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INVARIANTS OF FORMAL PSEUDODIFFERENTIAL OPERATOR ALGEBRAS AND ALGEBRAIC MODULAR FORMS

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Abstract. We study the question of extending an action of a group Γ on a commutative domain R to a formal pseudodifferential operator ring $B = R(\!(x\,;\,d)\!)$ with coefficients in R, as well as to some canonical quadratic extension $C = R(\!(x^{1/2}\,;\,\frac{1}{2}d)\!)_2$ of B. We give conditions for such an extension to exist and describe under suitable assumptions the invariant subalgebras B^Γ and C^Γ as Laurent series rings with coefficients in R^Γ. We apply this general construction to the numbertheoretical context of a subgroup Γ of $\mathrm{SL}(2,\mathbb{C})$ acting by homographies on an algebra R of functions in one complex variable. The subalgebra C_0^Γ of invariant operators of nonnegative order in C^Γ is then linearly isomorphic to the product space $\mathcal{M}_0 = \prod_{j\geq 0} M_j$, where M_j is the vector space of algebraic modular forms of weight j in R. We obtain a structure of noncommutative algebra on $\mathcal{M}_0$, which can be identified with a space of algebraic Jacobi forms. We study properties of the correspondence $\mathcal{M}_0 \to C_0^\Gamma$, whose restriction to even weights was previously known, using arithmetical arguments and the algebraic results of the first part of the article.

Introduction

Several studies in deformation or quantization theories have shown in various contexts that significant combinations of Rankin–Cohen brackets on modular forms of even weights correspond by isomorphic transfer to the noncommutative composition product in some associative algebras of invariant operators (see for instance [19, 4, 16, 2, 3, 14, 18, 11]). The main goal of this paper is to produce such a correspondence in a formal algebraic setting for modular forms of any weight (even or odd), which allows the construction to be applied to Jacobi forms.

The first part deals with the general algebraic problem of extending a group action from a ring R to a ring of formal pseudodifferential operators with coefficients in R, and to describe the subring of invariant operators. More precisely, let R be a commutative domain of characteristic zero containing the subfield of rational

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numbers, $U(R)$ its group of invertible elements, d a derivation of R, and Γ a group acting by automorphisms on R. We denote by $B = R((x;d))$ the ring of formal pseudodifferential operators with coefficients in R. The elements of B are the Laurent power series in one indeterminate x with coefficients in R, and the noncommutative product in B is defined from the commutation law

$$xf = \sum_{i\geq 0} d^i(f)x^{i+1} \quad \text{for any } f \in R.$$

We also consider some quadratic extension $C = R((y;\delta))_2$ of B, with $y^2 = x$, $\delta = \frac{1}{2}d$ and the commutation law

$$yf = \sum_{k\geq 0} \left(\tfrac{(2k)!}{2^k(k!)^2}\right) \delta^k(f)y^{2k+1} \quad \text{for any } f \in R.$$

This type of skew power series rings was already introduced in various ring-theoretical papers, see references in [6]. We prove (Theorem 1.2.5) that the action of Γ on R extends to an action by automorphisms on B if and only if there exists a multiplicative 1-cocycle $p : \Gamma \to U(R)$ such that $\gamma d\gamma^{-1} = p_\gamma.d$ for any $\gamma \in \Gamma$. We describe under this assumption all possible extensions of the action; they are parametrized by the arbitrary choice of a map $r : \Gamma \to R$ satisfying some compatibility condition related to p. We give (Theorem 1.3.1) sufficient conditions for the ring B^Γ of invariant operators to be described as a ring of formal pseudodifferential operators with coefficients in R^Γ. We prove (Theorems 1.2.9 and 1.3.2) similar results for the ring C; in this case a necessary and sufficient condition to extend the action of Γ from R to C is the existence of some multiplicative 1-cocycle $s : \Gamma \to U(R)$ satisfying $\gamma d\gamma^{-1} = s_\gamma^2.d$ for any $\gamma \in \Gamma$. The subring B is then necessarily stable under the extended actions, and B^Γ is a subring of C^Γ.

These general results are applied in the second part of the paper to the case of the complex homographic action. We fix a $\mathbb{C}$-algebra R of functions in one variable z stable under the standard derivation ∂_z and a subgroup Γ of $\mathrm{SL}(2,\mathbb{C})$ acting on R by

$$(f.\gamma)(z) = f\left(\tfrac{az+b}{cz+d}\right) \quad \text{for any } f \in R \text{ and } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma.$$

Applying the previous general construction for the derivation $d = -\partial_z$ and for the 1-cocycles $s : \gamma \mapsto cz+d$ and $p = s^2$, we define an action of Γ on $C = R((y;-\frac{1}{2}\partial_z))_2$. We give in Theorems 2.2.3 and 2.2.5 a combinatorial description of this action, whose restriction to $B = R((x;-\partial_z))$ with $x = y^2$ corresponds to the action already introduced in [4, Section 1]. We consider, for any $k \in \mathbb{Z}$, the subspace C_k^Γ of elements in C^Γ whose valuation related to y is greater or equal to k. It is easy to check that the image of the canonical projection $\pi_k : C_k^\Gamma \to R$ is included in the subspace M_k of algebraic modular forms of weight k. We construct a family $(\psi_k)_{k\geq 0}$ of splitting maps $\psi_k : M_k \to C_k^\Gamma$ which gives rise canonically (Theorem 2.3.2) to a vector space isomorphism

$$\Psi : \mathcal{M}_0 \to C_0^\Gamma, \quad \text{with } \mathcal{M}_0 = \prod_{k\geq 0} M_k.$$

This correspondence makes it possible, on the one hand, to identify C_k^Γ with a vector space of algebraic Jacobi forms of weight k (see Corollary 2.3.3), and on the other, to obtain by transfer a noncommutative product on $\mathcal{M}_0$. This multiplication on modular forms of nonnegative weights (even or odd) can be described in terms of linear combinations of Rankin–Cohen brackets with combinatorial rational coefficients

$$f \star g = \Psi^{-1}(\psi_k(f).\psi_\ell(g)) = \sum_{n\geq 0} \alpha_n(k,\ell)[f,g]_n$$

for any $f \in M_k$, $g \in M_\ell$, $k,\ell \geq 0$. The restriction to modular forms of even nonnegative weights gives the isomorphism

$$\Psi_2 : \mathcal{M}_0^{\mathrm{ev}} \to B_0^\Gamma, \quad \text{with } \mathcal{M}_0^{\mathrm{ev}} = \prod_{k\geq 0} M_{2k} \text{ and } B_0^\Gamma = B \cap C_0^\Gamma,$$

considered in [4, Section 3] (see also [16, 17]). We discuss at the end of Subsection 2.3 the obstructions for possible extensions of Theorem 2.3.2 to modular forms of negative weights. The purpose of Subsection 2.4 is to go further in the study of the isomorphism Ψ_2 using the structure of B_0^Γ deduced from the general Theorem 1.3.1. More precisely, assuming that R contains an invertible modular form χ of weight 2, B_0^Γ is the skew power series algebra $R^\Gamma[[u;D]]$ with coefficients in R^Γ, where u denotes the invariant operator $x\chi$ and D is the derivation $-\chi^{-1}\partial_z$ of R^Γ. A natural question is then to describe as a sequence of modular forms the inverse image by Ψ_2 of any invariant operator in B_0^Γ, or equivalently of any power u^k for $k \geq 0$. The value of $\Psi_2^{-1}(u^k)$ is entirely determined by some family $(g_{k,k+i})_{i,k\geq 0}$ of modular forms of weight $2k+2i$, which are zero if i is odd, and which can be calculated for even i's in terms of Rankin–Cohen brackets of powers of the modular form χ. Subsection 2.5 explores some ways of extending the previous results to negative odd weights. We finally mention in Subsection 2.6 as an open question the possibility of considering the whole study of the second part of this paper for other choices of the parameter r in the extension of the initial action from functions to operators.

1. Actions and invariants on pseudodifferential and quadratic pseudodifferential operator rings

1.1. **Definitions and preliminary results.** We fix a commutative domain R of characteristic zero containing $\mathbb{Q}$. We denote by $U(R)$ the group of invertible elements in R. We denote by $R\langle\langle x\rangle\rangle$ the left R-module of formal power series $\sum_{i\geq 0} f_i x^i$ with $f_i \in R$ for any $i \geq 0$. It is well known that, for any nonzero derivation d of R, we can define an associative noncommutative product $*_d$ on $R\langle\langle x\rangle\rangle$ by bilinear extension of the canonical identities $f *_d x^n = fx^n$ and $x *_d x^n = x^n *_d x = x^{n+1}$ for any $f \in R$ and $n \geq 0$, and from the commutation law

$$x *_d f = fx + d(f)x^2 + d^2(f)x^3 + \cdots = \sum_{n\geq 0} d^n(f)x^{n+1} \quad \text{for any } f \in R, \tag{1.1}$$

which implies, more generally, for any integer $i \geq 0$,

$$\begin{aligned} x^{i+1} *_d f &= fx^{i+1} + (i+1)d(f)x^{i+2} + \tfrac{(i+1)(i+2)}{2}d^2(f)x^{i+3} + \cdots \\ &= \textstyle\sum_{n\geq i} \binom{n}{i} d^{n-i}(f)x^{n+1}. \end{aligned} \tag{1.2}$$

We thus obtain a noncommutative ring denoted by $B_0 = R[[x;d]]$. The element x generates a two-sided ideal in B_0, and we can consider the localized ring $B = R((x;d))$ of B_0 with respect to the powers of x. The elements of B are the Laurent series $q = \sum_{i\geq m} f_i x^i$ with $m \in \mathbb{Z}$ and $f_i \in R$ for any $i \geq m$. The valuation $v_x(q)$ of q is defined by $v_x(q) = \min\{i \in \mathbb{Z} : f_i \neq 0\}$ for $q \neq 0$ and $v_x(0) = \infty$. The map v_x is a valuation function $B \to \mathbb{Z} \cup \{\infty\}$, and B_0 is the subring $\{q \in B : v_x(q) \geq 0\}$. It follows in particular that B_0 and B are domains. The invertible elements of B are the series $q = \sum_{i\geq m} f_i x^i$ such that $f_m \in U(R)$.

Definition 1.1.1. The noncommutative domain $B = R((x;d))$ is called *the ring of formal pseudodifferential operators in d with coefficients in R.*

Denoting by ∂ the derivation $-d$, relation (1.1) can be rewritten as

$$x^{-1} *_d f = fx^{-1} + \partial(f) \quad \text{for any } f \in R. \tag{1.3}$$

It is well known that this relation defines an associative noncommutative product in the subset A of polynomials in the variable $t = x^{-1}$ with coefficients in R (see for instance [10, pp. 11, 19]). This subset A is then a subring of B, denoted by $A = R[t;\partial]$.

Definition 1.1.2. The noncommutative domain $R[t;\partial]$ is called the ring of formal differential operators in ∂ with coefficients in R.

Denoting by y instead of x the variable in the left R-module $R\langle\langle y\rangle\rangle$, another structure of noncommutative ring (see [6, Example 1.3 (d)]) can be defined on $R\langle\langle y\rangle\rangle$ for any nonzero derivation δ of R from similar canonical identities and from the commutation law

$$\begin{aligned} y *_{\delta,2} f &= fy + \delta(f)y^3 + \tfrac{3}{2}\delta^2(f)y^5 + \tfrac{5}{2}\delta^3(f)y^7 + \tfrac{35}{8}\delta^4(f)y^9 + \cdots \\ &= \sum_{k\geq 0} \left(\tfrac{(2k)!}{2^k(k!)^2}\right)\delta^k(f)y^{2k+1}, \end{aligned} \tag{1.4}$$

which implies, more generally, for any integer $i \geq 0$,

$$y^{i+1} *_{\delta,2} f = fy^{i+1} + \sum_{k\geq 1}\left(\prod_{j=1}^{k} \tfrac{2(j-1)+i+1}{j}\right)\delta^k(f)y^{2k+i+1} \quad \text{for any } f \in R. \tag{1.5}$$

This ring is denoted by $C_0 = R[[y;\delta]]_2$. As in the previous case, C_0 can be embedded in the localization $C = R((y;\delta))_2$ of the Laurent series $q = \sum_{i\geq m} f_i y^i$ with $m \in \mathbb{Z}$ and $f_i \in R$ for any $i \geq m$. The valuation $v_y(q)$ is defined by $v_y(q) = \min\{i \in \mathbb{Z} : f_i \neq 0\}$ for $q \neq 0$ and $v_y(0) = \infty$. The map v_y is a valuation function $C \to \mathbb{Z} \cup \{\infty\}$, and C_0 is the subring $\{q \in B : v_y(q) \geq 0\}$. In particular, C_0 and C are domains. The invertible elements of C are the series $q = \sum_{i\geq m} f_i y^i$ such that $f_m \in U(R)$.

Definition 1.1.3. The noncommutative domain $C = R((y;\delta))_2$ is called *the ring of quadratic formal pseudodifferential operators in δ with coefficients in R.*

By elementary calculations it follows from (1.5) that

$$y^2 *_{\delta,2} f = fy^2 + 2\delta(f)y^4 + 4\delta^2(f)y^6 + \cdots = \sum_{k\geq 1} (2\delta)^{k-1}(f)y^{2k} \quad \text{for any } f \in R, \tag{1.6}$$

which is equivalent to

$$y^{-2} *_{\delta,2} f = fy^{-2} - 2\delta(f) \quad \text{for any } f \in R. \tag{1.7}$$

It follows from this observation that the submodule of Laurent series in even powers of y is a subring B of C, and the restriction to B of the product $*_{\delta,2}$ is the product $*_{2\delta}$ defined by (1.1) and (1.3). Then B can be identified with $R((x;d))$ setting $x = y^2$ and $d = 2\delta$.

Convention. In the rest of the paper, we no longer use the notations $*_{\delta,2}$ and $*_d$, and denote as usual by qq' the product of two elements q and q' of C (and, in particular, of B). We can then formulate some direct consequences of the embedding of B into C in the following proposition.

Proposition 1.1.4. *Let R be a commutative domain of characteristic zero containing $\mathbb{Q}$. Let d be a nonzero derivation of R.*

(i) *The ring of quadratic pseudodifferential operators $C = R((y;\delta))_2$ for $\delta = \frac{1}{2}d$ contains as a subring the ring of pseudodifferential operators $B = R((x;d))$ for $x = y^2$, and therefore the ring of differential polynomials $A = R[t;\partial]$ for $t = x^{-1} = y^{-2}$ and $\partial = -d$, i.e.,*

$$A = R[t;-d] \subset B = R((x;d)) \subset C = R((y;\tfrac{1}{2}d))_2.$$

(ii) *We have $v_y(q) = 2v_x(q)$ for any $q \in B$, and the subrings $B_0 = R[[x;d]]$ and $C_0 = R[[y;\delta]]_2$ satisfy $B_0 = B \cap C_0$.*

(iii) *We have $C = B \oplus By$, with $By = yB$, $(By)(By) = B$ and $(By)B = B(By) = By$.*

Proof. It follows by obvious calculations from identities (1.6) and (1.7). □

The following proposition shows that any automorphism of B (respectively C) stabilizing R is continuous with respect to the x-adic (respectively the y-adic) topology. A similar property is proved in [1] for different commutation laws and in the particular case where R a field. We start with a preliminary technical result.

