $d=1$, the (k,a)-generalized Fourier transform $\mathscr{F}_{k,a}$ provides a natural generalization of the Hankel transform. Indeed, if we set

$$\begin{aligned}B_{k,a}^{\text{even}}(x,y)&=\frac{1}{2}(B_{k,a}(x,y)+B_{k,a}(x,-y))\\ &=\Gamma\left(\frac{2k+a-1}{a}\right)\widetilde{J}_{\frac{2k-1}{a}}\left(\frac{2}{a}|xy|^{\frac{a}{2}}\right),\end{aligned}$$

then the transform $\mathscr{F}_{k,a}$ of an even function f on the real line specializes to a Hankel-type transform on $\mathbb{R}_+$. For this reason, the generalized Fourier transform $\mathscr{F}_{k,1}$ is called *k-Hankel transform.*

The following proposition plays a significant role in the next sections.

Proposition 2.7 ([2, 7, 21])**.** *Assume that $d \geq 1$, $k \geq 0$, $2\gamma + d + a - 2 > 0$, and that exactly one of the following additional assumptions holds:*

(i) $a \in \{1, 2\}$,
(ii) $d \geq 2$ *and* $a = \frac{2}{n}$, $n \in \mathbb{N}$.

Then, for all $x, y \in \mathbb{R}^d$, we have

$$|B_{k,a}(x, y)| \leq 1.$$

When $d = 1$ and $a > 0$, there exists a finite positive constant C, depending only on a and k, such that

$$\forall\, x, y \in \mathbb{R}, \quad |B_{k,a}(x, y)| \leq C.$$

Convention: When $d = 1$ and $a > 0$, we shall replace $B_{k,a}$ by the rescaled version $B_{k,a}/C$ but continue to use the same symbol $B_{k,a}$.

Remark 2.8. (i) The previous proposition implies that the (k, a)-generalized Fourier transform $\mathscr{F}_{k,a}$ is bounded on the space $L^1_{k,a}(\mathbb{R}^d)$, and we have

$$\|\mathscr{F}_{k,a}(f)\|_{L^\infty_{k,a}(\mathbb{R}^d)} \leq \frac{1}{c_{k,a}} \|f\|_{L^1_{k,a}(\mathbb{R}^d)},$$

for all f in $L^1_{k,a}(\mathbb{R}^d)$.

(ii) When $f(x) = F(\|x\|)$ is a radial function on $\mathbb{R}^d$ and belongs to $L^1_{k,a}(\mathbb{R}^d)$, we have

$$\forall \lambda \in \mathbb{R}^d, \quad \mathscr{F}_{k,a}(f)(\lambda) = a^{-\left(\frac{2\gamma+d-2}{a}\right)} \mathscr{F}_{B,a}^{\frac{2\gamma+d-2}{a}}(F)(\|\xi\|), \tag{2.8}$$

where $\mathscr{F}^\nu_{B,a}$ is the a-deformed Hankel transform of one variable defined by

$$\mathscr{F}^\nu_{B,a}(\psi)(s) := \int_0^\infty \psi(r) \widetilde{J}_\nu\Big(\frac{2}{a}(rs)^{\frac{a}{2}}\Big) r^{a(\nu+1)-1} dr,$$

for a function ψ defined on $\mathbb{R}_+$. Here, $\widetilde{J}_\nu$ is the normalized Bessel function given by (2.7).

In the rest of this paper, we assume that d, k and a meet the assumptions in Proposition 2.7. We now recall, clarify and improve the definition and the properties of the (k, a)-generalized translation operator given in [34].

Definition 2.9. Let $x \in \mathbb{R}^d$. The (k, a)*-generalized translation operator* $f \mapsto \tau_x f$ is defined on $L^2_{k,a}(\mathbb{R}^d)$ by

(i) When $d \geq 1$ and $\frac{2}{a} \in \mathbb{N}$,

$$\mathscr{F}_{k,a}(\tau_x f)(\xi) = \begin{cases} B_{k,a}(\xi, x)\mathscr{F}_{k,a}(f)(\xi) & \text{if } a = \frac{1}{r},\ r \in \mathbb{N}, \\ B_{k,a}(-\xi, x)\mathscr{F}_{k,a}(f)(\xi) & \text{if } a = \frac{2}{2r+1},\ r \in \mathbb{N}_0. \end{cases} \tag{2.9}$$

(ii) When $d=1$ and $\frac{2}{a}\in\mathbb{R}_+\backslash\mathbb{N}$,

$$\mathscr{F}_{k,a}(\tau_x f)(\xi)=\overline{B_{k,a}(\xi,x)}\mathscr{F}_{k,a}(f)(\xi). \tag{2.10}$$

It is useful to have a class of functions in which (2.10) holds pointwise. One such class is given by the generalized Wigner space $\mathscr{W}_{k,a}(\mathbb{R}^d)$:

$$\mathscr{W}_{k,a}(\mathbb{R}^d):=\left\{f\in L^1_{k,a}(\mathbb{R}^d):\mathscr{F}_{k,a}(f)\in L^1_{k,a}(\mathbb{R}^d)\right\}.$$

We now give several properties of the generalized translation operator.

Proposition 2.10. (i) *Let f be in $L^2_{k,n}(\mathbb{R}^d)$. We have*

$$\|\tau_x f\|_{L^2_{k,a}(\mathbb{R}^d)}\le\|f\|_{L^2_{k,a}(\mathbb{R}^d)},\quad \forall x\in\mathbb{R}^d.$$

(ii) *For all f in $\mathscr{W}_{k,a}(\mathbb{R}^d)$ or for all f in $L^2_{k,a}(\mathbb{R}^d)$ such that $\mathscr{F}_{k,a}(f)$ belongs to $L^1_{k,a}(\mathbb{R}^d)$ and $x\in\mathbb{R}^d$, we have, for almost every $y\in\mathbb{R}^d$,*

$$\tau_x f(y)=\begin{cases}\frac{1}{c_{k,a}}\displaystyle\int_{\mathbb{R}^d}B_{k,a}(\xi,x)B_{k,a}(\xi,y)\mathscr{F}_{k,a}(f)(\xi)\,d\nu_{k,a}(\xi) & \text{if } a=\frac{1}{r},\ r\in\mathbb{N},\\ \frac{1}{c_{k,a}}\displaystyle\int_{\mathbb{R}^d}B_{k,a}(-\xi,x)B_{k,a}(-\xi,y)\mathscr{F}_{k,a}(f)(\xi)\,d\nu_{k,a}(\xi) & \text{if } a=\frac{2}{2r+1},\ r\in\mathbb{N}_0.\end{cases}$$

(iii) *For all f in $\mathscr{W}_{k,a}(\mathbb{R}^d)$ and for all $x,y\in\mathbb{R}^d$, we have*

$$\tau_x f(y)=\tau_y(f)(x).$$

Remark 2.11. We note that the definition of the (k,a)-generalized translation operator given in [34] is valid when d, k and a meet the assumptions in Proposition 2.4 and $\frac{2}{a}\in\mathbb{N}$. Thus, Definition 2.9 generalizes and improves the above definition of the (k,a)-generalized translation operator given in [34].

Using the (k,a)-generalized translation operator, we define the (k,a)-generalized convolution product as follows.

Definition 2.12. (i) When $d\ge1$ and $\frac{2}{a}:=n\in\mathbb{N}$, the (k,a)-*generalized convolution product* of two suitable functions f and g is defined by

$$\forall\, x\in\mathbb{R}^d,\quad f*_{k,a}g(x)=\frac{1}{c_{k,a}}\int_{\mathbb{R}^d}\tau_x f((-1)^n y)g(y)\,d\nu_{k,a}(y). \tag{2.11}$$

(ii) When $d=1$ and $\frac{2}{a}\in\mathbb{Q}_+\backslash\mathbb{N}$, the (k,a)-*generalized convolution product* of $f\in L^2_{k,a}(\mathbb{R}^d)$ and $g\in L^1_{k,a}(\mathbb{R}^d)$ is the function $f*_{k,a}g$ of $L^2_{k,a}(\mathbb{R}^d)$ satisfying

$$\mathscr{F}_{k,a}(f*_{k,a}g)=\mathscr{F}_{k,a}(f)\mathscr{F}_{k,a}(g). \tag{2.12}$$

When $d\ge1$ and $\frac{2}{a}\in\mathbb{N}$, we have the following properties.

Proposition 2.13. (i) *Let f and g be in $L^2_{k,a}(\mathbb{R}^d)$. Then the function $f*_{k,a}g$ belongs to $L^2_{k,a}(\mathbb{R}^d)$ if and only if the function $\mathscr{F}_{k,a}(f)\mathscr{F}_{k,a}(g)$ is in $L^2_{k,a}(\mathbb{R}^d)$, and we have*

$$\mathscr{F}_{k,a}(f*_{k,a}g)=\mathscr{F}_{k,a}(f)\mathscr{F}_{k,a}(g),$$

in the L^2 case.

(ii) *Let f and g be in $L^2_{k,a}(\mathbb{R}^d)$. Then we have*

$$\int_{\mathbb{R}^d} |f *_{k,a} g(x)|^2 \, d\nu_{k,a}(x) = \int_{\mathbb{R}^d} |\mathscr{F}_{k,a}(f)(\xi)|^2 |\mathscr{F}_{k,a}(g)(\xi)|^2 \, d\nu_{k,a}(\xi),$$

whenever both sides are finite.

2.3. **Basic (k,a)-generalized wavelet theory.** In this subsection we recall, clarify and improve some results introduced and proved in [34].

Notation.

- $\mathbb{R}^{d+1}_+ = \{(b,y) = (b, y_1, \dots, y_d) \in \mathbb{R}^{d+1},\, b > 0\}$.
- $L^p_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)$, with $p \in [1,\infty]$, is the space of measurable functions f on $\mathbb{R}^{d+1}_+$ such that

$$\|f\|_{L^p_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)} := \left(\int_{\mathbb{R}^{d+1}_+} |f(b,y)|^p \, d\mu_{k,a}(b,y) \right)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty,$$

$$\|f\|_{L^\infty_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)} := \operatorname*{ess\,sup}_{(b,y)\in\mathbb{R}^{d+1}_+} |f(b,y)| < \infty,$$

where the measure $d\mu_{k,a}$ is defined by

$$\forall\, (b,y) \in \mathbb{R}^{d+1}_+, \quad d\mu_{k,a}(b,y) = \frac{d\nu_{k,a}(y)db}{b^{2\gamma+d+a-1}}. \tag{2.13}$$

Let $b > 0$. The *dilation operator* Δ_b of a measurable function h is defined by

$$\forall\, y \in \mathbb{R}^d, \quad \Delta_b h(y) := \frac{1}{b^{\frac{2\gamma+d+a-2}{2}}} h\left(\frac{y}{b}\right).$$

This operator satisfies the following

Proposition 2.14. (i) *For all b, c in $(0,\infty)$, we have*

$$\Delta_b \Delta_c = \Delta_{bc}.$$

(ii) *Let $b > 0$. For all h in $L^2_{k,a}(\mathbb{R}^d)$, the function $\Delta_b h$ belongs to $L^2_{k,a}(\mathbb{R}^d)$ and we have*

$$\|\Delta_b h\|_{L^2_{k,a}(\mathbb{R}^d)} = \|h\|_{L^2_{k,a}(\mathbb{R}^d)}$$

and

$$\forall\, \xi \in \mathbb{R}^d, \quad \mathscr{F}_{k,a}(\Delta_b h)(\xi) = b^{\frac{2\gamma+d+a-2}{2}} \mathscr{F}_{k,a}(h)(b\xi).$$

(iii) *Let $b > 0$. For all h, g in $L^2_{k,a}(\mathbb{R}^d)$, we have*

$$\langle \Delta_b h, g \rangle_{L^2_{k,a}(\mathbb{R}^d)} = \langle h, \Delta_{\frac{1}{b}} g \rangle_{L^2_{k,a}(\mathbb{R}^d)}.$$

(iv) *Let $b > 0$ and $y \in \mathbb{R}^d$. We have*

$$\Delta_b \tau_y = \tau_{by}\, \Delta_b.$$

Definition 2.15. A (k,a)-generalized wavelet on $\mathbb{R}^d$ is a measurable function h on $\mathbb{R}^d$ satisfying, for almost all $\xi \in \mathbb{R}^d$, the condition

$$0 < C_h := \int_0^\infty |\mathscr{F}_{k,a}(h)(b\xi)|^2 \, \frac{db}{b} < \infty. \tag{2.14}$$

Example 2.16. The function α_t, $t > 0$, defined on $\mathbb{R}^d$ by

$$\alpha_t(x) = \frac{1}{(2t)^{\frac{2\gamma+d+a-2}{a}}} e^{-\frac{\|x\|^a}{2at}},$$

satisfies

$$\forall\, \xi \in \mathbb{R}^d, \quad \mathscr{F}_{k,a}(\alpha_t)(\xi) = e^{-\frac{2t}{a}\|\xi\|^a}.$$

The function h, defined by $h(x) = -\dfrac{d}{dt}\alpha_t(x)$, is a (k,a)-generalized wavelet on $\mathbb{R}^d$.

Let $b > 0$ and let h be a (k,a)-generalized wavelet in $L^2_{k,a}(\mathbb{R}^d)$. When $d \geq 1$ and $\frac{2}{a} := n \in \mathbb{N}$, we consider the family $h_{b,y}$, $y \in \mathbb{R}^d$, of functions in $L^2_{k,a}(\mathbb{R}^d)$ defined by

$$\forall\, x \in \mathbb{R}^d, \quad h_{b,y}(x) := \tau_{(-1)^n y}(\Delta_b h)(x),$$

where τ_y, $y \in \mathbb{R}^d$, are the (k,a)-generalized translation operators given by (2.9).

