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POSITIVE PERIODIC SOLUTIONS OF A DISCRETE RATIO-DEPENDENT PREDATOR-PREY MODEL WITH IMPULSIVE EFFECTS

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Abstract. Studies of the dynamics of predator-prey systems are abundant in the literature. In an attempt to account for more realistic models, previous studies have opted for the use of discrete predator-prey systems with ratio-dependent functional response. In the present research we go a step further with the inclusion of impulsive effects in the dynamics. More concretely, by assuming that the coefficients involved in the system and the impulses are periodic, we obtain sufficient conditions for the existence of periodic solutions. We present some numerical examples to illustrate the effectiveness of our results.

1. Introduction

Competition represents one of the driving forces in biological and sociological systems. For instance, animals in an ecosystem may compete for food or mates; in a market economy, firms may compete over customers; sensory stimuli may compete for limited neural resources to achieve a particular task. In particular, the interaction between co-circulating pathogens and the immune system of a host may generate a complex competing dynamics [25], as well as two competing diseases spreading over the same population at the same time, where infection with either disease gives an individual subsequent immunity to both [19]. Furthermore, the decision of whether to get vaccinated or not might imply an implicit competition between cost and benefit of the vaccine by the end of the season of a seasonal disease [8]. From a more general viewpoint and based on spectral properties of complex networks, the competition between populations is studied and characterized [3]. In this paper we study an extension of the competition for food and survival in a predator-prey model. This competition is manifested specially when predators have to search for food and therefore have to share or compete for food. In this context, a more suitable general predator-prey theory should be based on the so-called ratio-dependent theory, which considers the per capita predator growth as a function of the ratio of prey to predator abundance (see for instance [4]

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and references therein). This is supported by numerous field and laboratory experiments and observations; see for instance [5]. Based on these observations, Arditi and Ginzburg [5] proposed the following ratio-dependent predator-prey model:

$$\begin{aligned} x' &= x(a - bx) - \frac{cxy}{my + x}, \\ y' &= y\left(-d + \frac{fx}{my + x}\right), \end{aligned} \tag{1.1}$$

where $x(t)$ and $y(t)$ stand for the densities of the prey and the predator, respectively; a, c, d and f are, respectively, the prey intrinsic growth rate, the capture rate, the death rate of the predator and the conversion rate; b measures the strength of competition among individuals of species x, a/b gives the carrying capacity of the prey and m is the half saturation constant. The predator consumes the prey with ratio-dependent functional response of Michaelis–Menten type $cuy/(m + u)$, $u = x/y$ and contributes to its growth with rate $fuy/(m + u)$. Predator-prey systems with ratio-dependent functional response have been studied extensively in the literature (see, for example, [21, 13, 16, 18, 20, 23, 29] and references therein).

However, in real-life situations it is possible to find highly heterogeneous environments that affect, for instance, birth rates, death rates, and other vital rates of populations. Therefore it is necessary to consider a non-autonomous model and take advantage of the properties of those varying parameters. For example, one may assume that these parameters are periodic or almost periodic for seasonal reasons. In this context, Fan et al. [12] incorporate the varying property of the parameters into the model and carry out systematic studies on the global dynamics of the following non-autonomous version of (1.1):

$$\begin{aligned} x' &= x(a(t) - b(t)x) - \frac{c(t)xy}{m(t)y + x}, \\ y' &= y\left(-d(t) + \frac{f(t)x}{m(t)y + x}\right), \end{aligned} \tag{1.2}$$

where all the variables and parameters have the same biological interpretation as in (1.1), except that in this case they are time dependent. In [12], the authors address some basic problems for (1.2), such as positive invariance, permanence, non-persistence, extinction, dissipativity and globally asymptotic stability of the system (1.2). Then the authors establish sufficient criteria for the existence of a unique positive periodic (almost periodic) solution of (1.2) that is globally asymptotically stable, when all parameters are periodic (almost periodic).