Lemma 1.1.5. *Data and notations are those of the previous proposition. Let ℓ be a positive integer.*

(i) *For any $q \in B$ of the form $q = 1 + \sum_{i\geq 1} f_i x^i$ with $f_i \in R$ for any $i \geq 1$, there exists an element r in B such that $q = r^\ell$.*

(ii) *For any $q \in C$ of the form $q = 1 + \sum_{i\geq 1} f_i y^i$ with $f_i \in R$ for any $i \geq 1$, there exists an element r in C such that $q = r^\ell$.*

Proof. The proof is the same in the two cases above and we write it for $B = R((x;d))$. We consider in B an element of nonnegative valuation $r = \sum_{i\geq 0} g_i x^i$ with $g_i \in R$ for any $i \geq 0$. For any integer $\ell \geq 1$, we set $r^\ell = \sum_{i\geq 0} g_{\ell,i} x^i$ with $g_{\ell,i} \in R$ for any $i \geq 0$. By a straightforward induction using (1.2), we check that $g_{\ell,i} =$

$\ell g_0^{\ell-1} g_i + h_{\ell,i}$, where the remainder $h_{\ell,i}$ depends only on previous $g_0, g_1, \dots, g_{i-1}$ and their images by some powers of d. Then, for any sequence $(f_i)_{i\geq 1}$ of elements in R, we can determine inductively a unique sequence $(g_i)_{i\geq 0}$ such that $g_0 = 1$ and $g_{\ell,i} = f_i$ for any $i \geq 1$, that is, $\big(\sum_{i\geq 0} g_i x^i\big)^\ell = 1 + \sum_{i\geq 1} f_i x^i$. ☐

Proposition 1.1.6. *Data and notations are those of the previous proposition.*

- (i) *Let γ be an automorphism of B whose restriction to R is an automorphism of R. Then $v_x(\gamma(q)) = v_x(q)$ for any $q \in B$. In particular, the restriction of γ to B_0 determines an automorphism of B_0.*
- (ii) *Let γ be an automorphism of C whose restriction to R is an automorphism of R. Then $v_y(\gamma(q)) = v_y(q)$ for any $q \in C$.*
- (iii) *Let γ be an automorphism of C whose restriction to R is an automorphism of R. Then the restriction of γ to B determines an automorphism of B.*

Proof. Let γ be an automorphism of B such that $\gamma(R) = R$. This implies $\gamma^{-1}(R) = R$. We introduce $m = v_x(\gamma(x)) \in \mathbb{Z}$ and suppose that $m < 0$. Then the element $z = 1 + x^{-1}$ satisfies $\gamma(z) = 1 + \gamma(x)^{-1} \in B_0$. Consider by point (i) of Lemma 1.1.5 an element $r \in B_0$ such that $r^2 = \gamma(z)$. Applying the automorphism γ^{-1} we have $v_x(z) = 2v_x(\gamma^{-1}(r))$, which gives a contradiction because $v_x(z) = -1$ by definition. Hence we have proved that $m \geq 0$. It follows in particular that $\gamma(B_0) \subset B_0$. Using the same argument for γ^{-1}, we conclude that $\gamma(B_0) = B_0$. In other words, the restrictions of γ and γ^{-1} to B_0 determine automorphisms of B_0.

We prove now that $m = 1$. In B_0, we can write $\gamma(x) = f(1 + w)x^m$, with $f \in U(R)$ (because $\gamma(x)$ is invertible in B) and $w \in B_0$ such that $v_x(w) \geq 1$. It follows that

$$x = \gamma^{-1}(f)\gamma^{-1}(1 + w)\gamma^{-1}(x)^m,$$

and then $v_x(\gamma^{-1}(f)) + v_x(\gamma^{-1}(1 + w)) + mv_x(\gamma^{-1}(x)) = 1$. Since $\gamma^{-1}(R) = R$, we have $v_x(\gamma^{-1}(f)) = 0$. The element $1 + w$ is invertible in B_0, hence $\gamma^{-1}(1 + w)$ is invertible in B_0, therefore $v_x(\gamma^{-1}(1+w)) = 0$. We conclude that $mv_x(\gamma^{-1}(x)) = 1$, then $m = 1$ and the proof of point (i) is complete. The proof of point (ii) is similar replacing B_0 by C_0, x by y and v_x by v_y.

Let γ be an automorphism of C such that $\gamma(R) = R$. We deduce from relation (1.7) that $\gamma(y)^{-2}\gamma(f) - \gamma(f)\gamma(y)^{-2} = -\gamma(d(f))$ for any $f \in R$. With the notation $z = \gamma(y)^{-2}$, we obtain $zf - fz = -\gamma d\gamma^{-1}(f) \in R$ for any $f \in R$. By point (ii), we have $v_y(z) = -2$. Using point (iii) of Proposition 1.1.4, we have $z = q + q'y$, with $q, q' \in B$, $v_y(q) = -2$, $v_y(q') \geq -2$. Since $qf - fq \in B$ for any $f \in R$, we deduce from previous identity that $q'yf - fq'y \in B$ for any $f \in R$. Suppose that $q' \neq 0$ and set $q'y = \sum_{i\geq \ell} f_{2i+1}y^{2i+1}$, with $\ell \geq -1$, $f_{2i+1} \in R$ for any $i \geq -1$ and $f_{2\ell+1} \neq 0$. Using (1.5), we have in C, for any $f \in R$, the expansion $q'yf - fq'y = (2\ell+1)f_{2\ell+1}\delta(f)y^{2\ell+3} + \cdots$, which is incompatible with the fact that $q'yf - fq'y \in B$ and δ is a nonzero derivation. Then we have necessarily $q' = 0$, that is, $z \in B$. In other words, $\gamma(x^{-1}) \in B$. Hence $\gamma(x) \in B$, and the proof is complete. ☐

1.2. **Extension to B and C of actions by automorphisms on R.**

Notations 1.2.1. We take all data and notations of Proposition 1.1.4. We consider a group Γ acting by automorphisms on the ring R. We denote this action on the right as follows:

$$(f \cdot \gamma) \cdot \gamma' = f \cdot \gamma\gamma' \quad \text{for all } f \in R,\ \gamma, \gamma' \in \Gamma. \tag{1.8}$$

A 1-cocycle for the action of Γ on the group $U(R)$ is a map $s : \Gamma \to U(R)\,,\, \gamma \mapsto s_\gamma$ satisfying

$$s_{\gamma\gamma'} = (s_\gamma \cdot \gamma')s_{\gamma'} \quad \text{for all } \gamma, \gamma' \in \Gamma. \tag{1.9}$$

We denote by $Z^1(\Gamma, U(R))$ the multiplicative abelian group of such 1-cocycles.

We answer the following two questions: give necessary and sufficient conditions for the existence of an action of Γ by automorphisms on B or C extending the given action on R, and describe all possible extended actions. We need some definitions.

Definitions 1.2.2. Data and notations are those of 1.2.1.

(i) The action of Γ on R is said to be d-compatible when, for any $\gamma \in \Gamma$, there exists $p_\gamma \in U(R)$ such that

$$d(f) \cdot \gamma = p_\gamma d(f \cdot \gamma) \quad \text{for any } f \in R. \tag{1.10}$$

This condition defines uniquely a map $p : \Gamma \to U(R)\,,\, \gamma \mapsto p_\gamma$, and $p \in Z^1(\Gamma, U(R))$. This map p is called the 1-cocycle associated to the d-compatibility.

(ii) The action of Γ on R is said to be quadratically d-compatible when there exists an element $s \in Z^1(\Gamma, U(R))$ such that

$$d(f) \cdot \gamma = s_\gamma^2 d(f \cdot \gamma) \quad \text{for any } \gamma \in \Gamma \text{ and any } f \in R. \tag{1.11}$$

Such a map s is called a 1-cocycle associated to the quadratic d-compatibility. It is not necessarily unique since any map $s' = \epsilon s$, where ϵ is a multiplicative function $\Gamma \to \{-1, +1\}$, is another element of $Z^1(\Gamma, U(R))$ satisfying (1.11).

Remark 1.2.3. Any quadratically d-compatible action is also d-compatible. Since $\delta = \frac{1}{2}d$ and $\partial = -d$, the d-compatibility is obviously equivalent to the δ-compatibility or the ∂-compatibility.

Examples 1.2.4. 1. For any commutative field $\Bbbk$ of characteristic zero, the group $\Gamma = \Bbbk \rtimes \Bbbk^\times$ for the product $(\mu, \lambda)(\mu', \lambda') = (\lambda\mu' + \mu, \lambda\lambda')$ acts by $\Bbbk$-automorphisms on the domain $R = \Bbbk[z]$ by $(f \cdot \gamma)(z) = f(\lambda z + \mu)$ for $\gamma = (\mu, \lambda) \in \Gamma$. It is clear that $\partial_z(f) \cdot \gamma = \lambda^{-1}\partial_z(f \cdot \gamma)$ for any polynomial $f \in R$. Hence this action is ∂_z-compatible, with $p \in Z^1(\Gamma, \Bbbk^\times)$ defined by $p : (\mu, \lambda) \mapsto \lambda^{-1}$. If $\Bbbk$ is not algebraically closed, the action is not necessarily quadratically ∂_z-compatible.

2. Let $\Bbbk$ be any commutative field of characteristic zero and $R = \Bbbk(z)$ the field of rational functions in one variable with coefficients in $\Bbbk$. The group $\Gamma = \mathrm{SL}(2, \Bbbk)$ acts by $\Bbbk$-automorphisms on R by $(f \cdot \gamma)(z) = f\left(\frac{\lambda z + \mu}{\eta z + \xi}\right)$, where $\gamma = \left(\begin{smallmatrix} \lambda & \mu \\ \eta & \xi \end{smallmatrix}\right) \in \Gamma$. We have $\partial_z(f) \cdot \gamma = (\eta z + \xi)^2 \partial_z(f \cdot \gamma)$ for any rational function $f \in R$. Hence this action

is quadratically ∂_z-compatible, with $s \in Z^1(\Gamma, R^\times)$ defined by $s : \left(\begin{smallmatrix}\lambda & \mu \\ \eta & \xi\end{smallmatrix}\right) \mapsto \eta z + \xi$. This kind of action is the main object of the second part of the paper.

We are now able to answer for B the questions formulated at the beginning of this subsection.

Theorem 1.2.5. *The action of Γ by automorphisms on R extends to an action of Γ by automorphisms on $B = R((x;d))$ if and only if it is d-compatible. We then have*

$$x^{-1} \cdot \gamma = p_\gamma x^{-1} + p_\gamma r_\gamma \quad \text{for any } \gamma \in \Gamma, \tag{1.12}$$

where $p \in Z^1(\Gamma, U(R))$ is the 1-cocycle associated to the d-compatibility and r is an arbitrary map $\Gamma \to R$ satisfying the identity

$$r_{\gamma\gamma'} = r_{\gamma'} + p_{\gamma'}^{-1}(r_\gamma \cdot \gamma') \quad \text{for all } \gamma, \gamma' \in \Gamma. \tag{1.13}$$

In particular, this action extends to an action on B if and only if it extends to an action on the subring $A = R[x^{-1}; -d]$.

Proof. Suppose that Γ acts by automorphisms on B with $R \cdot \gamma = R$ for any $\gamma \in \Gamma$. We can apply point (i) of Proposition 1.1.6 to write

$$x^{-1} \cdot \gamma = g_{-1} x^{-1} + g_0 + \sum_{j \geq 1} g_j x^j,$$

with $g_j \in R$ for any $j \geq -1$ and $g_{-1} \neq 0$. Moreover, $x^{-1} \in U(B)$ implies that $(x^{-1} \cdot \gamma) \in U(B)$, which is equivalent to $g_{-1} \in U(R)$. Applying γ to (1.3), we obtain $(x^{-1} \cdot \gamma)(f \cdot \gamma) - (f \cdot \gamma)(x^{-1} \cdot \gamma) = -d(f) \cdot \gamma$ for any $f \in R$. Since $f \cdot \gamma \in R$, we can derive the identity

$$\begin{aligned}[g_{-1}x^{-1}(f \cdot \gamma) - (f \cdot \gamma)g_{-1}x^{-1}] + [g_0(f \cdot \gamma) - (f \cdot \gamma)g_0] \\ + \sum_{j \geq 1} [g_j x^j (f \cdot \gamma) - (f \cdot \gamma) g_j x^j] = -d(f) \cdot \gamma.\end{aligned}$$

The first term is $g_{-1}[x^{-1}(f \cdot \gamma) - (f \cdot \gamma)x^{-1}] = -g_{-1} d(f \cdot \gamma) \in R$. The second term is zero by commutativity of R. The third term is of valuation ≥ 1. So we deduce that

$$-g_{-1} d(f \cdot \gamma) = -d(f) \cdot \gamma \quad \text{and} \quad \sum_{j \geq 1} [g_j x^j (f \cdot \gamma) - (f \cdot \gamma) g_j x^j] = 0 \qquad \text{for any } f \in R.$$

Setting $p_\gamma = g_{-1}$, we have $p_\gamma \in U(R)$ and the first equality means by definition that the action of Γ is d-compatible. We claim that the second assertion implies $g_j = 0$ for all $j \geq 1$. Suppose that there exists a minimal index $m \geq 1$ such that $g_m \neq 0$. Calculating with relation (1.2), we have $\sum_{j \geq m} [g_j x^j (f \cdot \gamma) - (f \cdot \gamma) g_j x^j] = 0$, then $m g_m d(f \cdot \gamma) x^{m+1} + \cdots = 0$ and therefore $d(f \cdot \gamma) = 0$ for any $f \in R$. It follows by d-compatibility of the action that $d = 0$, hence we get a contradiction. We have proved that $x^{-1} \cdot \gamma = g_{-1} x^{-1} + g_0$ with $p_\gamma = g_{-1} \in U(R)$ satisfying Definition 1.2.2 (i). Then we set $r_\gamma = (g_{-1})^{-1} g_0$. We have $x^{-1} \cdot \gamma = p_\gamma x^{-1} + p_\gamma r_\gamma$. Relations (1.9) for p and (1.13) for r follow from relation $(x^{-1} \cdot \gamma) \cdot \gamma' = x^{-1} \cdot \gamma\gamma'$.