Remark 2.17. Let $b > 0$ and let h be in $L^2_{k,a}(\mathbb{R}^d)$. We have

$$\forall\, (b,y) \in \mathbb{R}^{d+1}_+, \quad \|h_{b,y}\|_{L^2_{k,a}(\mathbb{R}^d)} \leq \|h\|_{L^2_{k,a}(\mathbb{R}^d)}. \tag{2.15}$$

Definition 2.18. Let h be a (k,a)-generalized wavelet in $L^2_{k,a}(\mathbb{R}^d)$. When $d \geq 1$ and $\frac{2}{a} \in \mathbb{N}$, the (k,a)-generalized continuous wavelet transform $\Phi^{k,a}_h$ is defined for regular functions f on $\mathbb{R}^d$ by

$$\Phi^{k,a}_h(f)(b,y) = \frac{1}{c_{k,a}} \int_{\mathbb{R}^d} f(y)\overline{h_{b,y}(x)}\, d\nu_{k,a}(x) = \frac{1}{c_{k,a}} \langle f, h_{b,y}\rangle_{L^2_{k,a}(\mathbb{R}^d)}, \tag{2.16}$$

for all $(b,y) \in \mathbb{R}^{d+1}_+$.

This transform can also be written in the form

$$\Phi^{k,a}_h(f)(b,y) = f *_{k,a} \overline{\Delta_b h}(y),$$

where $*_{k,a}$ is the (k,a)-generalized convolution product given by (2.11).

Similarly, when $d = 1$, $\frac{2}{a} \in \mathbb{Q}_+\backslash\mathbb{N}$, and h is a (k,a)-generalized wavelet in $L^1_{k,a}(\mathbb{R}^d) \cap L^2_{k,a}(\mathbb{R}^d)$, the (k,a)-generalized continuous wavelet transform $\Phi^{k,a}_h$ is defined for regular functions $f \in L^2_{k,a}(\mathbb{R}^d)$ by

$$\Phi^{k,a}_h(f)(b,y) = f *_{k,a} \overline{\Delta_b h}(y),$$

where $*_{k,a}$ is the (k,a)-generalized convolution product given by (2.12).

Remark 2.19. We note that the definition of the (k,a)-generalized continuous wavelet transform given in [34] is valid when d, k and a meet the assumptions in Proposition 2.4 and $\frac{2}{a} \in \mathbb{N}$. Thus, Definition 2.18 generalizes and improves the above definition of the (k,a)-generalized continuous wavelet transform given in [34]. We also note that if we consider the (k,a)-generalized wavelet transform given above, the proofs (with slight changes) and results of [34, Subsection 3.2] are true.

Henceforth, h will denote an arbitrary (k,a)-generalized wavelet, in $L^2_{k,a}(\mathbb{R}^d)$ when $d \geq 1$ and $\frac{2}{a} \in \mathbb{N}$, and in $L^1_{k,a}(\mathbb{R}^d) \cap L^2_{k,a}(\mathbb{R}^d)$ when $d = 1$ and $\frac{2}{a} \in \mathbb{Q}_+\backslash\mathbb{N}$.

Theorem 2.20 (Parseval's formula for $\Phi_h^{k,a}$)**.** *Let h be a (k,a)-generalized wavelet. Then, for all f and g in $L^2_{k,a}(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^{d+1}_+} \Phi_h^{k,a}(f)(b,y)\overline{\Phi_h^{k,a}(g)(b,y)}\, d\mu_{k,a}(b,y) = C_h \int_{\mathbb{R}^d} f(x)\overline{g(x)}\, d\nu_{k,a}(x). \tag{2.17}$$

Proof. With slight changes, the proof of inequality (2.17) can be obtained along the lines of [34]. □

Corollary 2.21 (Plancherel's formula for $\Phi_h^{k,a}$)**.** *Let h be a (k,a)-generalized wavelet. Then, for all f in $L^2_{k,a}(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^{d+1}_+} |\Phi_h^{k,a}(f)(b,y)|^2\, d\mu_{k,a}(b,y) = C_h \int_{\mathbb{R}^d} |f(x)|^2\, d\nu_{k,a}(x). \tag{2.18}$$

By simple calculations we prove the following

Lemma 2.22. *For any $f \in L^2_{k,a}(\mathbb{R}^d)$, we have*

$$\mathscr{F}_{k,a}\big(\Phi_h^{k,a}(f)(b,\cdot)\big)(\xi) = b^{\frac{2\gamma+d+a-2}{2}} \mathscr{F}_{k,a}(\overline{h})(b\xi)\mathscr{F}_{k,a}(f)(\xi). \tag{2.19}$$

We close this section with the following result when $d \geq 1$ and $\frac{2}{a} \in \mathbb{N}$:

Proposition 2.23. *Let h be a (k,a)-generalized wavelet in $L^2_{k,a}(\mathbb{R}^d)$. For any f in $L^2_{k,a}(\mathbb{R}^d)$ and for any $t > 0$, we have*

$$\forall\, (b,y) \in \mathbb{R}^{d+1}_+, \quad \Phi_h^{k,a}(\Delta_t f)(b,y) = \Phi_h^{k,a}(f)\left(\frac{b}{t}, \frac{y}{t}\right). \tag{2.20}$$

Proof. We assume that $d \geq 1$ and $\frac{2}{a} := n \in \mathbb{N}$. Using Proposition 2.14 and formula (2.16), we have

$$\begin{aligned}
\Phi_h^{k,a}(\Delta_t f)(b,y) &= \frac{1}{c_{k,a}} \left\langle \Delta_t f, \tau_{(-1)^n y}\Delta_b h\right\rangle = \frac{1}{c_{k,a}} \left\langle f, \Delta_{\frac{1}{t}}\, \tau_{(-1)^n y}\Delta_b h\right\rangle \\
&= \frac{1}{c_{k,a}} \left\langle f, \tau_{\frac{(-1)^n y}{t}}\, \Delta_{\frac{1}{t}}\Delta_b h\right\rangle = \frac{1}{c_{k,a}} \left\langle f, \tau_{\frac{(-1)^n y}{t}}\, \Delta_{\frac{b}{t}} h\right\rangle \\
&= \Phi_h^{k,a}(f)\left(\frac{b}{t}, \frac{y}{t}\right).
\end{aligned}$$

□

Remark 2.24. Let h be a (k,a)-generalized wavelet in $L^2_{k,a}(\mathbb{R}^d)$. Using (2.16), the Cauchy–Schwarz inequality and (2.15), we get, for all f in $L^2_{k,a}(\mathbb{R}^d)$,

$$\|\Phi_h^{k,a}(f)\|_{L^\infty_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)} \leq \frac{1}{c_{k,a}} \|f\|_{L^2_{k,a}(\mathbb{R}^d)} \|h\|_{L^2_{k,a}(\mathbb{R}^d)}. \tag{2.21}$$

By the Riesz–Thorin interpolation theorem we obtain the following

Proposition 2.25. *Let h be a (k,a)-generalized wavelet in $L^2_{k,a}(\mathbb{R}^d)$, $f \in L^2_{k,a}(\mathbb{R}^d)$ and $p \in [2,\infty]$. We have*

$$\|\Phi_h^{k,a}(f)\|_{L^p_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)} \leq (C_h)^{\frac{1}{p}} \left(\frac{\|h\|_{L^2_{k,a}(\mathbb{R}^d)}}{c_{k,a}}\right)^{\frac{p-2}{p}} \|f\|_{L^2_{k,a}(\mathbb{R}^d)}.$$

3. Heisenberg-type uncertainty inequalities for the (k,a)-generalized wavelet transform

In this section, we establish many versions of Heisenberg-type inequalities for the (k,a)-generalized wavelet transform.

3.1. L^2 Heisenberg-type uncertainty inequalities for $\Phi_h^{k,a}$. We recall the uncertainty principle of Heisenberg type for the (k,a)-generalized Fourier transform.

Proposition 3.1 (see [21])**.** *If $s,t>0$, then there exists a positive constant $\mathscr{C}_{k,a}(s,t)$ such that, for every $f\in L^2_{k,a}(\mathbb{R}^d)$, we have the inequality*

$$\Big\|\|\xi\|^t\mathscr{F}_{k,a}(f)\Big\|^{\frac{s}{s+t}}_{L^2_{k,a}(\mathbb{R}^d)}\Big\|\|x\|^s f\Big\|^{\frac{t}{s+t}}_{L^2_{k,a}(\mathbb{R}^d)}\geq \mathscr{C}_{k,a}(s,t)\|f\|_{L^2_{k,a}(\mathbb{R}^d)}. \tag{3.1}$$

For $s,t\geq\frac{a}{2}$, we have $\mathscr{C}_{k,a}(s,t)=\left(\frac{2\gamma+d+a-2}{2}\right)^{\frac{2st}{a(s+t)}}$.

Theorem 3.2. *Let $s,t>0$ and let h be a (k,a)-generalized wavelet on $\mathbb{R}^d$ in $L^2_{k,a}(\mathbb{R}^d)$. Then, for all f in $L^2_{k,a}(\mathbb{R}^d)$, we have the inequality*

$$\Big\|\|y\|^s\Phi_h^{k,a}(f)\Big\|^{\frac{t}{t+s}}_{L^2_{\mu_{k,a}}\left(\mathbb{R}^{d+1}_+\right)}\Big\|\|\xi\|^t\mathscr{F}_{k,a}(f)\Big\|^{\frac{s}{s+t}}_{L^2_{k,a}(\mathbb{R}^d)}\geq \mathscr{C}_{k,a}(s,t)(C_h)^{\frac{t}{2(s+t)}}\|f\|_{L^2_{k,a}(\mathbb{R}^d)}, \tag{3.2}$$

where $\mathscr{C}_{k,a}(s,t)$ is the positive constant given in Proposition 3.1.

Proof. Let us assume the non-trivial case that both integrals on the left hand side of (3.2) are finite. We get from the condition (2.14) for h that

$$\int_{\mathbb{R}^{d+1}_+}\|\xi\|^{2t}|\mathscr{F}_{k,a}(h)(b\xi)|^2\,|\mathscr{F}_{k,a}(f)(\xi)|^2\,\frac{d\nu_{k,a}(\xi)db}{b} = C_h\int_{\mathbb{R}^d}\|\xi\|^{2t}|\mathscr{F}_{k,a}(f)(\xi)|^2\,d\nu_{k,a}(\xi).$$

Using the relation (2.19), we obtain

$$\int_0^\infty\int_{\mathbb{R}^d}\|\xi\|^{2t}|\mathscr{F}_{k,a}[\Phi_h^{k,a}(f)(b,\cdot)](\xi)|^2\,d\mu_{k,a}(b,\xi) = C_h\int_{\mathbb{R}^d}\|\xi\|^{2t}|\mathscr{F}_{k,a}(f)(\xi)|^2\,d\nu_{k,a}(\xi). \tag{3.3}$$

On the other hand, Heisenberg's inequality for the (k,a)-generalized Fourier transform (3.1) implies that for any $b>0$, we have

$$\left(\int_{\mathbb{R}^d}\|\xi\|^{2t}|\mathscr{F}_{k,a}\big[\Phi_h^{k,a}(f)(b,\cdot)\big](\xi)|^2\,d\nu_{k,a}(\xi)\right)^{\frac{s}{s+t}}$$
$$\times\left(\int_{\mathbb{R}^d}\|y\|^{2s}|\Phi_h^{k,a}(f)(b,y)|^2\,d\nu_{k,a}(y)\right)^{\frac{t}{s+t}}$$
$$\geq(\mathscr{C}_{k,a}(s,t))^2\int_{\mathbb{R}^d}|\Phi_h^{k,a}(f)(b,y)|^2\,d\nu_{k,a}(y).$$

Integrating with respect to $\dfrac{db}{b^{2\gamma+d+a-1}}$, we obtain

$$\int_0^\infty\left[\left(\int_{\mathbb{R}^d}\|\xi\|^{2t}|\mathscr{F}_{k,a}\big[\Phi_h^{k,a}(f)(b,\cdot)\big](\xi)|^2\,d\nu_{k,a}(\xi)\right)^{\frac{s}{s+t}}\right.$$
$$\left.\times\left(\int_{\mathbb{R}^d}\|y\|^{2s}|\Phi_h^{k,a}(f)(b,y)|^2\,d\nu_{k,a}(y)\right)^{\frac{t}{s+t}}\right]\frac{db}{b^{2\gamma+d+a-1}}$$
$$\geq(\mathscr{C}_{k,a}(s,t))^2\int_0^\infty\int_{\mathbb{R}^d}|\Phi_h^{k,a}(f)(b,y)|^2\,d\nu_{k,a}(y)\frac{db}{b^{2\gamma+d+a-1}}.$$

The left hand side of this inequality may be estimated from above by using Hölder's inequality. The right hand side can be rewritten by the Plancherel formula for $\Phi_h^{k,a}$. Therefore, from (3.3) we get

$$\left(\int_{\mathbb{R}_+^{d+1}}\|\xi\|^{2t}|\mathscr{F}_{k,a}\big[\Phi_h^{k,a}(f)(b,\cdot)\big](\xi)|^2\,d\mu_{k,a}(b,\xi)\right)^{\frac{s}{s+t}}\left\|\|y\|^s\Phi_h^{k,a}(f)\right\|_{L^2_{\mu_{k,a}}(\mathbb{R}_+^{d+1})}^{\frac{2t}{t+s}}$$
$$=(C_h)^{\frac{s}{s+t}}\left\|\|y\|^s\Phi_h^{k,a}(f)\right\|_{L^2_{\mu_{k,a}}(\mathbb{R}_+^{d+1})}^{\frac{2t}{t+s}}\left\|\|\xi\|^t\mathscr{F}_{k,a}(f)\right\|_{L^2_{k,a}(\mathbb{R}^d)}^{\frac{2s}{s+t}}$$
$$\geq(\mathscr{C}_{k,a}(s,t))^2C_h\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}.$$

This proves the result. □

We consider a radial (k,a)-generalized wavelet h in $L^2_{k,a}(\mathbb{R}^d)$ and introduce the modified (k,a)-generalized continuous wavelet transform $\widetilde{\Phi_h^{k,a}}$, given by

$$\mathscr{F}_{k,a}\big(\widetilde{\Phi_h^{k,a}}(f)(b,\cdot)\big)(\xi)=b^{2\gamma+d+a-2}\mathscr{F}_{k,a}(f)(b\xi)\overline{\mathscr{F}_{k,a}(h)(\xi)},\quad(b,\xi)\in\mathbb{R}_+^{d+1}.$$

By using this transform, the following theorems give uncertainty principles of Heisenberg type for $\widetilde{\Phi_h^{k,a}}(f)(b,y)$ with respect to b and (b,y). In the rest of this subsection, h will be a radial (k,a)-generalized wavelet defined as above.