On the other hand, many authors [1, 2, 9, 11, 14, 17, 22, 24, 30, 31] have argued that the discrete time models governed by difference equations are more suitable than the continuous ones when populations have non-overlapping generations. In addition, discrete time models can also provide more efficient computational performance, in contrast to their continuous counterpart for numerical simulations. In particular, Fan and Wang [11] considered the discrete analogous of (1.2) which is given by the system

$$\begin{cases} x(k+1) = x(k)\exp\left[a(k) - b(k)x(k) - \dfrac{c(k)y(k)}{m(k)y(k)+x(k)}\right], \\ y(k+1) = y(k)\exp\left[-d(k) + \dfrac{f(k)x(k)}{m(k)y(k)+x(k)}\right], \quad k = 0,1,2,\dots, \end{cases}$$

and with the assumption that all coefficients are periodic, the authors studied the existence of positive periodic solutions.

On the other hand, there exists the possibility that the biological species could undergo sudden and drastic changes of their state at given times. These alterations in state might be due to external factors, such as partial destruction of their habitat, indiscriminate hunting of the species or artificial breeding. These changes can be represented in mathematical notation in the form of impulses, which cannot be well described by pure continuous-time or discrete-time models. In this case, the influence of these impulses on the system could be investigated by introducing impulsive effects (see, for instance, [6, 10, 26, 27, 28] and references therein).

In the present paper we will consider the following discrete ratio-dependent predator-prey system with impulses:

$$\begin{cases} x(k+1) = x(k)\exp\left[a(k) - b(k)x(k) - \dfrac{c(k)y(k)}{m(k)y(k)+x(k)}\right], \\ y(k+1) = y(k)\exp\left[-d(k) + \dfrac{f(k)x(k)}{m(k)y(k)+x(k)}\right], \quad k = 0,1,2,\dots, \\ x(k_m^+) = x(k_m) + I_{1m}(x(k_m), y(k_m)), \\ y(k_m^+) = y(k_m) + I_{2m}(x(k_m), y(k_m)), \quad m = 1,2,\dots, \end{cases} \tag{1.3}$$

where $I_{im} : \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^2$, $\Omega \neq \emptyset$, $i = 1,2$, $0 = k_0 < k_1 < k_2 < \cdots < k_m < \cdots$, are fixed impulsive points, and $\lim_{m\to\infty} k_m = +\infty$; $x(k_m^+)$ and $y(k_m^+)$ represent the new values of $x(k_m)$ and $y(k_m)$, which are used to find $x(k_m+1)$ and $y(k_m+1)$ in the first and second equation of (1.3).

The main goal of this paper is to give sufficient conditions for the existence of periodic solutions for system (1.3). To accomplish this purpose we will assume the following hypotheses:

(H1) $a(k), b(k), c(k), d(k), f(k), m(k) : \mathbb{N} \to \mathbb{R}^+$ are ω-periodic functions.
(H2) There exists an integer $q \in \mathbb{N}$ such that $k_{m+q} = k_m + \omega$, $I_\omega \cap \{k_m\}_{m=1}^{\infty} = \{k_1, \dots, k_{q-1}\}$, where $I_\omega = \{0, 1, \dots, \omega - 1\}$.
(H3) $I_{i(m+q)}(x,y) = I_{im}(x,y)$, $i = 1,2$.
(H4) There exist constants $M \geq 0$ and $0 < l < 1$ such that

$$0 \leq I_{1m}(x_1(k_m), x_2(k_m)) \leq M x_1(k_m)$$

and

$$-l x_2(k_m) \leq I_{2m}(x_1(k_m), x_2(k_m)) \leq 0, \quad \text{for } m = 1, \dots, q,$$

The organization of this paper shall be as follows. In the next section we present the Continuation Theorem that is the main tool that will be used to show our main result. In Section 3 we derive the conditions for the existence of positive periodic solutions. In Section 4 we show some numerical examples and the paper ends with the conclusion.

2. Preliminaries

Before showing the existence of periodic solutions of the system (1.3), we will need the following results.

Let X, Y be normed vector spaces, let $L : \operatorname{Dom} L \subset X \to Y$ be a linear mapping, and let $N : X \to Y$ be a continuous mapping. The mapping L will be called a *Fredholm mapping of index zero* if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Y. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$, it follows that $L|_{\operatorname{Dom} L \cap \operatorname{Ker} P} : \operatorname{Dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is invertible; its inverse is denoted by K_P. If Ω is an open bounded subset of X, the mapping N will be called *L-compact* on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-P)N : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Lemma 2.1 (Continuation Theorem [15])**.** *Let L be a Fredholm mapping of index zero and let N be L-compact. Suppose that*

(a) for each $\lambda \in (0, 1)$, every solution z of $Lz = \lambda Nz$ is such that $z \notin \partial\Omega$;
(b) $QNz \neq 0$ for each $z \in \partial\Omega \cap \operatorname{Ker} L$;
(c) the Brouwer degree $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then the operator equation $Lz = Nz$ has at least one solution lying in $\operatorname{Dom} L \cap \overline{\Omega}$.