Conversely, let us assume that the action of Γ on R is d-compatible. Denote by p the 1-cocycle associated to the d-compatibility. Let us choose a map $r : \Gamma \to R$

satisfying (1.13); the existence of such maps is discussed in Examples 1.2.7 below. For any $\gamma \in \Gamma$, we set $q_\gamma = p_\gamma r_\gamma$. To prove that the action on R can be extended to B, it is enough to check that we have $(x^{-1}f)\cdot\gamma = (x^{-1}\cdot\gamma)(f\cdot\gamma)$ for any $f \in R$, where $x^{-1}\cdot\gamma$ is defined by formula (1.12). So we calculate, for $f \in R$,

$$(p_\gamma x^{-1} + q_\gamma)(f\cdot\gamma) - (f\cdot\gamma)(p_\gamma x^{-1} + q_\gamma) \\ = p_\gamma(x^{-1}(f\cdot\gamma) - (f\cdot\gamma)x^{-1}) = -p_\gamma d(f\cdot\gamma).$$

Using (1.10) we obtain $(p_\gamma x^{-1} + q_\gamma)(f\cdot\gamma) - (f\cdot\gamma)(p_\gamma x^{-1} + q_\gamma) = -d(f)\cdot\gamma$ for any $f \in R$. Hence by (1.3) we obtain an action of Γ by automorphisms on the polynomial algebra A extending the initial action on R. The element $p_\gamma + q_\gamma x$ is invertible in B_0 and the element $x^{-1}\cdot\gamma = p_\gamma x^{-1} + q_\gamma$ is invertible in B because $p_\gamma \in U(R)$. Then we define an action of Γ by automorphisms on B extending the action on A by setting $x\cdot\gamma = (x^{-1}\cdot\gamma)^{-1} = x(p_\gamma + q_\gamma x)^{-1}$. The condition $(u\cdot\gamma)\cdot\gamma' = u\cdot\gamma\gamma'$ for any $\gamma,\gamma' \in \Gamma$ and any $u \in B$ follows from (1.9) for p and (1.13) for r. ☐

Corollary 1.2.6. *If the action of Γ by automorphisms on R is d-compatible, then it extends to an action of Γ by automorphisms on B defined by*

$$x^{-1}\cdot\gamma = p_\gamma x^{-1}, \text{ or equivalently } x\cdot\gamma = xp_\gamma^{-1} = \sum_{j\geq 0} d^j(p_\gamma^{-1})x^{j+1} \quad \text{for any } \gamma \in \Gamma,$$

where $p \in Z^1(\Gamma, U(R))$ is the 1-cocycle associated to the d-compatibility.

Proof. We just apply Theorem 1.2.5 for the trivial map r defined by $r_\gamma = 0$ for any $\gamma \in \Gamma$, which obviously satisfies (1.13). ☐

Examples 1.2.7. We consider a d-compatible action of Γ on R. Let p be the 1-cocycle associated to the d-compatibility. Relation (1.13) can be interpreted as a 1-cocycle condition for the right action of Γ on R defined by $\langle f \mid \gamma\rangle = p_\gamma^{-1}(f\cdot\gamma)$ for any $\gamma \in \Gamma$ and $f \in R$. We denote by $Z_p^1(\Gamma, R)$ the additive group of maps $r : \Gamma \to R$ satisfying (1.13). We consider here various examples for the choice of $r \in Z_p^1(\Gamma, R)$.

1. The case $r = 0$ corresponds to the extension described in Corollary 1.2.6. If r is a coboundary (i.e., there exists $f \in R$ such that $r_\gamma = p_\gamma^{-1}(f\cdot\gamma) - f$ for any $\gamma \in \Gamma$), then we can suppose up to a change of variables that $r = 0$, because the element $x' = (x^{-1} - f)^{-1}$ satisfies $B = R((x'; d)) = R((x; d))$ and $x'^{-1}\cdot\gamma = p_\gamma x'^{-1}$ for any $\gamma \in \Gamma$.
2. A straightforward calculation proves that the map $r : \Gamma \to R$, defined by $r_\gamma = -p_\gamma^{-1}d(p_\gamma)$ for any $\gamma \in \Gamma$, is an element of $Z_p^1(\Gamma, R)$. The corresponding action of Γ on B is given by $x^{-1}\cdot\gamma = p_\gamma x^{-1} - d(p_\gamma) = x^{-1}p_\gamma$ for any $\gamma \in \Gamma$.
3. For any $r \in Z_p^1(\Gamma, R)$ and any $\kappa \in R^\Gamma$, κr is an element of $Z_p^1(\Gamma, R)$. The corresponding action of Γ on B is given by $x^{-1}\cdot\gamma = p_\gamma x^{-1} + \kappa p_\gamma r_\gamma$ for any $\gamma \in \Gamma$. If we suppose moreover that $\kappa \in U(R)$, then $x' = (\kappa^{-1}x^{-1})^{-1}$ satisfies $B = R((x; d)) = R((x'; \kappa^{-1}d))$, and we obtain $x'^{-1}\cdot\gamma = p_\gamma x'^{-1} +$

$p_\gamma r_\gamma$ for any $\gamma \in \Gamma$. Up to a change of variables, these cases where $\kappa \in U(R)$ reduce to the case $\kappa = 1$.

In order to obtain for C an extension result similar to Theorem 1.2.5, we need the following technical lemma about square roots in C.

Lemma 1.2.8. *Let $q = \sum_{i\geq 2} g_i y^i$ be an element of C, with $g_i \in R$ for any $i \geq 2$, $g_2 \neq 0$. We suppose that there exists $e \in U(R)$ such that $g_2 = e^2$.*

(i) *There exists a unique element $z \in C$ of the form $z = ey + \sum_{i\geq 2} e_i y^i$, with $e_i \in R$ for any $i \geq 2$, such that $q = z^2$.*
(ii) *The only other series z' satisfying $z'^{\,2} = q$ is $z' = -z$.*
(iii) *Moreover, if $q \in B$, then $z \in By$.*

Proof. We compute inductively the coefficients e_i of z for $i \geq 2$ by identification in the equality $\sum_{i\geq 2} g_i y^i = \big(ey + \sum_{i\geq 2} e_i y^i\big)^2$. Using (1.5) in the expansion of the right hand side, we observe that $z^2 = e^2y^2 + (2e_2e)y^3 + (2e_3e + e_2^2 + e\delta(e))y^4 + \cdots = e^2y^2 + \sum_{i\geq 3}(2ee_{i-1} + h_{i-2})y^i$, where the remainder $h_{i-2} \in R$ depends only on previous elements $e, e_2, \ldots, e_{i-2}$ and their images by δ. Therefore $e_{i-1} = (2e)^{-1}(g_i - h_{i-2})$ and the proof of (i) follows by induction.

Since R is a domain, the only other element $e' \in R$ satisfying $e'^2 = g_2$ is $e' = -e$, then point (ii) follows from point (i).

By point (iii) of Proposition 1.1.4, we have $z = u + u'y$, with $u, u' \in B$, $v_y(u) \geq 2$, $v_y(u') = 0$. Suppose that $u \neq 0$ and set $u = \sum_{i\geq m} f_{2i} y^{2i}$ and $u' = \sum_{i\geq 0} f'_{2i} y^{2i}$, with $m \geq 1$, $f_{2i}, f'_{2i} \in R$, $f_{2m} \neq 0$ and $f'_0 = e$. Then $q = (u + u'y)^2 = (u^2 + u'yu'y) + (uu'y + u'yu)$ and the assumption $q \in B$ implies $uu'y + u'yu = 0$ using point (iii) of Proposition 1.1.4. By (1.4) and (1.6) we have

$$\begin{aligned} uu'y + u'yu &= (f_{2m}y^{2m} + \cdots)(ey + \cdots) + (ey + \cdots)(f_{2m}y^{2m} + \cdots) \\ &= 2f_{2m}ey^{2m+1} + \cdots \end{aligned}$$

with $2f_{2m}e \neq 0$. Hence we get a contradiction. □

Theorem 1.2.9. *The action of Γ by automorphisms on R extends to an action of Γ by automorphisms on $C = R((y; \frac{1}{2}d))_2$ if and only if it is quadratically d-compatible. In this case $B = R((x; d))$ is stable under the extended action.*

Proof. Suppose that the action extends to an action by automorphisms on C. By point (ii) of Proposition 1.1.6, we can consider the map $\Gamma \to U(R)$, $\gamma \mapsto s_\gamma$ defined by $y \cdot \gamma = s_\gamma^{-1} y + \cdots$. Writing $(y \cdot \gamma) \cdot \gamma' = (s_\gamma^{-1} \cdot \gamma') s_{\gamma'}^{-1} y + \cdots$, we observe with (1.9) that $s \in Z^1(\Gamma, U(R))$. Moreover it follows from point (iii) of Proposition 1.1.6 that the action restricts in an action of B. Since $x^{-1} \cdot \gamma = (y^{-1} \cdot \gamma)^2 = (s_\gamma y^{-1} + \cdots)^2 = s_\gamma^2 x^{-1} + \cdots$, we deduce from Theorem 1.2.5 that s_γ satisfies condition (1.11); therefore the action is quadratically d-compatible.

Conversely, assume now that the action of Γ on R is quadratically d-compatible. There exists some $s \in Z^1(\Gamma, U(R))$ satisfying (1.11). Then the map $p : \gamma \mapsto s_\gamma^2$ lies in $Z^1(\Gamma, U(R))$ and satisfies (1.10). Let r be any map $\Gamma \to R$ satisfying

(1.13). By Theorem 1.2.5, we define an action by automorphisms on B by setting $x^{-1}\cdot\gamma = p_\gamma x^{-1} + p_\gamma r_\gamma$ for any $\gamma\in\Gamma$. The expansion of its inverse in B is of the form $x\cdot\gamma = s_\gamma^{-2}x+\cdots$. By Lemma 1.2.8, there exists a unique element $\overline{y}_\gamma \in C$ such that $\overline{y}_\gamma^2 = x\cdot\gamma$ and whose expansion is of the form $\overline{y}_\gamma = s_\gamma^{-1}y+\cdots$. In particular, it follows from the action of γ on relation (1.6) that

$$\overline{y}_\gamma^2 f = f\overline{y}_\gamma^2 + \sum_{k\geq 2} 2^{k-1}((\delta^{k-1}(f\cdot\gamma^{-1}))\cdot\gamma)\,\overline{y}_\gamma^{2k} \quad \text{for any } f\in R. \tag{1.14}$$

We extend the action of γ to C by setting $y\cdot\gamma = \overline{y}_\gamma$. To prove that $q\mapsto q\cdot\gamma$ defines an automorphism of C, it is sufficient by (1.4) to prove that

$$\overline{y}_\gamma f = f\overline{y}_\gamma + \sum_{k\geq 1} \tfrac{(2k)!}{2^k(k!)^2}\,((\delta^{k}(f\cdot\gamma^{-1}))\cdot\gamma)\,\overline{y}_\gamma^{2k+1} \quad \text{for any } f\in R. \tag{1.15}$$

Since $\overline{y}_\gamma = s_\gamma^{-1}y+\cdots$ with $s_\gamma^{-1}\in U(R)$, any element of C can be written as a series in the variable $\overline{y}_\gamma$ with coefficients in R. In particular, for any $f\in R$, we have $\overline{y}_\gamma f = f\overline{y}_\gamma + \sum_{n\geq 1}\delta_n(f)\overline{y}_\gamma^{n+1}$, where $(\delta_n)_{n\geq 1}$ is a sequence of additive maps $R\to R$. Hence $\overline{y}_\gamma^2 f = f\overline{y}_\gamma^2 + \sum_{n\geq 2}\Delta_n(f)\overline{y}_\gamma^{n+1}$ with the notation $\Delta_n = \sum_{j=0}^{n-1}\delta_j\circ\delta_{n-j-1}$, where $\delta_0 = \mathrm{id}_R$. The identification with (1.14) leads to $\Delta_n = 0$ for even n, and $\Delta_n = (2\delta')^{k-1}$ for odd $n = 2k-1$, where δ' is defined by $\delta'(f) = \delta(f\cdot\gamma^{-1})\cdot\gamma$ for any $f\in R$. It follows by a straightforward induction that $\delta_n = 0$ for any odd n, then $\Delta_{2k-1} = \sum_{i+j=k-1}\delta_{2j}\circ\delta_{2i}$, and finally $\delta_{2k} = \frac{(2k)!}{2^k(k!)^2}\delta'^k$. So relation (1.15) is satisfied. Because $(s_\gamma^{-1}\cdot\gamma')s_{\gamma'}^{-1} = s_{\gamma\gamma'}^{-1}$ for all $\gamma,\gamma'\in\Gamma$, we deduce from $(x\cdot\gamma)\cdot\gamma' = x\cdot\gamma\gamma'$ with Lemma 1.2.8 that $(y\cdot\gamma)\cdot\gamma' = y\cdot\gamma\gamma'$. We conclude that this construction defines an action of Γ by automorphisms on C. ☐

Corollary 1.2.10. *We suppose that the action of Γ by automorphisms on R is quadratically d-compatible.*

(i) *Let $s\in Z^1(\Gamma, U(R))$ be a 1-cocycle associated to the quadratic d-compatibility. For $\gamma\in\Gamma$, let $\overline{y}_\gamma$ be the square root of xs_γ^{-2}, whose coefficient of minimal valuation in its expansion as a series in the variable y is s_γ^{-1}. Then the action extends to an action of Γ by automorphisms on C defined from*

$$y\cdot\gamma = \overline{y}_\gamma = s_\gamma^{-1}y+\cdots \quad \textit{for any } \gamma\in\Gamma.$$

(ii) *The other extensions of the action are given by $y\cdot\gamma = \epsilon_\gamma\overline{y}_\gamma$ for any $\gamma\in\Gamma$, where ϵ is a multiplicative map $\Gamma\to\{-1,+1\}$.*

Proof. We apply the second part of the proof of Theorem 1.2.9 to the case where r is defined by $r_\gamma = 0$ for any $\gamma\in\Gamma$. ☐

1.3. **Invariants in B and C for the extensions of actions on R.** We take all data and notations of Proposition 1.1.4. For Γ a group acting by automorphisms on B (respectively on C) stabilizing R, we give sufficient conditions for the invariant ring B^Γ (respectively C^Γ) to be described as a ring of pseudodifferential (respectively quadratic pseudodifferential) operators with coefficients in R^Γ.

Theorem 1.3.1. *Let* Γ *be a group acting by automorphisms on* $B = R((x; d))$ *stabilizing* R*. We assume that there exists in* B^Γ *an element* $w = gx^{-1} + h$ *with* $h \in R$ *and* $g \in U(R)$*. Then the derivation* $D = gd$ *restricts to a derivation of* R^Γ*, and we have* $A^\Gamma = R^\Gamma[w; -D]$*,* $B_0^\Gamma = R^\Gamma[[w^{-1}; D]]$ *and* $B^\Gamma = R^\Gamma((w^{-1}; D))$*.*

Proof. Since g is invertible in R, any polynomial into x^{-1} can be written as a polynomial into w. We have $wf - fw = gx^{-1}f - fgx^{-1} = -gd(f)$. Hence $A = R[w; -D]$ denoting $D = gd$. For any $f \in R^\Gamma$, $wf - fw \in R^\Gamma$ because $w \in B^\Gamma$. Hence the restriction of D to R^Γ is a derivation of R^Γ. Since w is invariant, an element of A written as a polynomial in w with coefficients in R is invariant if and only if any coefficient lies in R^Γ. We conclude that $A^\Gamma = R^\Gamma[w; -D]$.