Theorem 3.3. *For $s,t>0$ and for all f in $L^2_{k,a}(\mathbb{R}^d)$, the following inequality holds:*

$$\left\| b^t \widetilde{\Phi^{k,a}_h}(f) \right\|^{\frac{s}{s+t}}_{L^2_{\mu_{k,a}}\left(\mathbb{R}^{d+1}_+\right)} \left\| \|x\|^s f \right\|^{\frac{t}{s+t}}_{L^2_{k,a}(\mathbb{R}^d)} \geq \frac{\mathscr{C}_{k,a}(s,t)}{a^{\frac{s(2\gamma+d-2)}{a(s+t)}}} \left(\mathscr{M}\left(\left| \mathscr{F}^{\frac{2\gamma+d-2}{a}}_{B,a}(H) \right|^2 \right)(2t) \right)^{\frac{s}{2(s+t)}} \|f\|_{L^2_{k,a}(\mathbb{R}^d)},$$

where

$$\mathscr{M} : G \mapsto \mathscr{M}(G)(z) = \int_0^\infty G(r) \frac{dr}{r^{z+1}}$$

is the classical Mellin transform and $h(x) = H(r)$, with $r = \|x\|$.

Proof. In the following we assume that

$$\int_0^\infty \int_{\mathbb{R}^d} b^{2t} \left| \widetilde{\Phi^{k,a}_h}(f)(b,y) \right|^2 d\mu_{k,a}(b,y) < \infty$$

and

$$\int_{\mathbb{R}^d} \|x\|^{2s} |f(x)|^2 \, d\nu_{k,a}(x) < \infty;$$

otherwise, the inequality is trivially satisfied. Using Fubini's theorem and the Plancherel formula given by the relation (2.5), we get

$$\begin{aligned} \left\| b^t \widetilde{\Phi^{k,a}_h}(f) \right\|^2_{L^2_{\mu_{k,a}}\left(\mathbb{R}^{d+1}_+\right)} &= \int_{\mathbb{R}^d} |\mathscr{F}_{k,a}(f)(\xi)|^2 \left(\int_0^\infty b^{2t} \left| \mathscr{F}_{k,a}(h)\left(\frac{\xi}{b}\right) \right|^2 \frac{db}{b} \right) d\nu_{k,a}(\xi) \\ &= \int_{\mathbb{R}^d} \Lambda(\xi) |\mathscr{F}_{k,a}(f)(\xi)|^2 \, d\nu_{k,a}(\xi), \end{aligned}$$

with

$$\Lambda(\xi) = \int_0^\infty b^{2t} \left| \mathscr{F}_{k,a}(h)\left(\frac{\xi}{b}\right) \right|^2 \frac{db}{b}.$$

Introducing the Mellin transform, we see that $\Lambda(\xi)$ is just a function of $\|\xi\|^{2t}$. Indeed, from the relation (2.8) we obtain

$$\begin{aligned} \Lambda(\xi) &= \int_0^\infty b^{2t} \left| a^{-\left(\frac{2\gamma+d-2}{a}\right)} \mathscr{F}^{\frac{2\gamma+d-2}{a}}_{B,a}(H)\left(\frac{\|\xi\|}{b}\right) \right|^2 \frac{db}{b} \\ &= a^{-2\left(\frac{2\gamma+d-2}{a}\right)} \left(\int_0^\infty \left| \mathscr{F}^{\frac{2\gamma+d-2}{a}}_{B,a}(H)(r) \right|^2 \frac{dr}{r^{1+2t}} \right) \|\xi\|^{2t}, \\ &= a^{-2\left(\frac{2\gamma+d-2}{a}\right)} \left(\mathscr{M}\left(\left| \mathscr{F}^{\frac{2\gamma+d-2}{a}}_{B,a}(H) \right|^2 \right)(2t) \right) \|\xi\|^{2t}. \end{aligned}$$

Thus

$$\left(\int_{\mathbb{R}_+^{d+1}} b^{2t}\left|\widetilde{\Phi_h^{k,a}}(f)(b,y)\right|^2 d\mu_{k,a}(b,y)\right)^{\frac{s}{s+t}}\left(\int_{\mathbb{R}^d}\|x\|^{2s}|f(x)|^2\,d\nu_{k,a}(x)\right)^{\frac{t}{s+t}}$$
$$=\left(a^{-2\left(\frac{2\gamma+d-2}{a}\right)}\mathscr{M}\left(\left|\mathscr{F}_{B,a}^{\frac{2\gamma+d-2}{a}}(H)\right|^2\right)(2t)\right)^{\frac{s}{s+t}}$$
$$\times\left(\int_{\mathbb{R}^d}\|\xi\|^{2t}|\mathscr{F}_{k,a}(f)(\xi)|^2\,d\nu_{k,a}(\xi)\right)^{\frac{s}{s+t}}\left(\int_{\mathbb{R}^d}\|x\|^{2s}|f(x)|^2\,d\nu_{k,a}(x)\right)^{\frac{t}{s+t}}.$$

Now, the result is obtained from Proposition 3.1. □

Corollary 3.4. *For $s,t>0$ and for all f in $L^2_{k,a}(\mathbb{R}^d)$, the following inequality holds:*

$$\left\|\|y\|^s\Phi_h^{k,a}(f)\right\|^{\frac{2t}{s+t}}_{L^2_{\mu_{k,a}}(\mathbb{R}_+^{d+1})}\left\|b^t\widetilde{\Phi_h^{k,a}}(f)\right\|^{\frac{2s}{s+t}}_{L^2_{\mu_{k,a}}(\mathbb{R}_+^{d+1})}$$
$$\geq\frac{(\mathscr{C}_{k,a}(s,t))^2}{a^{\frac{2s(2\gamma+d-2)}{a(s+t)}}}(C_h)^{\frac{t}{s+t}}\left(\mathscr{M}\left(\left|\mathscr{F}_{B,a}^{\frac{2\gamma+d-2}{a}}(H)\right|^2\right)(2t)\right)^{\frac{s}{s+t}}\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}. \quad (3.4)$$

Proof. From above we have

$$\left(\int_{\mathbb{R}_+^{d+1}} b^{2t}\left|\widetilde{\Phi_h^{k,a}}(f)(b,y)\right|^2 d\mu_{k,a}(b,y)\right)^{\frac{s}{s+t}}$$
$$\times\left(\int_{\mathbb{R}_+^{d+1}}\|y\|^{2s}|\Phi_h^{k,a}(f)(b,y)|^2\,d\mu_{k,a}(b,y)\right)^{\frac{t}{s+t}}$$
$$=a^{\frac{-2s(2\gamma+d-2)}{a(s+t)}}\left(\mathscr{M}\left(\left|\mathscr{F}_{B,a}^{\frac{2\gamma+d-2}{a}}(H)\right|^2\right)(2t)\right)^{\frac{s}{s+t}}$$
$$\times\left(\int_{\mathbb{R}^d}\|\xi\|^{2t}|\mathscr{F}_{k,a}(f)(\xi)|^2\,d\nu_{k,a}(\xi)\right)^{\frac{s}{s+t}}$$
$$\times\left(\int_0^\infty\int_{\mathbb{R}^d}\|y\|^{2s}\left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y)\right)^{\frac{t}{s+t}}.$$

The relation (3.4) follows from (3.2). □

Corollary 3.5. *For $s,t>0$ and for all f in $L^2_{k,a}(\mathbb{R}^d)$, the following inequality holds:*

$$\left\|\,\|(b,y)\|^s\Phi_h^{k,a}(f)\right\|^{\frac{2t}{s+t}}_{L^2_{\mu_{k,a}}(\mathbb{R}_+^{d+1})}\left\|\,\|(b,y)\|^t\widetilde{\Phi_h^{k,a}}(f)\right\|^{\frac{2s}{s+t}}_{L^2_{\mu_{k,a}}(\mathbb{R}_+^{d+1})}$$
$$\geq\frac{(\mathscr{C}_{k,a}(s,t))^2}{a^{\frac{2s(2\gamma+d-2)}{a(s+t)}}}(C_h)^{\frac{t}{s+t}}\left(\mathscr{M}\left(\left|\mathscr{F}_{B,a}^{\frac{2\gamma+d-2}{a}}(H)\right|^2\right)(2t)\right)^{\frac{s}{s+t}}\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}.$$

Proof. The result follows from the relation (3.4) and the fact that

$$\Big\| \|(b,y)\|^s \Phi_h^{k,a}(f)(b,y)\Big\|_{L^2_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)}^{\frac{2t}{s+t}} \Big\| \|(b,y)\|^t \widetilde{\Phi_h^{k,a}}(f)(b,y)\Big\|_{L^2_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)}^{\frac{2s}{s+t}} \geq \Big\| \|y\|^s \Phi_h^{k,a}(f)(b,y)\Big\|_{L^2_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)}^{\frac{2t}{s+t}} \Big\| b^t \widetilde{\Phi_h^{k,a}}(f)(b,y)\Big\|_{L^2_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)}^{\frac{2s}{s+t}}. \qquad \square$$

As a consequence of the previous corollary, we have the following local-type uncertainty principle when $d \geq 1$ and $\frac{2}{a} \in \mathbb{N}$.

Corollary 3.6. *Let $s,t>0$ and let $U \subset \mathbb{R}^{d+1}_+$ be such that*

$$0 < \mu_{k,a}(U) := \int_U d\mu_{k,a}(b,y) < \infty.$$

For all f in $L^2_{k,a}(\mathbb{R}^d)$, the inequality

$$\int_U |\Phi_h^{k,a}(f)(b,y)|^2\, d\mu_{k,a}(b,y) \leq \mathscr{C}(k,a,s,t) \Big\| \|(b,y)\|^s \Phi_h^{k,a}(f)\Big\|_{L^2_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)}^{\frac{2t}{s+t}} \Big\| \|(b,y)\|^t \widetilde{\Phi_h^{k,a}}(f)\Big\|_{L^2_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)}^{\frac{2s}{s+t}}$$

holds, where

$$\mathscr{C}(k,a,s,t) := \frac{a^{\frac{2s(2\gamma+d-2)}{a(s+t)}} \mu_{k,a}(U) \|h\|^2_{L^2_{k,a}(\mathbb{R}^d)}}{c^2_{k,a} (\mathscr{C}_{k,a}(s,t))^2 (C_h)^{\frac{t}{s+t}} \Big(\mathscr{M}\Big(\Big|\mathscr{F}_{B,a}^{\frac{2\gamma+d-2}{a}}(H)\Big|^2\Big)(2t)\Big)^{\frac{s}{s+t}}}.$$

Proof. From the relation (2.21), we have

$$\int_0^\infty \int_{\mathbb{R}^d} \chi_U(b,y) \big|\Phi_h^{k,a}(f)(b,y)\big|^2 d\mu_{k,a}(b,y) \leq \frac{\mu_{k,a}(U)}{c^2_{k,a}} \|h\|^2_{L^2_{k,a}(\mathbb{R}^d)} \|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}.$$

On the other hand, from Corollary 3.5 we have

$$\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)} \leq \frac{a^{\frac{2s(2\gamma+d-2)}{a(s+t)}} \Big\| \|(b,y)\|^s \Phi_h^{k,a}(f)\Big\|_{L^2_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)}^{\frac{2t}{s+t}} \Big\| \|(b,y)\|^t \widetilde{\Phi_h^{k,a}}(f)\Big\|_{L^2_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)}^{\frac{2s}{s+t}}}{(\mathscr{C}_{k,a}(s,t))^2 (C_h)^{\frac{t}{s+t}} \Big(\mathscr{M}\Big(\Big|\mathscr{F}_{B,a}^{\frac{2\gamma+d-2}{a}}(H)\Big|^2\Big)(2t)\Big)^{\frac{s}{s+t}}}.$$

Thus the result is immediate. $\square$

3.2. **Heisenberg-type uncertainty inequalities via the (k,a)-entropy.** In this subsection we assume that $d \geq 1$ and $\frac{2}{a} \in \mathbb{N}$. A probability density function ρ on $\mathbb{R}^{d+1}_+$ is a positive measurable function on $\mathbb{R}^{d+1}_+$ satisfying

$$\int_{\mathbb{R}^{d+1}_+} \rho(b,y)\, d\mu_{k,a}(b,y) = 1.$$

Following Shannon [43], the (k,a)-entropy of a probability density function ρ on $\mathbb{R}_+^{d+1}$ is defined by

$$E_{k,a}(\rho) := -\int_{\mathbb{R}_+^{d+1}} \ln(\rho(b,y))\rho(b,y)\, d\mu_{k,a}(b,y).$$

Henceforth, we extend the definition of the (k,a)-entropy of a positive measurable function ρ on $\mathbb{R}_+^{d+1}$ whenever the previous integral on the right hand side is well defined.