Lemma 2.2 ([11])**.** *Let $g : \mathbb{N} \to \mathbb{R}$ be an ω-periodic function, i.e., $g(k+\omega) = g(k)$. Then for any fixed $k_1, k_2 \in I_\omega$ and any $k \in \mathbb{N}$, one has*

$$g(k) \leq g(k_1) + \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|,$$

$$g(k) \geq g(k_2) - \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|.$$

For convenience, we will use the notation

$$\overline{g} = \frac{1}{\omega} \sum_{k=0}^{\omega-1} g(k).$$

Define $l_2 = \{y = \{y(k)\} : y(k) \in \mathbb{R}^2,\, k \in \mathbb{N}\}$. Let $X \subset l_2$ denote the subspace of all ω-periodic sequences with the norm

$$\|y\|_X = \max_{k \in I_\omega} |y_1(k)| + \max_{k \in I_\omega} |y_2(k)|, \quad \text{for any } y = \{y(k) : k \in \mathbb{N}\} \in X.$$

Then $(X, \|\cdot\|_X)$ is a Banach space. Let

$$Y = \left\{ z = \left(\begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix} \begin{pmatrix} r_1 \\ s_1 \end{pmatrix}, \ldots, \begin{pmatrix} r_q \\ s_q \end{pmatrix} \right) \right\} = X \times \mathbb{R}^{2q},$$

where $y(k) = (y_1(k), y_2(k))^\top \in X$ and $(r_i, s_i)^\top \in \mathbb{R}^2$ are constant vectors, $i = 1, 2, \dots, q$. Let $\|z\|_Y = \|y\|_X + \sum_{i=1}^{q}(|r_i| + |s_i|)$. Then $(Y, \|\cdot\|_Y)$ is also a Banach space.

3. Main result

In the following result we give sufficient conditions for the existence of periodic solutions of the system (1.3).

Theorem 3.1. *Suppose that conditions* (H1)–(H4) *hold. If* $\overline{a} > \overline{c/m}$ *and* $\omega(\overline{f}-\overline{d}) > -q\ln(1-l)$*, then the system* (1.3) *has at least one positive* ω*-periodic solution.*

Proof. Let

$$u(k) = \ln\{x(k)\} \quad \text{and} \quad v(k) = \ln\{y(k)\};$$

then the system (1.3) is transformed to

$$\begin{cases} u(k+1) - u(k) = a(k) - b(k)\exp\{u(k)\} - \dfrac{c(k)\exp\{v(k)\}}{m(k)\exp\{v(k)\} + \exp\{u(k)\}} \equiv \phi_1(k), \\ v(k+1) - v(k) = -d(k) + \dfrac{f(k)\exp\{u(k)\}}{m(k)\exp\{v(k)\} + \exp\{u(k)\}} \equiv \phi_2(k), \quad k = 1, 2, \dots, \\ u(k_m^+) - u(k_m) = \ln\left[1 + \varphi_{1m}(u(k_m), v(k_m))\right], \\ v(k_m^+) - v(k_m) = \ln\left[1 + \varphi_{2m}(u(k_m), v(k_m))\right], \quad m = 1, 2, \dots, \end{cases} \tag{3.1}$$

where

$$\varphi_{1m}(u(k_m), v(k_m)) = \frac{I_{1m}(\exp\{u(k_m)\}, \exp\{v(k_m)\})}{\exp\{u(k_m)\}},$$
$$\varphi_{2m}(u(k_m), v(k_m)) = \frac{I_{2m}(\exp\{u(k_m)\}, \exp\{v(k_m)\})}{\exp\{v(k_m)\}}.$$