The element $u = w^{-1} \in B_0^\Gamma$ satisfies $v_x(u) = 1$ and its dominant coefficient g^{-1} is invertible in R. We have $x = g(1 + z)u$ for some $z \in B_0$ satisfying $v_x(z) \geq 1$. Then u is a uniformizer of the valuation v_x in B_0, which means that any element of B_0 can be written as a power series in the variable u with coefficients in R. In particular, for $q = \sum_{i\geq 0} f_i x^i$ an element of B_0^Γ with $f_i \in R$, it follows from point (i) of Proposition 1.1.6 that $f_0 \in R^\Gamma$ and $q - f_0 \in B_0^\Gamma$. We have $q - f_0 = q'u$ with $q' = \big(\sum_{i\geq 0} f_{i+1}x^i\big)g(1 + z) \in B_0$. Since q, f_0 and u belong to B_0^Γ, we deduce that $q' \in B_0^\Gamma$. We have proved that for any $q \in B_0^\Gamma$, there exist $f_0 \in R^\Gamma$ and $q' \in B_0^\Gamma$ such that $q = f_0 + q'u$. Applying this process for q', there exist $f_0' \in R^\Gamma$ and $q'' \in B_0^\Gamma$ such that $q = f_0 + f_0'u + q''u^2$. It follows by induction that q lies in the left R^Γ-module $R^\Gamma\langle\langle u\rangle\rangle$ of power series in the variable u with coefficients in R^Γ. We conclude that $B_0^\Gamma \subset R^\Gamma\langle\langle u\rangle\rangle$. The converse inclusion is clear, so $B_0^\Gamma = R^\Gamma\langle\langle u\rangle\rangle$. In particular, $R^\Gamma\langle\langle u\rangle\rangle$ is a subring of B_0. Hence, for any $f \in R^\Gamma$, there exist a sequence $(\delta_n(f))_{n\geq 0}$ of elements of R^Γ such that $uf = \sum_{n\geq 0} \delta_n(f)u^{n+1}$. The commutation relation $wf - fw = -D(f)$ becomes $uf - fu = uD(f)u$. We compute $uf = fu + uD(f)u = fu + [D(f)u + uD^2(f)u]u = fu + D(f)u^2 + [D^2(f)u + uD^3(f)u]u^2$, and conclude by iteration that $\delta_n(f) = D^n(f)$ for any $n \geq 0$. In other words, $B_0^\Gamma = R^\Gamma[[u; D]]$.

Let $R^\Gamma((u; D))$ be the localized ring of B_0^Γ with respect to the powers of u. Since u is invertible in B, we have $R^\Gamma((u; D)) \subset B$, and therefore $R^\Gamma((u; D)) \subset B^\Gamma$. Conversely, for any $f \in B^\Gamma$, there exists an integer $n \geq 0$ such that $fu^n \in B_0$; since $fu^n \in B_0^\Gamma = R^\Gamma[[u; D]]$, we deduce that $f \in R^\Gamma((u; D))$. Hence $B^\Gamma = R^\Gamma((u; D))$ and the proof is complete. □

Theorem 1.3.2. *Let Γ be a group acting by automorphisms on $C = R(\!(y;\delta)\!)_2$ stabilizing R. Denote by s the 1-cocycle in $Z^1(\Gamma, U(R))$ defined by $y\cdot\gamma = s_\gamma^{-1}y+\cdots$ for any $\gamma\in\Gamma$. We assume that there exists in C^Γ an element $w = e^2y^{-2}+h$ with $h\in R$ and $e\in U(R)$ such that $e\cdot\gamma = s_\gamma^{-1}e$ for any $\gamma\in\Gamma$. Then the derivation $\Delta = e^2\delta$ restricts to a derivation of R^Γ, and denoting by v the square root of w^{-1} whose expansion starts with $v = e^{-1}y+\cdots$, we have $C^\Gamma = R^\Gamma(\!(v;\Delta)\!)_2$.*

Proof. Let us recall that the existence of the map s follows from point (ii) of Proposition 1.1.6. We know by point (iii) of Proposition 1.1.6 that B is stable under the action of Γ on C. We can apply Theorem 1.3.1 with $g = e^2$ to deduce that $D = e^2d = 2e^2\delta$ restricts to a derivation of R^Γ and $B^\Gamma = R^\Gamma[[u;D]]$, where $u = w^{-1}\in B_0^\Gamma$. Since $u = e^{-2}y^2+\cdots$ lies in B, it follows from Lemma 1.2.8 that there exist in C two elements $v = \sum_{j\geq 1} g_jy^j$ and $v' = -v$ such that $v^2 = v'^2 = u$, with $g_j\in R$ for any $j\geq 1$, $g_j = 0$ for any even index j and $g_1 = e^{-1}$. For any $\gamma\in\Gamma$, we have $u = u\cdot\gamma = (v\cdot\gamma)^2$. Then $v\cdot\gamma = \pm v$. Since $v\cdot\gamma = (e^{-1}y+\cdots)\cdot\gamma = s_\gamma e^{-1}(s_\gamma^{-1}y+\cdots)+\cdots = e^{-1}y+\cdots$, we are necessarily in the case $v\cdot\gamma = v$. Therefore $v\in C^\Gamma$. Let us set $v = e^{-1}(1+z)y$, where $z = \sum_{j\geq 2} eg_jy^{j-1}\in C_0$ with $v_y(z)\geq 1$. Similarly there exists $z'\in C_0$ with $v_y(z')\geq 1$ such that $y = e(1+z')v$. Then v is a uniformizer of the valuation v_y in C_0, which means that any element of C_0 can be written as a power series in the variable v with coefficients in R. Using inductively point (ii) of Proposition 1.1.6, we can prove, in the same way as in the proof of Theorem 1.3.1, that $C_0^\Gamma = R^\Gamma\langle\!\langle v\rangle\!\rangle$. In particular, $R^\Gamma\langle\!\langle v\rangle\!\rangle$ is a subring of C_0. Hence, for any $f\in R^\Gamma$, there exists a sequence $(\delta_n(f))_{n\geq 0}$ of elements of R^Γ such that $vf = \sum_{n\geq 0}\delta_n(f)v^{n+1}$. Comparing with relation $v^{-2}f - fv^{-2} = -D(f)$, which follows from Theorem 1.3.1, direct calculations show that $\delta_i = 0$ for any odd index i, and $\delta_{2k} = \frac{(2k)!}{2^k(k!)^2}(\frac{D}{2})^k$. In other words, $C^\Gamma = R^\Gamma(\!(v;\Delta)\!)_2$, where $\Delta = \frac{1}{2}D$. ☐

Corollary 1.3.3. *Under the assumptions of the previous theorem, and defining in R^Γ the derivations $\Delta = e^2\delta$ and $D = 2\Delta = e^2d$, we have the following ring embeddings:*

$$\begin{array}{ccccc}
A = R[x^{-1};-d] & \hookrightarrow & B = R(\!(x;d)\!) & \xhookrightarrow[x=y^2]{} & C = R(\!(y;\delta)\!)_2 \\
\uparrow & & \uparrow & & \uparrow \\
A^\Gamma = R^\Gamma[w;-D] & \hookrightarrow & B^\Gamma = R^\Gamma(\!(w^{-1};D)\!) & \xhookrightarrow[w^{-1}=v^2]{} & C^\Gamma = R^\Gamma(\!(v;\Delta)\!)_2.
\end{array}$$

Proof. It is a joint formulation of Theorems 1.3.1 and 1.3.2. ☐

1.4. **Weighted invariants in R and equivariant splitting maps.** The data and notations are those of 1.2.1. We introduce, for any quadratically d-compatible action of a group Γ on R, a natural link between the invariants in C or B and some weighted invariants in R.

Definitions 1.4.1. For any 1-cocycle s in $Z^1(\Gamma, U(R))$ and any integer k, we define

$$(f|_k\gamma) = s_\gamma^{-k}(f\cdot\gamma)\quad\text{for any } f\in R,\ \gamma\in\Gamma. \tag{1.16}$$

It follows from relations (1.8) and (1.9) that

$$((f|_k\gamma)|_k\gamma') = (f|_k\gamma\gamma') \quad \text{for any } f \in R,\ \gamma, \gamma' \in \Gamma.$$

Hence relation (1.16) defines a right action of Γ on R, named the *weight k action* associated to s. In particular, the weight zero action is just the original action of Γ by automorphisms on R:

$$(f|_0\gamma) = f \cdot \gamma \quad \text{for any } f \in R,\ \gamma \in \Gamma.$$

We introduce the additive subgroup of weight k invariants

$$M_k = \{f \in R : (f|_k\gamma) = f \text{ for any } \gamma \in \Gamma\} = \{f \in R : f \cdot \gamma = s_\gamma^k f \text{ for any } \gamma \in \Gamma\}. \tag{1.17}$$

We have $M_0 = R^\Gamma$ and $M_k M_\ell \subseteq M_{k+\ell}$ for all $k, \ell \in \mathbb{Z}$.

Notations 1.4.2. We suppose that the action of Γ is quadratically d-compatible on R. Let $s \in Z^1(\Gamma, U(R))$ be a 1-cocycle associated to the quadratic d-compatibility. We define $p = s^2 \in Z^1(\Gamma, U(R))$ and we choose a map $r : \Gamma \to R$ satisfying (1.13). We consider the actions of Γ on B and C determined by Theorem 1.2.5 and Corollary 1.2.10. We denote by B^Γ and C^Γ the corresponding invariant rings. For any integer k, we introduce:

$$C_k = \{q \in C : v_y(q) \geq k\}, \qquad C_k^\Gamma = C_k \cap C^\Gamma,$$
$$B_k = \{q \in B : v_x(q) \geq k\}, \qquad B_k^\Gamma = B_k \cap B^\Gamma.$$

Proposition 1.4.3. *Let k be an integer.*

(i) *Let $\pi_k : C_k \to R$ be the canonical projection $\sum_{i\geq k} f_i y^i \mapsto f_k$. The restriction of π_k to C_k^Γ defines an additive map $\pi_k : C_k^\Gamma \to M_k$.*

(ii) *Let $\widehat{\pi}_k : B_k \to R$ be the canonical projection $\sum_{i\geq k} f_i x^i \mapsto f_k$. Then $\widehat{\pi}_k$ is the restriction of π_{2k} to B_k, and its restriction to B_k^Γ defines an additive map $\widehat{\pi}_k : B_k^\Gamma \to M_{2k}$.*

Proof. For any $q = \sum_{i\geq k} f_i y^i \in C_k$ with $f_i \in R$, $f_k \neq 0$, and for any $\gamma \in \Gamma$, it follows from Theorem 1.2.9 that

$$q \cdot \gamma = (f_k \cdot \gamma)(s_\gamma^{-1} y + \cdots)^k + (f_{k+1} \cdot \gamma)(s_\gamma^{-1} y + \cdots)^{k+1} + \cdots.$$

Then $q \cdot \gamma = (f_k \cdot \gamma) s_\gamma^{-k} y^k + \cdots$. Hence $q \cdot \gamma = q$ implies $(f_k \cdot \gamma) s_\gamma^{-k} = f_k$, or equivalently $f_k \in M_k$ by (1.17). Point (ii) follows from the second part of Theorem 1.2.9. □

Problem 1.4.4. Point (i) of this proposition leads to the natural question of finding additive right splitting maps $\psi_k : M_k \to C_k^\Gamma$ such that $\pi_k \circ \psi_k = \mathrm{id}_{M_k}$. Solutions arise by restriction if we can construct $\psi_k : R \to C_k$ satisfying $\pi_k \circ \psi_k = \mathrm{id}_R$ and the equivariance condition

$$\psi_k((f|_k\gamma)) = \psi_k(f) \cdot \gamma \quad \text{for any } \gamma \in \Gamma,\ f \in R.$$

In the particular case of even weights the corresponding question of finding splitting maps $\psi_{2k} : M_{2k} \to B_k^\Gamma$ was solved in [4] and [16] in various contexts involving modular forms.

2. Application to algebraic modular forms

2.1. **Homographic action on R.**

Notations 2.1.1. In order to apply the previous algebraic results in the arithmetical context of modular forms, we specialize the data and notations of 1.2.1. From now on Γ is a subgroup of $\mathrm{SL}(2,\mathbb{C})$, and R is a commutative $\mathbb{C}$-algebra of functions in one variable z, which is a domain. In the following we suppose that:

(i) Γ acts on the right by homographic automorphisms on R:

$$(f\cdot\gamma)(z) = f\left(\tfrac{az+b}{cz+d}\right) \quad \text{for any } f\in R \text{ and } \gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma; \tag{2.1}$$

(ii) $z\mapsto cz+d\in U(R)$ for any $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma$;
(iii) R is stable under the standard derivation ∂_z with respect to z.

Examples 2.1.2. A formal algebraic example of such a situation is the case where $R=\mathbb{C}(z)$, the field of complex rational functions in one indeterminate. Number-theoretical examples can arise from the following construction. Assume that Γ is a subgroup of $\mathrm{SL}(2,\mathbb{R})$ and denote by $\mathcal{H}=\{z\in\mathbb{C} : \mathrm{Im}(z)>0\}$ the Poincaré upper half-plane. Then $\mathcal{H}$ is stable by the homographic action of Γ, and various subalgebras R of holomorphic or meromorphic functions on $\mathcal{H}$ satisfy the previous conditions. For instance:

(1) $R_1=\mathrm{Hol}(\mathcal{H})$, the algebra of holomorphic functions on $\mathcal{H}$.
(2) $R_2=\mathrm{Mer}(\mathcal{H})$, the field of meromorphic functions on $\mathcal{H}$.
(3) For $\Gamma\subset\mathrm{SL}(2,\mathbb{Z})$, R_3 is the field of $f\in\mathrm{Mer}(\mathcal{H})$ such that, for any $p\in\mathbb{P}_1(\mathbb{Q})$, there exists an $\mathcal{H}$-neighborhood V_p of p such that f has neither zero nor pole in V_p. Here we consider the hyperbolic topology on $\overline{\mathcal{H}}=\mathcal{H}\cup\mathbb{P}_1(\mathbb{Q})$, where a fundamental system of $\mathcal{H}$-neighborhoods of $p\in\mathbb{P}_1(\mathbb{Q})$ is given by the upper half-planes $\{\mathrm{Im}(z)>M\}$ for $p=\infty$ and by the open disks $D_r=\{z\in\mathcal{H} : |z-(p+ir)|<r\}$ for $p\in\mathbb{Q}$. We can prove (see for instance [9, Theorem 2.11 of Chapter VI]) that $R_3^{\mathrm{SL}(2,\mathbb{Z})}=\mathbb{C}(j)$, where j is the modular invariant. It follows in particular that the field R_3 satisfies $R_3^{\Gamma}\neq\mathbb{C}$ for any subgroup Γ of $\mathrm{SL}(2,\mathbb{Z})$.

Notations 2.1.3. With data and notations from 2.1.1, we introduce, according to Proposition 1.1.4, the noncommutative $\mathbb{C}$-algebras

$$A=R[x^{-1};\partial_z]\subset B=R((x;d_z))\subset C=R((y;\delta_z))$$

with $x=y^2$, $d_z=-\partial_z$ and $\delta_z=\frac{1}{2}d_z$.

Lemma 2.1.4. *The map $s:\Gamma\to U(R),\gamma\mapsto s_\gamma$ defined by*

$$s_\gamma(z)=cz+d \quad \textit{for any } \gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma \tag{2.2}$$

is a 1-cocycle for the action of Γ.