The aim of this part is to study the localization of the (k,a)-entropy of the (k,a)-generalized wavelet transform over the space $\mathbb{R}_+^{d+1}$; indeed we have the following result.

Proposition 3.7. *Let $f \in L^2_{k,a}(\mathbb{R}^d)$ be a non-zero function. Then*

$$E_{k,a}\big(|\Phi_h^{k,a}(f)|^2\big) \geq -2C_h \|f\|^2_{L^2_{k,a}(\mathbb{R}^d)} \ln\left(\frac{\|f\|_{L^2_{k,a}(\mathbb{R}^d)}\|h\|_{L^2_{k,a}(\mathbb{R}^d)}}{c_{k,a}}\right). \tag{3.5}$$

Proof. Assume that $\|f\|_{L^2_{k,a}(\mathbb{R}^d)} = \|h\|_{L^2_{k,a}(\mathbb{R}^d)} = \sqrt{c_{k,a}}$; then by the relation (2.21) we deduce that

$$\forall (b,y) \in \mathbb{R}_+^{d+1}, \quad \big|\Phi_h^{k,a}(f)(b,y)\big| \leq \frac{1}{c_{k,a}}\|f\|_{L^2_{k,a}(\mathbb{R}^d)}\|h\|_{L^2_{k,a}(\mathbb{R}^d)} = 1.$$

In particular, $E_{k,a}(|\Phi_h^{k,a}(f)|^2) \geq 0$ and therefore if the entropy $E_{k,a}(|\Phi_h^{k,a}(f)|^2)$ is infinite, the inequality (3.5) holds obviously.

Suppose now that the entropy $E_{k,a}(|\Phi_h^{k,a}(f)|^2)$ is finite. We return now to the general case, so let $f \in L^2_{k,a}(\mathbb{R}^d)$ and $h \in L^2_{k,a}(\mathbb{R}^d)$ be non-zero functions and let

$$\varphi = \sqrt{c_{k,a}}\frac{f}{\|f\|_{L^2_{k,a}(\mathbb{R}^d)}} \quad \text{and} \quad \psi = \sqrt{c_{k,a}}\frac{h}{\|h\|_{L^2_{k,a}(\mathbb{R}^d)}}.$$

Then, $\varphi \in L^2_{k,a}(\mathbb{R}^d)$, $\psi \in L^2_{k,a}(\mathbb{R}^d)$ and $\|\varphi\|_{L^2_{k,a}(\mathbb{R}^d)} = \|\psi\|_{L^2_{k,a}(\mathbb{R}^d)} = \sqrt{c_{k,a}}$; hence

$$E_{k,a}(|\Phi_\psi^{k,a}(\varphi)|^2) \geq 0.$$

However,

$$\Phi_\psi^{k,a}(\varphi) = \frac{c_{k,a}}{\|f\|_{L^2_{k,a}(\mathbb{R}^d)}\|h\|_{L^2_{k,a}(\mathbb{R}^d)}}\Phi_h^{k,a}(f)$$

and

$$\begin{aligned} E_{k,a}(|\Phi_\psi^{k,a}(\varphi)|^2) = {} & \frac{c^2_{k,a}}{\|h\|^2_{L^2_{k,a}(\mathbb{R}^d)}\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}} E_{k,a}(|\Phi_h^{k,a}(f)|^2) \\ & + \ln\left(\frac{\|f\|_{L^2_{k,a}(\mathbb{R}^d)}\|h\|_{L^2_{k,a}(\mathbb{R}^d)}}{c_{k,a}}\right)\frac{2C_h c^2_{k,a}}{\|h\|^2_{L^2_{k,a}(\mathbb{R}^d)}}. \end{aligned}$$

So,

$$E_{k,a}(|\Phi_h^{k,a}(f)|^2) \geq -2C_h\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)} \ln\left(\frac{\|f\|_{L^2_{k,a}(\mathbb{R}^d)}\|h\|_{L^2_{k,a}(\mathbb{R}^d)}}{c_{k,a}}\right). \qquad \square$$

Using the (k,a)-entropy of the (k,a)-generalized wavelet transform, we can obtain the following Heisenberg uncertainty principle for $\Phi_h^{k,a}$. In this regard our proof uses some techniques employed in [44] for the L^2 Heisenberg uncertainty inequality for the classical Fourier transform.

Theorem 3.8. *Let p and q be two positive real numbers. Then there exists a positive constant $M_{p,q}(k,a)$ such that for every $f \in L^2_{k,a}(\mathbb{R}^d)$ we have*

$$\left(\int_{\mathbb{R}_+^{d+1}} \|y\|^p |\Phi_h^{k,a}(f)(b,y)|^2 \, d\mu_{k,a}(b,y)\right)^{\frac{q}{p+q}} \times \left(\int_{\mathbb{R}_+^{d+1}} b^{-q} |\Phi_h^{k,a}(f)(b,y)|^2 \, d\mu_{k,a}(b,y)\right)^{\frac{p}{p+q}} \geq M_{p,q}(k,a) C_h \|f\|^2_{L^2_{k,a}(\mathbb{R}^d)},$$

where

$$M_{p,q}(k,a) = \frac{d+2\gamma+a-2}{p^{\frac{q}{p+q}} q^{\frac{p}{p+q}}} e^{A_{p,q}(k,a)};$$

here

$$A_{p,q}(k,a) := pq \frac{\ln\left(\frac{pq}{d_k \Gamma\left(\frac{d+2\gamma+a-2}{p}\right)\Gamma\left(\frac{d+2\gamma+a-2}{q}\right)}\right)}{(d+2\gamma+a-2)(p+q)} - 1.$$

Proof. Assume that $\|f\|_{L^2_{k,a}(\mathbb{R}^d)} = \|h\|_{L^2_{k,a}(\mathbb{R}^d)} = \sqrt{c_{k,a}}$. For any positive real numbers t, p, q, let $\eta_{t,p,q}^{k,a}$ be the function defined on $\mathbb{R}_+^{d+1}$ by

$$\eta_{t,p,q}^{k,a}(b,y) = \frac{pq}{d_k \Gamma\left(\frac{d+2\gamma+a-2}{p}\right)\Gamma\left(\frac{d+2\gamma+a-2}{q}\right)} \frac{e^{-\frac{\|y\|^p + b^{-q}}{t}}}{t^{\frac{(d+2\gamma+a-2)(p+q)}{pq}}}.$$

By simple calculations we can see that

$$\int_{\mathbb{R}_+^{d+1}} \eta_{t,p,q}^{k,a}(b,y) \, d\mu_{k,a}(b,y) = 1;$$

in particular, the measure $d\sigma_{t,p,q}^{k,a}(b,y) = \eta_{t,p,q}^{k,a}(b,y) d\mu_{k,a}(b,y)$ is a probability measure on $\mathbb{R}_+^{d+1}$. Since the function $\varphi(t) = t\ln(t)$ is convex over $(0,\infty)$, by using Jensen's inequality for convex functions we get

$$\int_{\mathbb{R}_+^{d+1}} \frac{|\Phi_h^{k,a}(f)(b,y)|^2}{\eta_{t,p,q}^{k,a}(b,y)} \ln\left(\frac{|\Phi_h^{k,a}(f)(b,y)|^2}{\eta_{t,p,q}^{k,a}(b,y)}\right) d\sigma_{t,p,q}^{k,a}(b,y) \geq 0,$$

which implies in terms of the (k,a)-entropy that for any positive real numbers t, p, q, we have

$$E_{k,a}(|\Phi_h^{k,a}(f)|^2) + \ln\left(\frac{pq}{d_k\Gamma\left(\frac{d+2\gamma+a-2}{p}\right)\Gamma\left(\frac{d+2\gamma+a-2}{q}\right)}\right) C_h\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}$$
$$\leq \ln\left(t^{\frac{(d+2\gamma+a-2)(p+q)}{pq}}\right) C_h\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)} + \frac{1}{t}\int_{\mathbb{R}^{d+1}_+}(\|y\|^p + b^{-q})|\Phi_h^{k,a}(f)(b,y)|^2\, d\mu_{k,a}(b,y).$$

Therefore, by Proposition 3.7 we get

$$\int_{\mathbb{R}^{d+1}_+}(\|y\|^p + b^{-q})|\Phi_h^{k,a}(f)(b,y)|^2\, d\mu_{k,a}(b,y)$$
$$\geq t\left(\ln\left(\frac{pq}{d_k\Gamma\left(\frac{d+2\gamma+a-2}{p}\right)\Gamma\left(\frac{d+2\gamma+a-2}{q}\right)}\right) - \ln\left(t^{\frac{(d+2\gamma+a-2)(p+q)}{pq}}\right)\right) \times \|\Phi_h^{k,a}(f)\|^2_{L^2_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)}.$$

However, the expression

$$t\left(\ln\left(\frac{pq}{d_k\Gamma\left(\frac{d+2\gamma+a-2}{p}\right)\Gamma\left(\frac{d+2\gamma+a-2}{q}\right)}\right) - \ln\left(t^{\frac{(d+2\gamma+a-2)(p+q)}{pq}}\right)\right) \times \|\Phi_h^{k,a}(f)\|^2_{L^2_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)}$$

attains its upper bound at $t_0 = e^{A_{p,q}(k,a)}$, and consequently

$$\int_{\mathbb{R}^{d+1}_+}(\|y\|^p + b^{-q})|\Phi_h^{k,a}(f)(b,y)|^2\, d\mu_{k,a}(b,y) \geq C_{p,q}(k,a)C_h\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)},$$

where

$$C_{p,q}(k,a) = \frac{(d+2\gamma+a-2)(p+q)}{pq}e^{A_{p,q}(k,a)}.$$

Now, the general formula follows from above by substituting f by $\frac{\sqrt{c_{k,a}}f}{\|f\|_{L^2_{k,a}(\mathbb{R}^d)}}$ and h by $\frac{\sqrt{c_{k,a}}h}{\|h\|_{L^2_{k,a}(\mathbb{R}^d)}}$.

Therefore, for every $f \in L^2_{k,a}(\mathbb{R}^d)$ and $h \in L^2_{k,a}(\mathbb{R}^d)$, we get

$$\int_{\mathbb{R}^{d+1}_+}\|y\|^p|\Phi_h^{k,a}(f)(b,y)|^2\, d\mu_{k,a}(b,y) + \int_{\mathbb{R}^{d+1}_+} b^{-q}|\Phi_h^{k,a}(f)(b,y)|^2\, d\mu_{k,a}(b,y)$$
$$\geq C_{p,q}(k,a)C_h\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}.$$

Now, for every $\lambda > 0$ the dilates $\Delta_{\frac{1}{\lambda}} f \in L^2_{k,a}(\mathbb{R}^d)$; then by the last inequality we have

$$\int_{\mathbb{R}^{d+1}_+} \|y\|^p |\Phi^{k,a}_h(\Delta_{\frac{1}{\lambda}} f)(b,y)|^2 \, d\mu_{k,a}(b,y) + \int_{\mathbb{R}^{d+1}_+} b^{-q} |\Phi^{k,a}_h(\Delta_{\frac{1}{\lambda}} f)(b,y)|^2 \, d\mu_{k,a}(b,y) \geq C_{p,q}(k,a) C_h \|\Delta_{\frac{1}{\lambda}} f\|^2_{L^2_{k,a}(\mathbb{R}^d)}.$$

Thus using the fact that $\|\Delta_{\frac{1}{\lambda}} f\|^2_{L^2_{k,a}(\mathbb{R}^d)} = \|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}$ and (2.20) we get, for every positive real number λ,

$$\lambda^{-p} \int_{\mathbb{R}^{d+1}_+} \|y\|^p |\Phi^{k,a}_h(f)(b,y)|^2 \, d\mu_{k,a}(b,y) + \lambda^q \int_{\mathbb{R}^{d+1}_+} b^{-q} |\Phi^{k,a}_h(f)(b,y)|^2 \, d\mu_{k,a}(b,y) \geq C_{p,q}(k,a) C_h \|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}.$$

In particular, the inequality holds at the critical point

$$\lambda = \left(\frac{p \int_{\mathbb{R}^{d+1}_+} \|y\|^p |\Phi^{k,a}_h(f)(b,y)|^2 \, d\mu_{k,a}(b,y)}{q \int_{\mathbb{R}^{d+1}_+} b^{-q} |\Phi^{k,a}_h(f)(b,y)|^2 \, d\mu_{k,a}(b,y)} \right)^{\frac{1}{p+q}},$$

which implies that

$$\left(\int_{\mathbb{R}^{d+1}_+} \|y\|^p |\Phi^{k,a}_h(f)(b,y)|^2 \, d\mu_{k,a}(b,y) \right)^{\frac{q}{p+q}} \times \left(\int_{\mathbb{R}^{d+1}_+} b^{-q} |\Phi^{k,a}_h(f)(b,y)|^2 \, d\mu_{k,a}(b,y) \right)^{\frac{p}{p+q}} \geq M_{p,q}(k,a) C_h \|f\|^2_{L^2_{k,a}(\mathbb{R}^d)},$$

where

$$M_{p,q}(k,a) = C_{p,q}(k,a) \frac{p^{\frac{p}{p+q}} q^{\frac{q}{p+q}}}{p+q} = \frac{d+2\gamma+a-2}{p^{\frac{q}{p+q}} q^{\frac{p}{p+q}}} e^{A_{p,q}(k,a)}. \qquad \square$$

Remark 3.9. When $p = q = 2$, we get

$$\big\| \|y\| \, \Phi^{k,a}_h(f) \big\|_{L^2_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)} \big\| b^{-1} \Phi^{k,a}_h(f) \big\|_{L^2_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)} \geq \left(\frac{4}{d_k \left(\Gamma\left(\frac{d+2\gamma+a-2}{2} \right) \right)^2} \right)^{\frac{1}{d+2\gamma+a-2}} \frac{d+2\gamma+a-2}{2e} C_h \|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}.$$

4. Weighted inequalities for the (k,a)-generalized wavelet transform

The sharp Pitt inequality for the (k,a)-generalized Fourier transform was studied by Gorbachev et al. in [16]; they proved that, for every function f belonging to $\mathscr{S}(\mathbb{R}^d) \subseteq L^2_{k,a}(\mathbb{R}^d)$, we have that

$$\int_{\mathbb{R}^d} \|\xi\|^{-2\lambda} |\mathscr{F}_{k,a}(f)(\xi)|^2 \, d\nu_{k,a}(\xi) \leq C_{k,a}(\lambda) \int_{\mathbb{R}^d} \|x\|^{2\lambda} |f(x)|^2 \, d\nu_{k,a}(x) \tag{4.1}$$

holds with the sharp constant

$$C_{k,a}(\lambda) := a^{\frac{-4\lambda}{a}} \left[\frac{\Gamma\left(\frac{2\gamma+d+a-2-2\lambda}{2a}\right)}{\Gamma\left(\frac{2\gamma+d+a-2+2\lambda}{2a}\right)} \right]^2, \tag{4.2}$$

provided that

$$0 \leq \lambda < \frac{2\gamma+d+a-2}{2}, \quad 4\gamma+2d+a-4 \geq 0. \tag{4.3}$$

In the remainder of this section, we assume that condition (4.3) is satisfied.