Set $L : X \to Y$ as

$$L\begin{pmatrix} u(k) \\ v(k) \end{pmatrix} = \left(\begin{pmatrix} u(k+1) - u(k) \\ v(k+1) - v(k) \end{pmatrix}, \left\{\begin{pmatrix} u(k_m^+) - u(k_m) \\ v(k_m^+) - v(k_m) \end{pmatrix}\right\}_{m=1}^{q}\right),$$

and $N : X \to Y$ as

$$N\begin{pmatrix} u(k) \\ v(k) \end{pmatrix} = \left(\begin{pmatrix} \phi_1(k) \\ \phi_2(k) \end{pmatrix}, \left\{\begin{pmatrix} \ln\left[1 + \varphi_{1m}(u(k_m), v(k_m))\right] \\ \ln\left[1 + \varphi_{2m}(u(k_m), v(k_m))\right] \end{pmatrix}\right\}_{m=1}^{q}\right).$$

With this notation the system (3.1) can be written as $Lz = Nz$. Obviously,

$$\operatorname{Ker} L = \left\{\begin{pmatrix} u(k) \\ v(k) \end{pmatrix} : \begin{pmatrix} u(k) \\ v(k) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\right\} = \mathbb{R}^2$$

and

$$\operatorname{Im} L = \left\{ z = \left(\begin{pmatrix} \phi_1(k) \\ \phi_2(k) \end{pmatrix}, \left\{\begin{pmatrix} \varphi_{1m} \\ \varphi_{2m} \end{pmatrix}\right\}_{m=1}^{q}\right) \in Y \,\middle|\, \sum_{k=0}^{\omega-1} \phi_i(k) + \sum_{m=1}^{q} \varphi_{im} = 0, \ i = 1, 2 \right\}$$

is closed in Y, and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 2$. Therefore L is a Fredholm mapping of index zero.

Define the projectors $P : X \to X$ as

$$P \begin{pmatrix} u(k) \\ v(k) \end{pmatrix} = \frac{1}{\omega} \begin{pmatrix} \sum_{s=0}^{\omega-1} u(s) \\ \sum_{s=0}^{\omega-1} v(s) \end{pmatrix}$$

and $Q : Y \to Y$ as

$$\begin{aligned} Qz &= Q\left(\begin{pmatrix} \phi_1(k) \\ \phi_2(k) \end{pmatrix}, \left\{ \begin{pmatrix} \varphi_{1m} \\ \varphi_{2m} \end{pmatrix} \right\}_{m=1}^{q} \right) \\ &= \left(\frac{1}{\omega} \begin{pmatrix} \sum_{s=0}^{\omega-1} \phi_1(s) + \sum_{m=1}^{q} \varphi_{1m} \\ \sum_{s=0}^{\omega-1} \phi_2(s) + \sum_{m=1}^{q} \varphi_{2m} \end{pmatrix}, \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}_{m=1}^{q} \right); \end{aligned}$$

then P and Q are continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$. Furthermore, the generalized inverse of L, $K_P : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ exists and is given by

$$K_P z = \sum_{s=0}^{k-1} v(s) + \sum_{s > k_m} w_m + \frac{1}{\omega} \sum_{s=0}^{\omega-1} s v(s) - \sum_{m=1}^{q} w_m + \frac{1}{\omega} \sum_{m=1}^{q} k_m w_m - \sum_{s=0}^{\omega-1} v(s),$$

where $z = (\xi(k), w_1, \ldots, w_q)^\top \in Y$, $\xi(k) = (\phi_1(k), \phi_2(k))^\top$ and $w_m = (\varphi_{1m}, \varphi_{2m})^\top$, $m = 1, \ldots, q$. Thus,

$$QNz = Q\left(\begin{pmatrix} \phi_1(k) \\ \phi_2(k) \end{pmatrix}, \left\{ \begin{pmatrix} \ln(1 + \varphi_{1m}(u(k_m), v(k_m))) \\ \ln(1 + \varphi_{2m}(u(k_m), v(k_m))) \end{pmatrix} \right\}_{m=1}^{q} \right)$$

$$= \left(\frac{1}{\omega} \begin{pmatrix} \sum_{s=0}^{\omega-1} \phi_1(s) + \sum_{m=1}^{q} \ln(1 + \varphi_{1m}(u(k_m), v(k_m))) \\ \sum_{s=0}^{\omega-1} \phi_2(s) + \sum_{m=1}^{q} \ln(1 + \varphi_{2m}(u(k_m), v(k_m))) \end{pmatrix}, \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}_{m=1}^{q} \right)$$