Proof. A straightforward calculation using (2.1) proves that (1.9) is satisfied. □

Definitions 2.1.5. Since $s \in Z^1(\Gamma, U(R))$, we can apply Definitions 1.4.1 to introduce, for any integer k, the *weight k action of Γ on R* defined by

$$(f|_k\gamma)(z) = (cz+d)^{-k} f\left(\tfrac{az+b}{cz+d}\right) \quad \text{for any } f \in R \text{ and } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma,$$

and the $\mathbb{C}$*-vector space of algebraic weight k modular forms on R*:

$$M_k = \{f \in R : (f|_k\gamma) = f \text{ for any } \gamma \in \Gamma\}.$$

In particular, for $k = 0$,

$$(f|_0\gamma) = f \cdot \gamma \quad \text{and} \quad M_0 = R^\Gamma.$$

Notations 2.1.6. In the following we use the notations

$$\mathcal{M}_j = \prod_{k \geq j} M_k, \quad \mathcal{M}_j^{\text{ev}} = \prod_{k \geq j} M_{2k} \quad \text{for any } j \in \mathbb{Z},$$
$$\mathcal{M}_* = \bigcup_{j \in \mathbb{Z}} \mathcal{M}_j, \quad \mathcal{M}_*^{\text{ev}} = \bigcup_{j \in \mathbb{Z}} \mathcal{M}_j^{\text{ev}},$$

with the convention $\mathcal{M}_{j_1} \subset \mathcal{M}_{j_2}$ for $j_1 \geq j_2$. Throughout the rest of the paper, we suppose that $M_k \cap M_\ell = \{0\}$ for all integers $k \neq \ell$. In the classical situations, it is sufficient, for this additional hypothesis, to assume that Γ contains at least one matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $(c, d) \notin \{0\} \times \mathbb{U}_\infty$, where $\mathbb{U}_\infty$ is the group of roots of 1 in $\mathbb{C}$. Observe that this excludes the unipotent cases where $\Gamma \subset \{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) : a \in \mathbb{R}\}$. Then an element $\widetilde{f}$ of $\mathcal{M}_j$ can be denoted unambiguously by $\widetilde{f} = \sum_{k \geq j} f_k$, where $f_k \in M_k$.

Remark 2.1.7. For k and m nonnegative integers, denote by $\mathcal{J}_{k,m}$ the space of algebraic Jacobi forms on R of weight k and index m, defined as functions $\Phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ satisfying the Jacobi transformation equation

$$\Phi\left(\tfrac{az+b}{cz+d}, \tfrac{Z}{cz+d}\right) = (cz+d)^k e^{\frac{2i\pi mcZ^2}{cz+d}} \Phi(z, Z) \quad \text{for any } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma,$$

and admitting around $Z = 0$ a Taylor expansion $\Phi(z, Z) = \sum_{\nu \geq 0} X_\nu(z) Z^\nu$ with $X_\nu \in R$ for any $\nu \geq 0$. Note that $\mathcal{J}_{k,m}$ is isomorphic to $\mathcal{J}_{k,1}$ for any $m \geq 1$ via the map $Z \mapsto \sqrt{m} Z$. Then, for any $k \geq 0$ and $m \geq 1$, the vector space $\mathcal{J}_{k,m}$ is isomorphic to $\mathcal{M}_k$; an isomorphism is explicitly described in [7, p. 34], where the space $\mathcal{J}_{k,m}$ is denoted by $M_{k,m}$.

2.2. **Homographic action on B and C.** The data and notations are those of 2.1.3.

Lemma 2.2.1. *The homographic action of Γ on R is quadratically ∂_z-compatible, and an associated 1-cocycle is the map s defined by* (2.2)*.*

Proof. We have already observed in example 2 of 1.2.4 that, for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and any $f \in R$, we have $\partial_z(f \cdot \gamma)(z) = (cz+d)^{-2} \partial_z f(\frac{az+b}{cz+d})$. In other words, $\partial_z f \cdot \gamma = s_\gamma^2 \partial_z(f \cdot \gamma)$. From Definition 1.2.2 (ii) and Lemma 2.1.4, the lemma is proved. ☐

We can then apply the results of the first part of the paper: for the canonical choice $r=0$ (see example 1 of 1.2.7) we obtain the following results.

Proposition 2.2.2.

(i) *The homographic action of* Γ *on* R *extends to an action by automorphisms on* B *defined by*

$$(x^{-1}|\gamma)=(cz+d)^2x^{-1} \quad \textit{for any } \gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma. \tag{2.3}$$

Consequently, for any $q=\sum_{n>-\infty} f_n x^n \in B$ *with* $f_n\in R$, *we have*

$$(q|\gamma)=\sum_{n>-\infty}(f_n\cdot\gamma)(x^{-1}|\gamma)^{-n} \quad \textit{for any } \gamma\in\Gamma.$$

(ii) *The homographic action of* Γ *on* R *extends to an action by automorphisms on* C *defined by*

$$(y|\gamma)=(cz+d)^{-1}y+\cdots \quad \textit{for any } \gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma,$$

where this Laurent series is the square root in C *of* $(x|\gamma)=x(cz+d)^{-2}$, *whose term of minimal valuation is* $(cz+d)^{-1}y$. *Consequently, for any* $q=\sum_{n>-\infty} f_n y^n\in C$ *with* $f_n\in R$, *we have*

$$(q|\gamma)=\sum_{n>-\infty}(f_n\cdot\gamma)(y|\gamma)^n \quad \textit{for any } \gamma\in\Gamma.$$

(iii) *The subalgebra* B *of* C *is stable under the action* (ii), *and the action* (i) *is the restriction to* B *of the action* (ii).

Proof. We apply Theorem 1.2.5 and Theorem 1.2.9. □

The following two theorems give explicit formulas describing the action of Γ on B and C introduced in Proposition 2.2.2.

Theorem 2.2.3. *The extension to* B *and* C *of the homographic action of* Γ *on* R *given by Proposition 2.2.2 satisfy*

$$(x|\gamma)=\sum_{n\geq 0}(n+1)!(cz+d)^{-2}\left(\tfrac{c}{cz+d}\right)^n x^{n+1} \tag{2.4}$$

$$(y|\gamma)=\sum_{u\geq 0}\tfrac{(2u+1)!(2u)!}{16^u(u)!^3}(cz+d)^{-1}\left(\tfrac{c}{cz+d}\right)^u y^{2u+1} \tag{2.5}$$

for any $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma$.

Proof. From relations (1.1) and (2.3), we have

$$(x|\gamma)=(x^{-1}|\gamma)^{-1}=x(cz+d)^{-2}=\sum_{u\geq 1}d_z^{u-1}((cz+d)^{-2})x^u=\sum_{u\geq 1}\tfrac{u!}{c^2}\left(\tfrac{c}{cz+d}\right)^{u+1}x^u,$$

which proves formula (2.4). We find here the action of Γ described in [4, formula 1.7].

In order to prove (2.5), we introduce the expansion $(y|\gamma)=\sum_{i\geq 1}a_iy^i$ in C with $a_i\in R$. Using the identity $(y|\gamma)^2=(x|\gamma)$ we prove by induction on i that $a_i=0$

for even i and that there exists a sequence $(\rho_j)_{j\geq 1}$ of complex numbers such that $(y|\gamma) = \sum_{j\geq 1} \frac{\rho_j}{c}\left(\frac{c}{cz+d}\right)^j y^{2j-1}$. Then we have

$$(y|\gamma)^2 = \sum_{n,m\geq 1} \frac{\rho_n\rho_m}{c^2}\left(\frac{c}{cz+d}\right)^n \left[y^{2n-1}\left(\frac{c}{cz+d}\right)^m\right] y^{2m-1}.$$

From (1.5), and using the combinatorial identity

$$\prod_{i=0}^{s-1}(2m+1+2i) = 2^{-s}\frac{(2m+2s)!m!}{(m+s)!(2m)!} \quad \text{for any } m\geq 0 \text{ and } s\geq 1, \tag{2.6}$$

we have $y^{2n-1}f = \sum_{\ell\geq n-1} \frac{(2\ell)!(n-1)!}{\ell!(2n-2)!}\frac{d_z^{\ell-n+1}(f)}{4^{\ell-n+1}(\ell-n+1)!}y^{2\ell+1}$, and an easy computation gives $d_z^t\left(\left(\frac{c}{cz+d}\right)^m\right) = \frac{(m+t-1)!}{(m-1)!}\left(\frac{c}{cz+d}\right)^{m+t}$. We deduce that

$$\begin{aligned}(y|\gamma)^2 &= \sum_{n,m\geq 1}\frac{\rho_n\rho_m}{c^2}\sum_{\ell\geq n-1}\frac{(2\ell)!(n-1)!(m+\ell-n)!}{4^{\ell-n+1}(\ell-n+1)!(2n-2)!\ell!(m-1)!}\left(\frac{c}{cz+d}\right)^{\ell+m+1} y^{2\ell+2m}\\ &= \sum_{u\geq 1}\left(\sum_{m=1}^{u}\sum_{i=m}^{u}\frac{\rho_{u+1-i}\rho_m}{4^{i-m}(i-m)!}\frac{(2u-2m)!(u-i)!(i-1)!}{(2u-2i)!(u-m)!(m-1)!}\right)\frac{1}{c^2}\left(\frac{c}{cz+d}\right)^{u+1} y^{2u},\end{aligned}$$

with the change of variables $u = \ell + m$, and $i = u+1-n$. By identification in the equality $(y|\gamma)^2 = (x|\gamma)$, the coefficients ρ_j satisfy, for any $u\geq 1$, the equation

$$\sum_{m=1}^{u}\sum_{i=m}^{u}\frac{\rho_{u+1-i}\rho_m}{4^{i-m}(i-m)!}\frac{(2u-2m)!(u-i)!(i-1)!}{(2u-2i)!(u-m)!(m-1)!} = u! \tag{2.7}$$

The relation for $u=1$ gives $\rho_1^2 = 1$, then the choice of sign for ρ_1 determines inductively all the terms ρ_u, $u\geq 1$. Hence relation (2.7) determines uniquely up to sign the sequence $(\rho_u)_{u\geq 1}$. Therefore the proof will be complete if we check that the coefficients $\rho_u = \frac{(2u-1)!(2u-2)!}{16^{u-1}(u-1)!^3}$ satisfy relation (2.7). Then we have to prove that, for any $u\geq 1$,

$$\sum_{m=1}^{u}\frac{(2u-2m)!(2m-1)!(2m-2)!}{4^{2u+m-2}(m-1)!^4(u-m)!}A(u,m), = u! \tag{2.8}$$

with the notation $A(u,m) = \sum_{i=m}^{u} G(u,m,i)$ and $G(u,m,i) = \frac{4^i(i-1)!(2u-2i+1)!}{(i-m)!(u-i)!^2}$. Using Zeilberger's algorithm (see [13]) we obtain the following relation, which is easy to check directly:

$$(4u+6)G(u,m,i) - (u+1-m)G(u+1,m,i) = H(u,m,i+1) - H(u,m,i),$$

where $H(u,m,i) = \frac{4^i(i-1)!(2u+3-2i)!}{(i-m-1)!(u+1-i)!^2}$. Then the summation on i gives the relation $(4u+6)A(u,m) - (u+1-m)A(u+1,m) = 0$, so $A(u,m) = A(m,m)\prod_{t=m}^{u-1}\frac{2(2t+3)}{t+1-m}$. Using $A(m,m) = 4^m(m-1)!$ and (2.6) we obtain $A(u,m) = 4^m\frac{(m-1)!(2u+1)!m!}{(u-m)!(2m+1)!u!}$. So equation (2.8) is equivalent to $T(u) = 1$, where $T(u) = \sum_{m=1}^{u} K(u,m)$ with

$K(u,m) = \frac{(2u+1)!}{16^{u-1}u!^2}\frac{(2m-2)!(2m-1)!m!(2u-2m)!}{(2m+1)!(m-1)!^3(u-m)!^2}$. Using again Zeilberger's algorithm we find the relation

$$K(u+1,m) - K(u,m) = J(u,m) - J(u,m+1)$$

with $J(u,m) = \frac{(2u-2m+2)!(2u+1)!(2m-2)!}{16^u u u!(u+1)!(m-1)!(m-2)!(u+1-m)!^2}$. The summation on m gives $T(u+1) - T(u) = 0$, so $T(u) = T(1) = 1$ for any $u \geq 1$, and the proof is complete. □

The next step is to describe the action of Γ on any power of y. We proceed depending on the parity and the sign of the exponent. In particular, relations (2.10) and (2.9) below are related to the action (2.4) on B and already appeared in previous studies about modular forms of even weight [4, 16].

Lemma 2.2.4. *For any $k \in \mathbb{N}$, we have*

$$(y^{-2k}|\gamma) = (x^{-k}|\gamma) = \sum_{u=0}^{k-1} u!\binom{k}{u}\binom{k-1}{u}(cz+d)^{2k}\left(\frac{c}{cz+d}\right)^u x^{u-k}, \tag{2.9}$$

$$(y^{2k}|\gamma) = (x^k|\gamma) = \sum_{u\geq 0} u!\binom{k+u-1}{u}\binom{k+u}{u}(cz+d)^{-2k}\left(\frac{c}{cz+d}\right)^u x^{u+k}, \tag{2.10}$$

$$(y^{2k+1}|\gamma) = \frac{(k+1)!k!}{(2k+2)!(2k)!}\sum_{u\geq 0}\frac{(2k+2u)!(2k+2u+2)!}{16^u u!(k+u)!(k+u+1)!}(cz+d)^{-2k-1}\left(\frac{c}{cz+d}\right)^u y^{2k+1+2u}, \tag{2.11}$$

$$(y^{-2k+1}|\gamma) = \frac{(2k)!(2k-2)!}{k!(k-1)!}(cz+d)^{2k-1}\Bigg[\sum_{u=0}^{k-1}\frac{(k-u)!(k-1-u)!}{16^u u!(2k-2u)!(2k-2-2u)!}\left(\frac{c}{cz+d}\right)^u y^{-2k+1+2u}$$
$$- \sum_{u\geq 0}\frac{(2u)!(2u+2)!}{u!(u+1)!}\frac{1}{16^{u+k}(u+k)!}\left(\frac{c}{cz+d}\right)^{u+k} y^{2u+1}\Bigg]. \tag{2.12}$$

Proof. For the negative even powers of y (the negative powers of x), we prove relation (2.9) inductively using the formulas $(x^{-1}|\gamma) = (cz+d)^2 x^{-1}$ and $(x^{-k-1}|\gamma) = (x^{-1}|\gamma)(x^{-k}|\gamma)$ for $k > 0$ with the relation $x^{-1}f = fx^{-1} + \partial_z f$.

For the positive even powers of y (the positive powers of x) set, for any $k \geq 1$, $(y^{2k}|\gamma) = \sum_{n\geq 0} a_k(n)(cz+d)^{-2k-n}c^n y^{2k+2n}$, with $a_k(n) \in \mathbb{C}$. Equations $(x^{-1}|\gamma) = (cz+d)^{-2}x^{-1}$ and $(y^{2k}|\gamma) = (x^{-1}|\gamma)(y^{2k+2}|\gamma)$ give, for $n \geq 1$, the relations $a_{k+1}(n) = (2k+n+1)a_{k+1}(n-1) + a_k(n)$. Then a double induction on k and n proves relation (2.10) using the expression of $a_1(n)$ for any $n \geq 0$ and $a_k(0) = 1$ for any $k > 0$.