The first objective of this section is to formulate an analogue of Pitt's inequality (4.1) for the (k,a)-generalized wavelet transform.

Theorem 4.1. *For any arbitrary $f \in \mathscr{S}(\mathbb{R}^d) \subseteq L^2_{k,a}(\mathbb{R}^d)$, the Pitt inequality for the (k,a)-generalized wavelet transform is given by*

$$C_h \int_{\mathbb{R}^d} \|\xi\|^{-2\lambda} \left|\mathscr{F}_{k,a}(f)(\xi)\right|^2 d\nu_{k,a}(\xi) \\ \leq C_{k,a}(\lambda) \int_{\mathbb{R}^{d+1}_+} \|y\|^{2\lambda} \left|\Phi^{k,a}_h(f)(b,y)\right|^2 d\mu_{k,a}(b,y), \tag{4.4}$$

where $C_{k,a}(\lambda)$ is given by (4.2).

Proof. As a consequence of the inequality (4.1), for any $b>0$ we have

$$\int_{\mathbb{R}^d} \|\xi\|^{-2\lambda} \left|\mathscr{F}_{k,a}\left[\Phi^{k,a}_h(f)(b,\cdot)\right](\xi)\right|^2 d\nu_{k,a}(\xi) \\ \leq C_{k,a}(\lambda) \int_{\mathbb{R}^d} \|y\|^{2\lambda} \left|\Phi^{k,a}_h(f)(b,y)\right|^2 d\nu_{k,a}(y),$$

which upon integration with respect to the Haar measure $\frac{db}{b^{2\gamma+d+a-1}}$ yields

$$\int_{\mathbb{R}^{d+1}_+} \|\xi\|^{-2\lambda} \left|\mathscr{F}_{k,a}\left[\Phi^{k,a}_h(f)(b,\cdot)\right](\xi)\right|^2 d\mu_{k,a}(b,\xi) \\ \leq C_{k,a}(\lambda) \int_{\mathbb{R}^{d+1}_+} \|y\|^{2\lambda} \left|\Phi^{k,a}_h(f)(b,y)\right|^2 d\mu_{k,a}(b,y). \tag{4.5}$$

Invoking Lemma 2.22, we can express the inequality (4.5) in the following manner:

$$\int_{\mathbb{R}_+^{d+1}} \|\xi\|^{-2\lambda} \left|\mathscr{F}_{k,a}(f)(\xi)\right|^2 b^{2\gamma+d+a-2} \left|\mathscr{F}_{k,a}(\overline{h})(b\xi)\right|^2 d\mu_{k,a}(b,\xi) \leq C_{k,a}(\lambda) \int_{\mathbb{R}_+^{d+1}} \|y\|^{2\lambda} \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y).$$

Equivalently, we have

$$\int_{\mathbb{R}^d} \|\xi\|^{-2\lambda} \left|\mathscr{F}_{k,a}(f)(\xi)\right|^2 \left\{ \int_0^\infty \left|\mathscr{F}_{k,a}(\overline{h})(b\xi)\right|^2 \frac{db}{b} \right\} d\nu_{k,a}(\xi) \leq C_{k,a}(\lambda) \int_{\mathbb{R}_+^{d+1}} \|y\|^{2\lambda} \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y).$$

Using the hypothesis on h and by simple calculations, we obtain

$$C_h \int_{\mathbb{R}^d} \|\xi\|^{-2\lambda} \left|\mathscr{F}_{k,a}(f)(\xi)\right|^2 d\nu_{k,a}(\xi) \leq C_{k,a}(\lambda) \int_{\mathbb{R}_+^{d+1}} \|y\|^{2\lambda} \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y),$$

which establishes the Pitt inequality for the (k,a)-generalized wavelet transform. □

Remark 4.2. For $\lambda = 0$, equality holds in (4.4), which is the Plancherel formula (2.18).

Theorem 4.3. *For any function $f \in \mathscr{S}(\mathbb{R}^d)$, the following inequality holds:*

$$\int_{\mathbb{R}_+^{d+1}} \log\|y\| \, |\Phi_h^{k,a}(f)(b,y)|^2 \, d\mu_{k,a}(b,y) + C_h \int_{\mathbb{R}^d} \log\|\xi\| \, |\mathscr{F}_{k,a}(f)(\xi)|^2 \, d\nu_{k,a}(\xi) \geq \frac{2}{a} \left[\frac{\Gamma'\left(\frac{2\gamma+d+a-2}{2a}\right)}{\Gamma\left(\frac{2\gamma+d+a-2}{2a}\right)} + \log a \right] C_h \|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}. \tag{4.6}$$

Proof. For every $0 \leq \lambda < \frac{2\gamma+d+a-2}{2}$, we define

$$P(\lambda) = C_h \int_{\mathbb{R}^d} \|\xi\|^{-2\lambda} \left|\mathscr{F}_{k,a}(f)(\xi)\right|^2 d\nu_{k,a}(\xi) - C_{k,a}(\lambda) \int_{\mathbb{R}_+^{d+1}} \|y\|^{2\lambda} \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y). \tag{4.7}$$

On differentiating (4.7) with respect to λ, we obtain

$$\begin{aligned}P'(\lambda) = &-2C_h \int_{\mathbb{R}^d} \|\xi\|^{-2\lambda} \log \|\xi\| \left|\mathscr{F}_{k,a}(f)(\xi)\right|^2 d\nu_{k,a}(\xi)\\ &- 2C_{k,a}(\lambda) \int_{\mathbb{R}^{d+1}_+} \|y\|^{2\lambda} \log \|y\| \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y)\\ &- C'_{k,a}(\lambda) \int_{\mathbb{R}^{d+1}_+} \|y\|^{2\lambda} \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y),\end{aligned}$$

where

$$C'_{k,a}(\lambda) = -\frac{2}{a} C_{k,a}(\lambda) \left(2\log a + \frac{\Gamma'\left(\frac{2\gamma+d+a-2-2\lambda}{2a}\right)}{\Gamma\left(\frac{2\gamma+d+a-2-2\lambda}{2a}\right)} + \frac{\Gamma'\left(\frac{2\gamma+d+a-2+2\lambda}{2a}\right)}{\Gamma\left(\frac{2\gamma+d+a-2+2\lambda}{2a}\right)} \right). \tag{4.8}$$

For $\lambda = 0$, equation (4.8) yields

$$C'_{k,a}(0) = -\frac{4}{a} \left[\log a + \frac{\Gamma'\left(\frac{2\gamma+d+a-2}{2a}\right)}{\Gamma\left(\frac{2\gamma+d+a-2}{2a}\right)} \right]. \tag{4.9}$$

By virtue of the (k,a)-generalized Pitt inequality (4.4), it follows that $P(\lambda) \le 0$ for all $\lambda \in \left[0, \frac{2\gamma+d+a-2}{2}\right)$, and

$$\begin{aligned}P(0) &= C_h \int_{\mathbb{R}^d} \left|\mathscr{F}_{k,a}(f)(\xi)\right|^2 d\nu_{k,a}(\xi) - C_{k,a}(0) \int_{\mathbb{R}^{d+1}_+} \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y)\\ &= C_h \|f\|^2_{L^2_{k,a}(\mathbb{R}^d)} - C_h \|f\|^2_{L^2_{k,a}(\mathbb{R}^d)} = 0.\end{aligned}$$

Therefore we deduce that

$$P'(0^+) := \lim_{\lambda \to 0^+} \frac{P(\lambda)}{\lambda} \le 0.$$

Equivalently, we have

$$\begin{aligned}&- 2C_h \int_{\mathbb{R}^d} \log \|\xi\| \left|\mathscr{F}_{k,a}(f)(\xi)\right|^2 d\nu_{k,a}(\xi)\\ &\quad - C'_{k,a}(0) \int_{\mathbb{R}^{d+1}_+} \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y)\\ &\quad - 2C_{k,a}(0) \int_{\mathbb{R}^{d+1}_+} \log \|y\| \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y) \le 0.\end{aligned}$$

Applying Plancherel's formula (2.18) and the estimate (4.9) of $C'_{k,a}(0)$, we get

$$-2C_h \int_{\mathbb{R}^d} \log\|\xi\| \left|\mathscr{F}_{k,a}(f)(\xi)\right|^2 d\nu_{k,a}(\xi) - 2\int_{\mathbb{R}_+^{d+1}} \log\|y\| \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y) + \frac{4}{a}\left[\frac{\Gamma'\left(\frac{2\gamma+d+a-2}{2a}\right)}{\Gamma\left(\frac{2\gamma+d+a-2}{2a}\right)} + \log a\right] C_h \|f\|^2_{L^2_{k,a}(\mathbb{R}^d)} \le 0,$$

or, equivalently,

$$\int_{\mathbb{R}_+^{d+1}} \log\|y\| \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y) + C_h \int_{\mathbb{R}^d} \log\|\xi\| \left|\mathscr{F}_{k,a}(f)(\xi)\right|^2 d\nu_{k,a}(\xi) \ge \frac{2}{a}\left[\frac{\Gamma'\left(\frac{2\gamma+d+a-2}{2a}\right)}{\Gamma\left(\frac{2\gamma+d+a-2}{2a}\right)} + \log a\right] C_h \|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}.$$

This completes the proof of Theorem 4.3. □

The Beckner inequality for the (k,a)-generalized Fourier transform [16] is given by

$$\int_{\mathbb{R}^d} \log\|y\| \, |f(y)|^2 d\nu_{k,a}(y) + \int_{\mathbb{R}^d} \log\|\xi\| \left|\mathscr{F}_{k,a}(f)(\xi)\right|^2 d\nu_{k,a}(\xi) \ge \frac{2}{a}\left[\frac{\Gamma'\left(\frac{2\gamma+d+a-2}{2a}\right)}{\Gamma\left(\frac{2\gamma+d+a-2}{2a}\right)} + \log a\right] \int_{\mathbb{R}^d} |f(y)|^2 d\nu_{k,a}(y) \tag{4.10}$$

for all $f \in \mathscr{S}(\mathbb{R}^d)$. This inequality is related to the Heisenberg uncertainty principle and for that reason it is often referred to as the logarithmic uncertainty principle. Considerable attention has been paid to this inequality for its various generalizations, improvements, analogues, and their applications [21].

We now present an alternative proof of Theorem 4.3. The strategy of the proof is different from the one above, and is obtained directly from the generalized Beckner's inequality (4.10).

Proof of Theorem 4.3. We replace f in (4.10) with $\Phi_h^{k,a}(f)(b,\cdot)$, so that

$$\int_{\mathbb{R}^d} \log\|y\| \, |\Phi_h^{k,a}(f)(b,y)|^2 d\nu_{k,a}(y) + \int_{\mathbb{R}^d} \log\|\xi\| \left|\mathscr{F}_{k,a}\left[\Phi_h^{k,a}(f)(b,\cdot)\right](\xi)\right|^2 d\nu_{k,a}(\xi) \ge \frac{2}{a}\left[\frac{\Gamma'\left(\frac{2\gamma+d+a-2}{2a}\right)}{\Gamma\left(\frac{2\gamma+d+a-2}{2a}\right)} + \log a\right] \int_{\mathbb{R}^d} \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\nu_{k,a}(y),$$

for all $b \in (0,\infty)$. Integrating this inequality with respect to the measure $\frac{db}{b^{2\gamma+d+a-1}}$, we obtain

$$\begin{aligned}\frac{2}{a}\left[\frac{\Gamma'\left(\frac{2\gamma+d+a-2}{2a}\right)}{\Gamma\left(\frac{2\gamma+d+a-2}{2a}\right)}+\log a\right]\int_{\mathbb{R}^{d+1}_+}\left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y)\\ \leq \int_{\mathbb{R}^{d+1}_+}\log\|y\|\left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y)\\ +\int_{\mathbb{R}^{d+1}_+}\log\|\xi\|\left|\mathscr{F}_{k,a}\left[\Phi_h^{k,a}(f)(b,\cdot)\right](\xi)\right|^2 d\mu_{k,a}(b,\xi).\end{aligned}$$

Using Plancherel's formula (2.18), we get

$$\begin{aligned}\frac{2}{a}\left[\frac{\Gamma'\left(\frac{2\gamma+d+a-2}{2a}\right)}{\Gamma\left(\frac{2\gamma+d+a-2}{2a}\right)}+\log a\right] C_h\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}\\ \leq \int_{\mathbb{R}^{d+1}_+}\log\|y\|\left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y)\\ +\int_{\mathbb{R}^{d+1}_+}\log\|\xi\|\,|\mathscr{F}_{k,a}[\Phi_h^{k,a}(f)(b,\cdot)](\xi)|^2 d\mu_{k,a}(b,y).\end{aligned} \tag{4.11}$$