and

$$
\begin{aligned}
&(K_P(I-Q)N)(z(k)) \\
&\quad= \sum_{s=0}^{k-1}\begin{pmatrix}\phi_1(s)\\ \phi_2(s)\end{pmatrix} + \sum_{s>k_m}\begin{pmatrix}\ln(1+\varphi_{1m}(u(k_m),v(k_m)))\\ \ln(1+\varphi_{2m}(u(k_m),v(k_m)))\end{pmatrix} \\
&\qquad - \left[\frac{k}{\omega}+\frac{\omega-1}{2\omega}\right]\left[\sum_{s=0}^{\omega-1}\begin{pmatrix}\phi_1(s)\\ \phi_2(s)\end{pmatrix} + \sum_{m=1}^{q}\begin{pmatrix}\ln(1+\varphi_{1m}(u(k_m),v(k_m)))\\ \ln(1+\varphi_{2m}(u(k_m),v(k_m)))\end{pmatrix}\right] \\
&\qquad + \frac{1}{\omega}\left[\sum_{s=0}^{\omega-1}\begin{pmatrix}s\phi_1(s)\\ s\phi_2(s)\end{pmatrix} + \sum_{m=1}^{q}\begin{pmatrix}k_m\ln(1+\varphi_{1m}(u(k_m),v(k_m)))\\ k_m\ln(1+\varphi_{2m}(u(k_m),v(k_m)))\end{pmatrix}\right].
\end{aligned}
$$

Clearly, QN and $K_P(I-Q)N$ are continuous. Since X is a finite dimensional Banach space, it is not difficult to prove that $K_P(I-Q)N(\overline{\Omega})$ is relatively compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Therefore the mapping N is L-compact on $\overline{\Omega}$.

Now, we need to find an appropiate open, bounded subset Ω for the application of the Continuation Theorem. Corresponding to equation $Lz = \lambda Nz$, $\lambda \in (0,1)$, we have

$$
\begin{cases}
u(k+1)-u(k) = \lambda\left[a(k) - b(k)\exp\{u(k)\} - \dfrac{c(k)\exp\{v(k)\}}{m(k)\exp\{v(k)\}+\exp\{u(k)\}}\right], \\
v(k+1)-v(k) = \lambda\left[-d(k) + \dfrac{f(k)\exp\{u(k)\}}{m(k)\exp\{v(k)\}+\exp\{u(k)\}}\right], \quad k=1,2,\dots, \\
u(k_m^+)-u(k_m) = \lambda\ln\left[1+\varphi_{1m}(u(k_m),v(k_m))\right], \\
v(k_m^+)-v(k_m) = \lambda\ln\left[1+\varphi_{2m}(u(k_m),v(k_m))\right], \quad m=1,2,\dots.
\end{cases}
\tag{3.2}
$$

Suppose that $(u(k),v(k))^\top \in X$ is an arbitrary solution of the system (3.2) for a certain $\lambda \in (0,1)$. Summing on both sides of (3.2) from 0 to $\omega-1$ with respect to k, we obtain

$$
\begin{aligned}
0 &= \sum_{k=0}^{\omega-1}[u(k+1)-u(k)] \\
&= \lambda\sum_{k=0}^{\omega-1}\left[a(k) - b(k)\exp\{u(k)\} - \frac{c(k)\exp\{v(k)\}}{m(k)\exp\{v(k)\}+\exp\{u(k)\}}\right] \\
&\qquad + \lambda\sum_{m=1}^{q}\ln\left[1+\varphi_{1m}(u(k_m),v(k_m))\right],
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \sum_{k=0}^{\omega-1}[v(k+1)-v(k)] \\
&= \lambda \sum_{k=0}^{\omega-1}\left[-d(k)+\frac{f(k)\exp\{u(k)\}}{m(k)\exp\{v(k)\}+\exp\{u(k)\}}\right] \\
&\quad + \lambda \sum_{m=1}^{q} \ln\left[1+\varphi_{2m}(u(k_m),v(k_m))\right],
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\omega\overline{a} + \sum_{m=1}^{q} \ln\left[1+\varphi_{1m}(u(k_m),v(k_m))\right] \\
&\qquad = \sum_{k=0}^{\omega-1}\left[b(k)\exp\{v(k)\}+\frac{c(k)\exp\{v(k)\}}{m(k)\exp\{v(k)\}+\exp\{u(k)\}}\right] \qquad (3.3)
\end{aligned}
$$