We consider now relation (2.11). Let $k \geq 0$; using (2.10), (2.5) and the relation $(y^{2k+1}|\gamma) = (x^k|\gamma)(y|\gamma)$ we obtain $(y^{2k+1}|\gamma) = \sum_{s\geq 0}\frac{\alpha_k(s)}{(cz+d)^{2k+1}}\left(\frac{c}{cz+d}\right)^s y^{2k+1+2s}$, where $\alpha_k(s) = \sum_{r=0}^{s}\frac{(2r+1)!(2r)!(s-k+r-1)!}{16^r r!^4(k-1)!k!}\beta_{k,s}(r)$, with $\beta_{k,s}(r) = \sum_{j=0}^{s-r}\frac{(k+s-r-j)!(r+j)!}{(s-r-j)!j!}$. By Zeilberger's algorithm we deduce $(k+s+2)\beta_{k,s}(r) - (s-r+1)\beta_{k,s+1}(r) = 0$, which proves that $\beta_{k,s}(r) = \frac{k!r!(k+s+1)!}{(s-r)!(k+r+1)!}$; more precisely, we have the relation

$$(k+s+2)b(s,j) - (s-r+1)b(s+1,j) = G(s,j+1) - G(s,j),$$

where $b(s,j) = \frac{(k+s-r-j)!(r+j)!}{(s-r-j)!j!}$ and $G(s,j) = \frac{(k+s-r-j+1)!(r+j)!}{(j-1)!(s-r-j+1)!}$.

Applying again Zeilberger's algorithm to $C(s,r) = \frac{(2r+1)!(2r)!(k+s-r-1)!}{16^r r!^3 (s-r)!(k+r+1)!}$, we obtain

$$(2s+2k+3)(2s+2k+1)C(s,r) - 4(s+1)(k+s+2)C(s+1,r) = H(s,r+1) - H(s,r),$$

with $H(s,r) = \frac{4(k+s-r)!(2s+1)!(2r)!}{16^r(s-r+1)!(k+r)!(r-1)!r!^2}$.

It follows that $\alpha_k(s) = \frac{(k+1)!k!(2k+2s)!(2k+2s+2)!}{(2k+2)!(2k)!16^s s!(k+s)!(k+s+1)!}$, which proves formula (2.11).

Finally, for formula (2.12), set

$$(y^{-2k+1}|\gamma) = \sum_{j\geq 0} \alpha_{-k}(j)(cz+d)^{2k-1}\left(\tfrac{c}{cz+d}\right)^j y^{-2k+1+2j},$$

with $\alpha_{-k}(j) \in \mathbb{C}$. The equality $(y^{-2k-1}|\gamma) = (x^{-1}|\gamma)(y^{-2k+1}|\gamma)$ gives, for any nonnegative j, the relation

$$\alpha_{-(k+1)}(j) = \alpha_{-k}(j) + (2k-j)\alpha_{-k}(j-1), \tag{2.13}$$

with the convention $\alpha_{-k}(-1) = 0$. We then proceed inductively on k. The expression of $\alpha_0(j)$ is given by equation (2.5). We have $\alpha_{-k}(0) = 1$ for any $k \geq 0$, and by (2.13), $\alpha_{-1}(j) = -4j^2\frac{(2j-1)!(2j-2)!}{16^j j!^3}$ for $j \geq 1$ proves formula (2.12) for $k = 1$ and gives the base case.

For the inductive step, we prove that $\alpha_{-(k+1)}(j)$ is equal to the corresponding term of formula (2.12) separating the three cases $j \leq k-1$, $j = k$ and $j \geq k+1$. We check by direct computations using induction hypothesis for k that the right hand side of formula (2.13) corresponds to the expected expression in formula (2.12) for $\alpha_{-(k+1)}(j)$. □

We are now in a position to give a unified formula for the action of Γ on any power of y.

Theorem 2.2.5. *For any $k \in \mathbb{Z}$, we have*

$$(y^k|\gamma) = \sum_{u\geq 0} \omega_k(u)(cz+d)^{-k}\left(\tfrac{c}{cz+d}\right)^u y^{2u+k}, \tag{2.14}$$

where $\omega_k(0) = 1$ and $\omega_k(u) = \frac{1}{4^u u!}\prod_{i=0}^{u-1}(k+2i)(k+2i+2)$ for any $u \geq 1$.

Proof. Using formula (2.6), we check that formula (2.14) corresponds in each case (k even or odd, positive or negative) to formulas (2.10)–(2.12). □

2.3. **Equivariant splitting maps and noncommutative product on modular forms.** The data and notations are those of previous Subsections 2.1 and 2.2. In order to simplify the notations, we will set $f^{(n)} = \partial_z^n f$ for any $f \in R$ and $n \geq 0$. According to problem 1.4.4 and in connection with the construction already known in the even case, we seek to construct, for any $m \in \mathbb{Z}$, a linear morphism $\psi_m : R \to C$ satisfying the following conditions:

(C1) $\psi_m((f|_m\gamma)) = (\psi_m(f)|\gamma)$ for any $f \in R, \gamma \in \Gamma$;
(C2) there exists some complex sequence $(\alpha_m(n))_{n\geq 0}$ such that

$$\psi_m(f) = \sum_{n\geq 0} \alpha_m(n) f^{(n)} y^{m+2n} \quad \text{for any } f \in R;$$

(C3) $\alpha_m(0) = 1$.

We prove in the following that such a map ψ_m exists and is unique for $m \geq 0$ (Proposition 2.3.1), exists but is not unique if m is negative and even (Remarks 2.3.5), and doesn't exist if m is negative and odd (Proposition 2.3.6).

Proposition 2.3.1. *For any nonnegative integer m, the linear map $\psi_m : R \to C_0$ defined by*

$$\psi_m(f) = \sum_{n\geq 0} \tfrac{(-1)^n}{4^n n!} \left(\prod_{i=0}^{n-1} \tfrac{(m+2i)(m+2i+2)}{(m+i)} \right) f^{(n)} y^{m+2n} \quad \textit{for any } f \in R \tag{2.15}$$

is the unique map satisfying the three conditions (C1), (C2) *and* (C3).

Proof. The aim is to prove, for any nonnegative integer k, the following formulas:

$$\psi_{2k}(f) = \sum_{n\geq 0} \tfrac{(2k-1)!\,(n+k-1)!}{(n+2k-1)!\,(k-1)!} \tbinom{-k-1}{n} f^{(n)} x^{k+n}, \tag{2.16}$$

$$\psi_{2k+1}(f) = \tfrac{k!^2}{(2k+1)!} \sum_{n\geq 0} \tfrac{(-1)^n}{16^n} \tfrac{(2k+1+2n)!(2k+2n)!}{n!(2k+n)!(k+n)!^2} f^{(n)} y^{2k+1+2n}. \tag{2.17}$$

For even weights, formula (2.16) and the unicity follow from [4, Props. 2 and 6] up to a multiplicative constant in order to assure condition (C3). We prove now formula (2.17) and show that this is the unique map satisfying the conditions (C1), (C2) and (C3) for odd positive weights.

We use at first the following equality for $m \geq 0$ and $n \geq 0$ integers:

$$((f|_n\gamma))^{(m)}(z) = \sum_{r=0}^{m} \tfrac{(-1)^{m-r} m!(m+n-1)!}{r!(m-r)!(n-1+r)!(cz+d)^{n+2r}} \left(\tfrac{c}{cz+d}\right)^{m-r} (f^{(r)} \cdot \gamma)(z), \tag{2.18}$$

which can be easily checked by induction on m. Let $(\alpha_{2k+1}(n))_{n\geq 0}$ be a sequence of complex numbers, and let $\phi_{2k+1}(f) = \sum_{n\geq 0} \alpha_{2k+1}(n) f^{(n)} y^{2k+1+2n}$. On the one hand, we have $(\phi_{2k+1}(f)|\gamma) = \sum_{n\geq 0} \alpha_{2k+1}(n)(f^{(n)} \cdot \gamma)(y^{2k+1+2n}|\gamma)$. Using (2.11) we obtain (with $u = r + n$):

$$\begin{aligned}(\phi_{2k+1}(f)|\gamma)(z) = \sum_{u\geq 0} \sum_{n=0}^{u} & \tfrac{\alpha_{2k+1}(n)(k+n+1)!(k+n)!(2k+2u)!(2k+2u+2)!}{(2k+2n)!(2k+2n+2)!(k+u)!(k+u+1)!16^{u-n}} \\ & \times \tfrac{(cz+d)^{-2k-1-2n}}{(u-n)!} \left(\tfrac{c}{cz+d}\right)^{u-n} (f^{(n)} \cdot \gamma)(z) y^{2k+1+2u}.\end{aligned}$$

On the other hand, using (2.18) we have

$$\begin{aligned}\phi_{2k+1}(f|_{2k+1}\gamma)(z) &= \sum_{u\geq 0} \alpha_{2k+1}(u)(f|_{2k+1}\gamma)^{(u)}(z) y^{2k+1+2u} \\ &= \sum_{u\geq 0} \sum_{n=0}^{u} \alpha_{2k+1}(u) \tfrac{u!(u+2k)!(-1)^{u-n}}{n!(n+2k)!} \\ &\quad \times \tfrac{(cz+d)^{-2k-1-2n}}{(u-n)!} \left(\tfrac{c}{cz+d}\right)^{u-n} (f^{(n)} \cdot \gamma)(z) y^{2k+1+2u}.\end{aligned}$$

Then we deduce that condition (C1) holds for ϕ_{2k+1} if and only if the sequence $(\alpha_{2k+1}(n))_{n\geq 0}$ satisfies, for any $u\geq 0$ and any $0\leq n\leq u$, the equality

$$\alpha_{2k+1}(u)!\tfrac{u!(u+2k)!(-1)^{u-n}}{n!(n+2k)!}=\alpha_{2k+1}(n)\tfrac{(k+n+1)!(k+n)!(2k+2u)!(2k+2u+2)!}{(2k+2n)!(2k+2n+2)!(k+u)!(k+u+1)!16^{u-n}}. \quad (2.19)$$

This equation is true for $n=u$ for any $u\geq 0$. Let $u\geq 1$, and consider $n=u-1$. Then relation (2.19) is equivalent to

$$\begin{aligned}-u(u+2k)\alpha_{2k+1}(u) &= \tfrac{\alpha_{2k+1}(u-1)}{16}\tfrac{(k+u)!(k+u-1)!(2k+2u)!(2k+2u+2)!}{(k+u)!(k+u+1)!(2k+2u)!(2k+2u-2)!}\\ &= \alpha_{2k+1}(u-1)(k+u+\tfrac{1}{2})(k+u-\tfrac{1}{2}).\end{aligned}$$

Assuming that $\alpha_{2k+1}(0)=1$ to satisfy condition (C3), we obtain as a necessary condition that, for any $u\geq 1$, $\frac{\alpha_{2k+1}(u)}{\alpha_{2k+1}(u-1)}=-\frac{(u+k+\frac{1}{2})(u+k-\frac{1}{2})}{u(u+2k)}$. Using formula (2.6), it follows that

$$\alpha_{2k+1}(u)=(-1)^u\prod_{i=1}^{u}\tfrac{(k+\frac{1}{2}+i)(k-\frac{1}{2}+i)}{i(2k+i)}=\tfrac{(-1)^u}{16^u}\tfrac{k!^2}{(2k+1)!}\tfrac{(2k+2u+1)!(2k+2u)!}{u!(u+2k)!(u+k)!^2}.$$

This proves consequently the unicity of the sequence $(\alpha_{2k+1}(u))_{u\geq 0}$.

Let now $u\geq 1$ and $0\leq n\leq u$, and let $\alpha_{2k+1}(u)=\frac{(-1)^u}{16^u}\frac{k!^2}{(2k+1)!}\frac{(2k+2u+1)!(2k+2u)!}{u!(u+2k)!(u+k)!^2}$. On the one hand, we have

$$\alpha_{2k+1}(u)\tfrac{u!(u+2k)!(-1)^{u-n}}{n!(u-n)!(n+2k)!}=\tfrac{(-1)^n}{16^u}\tfrac{k!^2}{(2k+1)!}\tfrac{(2k+2u+1)!(2k+2u)!}{n!(u-n)!(n+2k)!(u+k)!^2},$$

and, on the other hand,

$$\begin{aligned}&\alpha_{2k+1}(n)\tfrac{(k+n+1)!(k+n)!(2k+2u)!(2k+2u+2)!}{(2k+2n)!(2k+2n+2)!(k+u)!(k+u+1)!16^{u-n}(u-n)!}\\ &\qquad=\tfrac{(-1)^n}{16^u}\tfrac{k!^2}{(2k+1)!}\tfrac{(2k+2u+1)!(2k+2u)!}{n!(u-n)!(n+2k)!(u+k)!^2},\end{aligned}$$

which proves that the sequence $(\alpha_{2k+1}(u))_{u\geq 0}$ satisfies equation (2.19) for any $u\geq 0$ and $0\leq n\leq u$, and gives the existence of the lifting ψ_{2k+1}. ☐

According to the general principle of quantization by deformation, we can now transfer to modular forms the noncommutative product on operators in C_0^Γ. The case of even weights was considered in [4] and [16].

Theorem 2.3.2. *We use Notations 2.1.3 and 2.1.6, and we set* $\mathcal{M}_0=\prod_{j\geq 0}M_j$ *and* $\mathcal{M}_0^{\mathrm{ev}}=\prod_{j\geq 0}M_{2j}$. *We consider the algebras* $B_0=R[[x;d_z]]\subset C_0=R[[y;\delta_z]]_2$.

(i) *The map* $\Psi:\mathcal{M}_0\to C_0^\Gamma$, *defined by* $\Psi(\widetilde{f})=\sum_{m\geq 0}\psi_m(f_m)$ *for any* $\widetilde{f}=\sum_{m\geq 0}f_m$, *is a vector space isomorphism, and* $\mathcal{M}_0$ *is an associative algebra for the noncommutative product*

$$\widetilde{f}\star\widetilde{g}=\Psi^{-1}(\Psi(\widetilde{f})\cdot\Psi(\widetilde{g}))\quad\text{for all } \widetilde{f},\widetilde{g}\in\mathcal{M}_0. \quad (2.20)$$

(ii) *In particular, for any modular forms f,g of respective nonnegative weights k and ℓ (even or odd), the product $f\star g$ in $\mathcal{M}_0$ is*

$$f\star g=\Psi^{-1}(\psi_k(f)\cdot\psi_\ell(g))=\sum_{n\geq 0}\alpha_n(k,\ell)[f,g]_n\in\mathcal{M}_0, \quad (2.21)$$

where the coefficients $\alpha_n(k,\ell)$ are rational constants depending only on k,ℓ and n, and $[f,g]_n = \sum_{j=0}^{n}(-1)^j\binom{k+n-1}{n-j}\binom{\ell+n-1}{j}f^{(j)}g^{(n-j)} \in M_{k+\ell+2n}$ is the n-th Rankin–Cohen bracket of f and g.