We shall now simplify the second integral of (4.11). By using Lemma 2.22 we infer that

$$\begin{aligned}\int_{\mathbb{R}^{d+1}_+}\log\|\xi\|\,|\mathscr{F}_{k,a}\left[\Phi_h^{k,a}(f)(b,\cdot)\right](\xi)|^2 d\mu_{k,a}(b,y)\\ = C_h\int_{\mathbb{R}^d}\log\|\xi\|\,|\mathscr{F}_{k,a}(f)(\xi)|^2 d\nu_{k,a}(\xi).\end{aligned} \tag{4.12}$$

Plugging the relation (4.12) in (4.11) gives the desired inequality for the (k,a)-generalized wavelet transforms as

$$\begin{aligned}\int_{\mathbb{R}^{d+1}_+}\log\|y\|\left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y)+C_h\int_{\mathbb{R}^d}\log\|\xi\|\,|\mathscr{F}_{k,a}(f)(\xi)|^2 d\nu_{k,a}(\xi)\\ \geq \frac{2}{a}\left[\frac{\Gamma'\left(\frac{2\gamma+d+a-2}{2a}\right)}{\Gamma\left(\frac{2\gamma+d+a-2}{2a}\right)}+\log a\right] C_h\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}.\end{aligned}$$

The previous inequality is the desired Beckner's uncertainty principle for the (k,a)-generalized wavelet transform. □

Corollary 4.4. *Let h be a (k,a)-generalized wavelet on $\mathbb{R}^d$ in $L^2_{k,a}(\mathbb{R}^d)$ such that $C_h = 1$. For any function $f \in \mathscr{S}(\mathbb{R}^d)$, we have the inequality*

$$\left\{ \int_{\mathbb{R}^{d+1}_+} \|y\|^2 \left|\Phi^{k,a}_h(f)(b,y)\right|^2 d\mu_{k,a}(b,y)\right\}^{1/2} \left\{\int_{\mathbb{R}^d} \|\xi\|^2 |\mathscr{F}_{k,a}(f)(\xi)|^2 \, d\nu_{k,a}(\xi)\right\}^{1/2} \geq \exp\left(\frac{2}{a}\left[\frac{\Gamma'\left(\frac{2\gamma+d+a-2}{2a}\right)}{\Gamma\left(\frac{2\gamma+d+a-2}{2a}\right)} + \log a\right]\right) \|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}.$$

Proof. Using Jensen's inequality in (4.6) and the fact that $C_h = 1$ we obtain

$$\begin{aligned}
&\log\left\{\int_{\mathbb{R}^{d+1}_+} \|y\|^2 \frac{\left|\Phi^{k,a}_h(f)(b,y)\right|^2}{\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}} d\mu_{k,a}(b,y) \int_{\mathbb{R}^d} \|\xi\|^2 \frac{|\mathscr{F}_{k,a}(f)(\xi)|^2}{\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}} d\nu_{k,a}(\xi)\right\}^{1/2} \\
&\geq \int_{\mathbb{R}^{d+1}_+} \log\|y\| \frac{\left|\Phi^{k,a}_h(f)(b,y)\right|^2}{\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}} d\mu_{k,a}(b,y) + \int_{\mathbb{R}^d} \log\|\xi\| \frac{|\mathscr{F}_{k,a}(f)(\xi)|^2}{\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}} d\nu_{k,a}(\xi) \\
&\geq \frac{2}{a}\left[\frac{\Gamma'\left(\frac{2\gamma+d+a-2}{2a}\right)}{\Gamma\left(\frac{2\gamma+d+a-2}{2a}\right)} + \log a\right],
\end{aligned}$$

which upon simplification yields the result. □

Remark 4.5. (i) Gorbachev et al. in [16] state that Pitt's and Beckner's inequalities for the (k,a)-generalized Fourier transform are satisfied if we replace the condition $4\gamma + 2d + a - 4 \geq 0$ by $2\gamma + d + a - 2 > 0$. Thus, proceeding as above, we state that our results of this section are also true if we assume that the new condition is held.

(ii) Using the approximation identity (see [47])

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} - 2\int_0^\infty \frac{t}{(t^2+z^2)(e^{2\pi t}-1)} \, dt, \tag{4.13}$$

we infer that

$$\exp\left(\frac{2}{a}\left[\frac{\Gamma'\left(\frac{2\gamma+d+a-2}{2a}\right)}{\Gamma\left(\frac{2\gamma+d+a-2}{2a}\right)} + \log a\right]\right) \approx \left(\frac{2\gamma+d+a-2}{2}\right)^{\frac{2}{a}},$$

for $2\gamma+d+a-2 \gg 1$, which is the constant of the Heisenberg uncertainty principle for the (k,a)-generalized wavelet transform given in Theorem 3.2 when $a \leq 2$.

(iii) Proceeding as above in the logarithmic uncertainty inequality (4.10), we deduce the following Heisenberg uncertainty inequality:

$$\big\| \|y\| f \big\|_{L^2_{k,a}(\mathbb{R}^d)} \big\| \|\xi\| \mathscr{F}_{k,a}(f) \big\|_{L^2_{k,a}(\mathbb{R}^d)} \geq \exp\left(\frac{2}{a} \left[\frac{\Gamma'\left(\frac{2\gamma+d+a-2}{2a}\right)}{\Gamma\left(\frac{2\gamma+d+a-2}{2a}\right)} + \log a \right] \right) \|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}. \tag{4.14}$$

(iv) Using the approximation relation (4.13), we deduce that the constant in the right hand side of (4.14) satisfies the relation

$$\exp\left(\frac{2}{a} \left[\frac{\Gamma'\left(\frac{2\gamma+d+a-2}{2a}\right)}{\Gamma\left(\frac{2\gamma+d+a-2}{2a}\right)} + \log a \right] \right) \approx \left(\frac{2\gamma+d+a-2}{2} \right)^{\frac{2}{a}},$$

for $2\gamma+d+a-2 \gg 1$, which is the constant of the Heisenberg uncertainty principle for the (k,a)-generalized Fourier transform given in Proposition 3.1 when $a \leq 2$.

5. Concentration uncertainty principles for the (k,a)-generalized wavelet transforms

In this section, we derive some concentration uncertainty principles for the (k,a)-generalized wavelet transform as an analog of the Benedicks–Amrein–Berthier and local uncertainty principles in the time-frequency analysis.

5.1. **Benedicks–Amrein–Berthier's uncertainty principle.**

In [21], Johansen proved the Benedicks–Amrein–Berthier uncertainty principle for the (k,a)-generalized Fourier transform, which states that if E_1 and E_2 are two subsets of $\mathbb{R}^d$ with finite measure, then there exists a positive constant $C_{k,a}(E_1,E_2)$ such that, for any $f \in L^2_{k,a}(\mathbb{R}^d)$,

$$\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)} \leq C_{k,a}(E_1,E_2) \left\{ \int_{\mathbb{R}^d \setminus E_1} |f(y)|^2 \, d\nu_{k,a}(y) + \int_{\mathbb{R}^d \setminus E_2} |\mathscr{F}_{k,a}(f)(\xi)|^2 \, d\nu_{k,a}(\xi) \right\}. \tag{5.1}$$

Our primary interest in this subsection is to establish the Benedicks–Amrein–Berthier uncertainty principle for the (k,a)-generalized wavelet transforms in arbitrary space dimensions by employing the inequality (5.1). In this direction, we have the following main theorem.

Theorem 5.1. *For any arbitrary function* $f \in L^2_{k,a}(\mathbb{R}^d)$*, we have the uncertainty inequality*

$$\int_0^\infty \int_{\mathbb{R}^d\setminus E_1} \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y) + C_h \int_{\mathbb{R}^d\setminus E_2} |\mathscr{F}_{k,a}(f)(\xi)|^2\, d\nu_{k,a}(\xi) \geq \frac{C_h \|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}}{C_{k,a}(E_1,E_2)}, \quad (5.2)$$

where $C_{k,a}(E_1,E_2)$ *is the constant given in the relation* (5.1)*.*

Proof. Since, for all $b>0$, we have $\Phi_h^{k,a}(f)(b,\cdot) \in L^2_{k,a}(\mathbb{R}^d)$ whenever $f \in L^2_{k,a}(\mathbb{R}^d)$, we can replace the function f appearing in (5.1) with $\Phi_h^{k,a}(f)(b,\cdot)$ to get

$$\int_{\mathbb{R}^d} \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\nu_{k,a}(y) \leq C_{k,a}(E_1,E_2)\bigg\{ \int_{\mathbb{R}^d\setminus E_1} \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\nu_{k,a}(y) + \int_{\mathbb{R}^d\setminus E_2} \left|\mathscr{F}_{k,a}\left[\Phi_h^{k,a}(f)(b,\cdot)\right](\xi)\right|^2 d\nu_{k,a}(\xi)\bigg\}.$$

By integrating this inequality with respect to the measure $\dfrac{db}{b^{2\gamma+d+a-1}}$, we obtain

$$\int_{\mathbb{R}^{d+1}_+} \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y) \leq C_{k,a}(E_1,E_2)\bigg\{ \int_0^\infty \int_{\mathbb{R}^d\setminus E_1} \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y) + \int_0^\infty \int_{\mathbb{R}^d\setminus E_2} \left|\mathscr{F}_{k,a}\left[\Phi_h^{k,a}(f)(b,\cdot)\right](\xi)\right|^2 d\mu_{k,a}(b,\xi)\bigg\}.$$

Using Lemma 2.22 together with Plancherel's formula (2.18), the above inequality becomes

$$\frac{C_h \|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}}{C_{k,a}(E_1,E_2)} \leq \int_0^\infty \int_{\mathbb{R}^d\setminus E_1} \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y) + \int_{\mathbb{R}^d\setminus E_2} \int_0^\infty b^{2\gamma+d+a-2} |\mathscr{F}_{k,a}(f)(\xi)|^2\, |\mathscr{F}_{k,a}(\overline{h})(b\xi)|^2\, d\mu_{k,a}(b,\xi),$$

which further implies

$$\frac{C_h \|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}}{C_{k,a}(E_1,E_2)} \leq \int_0^\infty \int_{\mathbb{R}^d\setminus E_1} \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y) + \int_{\mathbb{R}^d\setminus E_2} |\mathscr{F}_{k,a}(f)(\xi)|^2 \left\{\int_0^\infty |\mathscr{F}_{k,a}(\overline{h})(b\xi)|^2 \frac{db}{b}\right\} d\nu_{k,a}(\xi).$$

Thus, using the fact that h is (k,a)-generalized wavelet on $\mathbb{R}^d$, we obtain

$$\int_0^\infty \int_{\mathbb{R}^d\setminus E_1} \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y) + C_h \int_{\mathbb{R}^d\setminus E_2} |\mathscr{F}_{k,a}(f)(\xi)|^2 d\nu_{k,a}(\xi) \geq \frac{C_h\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}}{C_{k,a}(E_1,E_2)},$$

which is the desired Benedicks–Amrein–Berthier's uncertainty principle for the (k,a)-generalized wavelet transforms in arbitrary space dimensions. □

Theorem 5.1 allows us to obtain a general form of Heisenberg-type uncertainty inequality for the (k,a)-generalized wavelet transforms when $d \geq 1$ and $\frac{2}{a} \in \mathbb{N}$.

Corollary 5.2. *Let $s,t > 0$. Then there exists a positive constant $\mathfrak{C}_{k,a}(s,t)$ such that for any arbitrary function $f \in L^2_{k,a}(\mathbb{R}^d)$, we have the uncertainty inequality*

$$\left\|\|y\|^s \Phi_h^{k,a}(f)\right\|^t_{L^2_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)} \left\|\|\xi\|^t \mathscr{F}_{k,a}(f)\right\|^s_{L^2_{k,a}(\mathbb{R}^d)} \geq \mathfrak{C}_{k,a}(s,t)(C_h)^{\frac{t}{2}} \|f\|^{s+t}_{L^2_{k,a}(\mathbb{R}^d)}.$$

Proof. Let $s,t > 0$ and let $f \in L^2_{k,a}(\mathbb{R}^d)$. Take $E_1 = E_2 = \mathscr{B}_d(0,1)$, the unit ball in $\mathbb{R}^d$. Then, by (5.2),

$$\int_{\mathscr{B}^c_d(0,1)} \int_0^\infty \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y) + C_h \int_{\mathscr{B}^c_d(0,1)} |\mathscr{F}_{k,a}(f)(\xi)|^2 d\nu_{k,a}(\xi) \geq \frac{C_h\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}}{C(k,a)};$$

here $C(k,a) \coloneqq C_{k,a}(E_1,E_2)$. It follows that

$$\int_{\mathbb{R}^{d+1}_+} \|y\|^{2s} \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y) + C_h \int_{\mathbb{R}^d} \|\xi\|^{2t} |\mathscr{F}_{k,a}(f)(\xi)|^2 d\nu_{k,a}(\xi) \geq \frac{C_h\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}}{C(k,a)}.$$

Now, replacing f by $\Delta_\lambda f$, we get by (2.20)

$$\int_{\mathbb{R}^{d+1}_+} \|y\|^{2s} \left|\Phi_h^{k,a}(f)\left(\tfrac{b}{\lambda},\tfrac{y}{\lambda}\right)\right|^2 d\mu_{k,a}(b,y) + \lambda^{2\gamma+d+a-2} C_h \int_{\mathbb{R}^d} \|\xi\|^{2t} |\mathscr{F}_{k,a}(f)(\lambda\xi)|^2 d\nu_{k,a}(\xi) \geq \frac{C_h\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}}{C(k,a)}.$$

Thus

$$\lambda^{2s} \int_{\mathbb{R}^{d+1}_+} \|y\|^{2s} \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y) + \lambda^{-2t} C_h \int_{\mathbb{R}^d} \|\xi\|^{2t} |\mathscr{F}_{k,a}(f)(\xi)|^2 d\nu_{k,a}(\xi) \geq \frac{C_h\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)}}{C(k,a)}.$$

The desired result follows by minimizing the right hand side over $\lambda > 0$. □

5.2. **Local-type uncertainty principles.** We begin this subsection by recalling the local uncertainty principle for the (k,a)-generalized Fourier transform.