and

$$
\omega\overline{d} - \sum_{m=1}^{q} \ln\left[1+\varphi_{2m}(u(k_m),v(k_m))\right] = \sum_{k=0}^{\omega-1}\left[\frac{c(k)\exp\{u(k)\}}{m(k)\exp\{v(k)\}+\exp\{u(k)\}}\right]. \qquad (3.4)
$$

It follows from (3.3), (3.4) and (H4) that

$$
\begin{aligned}
&\sum_{k=0}^{\omega-1}|u(k+1)-u(k)| \\
&\quad \le \sum_{k=0}^{\omega-1}\left[a(k)+b(k)\exp\{u(k)\}+\frac{c(k)\exp\{v(k)\}}{m(k)\exp\{v(k)\}+\exp\{u(k)\}}\right] \qquad (3.5) \\
&\qquad + \sum_{m=1}^{q} \ln[1+\varphi_{1m}(u(k_m),v(k_m))] \le 2[\omega\overline{a}+q\ln(1+M)] =: A_1,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k=0}^{\omega-1}|v(k+1)-v(k)| \\
&\quad \le \sum_{k=0}^{\omega-1}\left[d(k)+\frac{f(k)\exp\{u(k)\}}{m(k)\exp\{v(k)\}+\exp\{u(k)\}}\right] \qquad (3.6) \\
&\qquad - \sum_{m=1}^{q} \ln[1+\varphi_{2m}(u(k_m),v(k_m))] \le 2[\omega\overline{d}-q\ln(1-l)] =: A_2.
\end{aligned}
$$

Since $(u(k), v(k)) \in X$, there exist $\xi_i, \eta_i \in I_\omega$ such that

$$\begin{aligned} u(\xi_1) &= \min_{k\in I_\omega}\{u(k)\}, \quad v(\xi_2) = \min_{k\in I_\omega}\{v(k)\}, \\ u(\eta_1) &= \max_{k\in I_\omega}\{u(k)\}, \quad v(\eta_2) = \max_{k\in I_\omega}\{v(k)\}. \end{aligned} \tag{3.7}$$

It follows from (3.3) and (3.7) that

$$\begin{aligned} \omega\overline{a} + q\ln(1+M) &\geq w\overline{a} + \sum_{m=1}^{q} \ln[1+\varphi_{1m}(u(k_m), v(k_m))] \\ &\geq \sum_{k=0}^{\omega-1} b(k)\exp\{u(k)\} \geq \omega\overline{b}\exp\{u(\xi_1)\}; \end{aligned}$$

hence and from Lemma 2.1 and (3.5), it follows that

$$u(k) \leq u(\xi_1) + \sum_{s=0}^{\omega-1} |u(s+1) - u(s)| \leq L_1 + A_1. \tag{3.8}$$

From (3.3) and (3.7), we also obtain

$$\begin{aligned} \omega\overline{a} &\leq \omega\overline{a} + \sum_{m=1}^{q} \ln\left[1+\varphi_{1m}(u(k_m), v(k_m))\right] \leq \sum_{k=0}^{\omega-1}\left[b(k)\exp\{u(k)\} + \frac{c(k)}{m(k)}\right] \\ &\leq \omega\left[\overline{b}\exp\{u(\eta_1)\} + \overline{c/m}\right], \end{aligned}$$

and then

$$u(\eta_1) \geq \ln\left[\frac{\overline{a} - \overline{c/m}}{\overline{b}}\right] =: l_1.$$