(iii) *The restriction of Ψ to the subspace $\mathcal{M}_0^{\mathrm{ev}}$ determines a vector space isomorphism Ψ_2 between $\mathcal{M}_0^{\mathrm{ev}}$ and B_0^Γ.*

Proof. It is clear that Ψ is linear and injective. Let $q = \sum_{m=m_0}^{+\infty} h_m y^m$ be an element of valuation $m_0 \geq 0$ in C_0^Γ. Then $h_{m_0} \in M_{m_0}$, and by condition (C3) the element $q - \psi_{m_0}(h_{m_0})$ lies in C_0^Γ and its valuation is greater than m_0. It follows by induction that $q \in \Psi(\mathcal{M}_{m_0})$. Then Ψ is a vector space isomorphism, and point (i) is obtained by transfer of structures. Point (ii) is a consequence of condition (C2) satisfied by ψ_m, and of the property of Rankin–Cohen brackets to be (up to constant) the unique operator $\sum_{k=0}^{n} a_n f^{(k)}g^{(n-k)}$ with $a_n \in \mathbb{C}$ mapping $M_k \times M_l$ to M_{k+l+2n} (see for instance the end of Section 1 of [19]). Point (iii) is clear by condition (C2). ☐

Corollary 2.3.3. *The invariant subspace C_k^Γ is for any $k \geq 0$ isomorphic to the space $\mathcal{J}_{k,m}$ of algebraic Jacobi forms on R of weight k and positive index m. In particular, relation* (2.20) *defines a structure of noncommutative algebra on $\mathcal{J}_{k,m}$.*

Proof. It follows from Theorem 2.3.2 by Remark 2.1.7. ☐

Remark 2.3.4. The isomorphisms Ψ and Ψ^{-1} are explicit; we can separate even and odd weights cases. In the even case, relations between the coefficients of an element $\widetilde{f} = \sum_{n\geq 1} f_{2n}$ of $\mathcal{M}_0^{\mathrm{ev}}$ and its image $\Psi_2(\widetilde{f}) = \sum_{m\geq 1} h_m x^m$ in B_0^Γ are (as proved in [4])

$$\begin{aligned} h_m &= \sum_{r=0}^{m-1} \tfrac{(m-1)!(2m-2r-1)!}{(m-r-1)!(2m-r-1)!}\binom{-m+r-1}{r} f_{2m-2r}^{(r)}, \\ f_{2n} &= \tfrac{(n-1)!}{(2n-2)!}\sum_{r=0}^{n-1} \tfrac{(2n-2-r)!}{(n-1-r)!}\binom{n}{r} h_{n-r}^{(r)}, \end{aligned} \tag{2.22}$$

and the coefficients $\alpha_n(2k,2\ell)$ are computed for instance in [4, 16, 17]. In the odd case, technical calculations give the following relations between the coefficients of $\widetilde{f} = \sum_{n\geq 0} f_{2n+1}$ of $\prod_{n\geq 0} M_{2n+1}$ and its image $\Psi(\widetilde{f}) = \sum_{m\geq 0} h_{2m+1}y^{2m+1}$:

$$\begin{aligned} h_{2m+1} &= \tfrac{(2m+1)!(2m)!}{m!^2}\sum_{r=0}^{m} \tfrac{(-1)^r(m-r)!^2}{16^r r!(2m-2r+1)!(2m-r)!} f_{2m-2r+1}^{(r)}, \\ f_{2n+1} &= \tfrac{(2n)!(2n+1)!}{(2n-1)!n!^2}\sum_{r=0}^{n} \tfrac{(2n-1-r)!(n-r)!^2}{16^r(2n-2r)!r!(2n-2r+1)!} h_{2n-2r+1}^{(r)}. \end{aligned}$$

Remarks 2.3.5. (i) The coefficients $\alpha_m(n) = \frac{(-1)^n}{4^n n!}\prod_{i=0}^{n-1}\frac{(m+2i)(m+2i+2)}{(m+i)}$ appearing in Proposition 2.3.1 are also well defined for m even and non-positive. For $n \geq \frac{-m}{2}$, $\alpha_m(n) = 0$ since $\alpha_{-2k}(n) = \frac{1}{4^n n!}\prod_{i=0}^{n-1}\frac{(2k-2i)(2k-2-2i)}{(2k-i)}$ vanishes for $n \geq k$ because of the term $i = k-1$. Therefore relation (2.16) can be completed for $k>0$ by

$$\psi_{-2k}(f) = \sum_{n=0}^{k} \tfrac{k!\,(2k-n)!}{(k-n)!\,(2k)!}\tbinom{k-1}{n} f^{(n)} x^{-k+n}. \tag{2.23}$$

The map $\psi_{-2k} : R \to B$ satisfies the three conditions (C1), (C2) and (C3); see for instance [4]. Then we can build a vector space isomorphism $\overline{\Psi}_2 : \mathcal{M}_*^{\mathrm{ev}} \to B^{\Gamma}$ whose restriction to $\mathcal{M}_0^{\mathrm{ev}}$ is the isomorphism Ψ_2 of point (iii) of Theorem 2.3.2.

(ii) Observe, however, that the lifting ψ_{-2k} is not the only map satisfying the three conditions (C1), (C2) and (C3). Indeed, using the well-known Bol's identity

$$(f|_{2-h}\gamma)^{(h-1)} = (f^{(h-1)}|_h\gamma) \quad \text{for any } \gamma \in \mathrm{SL}(2,\mathbb{C}),\ f \in R,\ h>0,$$

the operator $\phi_c : f \mapsto \psi_{-2k}(f) + c\psi_{2k+2}(f^{(2k+1)})$ satisfies, for any $c \in \mathbb{C}$, conditions (C2) and (C3), as well as the equivariance condition (C1):

$$\begin{aligned}(\phi_c(f)|\gamma) &= (\psi_{-2k}(f)|\gamma) + c(\psi_{2k+2}(f^{(2k+1)})|\gamma)\\ &= \psi_{-2k}(f|_{-2k}\gamma) + c\psi_{2k+2}(f^{(2k+1)}|_{2k+2}\gamma)\\ &= \psi_{-2k}(f|_{-2k}\gamma) + c\psi_{2k+2}\big((f|_{-2k}\gamma)^{(2k+1)}\big)\\ &= \phi_c(f|_{-2k}\gamma).\end{aligned}$$

The lifting map ψ_{-2k} is canonical in the sense that it's the only one which is polynomial, that is, of the form $\psi_{-2k}(f) = \sum_{n=0}^{k} \alpha_n f^{(n)} x^{n-k} \in A$ for any $f \in R$.

The coefficients $\alpha_m(n)$ introduced in Remarks 2.3.5 (i) are not defined for $m = -2k+1$ with $k>0$ (for $n \geq 2k$, the denominator vanishes without vanishing of the numerator). In this negative and odd case, we have the following result.

Proposition 2.3.6. *Let k be a positive integer. There is no map $\psi_{-2k+1} : R \to C$ satisfying the three conditions* (C1), (C2) *and* (C3). *More precisely, the only map $\psi_{-2k+1} : R \to C$ satisfying conditions* (C1) *and* (C2) *is defined (up to constant) by $\psi_{-2k+1}(f) = \psi_{2k+1}(f^{(2k)})$ for any $f \in R$, and it doesn't satisfy condition* (C3).

Proof. Relation (2.18) can be extended to the cases where $m \geq 0$ and $n \in \mathbb{Z}$ by

$$(f|_n\gamma)^{(m)}(z) = \sum_{r=0}^{m} \tfrac{m!}{r!}\tbinom{m+n-1}{m-r}\tfrac{(-c)^{m-r}}{(cz+d)^{n+m+r}} f^{(r)}(\gamma z) \tag{2.24}$$

which gives Bol's identity in the particular case $m = 1-n$. Let $k>0$ and suppose that there exists a linear morphism $\psi_{-2k+1} : R \to C$ satisfying (C1) and (C2). Using (2.12) and (2.24), we can show as in the proof of Proposition 2.3.1 that the coefficients of the series $\psi_{-2k+1}(f) = \sum_{n\geq 0} \alpha_{-2k+1}(n) f^{(n)} y^{-2k+1+2n}$ must satisfy the equality

$$\alpha_{-2k+1}(n)\tfrac{n!}{r!}\tbinom{n-2k}{n-r}(-1)^{n-r}$$
$$= \begin{cases} \frac{\alpha_{-2k+1}(r)}{16^{n-r}(n-r)!}\,\frac{(2k-2r)!(2k-2r-2)!(k-n)!(k-n-1)!}{(k-r)!(k-r-1)!(2k-2n)!(2k-2n-2)!} & \text{if } r \leq n \leq k-1,\\ -\frac{\alpha_{-2k+1}(r)}{16^{n-r}(n-r)!}\,\frac{(2k-2r)!(2k-2r-2)!(2n-2k)!(2n-2k+2)!}{(k-r)!(k-r-1)!(n-k)!(n-k+1)!} & \text{if } r \leq k-1 < n,\\ \frac{\alpha_{-2k+1}(r)}{16^{n-r}(n-r)!}\,\frac{(r-k+1)!(r-k)!(2n-2k)!(2n-2k+2)!}{(2r-2k+2)!(2r-2k)!(n-k)!(n-k+1)!} & \text{if } k \leq r \leq n.\end{cases}$$

For $n = 2k$ and $r < 2k$, we deduce that $\alpha_{-2k+1}(r) = 0$ (separating the cases $r > k$ and $r \leq k$). Then condition (C3) cannot be satisfied. Moreover, if we fix $\alpha_{-2k+1}(2k) = 1$, it is easy to show that we have $\alpha_{-2k+1}(n) = \alpha_{2k+1}(n - 2k)$ for any $n \geq 2k$, which proves the second assertion of the proposition. □

2.4. **Invariants for the homographic action on B.** The data and notations are those of previous Subsections 2.1, 2.2 and 2.3.

Theorem 2.4.1. *We assume that there exists a weight 2 modular form $\chi \in R$, which is invertible in R. Then $B^\Gamma = R^\Gamma((u; D))$ and $B_0^\Gamma = R^\Gamma[[u; D]]$ for $u = x\chi \in B^\Gamma$ and $D = -\chi^{-1}\partial_z$.*

Proof. It is clear that $u = x\chi$ lies in B^Γ, since its inverse $w = \chi^{-1}x^{-1}$ is invariant by (2.3). Then we apply Theorem 1.3.1 with $g = \chi^{-1}$, $h = 0$ and $d = -\partial_z$. □

Remark 2.4.2. It follows from the previous theorem that the invariant algebra B^Γ is noncommutative if and only if the restriction of D to R^Γ is not trivial, or equivalently $R^\Gamma \neq \mathbb{C}$. Since the product of any modular form of even weight $2k$ by χ^{-k} lies in R^Γ, this condition is also equivalent to $M_{2k} \neq \mathbb{C}\chi^k$. In the degenerate case where $R^\Gamma = \mathbb{C}$, the product $\star$ defined by (2.21) is commutative and most of the following results become much simpler. We refer to Examples 2.1.2 to give examples of fields R such that R^Γ is different from $\mathbb{C}$, and containing an invertible weight 2 modular form.

Problem 2.4.3. The explicit description of B^Γ given by Theorem 2.4.1 and the isomorphism $\overline{\Psi}_2 : \mathcal{M}_*^{\mathrm{ev}} \to B^\Gamma$ introduced in Remarks 2.3.5 lead naturally to the question of finding relations between the terms of a sequence of modular forms in $\mathcal{M}_*^{\mathrm{ev}}$ and the coefficients in R^Γ of its image in B^Γ by $\overline{\Psi}_2$. We answer this problem for positive weights, that is, for the isomorphism $\Psi_2 : \mathcal{M}_0^{\mathrm{ev}} \to B_0^\Gamma$. Our first goal is to calculate, for any invariant pseudodifferential operator $q \in B_0^\Gamma$, the modular forms $f_{2m} \in M_{2m}$ appearing in $\Psi_2^{-1}(q) = \sum_{m\geq 0} f_{2m}$. For this we introduce the following notation for the powers of the generator u:

$$\text{for any } k \in \mathbb{N}, \quad \Psi_2^{-1}(u^k) = \sum_{n\geq k} g_{k,2n}, \text{ with } g_{k,2n} \in M_{2n}. \tag{2.25}$$

Proposition 2.4.4. *The assumptions are those of Theorem 2.4.1. Let $q \in B_0^\Gamma$ and $(a_k)_{k\geq 0}$ be the sequence of elements of R^Γ such that $q = \sum_{k\geq 0} a_k u^k$. Then the modular forms $f_{2m} \in M_{2m}$ appearing as the terms of $\Psi_2^{-1}(q) = \sum_{m\geq 0} f_{2m}$ are given by*

$$f_{2m} = (-1)^m \tfrac{(m-1)!}{(2m-2)!} \sum_{k=0}^{m} \sum_{n=k}^{m} (-1)^n \tfrac{(2n-1)!(m-n)!}{(n-1)!(m+n-1)} \tbinom{m}{m-n} [a_k, g_{k,2n}]_{m-n}.$$

Proof. The expression of any element of B_0^Γ as $q = \sum_{k\geq 0} a_k u^k$, with $a_k \in R^\Gamma$, follows directly from Theorem 2.4.1. By Theorem 2.3.2 we have

$$\begin{aligned}\Psi_2^{-1}(q) &= \sum_k \Psi_2^{-1}(a_k u^k) = \sum_k a_k \star \Psi_2^{-1}(u^k) \\ &= \sum_{k,n} a_k \star g_{k,2n} = \sum_{k,n,r} \alpha_r(0, 2n)[a_k, g_{k,2n}]_r.\end{aligned}$$

Then $f_{2m} = \sum_{k=0}^{m}\sum_{n=k}^{m} \alpha_{m-n}(0,2n)[a_k, g_{k,2n}]_{m-n}$ for any $m \geq 0$. By formula (2.16), we obtain $a_k \cdot \psi_{2n}(g_{k,2n}) = \sum_{t\geq n} h_t x^t$, with

$$h_t = \tfrac{(2n-1)!(t-1)!}{(t+n-1)!(n-1)!}\tbinom{-n-1}{t-n} a_k \left(g_{k,2n}\right)^{(t-n)}.$$

We deduce with formula (2.22) that, for any $m \geq n \geq k$,

$$\alpha_{m-n}(0,2n)[a_k, g_{k,2n}]_{m-n} = \tfrac{(m-1)!}{(2m-2)!}\sum_{r=0}^{m-1} \tfrac{(2m-2-r)!}{(m-1-r)!}\tbinom{m}{r} h_{m-r}^{(r)}.$$

The left hand side is equal to

$$\alpha_{m-n}(0,2n)\sum_{i=0}^{m-n}(-1)^i \tbinom{m-n-1}{m-n-i}\tbinom{m+n-1}{i} a_k^{(i)} g_{k,2n}^{(m-n-i)},$$

and expanding the right hand side gives the formula

$$\tfrac{(m-1)!(2n-1)!}{(2m-2)!(n-1)!}\sum_{i=0}^{m-n}\tfrac{1}{i!} a_k^{(i)} g_{k,2n}^{(m-n-i)}\left[\sum_{r=i}^{m-n}\tfrac{(2m-2-r)!r!}{(m-r+n-1)!(r-i)!}\tbinom{m}{r}\tbinom{-n-1}{m-r-n}\right].$$

Identifying the terms for $i = m-n$ of each side gives the identity

$$\alpha_{m-n}(0,2n) = (-1)^{m-n}\tfrac{(m-1)!(2n-1)!(m-n)!}{(2m-2)!(n-1)!(m+n-1)}\tbinom{m}{m-n},$$

which proves the proposition. □

The next step is to obtain an explicit expression of the modular forms $g_{k,2n}$ as functions of χ.