Proposition 5.3 ([15]). *Let $E \subset \mathbb{R}^d$ be such that $0 < \nu_{k,a}(E) := \int_E d\nu_{k,a}(x) < \infty$. For $0 < s < \frac{2\gamma+d+a-2}{2}$, there exists a positive constant $\mathfrak{C}(k,a,s)$ such that, for any $f \in L^2_{k,a}(\mathbb{R}^d)$,*

$$\int_E \left|\mathscr{F}_{k,a}(f)(\xi)\right|^2 d\nu_{k,a}(\xi) \leq \mathfrak{C}(k,a,s)\big(\nu_{k,a}(E)\big)^{\frac{2s}{2\gamma+d+a-2}} \| \|y\|^s f\|^2_{L^2_{k,a}(\mathbb{R}^d)}. \tag{5.3}$$

The first objective of this subsection is to establish the local uncertainty principle for the (k,a)-generalized wavelet transform in arbitrary space dimensions by employing the inequality (5.3).

Theorem 5.4. *Let E be a subset of $\mathbb{R}^d$ with finite measure. Then, for any $f \in L^2_{k,a}(\mathbb{R}^d)$ and any $s \in \left(0, \frac{2\gamma+d+a-2}{2}\right)$, we have*

$$\left\|\|y\|^s \Phi_h^{k,a}(f)\right\|^2_{L^2_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)} \geq \frac{C_h}{\mathfrak{C}(k,a,s)(\nu_{k,a}(E))^{\frac{2s}{2\gamma+d+a-2}}} \int_E |\mathscr{F}_{k,a}(f)(\xi)|^2 d\nu_{k,a}(\xi),$$

where $\mathfrak{C}(k,a,s)$ is the constant given in (5.3).

Proof. Let $b>0$. Since $\Phi_h^{k,a}(f)(b,\cdot) \in L^2_{k,a}(\mathbb{R}^d)$ whenever $f \in L^2_{k,a}(\mathbb{R}^d)$, we can replace the function f appearing in (5.3) with $\Phi_h^{k,a}(f)(b,\cdot)$ to get

$$\begin{aligned}\int_E &\left|\mathscr{F}_{k,a}\left[\Phi_h^{k,a}(f)(b,\cdot)\right](\xi)\right|^2 d\nu_{k,a}(\xi)\\ &\leq \mathfrak{C}(k,a,s)\big(\nu_{k,a}(E)\big)^{\frac{2s}{2\gamma+d+a-2}} \left\|\|y\|^s \Phi_h^{k,a}(f)(b,\cdot)\right\|^2_{L^2_{k,a}(\mathbb{R}^d)}.\end{aligned} \tag{5.4}$$

For an explicit expression of (5.4), we shall integrate this inequality with respect to the measure $\frac{db}{b^{2\gamma+d+a-1}}$ to get

$$\begin{aligned}\int_0^\infty \int_E &\left|\mathscr{F}_{k,a}\left[\Phi_h^{k,a}(f)(b,\cdot)\right](\xi)\right|^2 d\mu_{k,a}(b,\xi)\\ &\leq \mathfrak{C}(k,a,s)\big(\nu_{k,a}(E)\big)^{2s/2\gamma+d+a-2} \int_{\mathbb{R}^{d+1}_+} \|y\|^{2s}\left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y),\end{aligned}$$

which together with Lemma 2.22 and Fubini's theorem gives

$$\begin{aligned}\int_E &|\mathscr{F}_{k,a}(f)(\xi)|^2 \left(\int_0^\infty |\mathscr{F}_{k,a}(\overline{h})(b\xi)|^2 d\nu_{k,a}(\xi)\right)\frac{db}{b}\\ &\leq \mathfrak{C}(k,a,s)\big(\nu_{k,a}(E)\big)^{\frac{2s}{2\gamma+d+a-2}} \int_{\mathbb{R}^{d+1}_+} \|y\|^{2s}\left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y).\end{aligned} \tag{5.5}$$

Using the hypothesis on h, inequality (5.5) reduces to

$$C_h \int_E \left|\mathscr{F}_{k,a}(f)(\xi)\right|^2 d\nu_{k,a}(\xi) \\ \leq \mathfrak{C}(k,a,s)\big(\nu_{k,a}(E)\big)^{\frac{2s}{2\gamma+d+a-2}} \int_{\mathbb{R}_+^{d+1}} \|y\|^{2s}\left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y).$$

Or, equivalently, for any $0 < s < \frac{2\gamma+d+a-2}{2}$,

$$\int_{\mathbb{R}_+^{d+1}} \|y\|^{2s}\left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y) \\ \geq \frac{C_h}{\mathfrak{C}(k,a,s)\big(\nu_{k,a}(E)\big)^{\frac{2s}{2\gamma+d+a-2}}} \int_E |\mathscr{F}_{k,a}(f)(\xi)|^2 d\nu_{k,a}(\xi).$$

This completes the proof of Theorem 5.4. □

Let E be a subset of $\mathbb{R}^d$. We define the *Paley–Wiener space* $PW_{k,a}(E)$ as follows:

$$PW_{k,a}(E) := \Big\{f \in L^2_{k,a}(\mathbb{R}^d) : \operatorname{supp} \mathscr{F}_{k,a}(f) \subset E\Big\}.$$

Using Plancherel's formula (2.5), the definition of Paley–Wiener space and the previous theorem, we obtain the following

Corollary 5.5. *Let E be a subset of $\mathbb{R}^d$ with finite measure $0 < \nu_{k,a}(E) < \infty$. Let $0 < s < \frac{2\gamma+d+a-2}{2}$. For any $f \in PW_{k,a}(E)$, we have*

$$\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)} \leq \frac{\mathfrak{C}(k,a,s)(\nu_{k,a}(E))^{\frac{2s}{2\gamma+d+a-2}}}{C_h} \int_{\mathbb{R}_+^{d+1}} \|y\|^{2s}\left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y),$$

where $\mathfrak{C}(k,a,s)$ is the constant given in Proposition 5.3.

By interchanging the roles of f and $\mathscr{F}_{k,a}(f)$ in Proposition 5.3, we get the following

Corollary 5.6. *Let F be a subset of $\mathbb{R}^d$ with finite measure $0 < \nu_{k,a}(F) < \infty$. For $0 < t < \frac{2\gamma+d+a-2}{2}$ and for any $f \in L^2_{k,a}(\mathbb{R}^d)$, we have*

$$\int_F \left|f(y)\right|^2 d\nu_{k,a}(y) \leq \mathfrak{C}(k,a,t)(\nu_{k,a}(F))^{\frac{2t}{2\gamma+d+a-2}} \left\|\|\xi\|^t \mathscr{F}_{k,a}(f)\right\|^2_{L^2_{k,a}(\mathbb{R}^d)},$$

where $\mathfrak{C}(k,a,t)$ is the constant given in Proposition 5.3.

Applying Corollary 5.6 and using similar ideas to those in the proof of Theorem 5.4, we prove the following

Corollary 5.7. *Let F be a subset of $\mathbb{R}^d$ with finite measure $0 < \nu_{k,a}(F) < \infty$ and let $0 < t < \frac{2\gamma+d+a-2}{2}$. For any $f \in L^2_{k,a}(\mathbb{R}^d)$, we have*

$$\int_0^\infty \int_F \left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y) \\ \leq \mathfrak{C}(k,a,t)(\nu_{k,a}(F))^{\frac{2t}{2\gamma+d+a-2}} C_h \left\|\|\xi\|^t \mathscr{F}_{k,a}(f)\right\|^2_{L^2_{k,a}(\mathbb{R}^d)},$$

where $\mathfrak{C}(k,a,t)$ is the constant given in Proposition 5.3.

Let F be a subset of $\mathbb{R}^d$. The *generalized Paley–Wiener space* $GPW_{k,a}(F)$ is defined as follows:

$$GPW_{k,a}(F) := \left\{ f \in L^2_{k,a}(\mathbb{R}^d) : \forall\, b > 0,\ \operatorname{supp} \Phi_h^{k,a}(f)(b,\cdot) \subset F \right\}.$$

Applying Plancherel's formula (2.18), the definition of generalized Paley–Wiener space and the previous corollary we obtain the following

Corollary 5.8. *Let E and F be two subsets of $\mathbb{R}^d$ such that $0 < \nu_{k,a}(E), \nu_{k,a}(F) < \infty$. Let $0 < s, t < \frac{2\gamma+d+a-2}{2}$.*

(i) *For any $f \in GPW_{k,a}(F)$, we have*

$$\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)} \leq \mathfrak{C}(k,a,t)(\nu_{k,a}(F))^{\frac{2t}{2\gamma+d+a-2}} \left\| \|\xi\|^t \mathscr{F}_{k,a}(f) \right\|^2_{L^2_{k,a}(\mathbb{R}^d)}.$$

(ii) *For any $f \in PW_{k,a}(E) \cap GPW_{k,a}(F)$, we have*

$$\|f\|^{s+t}_{L^2_{k,a}(\mathbb{R}^d)} \leq \frac{(\mathfrak{C}(k,a,t))^{\frac{s}{2}}(\mathfrak{C}(k,a,s))^{\frac{t}{2}}}{(C_h)^{\frac{t}{2}}} (\nu_{k,a}(E)\nu_{k,a}(F))^{\frac{ts}{2\gamma+d+a-2}} \times \left\| \|\xi\|^t \mathscr{F}_{k,a}(f) \right\|^s_{L^2_{k,a}(\mathbb{R}^d)} \left\| \|y\|^s \Phi_h^{k,a}(f) \right\|^t_{L^2_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)}.$$

Using Theorem 5.4 we establish another version of the Heisenberg-type uncertainty inequality for the (k,a)-generalized wavelet transforms.

Theorem 5.9. *Let $0 < s < \frac{2\gamma+d+a-2}{2}$ and $t > 0$. Then for any $f \in L^2_{k,a}(\mathbb{R}^d)$, we have*

$$\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)} \leq \mathscr{C}(k,a,s,t) \left\| \|y\|^s \Phi_h^{k,a}(f) \right\|^{\frac{2t}{s+t}}_{L^2_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)} \left\| \|\xi\|^t \mathscr{F}_{k,a}(f) \right\|^{\frac{2s}{s+t}}_{L^2_{k,a}(\mathbb{R}^d)},$$

where

$$\mathscr{C}(k,a,s,t) = \left(\frac{\mathfrak{C}(k,a,s) \left(\frac{d_k}{2\gamma+d+a-2}\right)^{\frac{2s}{2\gamma+d+a-2}}}{C_h} \right)^{\frac{t}{s+t}} \left[\left(\frac{s}{t}\right)^{\frac{t}{s+t}} + \left(\frac{t}{s}\right)^{\frac{s}{s+t}} \right].$$

Proof. Let $r > 0$. Then

$$\|f\|^2_{L^2_{k,a}(\mathbb{R}^d)} = \int_{\mathscr{B}_d(0,r)} |\mathscr{F}_{k,a}(f)(\xi)|^2 \, d\nu_{k,a}(\xi) + \int_{\mathscr{B}^c_d(0,r)} |\mathscr{F}_{k,a}(f)(\xi)|^2 \, d\nu_{k,a}(\xi), \tag{5.6}$$

where $\mathscr{B}_d(0,r)$ denotes the ball in $\mathbb{R}^d$ of center 0 and radius r.