Proposition 2.4.5. *For any nonnegative integers k and i, we have*

$$g_{k,2k+2i} = (-1)^i\tfrac{(k+i)!(k+i-1)!}{(2k+2i-2)!k!}\sum_{\substack{(t_1,\dots,t_k)\in(\mathbb{Z}_{\geq 0})^k\\ t_1+\dots+t_k=i}}\gamma_i(t_1,\dots,t_k)\tfrac{\chi^{(t_1)}}{(t_1+1)!}\cdots\tfrac{\chi^{(t_k)}}{(t_k+1)!}, \quad (2.26)$$

where the coefficients $\gamma_i(t_1,\dots,t_k)$ are defined for $(t_1,\dots,t_k)\in(\mathbb{Z}_{\geq 0})^k$ such that $\sum_{j=1}^{k} t_j = i$ by

$$\gamma_i(t_1,\dots,t_k) = \sum_{r=0}^{i}(-1)^r\tfrac{(2k+2i-2-r)!}{(k+i-1-r)!}\sum_{\substack{b_1+\dots+b_k=r\\ 0\leq b_1\leq t_1\\ \dots\\ 0\leq b_k\leq t_k}}\tbinom{t_1+1}{b_1}\dots\tbinom{t_k+1}{b_k} \quad (2.27)$$

and satisfy

$$\begin{aligned}&\tfrac{\gamma_i(t_1,\dots,t_k)}{(k+i-1)!}\\ &= \sum_{a_1+\dots+a_k\leq k+i-1}\tbinom{k+i-2}{a_1+\dots+a_k-1}\prod_{j=1}^{k}\left[(-1)^{t_j}\delta(a_j=0)+\delta(a_j=t_j+1)\right],\end{aligned} \quad (2.28)$$

where δ stands for the Kronecker symbol.

Proof. Firstly, let us observe that the sum in the right hand side of (2.28) contains 2^k-2 terms corresponding to all k-tuples $(a_1,a_2,\dots,a_k)$ with $a_j\in\{0,t_j+1\}$ for any $1\leq j\leq k$, except the two k-tuples $(0,0,\dots,0)$ and $(t_1+1,t_2+1,\dots,t_k+1)$.

The proofs of identities (2.26), (2.27) and (2.28) are straightforward but quite technical, and we give only the main steps intentionally omitting some details. Applying relation (1.2) to $u = x\chi$ we prove by induction on k that

$$u^k = \sum_{n\geq 0}\left(\frac{(-1)^n(k+n)!}{k!}\sum_{s_1+\ldots+s_k=n}\frac{\chi^{(s_1)}}{(s_1+1)!}\cdots\frac{\chi^{(s_k)}}{(s_k+1)!}\right)x^{k+n}.$$

Then we apply (2.22) to obtain

$$g_{k,2n} = \frac{(-1)^{n-k}n!(n-1)!}{(2n-2)!k!} \times \sum_{t_1+\ldots+t_k=n-k}\left(\sum_{r=0}^{n-k}\frac{(-1)^r(2n-2-r)!}{(n-1-r)!}\beta_r(t_1,\ldots,t_k)\right)\frac{\chi^{(t_1)}}{(t_1+1)!}\cdots\frac{\chi^{(t_k)}}{(t_k+1)!},$$

where, for any $0\leq r\leq n-k$,

$$\beta_r(t_1,\ldots,t_k) = \sum_{\substack{b_1+\ldots+b_k=r\\ b_1\neq t_1+1,\ldots,b_k\neq t_k+1}}\binom{t_1+1}{b_1}\cdots\binom{t_k+1}{b_k},$$

which proves (2.26) and (2.27) with

$$\gamma_i(t_1,\ldots,t_k) = \sum_{r=0}^{i}(-1)^r\frac{(2k+2i-2-r)!}{(k+i-1-r)!}\beta_r(t_1,\ldots,t_k).$$

Observe that $\beta_r(t_1,\ldots,t_k)$ is the X^r-coefficient in $A(X) = \prod_{j=1}^{k}[(X+1)^{t_j+1}-X^{t_j+1}]$. We have $\gamma_i(t_1,\ldots,t_k) = Q^{(k+i-1)}(1)$, with the notation

$$Q(X) = \sum_{r=0}^{i}(-1)^r\beta_r(t_1,\ldots,t_k)X^{2k+2i-2-r} = X^{2i+2k-2}A(-\tfrac{1}{X}).$$

We deduce that

$$\gamma_i(t_1,\ldots,t_k) = \left[X^{k+i-2}\prod_{j=1}^{k}\left((X-1)^{t_j+1}-(-1)^{t_j+1}\right)\right]^{(k+i-1)}\Bigg|_{X=1},$$

which leads to (2.28) using the relations

$$\left[X^{k+i-2}\right]^{(b)}\Big|_{X=1} = \begin{cases}\frac{(k+i-2)!}{(k+i-2-b)!} = b!\binom{k+i-2}{b} & \text{if } b\leq k+i-2,\\ 0 & \text{if } b=k+i-1,\end{cases}$$

and

$$\left[(X-1)^{t_j+1}-(-1)^{t_j+1}\right]^{(a_j)}\Big|_{X=1} = a_j!\left[(-1)^{t_j}\delta_{a_j=0}+\delta_{a_j=t_j+1}\right]. \qquad\square$$

Corollary 2.4.6. *We have $g_{k,2k+2i} = 0$ for any nonnegative integer k and any nonnegative odd integer i.*

Proof. We use the expression of the $g_{k,2k+2i}$ given in the previous proposition. Let i be an odd integer, and let $(t_1,\ldots,t_k)\in(\mathbb{Z}_{\geq 0})^k$ be such that $t_1+\ldots+t_k = i$. In the right hand side of (2.28), we can group into pairs the k-tuples $(a_1,\ldots,a_k)$ and $(t_1+1-a_1,\ldots,t_k+1-a_k)$. The associated summands are opposite numbers because i is odd. Hence $\gamma_i(t_1,\ldots,t_k) = 0$. $\square$

Remark 2.4.7. The result obtained in Corollary 2.4.6 as a consequence of Proposition 2.4.5 can also be proved by theoretical arguments using algebraic properties of the product $\star$ defined in (2.21) and some combinatorial identity conjectured in [4, relation (3.4)] (see [17] for a proof). We proceed by induction on $k \geq 2$. For $k = 2$, applying (2.21) with the notation from (2.25) we have

$$\Psi_2^{-1}(u^2) = \chi \star \chi = \sum_{n\geq 0} \alpha_n(2,2)[\chi,\chi]_n,$$

which proves by $(-1)^i$-symmetry of the Rankin–Cohen brackets that $g_{2,4+2i} = \alpha_i(2,2)[\chi,\chi]_i = 0$ for any odd i. Suppose now that $\Psi_2^{-1}(u^k) = \sum_{i\in 2\mathbb{Z}_{\geq 0}} g_{k,2k+2i}$ with $g_{k,2k+2i} \in M_{2k+2i}$. Then we compute

$$\begin{aligned}\Psi_2^{-1}(u^{k+1}) = \Psi_2^{-1}(u^k \cdot u) &= \sum_{n\geq k,\, 2|n-k} g_{k,2n} \star \chi \\ &= \sum_{n\geq k,\, 2|n-k}\ \sum_{m\geq 0} \alpha_m(2n,2)[g_{k,2n},\chi]_m,\end{aligned}$$

and similarly,

$$\begin{aligned}\Psi_2^{-1}(u^{k+1}) = \Psi_2^{-1}(u \cdot u^k) &= \sum_{n\geq k,\, 2|n-k} \chi \star g_{k,2n} \\ &= \sum_{n\geq k,\, 2|n-k}\ \sum_{m\geq 0} \alpha_m(2n,2)[\chi, g_{k,2n}]_m.\end{aligned}$$

Hence we deduce that, for each $r \geq k+1$,

$$\sum_{n+m=r} \alpha_m(2n,2)[g_{k,2n},\chi]_m = \sum_{n+m=r} \alpha_m(2,2n)(-1)^m[g_{k,2n},\chi]_m = g_{k+1,2r}.$$

It follows from formula (3.4) of [4] that $\alpha_m(2,2n) = \alpha_m(2n,2)$. Using this identity and the induction hypothesis $g_{k,2n} = 0$ if $n-k$ is odd, we conclude that

$$\begin{aligned}g_{k+1,2r} &= \sum_{\substack{m=0\\ m \text{ odd}}}^{r-k} \alpha_m(2r-2m,2)[g_{k,2r-2m},\chi]_m \\ &= -\sum_{\substack{m=0\\ m \text{ odd}}}^{r-k} \alpha_m(2,2r-2m)[g_{k,2r-2m},\chi]_m = 0.\end{aligned}$$

Remark 2.4.8. For even nonnegative integers i, the modular forms $g_{k,2k+2i}$ defined by (2.25) are described in Proposition 2.4.5 as multidifferential polynomials of the fundamental weight two modular form χ. Then we can necessarily express them in terms of Rankin–Cohen brackets. For small values of i, we compute

$$\begin{aligned}g_{k,2k} &= \chi^k,\\ g_{k,2k+4} &= \tfrac{(k+2)!}{72(2k+1)(k-2)!}\,\chi^{k-2}[\chi,\chi]_2,\\ g_{k,2k+8} &= 2\tfrac{(2k+1)!(k+4)!(k+5)!}{(2k+6)!(k-1)!(k-2)!}\,\chi^{k-4}\left(\tfrac{[\chi,\chi^3]_4}{16920} + \tfrac{47k^2-187k+282}{121824k}[\chi,\chi]_2^2\right).\end{aligned}$$

2.5. **Invariants for the homographic action on C.** The data and notations are those of previous Subsections 2.1, 2.2 and 2.3. A natural question is whether there exists, for the action of Γ on C studied in Subsection 2.2, a description of the invariant subalgebra C^Γ similar to the one obtained in Theorem 2.4.1 for B^Γ.

Theorem 2.5.1. *We suppose that there exists a weight 1 modular form ξ which is invertible in R. Then we have $C^\Gamma = R^\Gamma((v;\Delta))_2$ for $\Delta = -\frac{1}{2}\xi^{-2}\partial_z$ and $v = \xi y + \cdots$ a square root of $y^2\xi^2$.*

Proof. It is the direct application in the modular situation of Theorem 1.3.2. □

Remark 2.5.2. The previous theorem describes the elements of $C^\Gamma = R^\Gamma((v;\Delta))_2$ as series with coefficients in R^Γ, where the uniformizer v is choosen as a square root of $u = x\xi^2 = \psi_2(\xi^2)$. In particular, C^Γ contains the ring of differential operators $A^\Gamma = R^\Gamma[u^{-1};-2\Delta]$. The main obstacle to explicit calculations in this case is the complicated shape of the square root v.

Another more natural idea would be to consider the uniformizer $z = \psi_1(\xi)$. By reasoning as in Theorem 1.3.2, we can prove then that $C^\Gamma = R^\Gamma((z;S))$, where the product zf with $f \in R^\Gamma$ is twisted by some higher derivation $S = (\delta_k)_{k\geq 0}$ giving rise to commutation laws more general than (1.1) or (1.4) (see [6] for precise definitions). The main difficulty in this case lies in the complexity of these commutation laws.

Straightforward calculations show that $z^{-2} = u^{-1} - \frac{5}{64}\frac{[\xi,\xi]_2}{\xi^6}u - \frac{5}{64}\frac{[\xi,\xi^2]_3}{\xi^9}u^2 + \cdots$. In particular, $z^{-2} \neq fu^{-1} + g$ for any $f, g \in R^\Gamma$, $f \neq 0$. By uniqueness of the subring A^Γ in C^Γ (see [6, Section 5.3]), it follows that the higher derivation S is actually different from the sequence $\left(\frac{(2k)}{2^k(k!)^2}d^k\right)_{k\geq 0}$ of (1.4) for any derivation d of R^Γ.

Under the assumption of Theorem 2.5.1 we can extend part of Theorem 2.3.2 to the space $\mathcal{M}_*$ introduced in Notations 2.1.6.

Proposition 2.5.3. *We suppose that there exists a weight 1 modular form ξ which is invertible in R. For $k > 0$, let $\psi_{-k} : M_{-k} \to C^\Gamma_{-k}$ be the vector space morphism defined by*

$$\psi_{-k}(f) = \psi_{2k}(\xi^{2k})^{-1}\psi_k(f\xi^{2k}). \tag{2.29}$$

Then the morphisms ψ_m defined by (2.15) if $m \geq 0$ and by (2.29) if $m < 0$ induce canonically a vector space isomorphism $\overline{\Psi} : \mathcal{M}_* \to C^\Gamma$, which defines by transfer a structure of associative algebra on $\mathcal{M}_*$.

Proof. Let $k > 0$ and let $f \in M_{-k}$, $f \neq 0$. Then $f\xi^{2k} \in M_k$ and $\psi_k(f\xi^{2k}) \in C^\Gamma_k$. In the same way $\xi^{2k} \in M_{2k}$, so $\psi_{2k}(\xi^{2k}) \in C^\Gamma_{2k}$. Since its $2k$-valuation term is $\xi^{2k}y^{2k}$ and $\xi \in U(R)$ by assumption, we deduce that $\psi_{2k}(\xi^{2k}) \in U(C^\Gamma)$ and so $\psi_{2k}(\xi^{2k})^{-1} \in C^\Gamma_{-2k}$. We have consequently $\psi_{-k}(f) \in C^\Gamma_{-k}$, and it is easily checked that its term of valuation $-k$ is fy^{-k}. The map ψ_{-k} is clearly linear, and we prove the surjectivity of $\overline{\Psi}$ recursively as in the proof of Theorem 2.3.2. □

Remarks 2.5.4. (i) Notice that we don't have in the previous proposition an equivalent of point (ii) of Theorem 2.3.2, because the morphism ψ_k doesn't satisfy condition (C2) for $k < 0$.

(ii) For $k < 0$ an even integer, the map ψ_k defined by (2.29) can be replaced by the more canonical one introduced previously in formula (2.23), which satisfies condition (C2). In this case, the isomorphism $\overline{\Psi}_2$ in Remarks 2.3.5 is the restriction of $\overline{\Psi}$ to $\mathcal{M}_*^{\mathrm{ev}}$.

2.6. **Additional comment.** The action of Γ on B and C described in Proposition 2.2.2 and studied throughout the rest of the article is based on the choice $r = 0$ in formula (1.12). For the same 1-cocycle s defined by (2.2), we know by example 2 of 1.2.7 that another choice for r could be the map $r' : \Gamma \to R$ defined by $r'_\gamma = -s_\gamma^{-2} d(s_\gamma^2)$. Using (2.2), we have $r'_\gamma = (cz+d)^{-2}\partial_z((cz+d)^2) = \frac{2c}{cz+d}$ for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. In other words, the homographic action of Γ on R extends to an action by automorphisms on B defined by

$$(x^{-1}|\gamma)_1 = (cz+d)^2\left(x^{-1} + \tfrac{2c}{cz+d}\right) \quad \text{for any } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma.$$

As explained in example 3 of 1.2.7, we can extend, for any $\kappa \in \mathbb{C}^*$, the homographic action of Γ on R by the action by automorphisms on B defined by

$$(x^{-1}|\gamma)_\kappa = (cz+d)^2\left(x^{-1} + \kappa\tfrac{2c}{cz+d}\right) \quad \text{for any } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma,$$

but the algebraic study of these actions and associated invariant algebras reduces to the case $\kappa = 1$. Arithmetical interpretations of these actions were studied in [4].

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