From Theorem 5.4 and by simple calculations, we have

$$\int_{\mathscr{B}_d(0,r)} |\mathscr{F}_{k,a}(f)(\xi)|^2 \, d\nu_{k,a}(\xi) \leq \frac{\mathfrak{C}(k,a,s) \left(\frac{d_k}{2\gamma+d+a-2}\right)^{\frac{2s}{2\gamma+d+a-2}}}{C_h} r^{2s} \left\| \|y\|^s \Phi_h^{k,a}(f) \right\|^2_{L^2_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)}. \tag{5.7}$$

Moreover, it is easy to see that

$$\int_{\mathscr{B}_d^c(0,r)} |\mathscr{F}_{k,a}(f)(\xi)|^2\, d\nu_{k,a}(\xi) \leq r^{-2t} \int_{\mathbb{R}^d} \|\xi\|^{2t} |\mathscr{F}_{k,a}(f)(\xi)|^2\, d\nu_{k,a}(\xi). \tag{5.8}$$

Combining the relations (5.6), (5.7) and (5.8), we get

$$\begin{aligned} \|f\|^2_{L^2_{k,a}(\mathbb{R}^d)} \leq \frac{\mathfrak{C}(k,a,s)\left(\frac{d_k}{2\gamma+d+a-2}\right)^{\frac{2s}{2\gamma+d+a-2}}}{C_h} r^{2s} \big\| \|y\|^s \Phi_h^{k,a}(f)\big\|^2_{L^2_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)} \\ + r^{-2t} \big\| \|\xi\|^t \mathscr{F}_{k,a}(f)\big\|^2_{L^2_{k,a}(\mathbb{R}^d)}. \end{aligned}$$

If we choose

$$r = \left[\frac{t}{s} \frac{C_h}{\mathfrak{C}(k,a,s)\left(\frac{d_k}{2\gamma+d+a-2}\right)^{\frac{2s}{2\gamma+d+a-2}}}\right]^{\frac{1}{2s+2t}} \left(\frac{\big\| \|\xi\|^t \mathscr{F}_{k,a}(f)\big\|_{L^2_{k,a}(\mathbb{R}^d)}}{\big\| \|y\|^s \Phi_h^{k,a}(f)\big\|_{L^2_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)}}\right)^{\frac{1}{s+t}}$$

we obtain the desired inequality. □

We close this subsection with the following local uncertainty principle version:

Theorem 5.10 (Faris–Price's uncertainty principle for $\Phi_h^{k,a}$)**.** *Let η, p be two real numbers such that $0 < \eta < 2\gamma + d + a - 2$ and $p \geq 1$. Then there is a positive constant $C_{k,a}(\eta,p)$ such that for every f in $L^2_{k,a}(\mathbb{R}^d)$ and for every measurable subset $T \subset \mathbb{R}^{d+1}_+$ such that $0 < \mu_{k,a}(T) < \infty$, we have*

$$\begin{aligned} &\left(\int_T |\Phi_h^{k,a}(f)(b,y)|^p\, d\mu_{k,a}(b,y)\right)^{\frac{1}{p}} \\ &\quad \leq C_{k,a}(\eta,p)\big(\mu_{k,a}(T)\big)^{\frac{1}{p(p+1)}} \left\| \|\left(\tfrac{1}{b}, y\right)\|^\eta \Phi_h^{k,a}(f)\right\|^{\frac{4\gamma+2d+2a-4}{(2\gamma+d+a-2+\eta)(p+1)}}_{L^2_{\mu_{k,a}}(\mathbb{R}^{d+1}_+)} \\ &\quad \times \left(\|f\|_{L^2_{k,a}(\mathbb{R}^d)} \|h\|_{L^2_{k,a}(\mathbb{R}^d)}\right)^{\frac{(2\gamma+d+a-2+\eta)(p+1)-(4\gamma+2d+2a-4)}{(2\gamma+d+a-2+\eta)(p+1)}}. \end{aligned}$$

Proof. One can assume that $\|f\|_{L^2_{k,a}(\mathbb{R}^d)} = \|h\|_{L^2_{k,a}(\mathbb{R}^d)} = \sqrt{c_{k,a}}$; then, for every positive real number $r > 1$, we have

$$\|\Phi_h^{k,a}(f)\|_{L^p_{\mu_{k,a}}(T)} \leq \|\Phi_h^{k,a}(f)\mathbb{1}_{V_r}\|_{L^p_{\mu_{k,a}}(T)} + \|\Phi_h^{k,a}(f)\mathbb{1}_{V_r^c}\|_{L^p_{\mu_{k,a}}(T)},$$

where V_r denotes the subset of $\mathbb{R}^{d+1}_+$ given by

$$V_r := \left\{(b,y) \in \mathbb{R}^{d+1}_+ : \left\|\left(\tfrac{1}{b}, y\right)\right\| \leq r\right\}.$$

However, Hölder's inequality and (2.21) give that, for every $\eta \in (0, 2\gamma + d + a - 2)$,

$$\begin{aligned}
&\left\| \Phi_h^{k,a}(f)\mathbb{1}_{V_r} \right\|_{L^p_{\mu_{k,a}}(T)} \\
&\quad = \left(\int_{\mathbb{R}^{d+1}_+} \left| \Phi_h^{k,a}(f)(b,y) \right|^p \mathbb{1}_{V_r}(b,y)\mathbb{1}_T(b,y)\, d\mu_{k,a}(b,y) \right)^{\frac{1}{p}} \\
&\quad \leq \left(\int_{\mathbb{R}^{d+1}_+} \left| \Phi_h^{k,a}(f)(b,y) \right|^{\frac{p}{p+1}} \mathbb{1}_{V_r}(b,y)\mathbb{1}_T(b,y)\, d\mu_{k,a}(b,y) \right)^{\frac{1}{p}} \\
&\quad \leq \left(\mu_{k,a}(T)\right)^{\frac{1}{p(p+1)}} \left\| \Phi_h^{k,a}(f)\mathbb{1}_{V_r} \right\|^{\frac{1}{p+1}}_{L^1_{\mu_{k,a}}\left(\mathbb{R}^{d+1}_+\right)} \\
&\quad \leq \left(\mu_{k,a}(T)\right)^{\frac{1}{p(p+1)}} \left\| \|(\tfrac{1}{b}, y)\|^{\eta} \Phi_h^{k,a}(f) \right\|^{\frac{1}{p+1}}_{L^2_{\mu_{k,a}}\left(\mathbb{R}^{d+1}_+\right)} \left\| \|(\tfrac{1}{b}, y)\|^{-\eta} \mathbb{1}_{V_r} \right\|^{\frac{1}{p+1}}_{L^2_{\mu_{k,a}}\left(\mathbb{R}^{d+1}_+\right)}.
\end{aligned}$$

On the other hand, by simple calculations we see that

$$\begin{aligned}
&\left\| \|(\tfrac{1}{b}, y)\|^{-\eta} \mathbb{1}_{V_r} \right\|_{L^2_{\mu_{k,a}}\left(\mathbb{R}^{d+1}_+\right)} \\
&\qquad\qquad \leq \frac{\sqrt{d_k}\,\Gamma\left(\gamma + \frac{d+a-2}{2}\right)}{\sqrt{(4\gamma + 2d + 2a - 4 - 2\eta)\Gamma(2\gamma + d + a - 2)}}\, r^{2\gamma + d + a - 2 - \eta}.
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\left\| \Phi_h^{k,a}(f)\mathbb{1}_{V_r} \right\|_{L^p_{\mu_{k,a}}(T)} \\
&\quad \leq \left(\mu_{k,a}(T)\right)^{\frac{1}{p(p+1)}} \left(\frac{\sqrt{d_k}\,\Gamma\left(\gamma + \frac{d+a-2}{2}\right)}{\sqrt{(4\gamma + 2d + 2a - 4 - 2\eta)\Gamma(2\gamma + d + a - 2)}} \right)^{\frac{1}{p+1}} \\
&\qquad \times r^{\frac{2\gamma + d + a - 2 - \eta}{p+1}} \left\| \left\|(\tfrac{1}{b}, y)\right\|^{\eta} \Phi_h^{k,a}(f) \right\|^{\frac{1}{p+1}}_{L^2_{\mu_{k,a}}\left(\mathbb{R}^{d+1}_+\right)}.
\end{aligned}$$

On the other hand, and again by Hölder's inequality and the relation (2.21), we deduce that

$$\begin{aligned}
&\left\| \Phi_h^{k,a}(f)\mathbb{1}_{V_r^c} \right\|_{L^p_{\mu_{k,a}}(T)} \\
&\quad \leq \left(\mu_{k,a}(T)\right)^{\frac{1}{p(p+1)}} \left(\int_{\mathbb{R}^{d+1}_+} \left| \Phi_h^{k,a}(f)(b,y) \right|^2 \mathbb{1}_{V_r^c}(b,y)\, d\mu_{k,a}(b,y) \right)^{\frac{1}{p+1}} \\
&\quad \leq \left(\mu_{k,a}(T)\right)^{\frac{1}{p(p+1)}} \left\| \left\|(\tfrac{1}{b}, y)\right\|^{\eta} \Phi_h^{k,a}(f) \right\|^{\frac{2}{p+1}}_{L^2_{\mu_{k,a}}\left(\mathbb{R}^{d+1}_+\right)} r^{-\frac{2\eta}{p+1}}.
\end{aligned}$$

Hence, for every $\eta \in (0, 2\gamma+d+a-2)$,

$$\left(\int_T \left|\Phi_h^{k,a}(f)(b,y)\right|^p d\mu_{k,a}(b,y)\right)^{\frac{1}{p}}$$

$$\leq \left(\mu_{k,a}(T)\right)^{\frac{1}{p(p+1)}} \left\|\left\|\left(\tfrac{1}{b},y\right)\right\|^\eta \Phi_h^{k,a}(f)\right\|_{L^2_{\mu_{k,a}}\left(\mathbb{R}^{d+1}_+\right)}^{\frac{1}{p+1}}$$

$$\times \left(\left(\frac{\sqrt{d_k}\,\Gamma\left(\gamma+\frac{d+a-2}{2}\right)}{\sqrt{(4\gamma+2d+2a-4-2\eta)\Gamma(2\gamma+d+a-2)}}\right)^{\frac{1}{p+1}} r^{\frac{2\gamma+d+a-2-\eta}{p+1}} + \left\|\left\|\left(\tfrac{1}{b},y\right)\right\|^\eta \Phi_h^{k,a}(f)\right\|_{L^2_{\mu_{k,a}}\left(\mathbb{R}^{d+1}_+\right)}^{\frac{1}{p+1}} r^{-\frac{2\eta}{p+1}}\right).$$

In particular, the inequality holds for

$$r_0 = \frac{\left(\frac{2\eta}{2\gamma+d+a-2-\eta}\right)^{\frac{p+1}{2\gamma+d+a-2+\eta}}}{\left(\frac{\sqrt{d_k}\,\Gamma\left(\gamma+\frac{d+a-2}{2}\right)}{\sqrt{(4\gamma+2d+2a-4-2\eta)\Gamma(2\gamma+d+a-2)}}\right)^{\frac{1}{2\gamma+d+a-2+\eta}}} \left\|\left\|\left(\tfrac{1}{b},y\right)\right\|^\eta \Phi_h^{k,a}(f)\right\|_{L^2_{\mu_{k,a}}\left(\mathbb{R}^{d+1}_+\right)}^{\frac{1}{2\gamma+d+a-2+\eta}},$$

and therefore

$$\left(\int_T |\Phi_h^{k,a}(f)(b,y)|^p d\mu_{k,a}(b,y)\right)^{\frac{1}{p}}$$

$$\leq \left(\mu_{k,a}(T)\right)^{\frac{1}{p(p+1)}} \left(\frac{\sqrt{d_k}\,\Gamma\left(\gamma+\frac{d+a-2}{2}\right)}{\sqrt{(4\gamma+2d+2a-4-2\eta)\Gamma(2\gamma+d+a-2)}}\right)^{\frac{2\eta}{(2\gamma+d+a-2+\eta)(p+1)}}$$

$$\times \left\|\left\|\left(\tfrac{1}{b},y\right)\right\|^\eta \Phi_h^{k,a}(f)\right\|_{L_{\mu_{k,a}}\left(\mathbb{R}^{d+1}_+\right)}^{\frac{4\gamma+2d+2a-4}{(2\gamma+d+a-2+\eta)(p+1)}}$$

$$\times \left(\frac{2\eta}{2\gamma+d+a-2-\eta}\right)^{\frac{-2\eta}{2\gamma+d+a-2+\eta}} \left(\frac{2\gamma+d+a-2+\eta}{2\gamma+d+a-2-\eta}\right).$$

Now, the general formula follows from above by substituting f by $\frac{\sqrt{c_{k,a}}f}{\|f\|_{L^2_{k,a}(\mathbb{R}^d)}}$ and h by $\frac{\sqrt{c_{k,a}}h}{\|h\|_{L^2_{k,a}(\mathbb{R}^d)}}$. □

Remark 5.11. Proceeding as in Theorem 3.3, we prove that, if h is a radial (k,a)-generalized wavelet, then

$$\left\|\|\xi\|^t \mathscr{F}_{k,a}(f)\right\|^2_{L^2_{k,a}(\mathbb{R}^d)} = \frac{a^{\frac{2(2\gamma+d-2)}{a}}}{\mathscr{M}\left(\left|\mathscr{F}_{B,a}^{\frac{2\gamma+d-2}{a}}(H)\right|^2\right)(2t)} \int_{\mathbb{R}^{d+1}_+} b^{-2t}\left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y).$$

Thus, if we use this formula, we derive new uncertainty inequalities of the Heisenberg type written according to the terms

$$\int_{\mathbb{R}_+^{d+1}} b^{-2t}\left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y) \quad \text{and} \quad \int_{\mathbb{R}_+^{d+1}} \|y\|^{2s}\left|\Phi_h^{k,a}(f)(b,y)\right|^2 d\mu_{k,a}(b,y).$$

6. Perspectives

It is worth mentioning that in [16] the authors formulate an interesting conjecture stating that, for arbitrary $a > 0$, the modulus of the kernel $B_{k,a}$ on $\mathbb{R}^d$ is bounded above by 1 whenever the positive multiplicity function k satisfies $2\gamma + d + a - 3 \geq 0$ (in the authors' notation). This conjecture was proved in [16, pp. 13–14] for $d = 1$ and $a = 1$, i.e., $|B_{k,1}(\lambda, x)| \leq 1$ for all $\lambda, x \in \mathbb{R}$ and $\gamma \geq 1/2$. For $a = 2$, it is well known that the conjecture holds true for all $d \geq 1$. We also mention that in [7] it was proved that for $d \geq 2$ and for arbitrary $a = 2/n$, with $n \in \mathbb{N}$, the modulus of the kernel $B_{k,a}$ is bounded above by 1, without additional conditions on the multiplicity function k.

So, we note that if the previous conjecture is true, the following holds: If we extend the definition of the (k, a)-generalized translation operator given by (2.10) for any $d \geq 1$, and if we extend the definition of the (k, a)-generalized wavelet transform by (2.16) when $d \geq 1$ and $\frac{2}{a} \in \mathbb{N}$, and by (2.12) when $d \geq 1$, $a \in \mathbb{Q}_+$ and $\frac{2}{a} \in \mathbb{Q}_+ \setminus \mathbb{N}$, then the proofs and results of this paper are true for arbitrary $a \in \mathbb{Q}_+$ and positive multiplicity function k satisfying $2\gamma + d + a - 3 \geq 0$. The same conclusion of the Calderón reproducing formula of [34] is also valid if the previous conjecture is true.

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