By using Lemma 2.2 and (3.5) we have

$$u(k) \geq u(\eta_1) - \sum_{s=0}^{\omega-1} |u(s+1) - u(s)| \geq l_1 - A_1, \tag{3.9}$$

so

$$|u(k)| \leq \max\{|l_1 - A_1|, |L_1 + A_1|\} =: B_1. \tag{3.10}$$

Similarly, from (3.4), (3.7) and (3.8) we derive

$$\begin{aligned} \omega\overline{d} &\leq \omega\overline{d} - \sum_{m=1}^{q} \ln\left[1+\varphi_{2m}(u(k_m), v(k_m))\right] \leq \sum_{k=0}^{\omega-1} \frac{f(k)\exp\{u(k)\}}{m(k)\exp\{v(k)\}} \\ &\leq \sum_{k=0}^{\omega-1} \frac{f(k)\exp\{L_1+A_1\}}{m(k)\exp\{v(\xi_2)\}} \leq \omega\left(\overline{\frac{f}{m}}\right)\frac{\exp\{L_1+A_1\}}{\exp\{v(\xi_2)\}}; \end{aligned}$$

therefore

$$v(\xi_2) \leq \ln\left[\frac{(\overline{f/m})\exp\{L_1+A_1\}}{\overline{d}}\right] =: L_2.$$

From Lemma 2.2 and (3.6) we have

$$v(k) \leq v(\xi_2) + \sum_{s=0}^{\omega-1} |v(s+1) - v(s)| \leq L_2 + A_2. \tag{3.11}$$

On the other hand, from (3.4), (3.7) and (3.9) we obtain

$$\begin{aligned}\omega\overline{d} - q\ln(1-l) &\geq \omega\overline{d} - \sum_{m=1}^{q} \ln[1+\varphi_{2m}(u(k_m), v(k_m))]\\ &\geq \sum_{k=0}^{\omega-1} \frac{f(k)\exp\{l_1 - A_1\}}{m^u \exp\{v(\eta_2)\} + \exp\{l_1 - A_1\}}\\ &\geq \frac{\omega\overline{f}\exp\{l_1 - A_1\}}{m^u \exp\{v(\eta_2)\} + \exp\{l_1 - A_1\}},\end{aligned}$$

where $m^u = \max_{k\in I_\omega} m(k)$. Then it follows that

$$v(\eta_2) \geq \ln\left[\frac{1}{m^u}\left(\frac{\omega\overline{f}}{\omega\overline{d} - q\ln(1-l)} - 1\right)\right] + l_1 - A_1 =: l_2;$$

consequently

$$v(k) \geq v(\eta_2) - \sum_{s=0}^{\omega-1} |v(s+1) - v(s)| \geq l_2 - A_2. \tag{3.12}$$

From (3.11) and (3.12) we obtain

$$|v(k)| \leq \max\{|l_2 - A_2|, |L_2 + A_2|\} =: B_2. \tag{3.13}$$

Clearly B_1 and B_2 are independent of λ. Take $B = B_1 + B_2 + B_0$, where B_0 is taken sufficiently large such that $B_0 > |l_1| + |L_1| + |l_2| + |L_2|$, and define $\Omega = \{z = (u,v)^\top \in X : \|z\| < B\}$.

Now we can verify the conditions of Lemma 2.1.

(a) From (3.10) and (3.13), one can conclude that $Lz \neq \lambda Nz$, for each $\lambda \in (0,1)$ and $z \in \partial\Omega \cap \operatorname{Dom} L$.

(b) When $(u(k), v(k))^\top \in \partial\Omega \cap \operatorname{Ker} L$, $(u(k), v(k))^\top$ is a constant vector in $\mathbb{R}^2$ with $\|(u,v)\| = B$; then $QNz \neq 0$.

(c) In order to verify condition (c) in Lemma 2.1, we consider the algebraic equations

$$\begin{cases}\overline{a} - \overline{b}\exp\{u\} - \dfrac{1}{\omega}\displaystyle\sum_{k=0}^{\omega-1} \frac{c(k)\exp\{v\}}{m(k)\exp\{v\} + \exp\{u\}} + \dfrac{1}{\omega}\sum_{m=1}^{q} \mu\ln[1+\varphi_{1m}(u,v)] = 0\\ -\overline{d} + \dfrac{1}{\omega}\displaystyle\sum_{k=0}^{\omega-1} \frac{f(k)\exp\{u\}}{m(k)\exp\{v\} + \exp\{u\}} + \dfrac{1}{\omega}\sum_{m=1}^{q} \mu\ln[1+\varphi_{2m}(u,v)] = 0,\end{cases} \tag{3.14}$$

where $\mu \in [0,1]$ is a parameter. Now, if (u^*, v^*) is a solution of (3.14) then it is easy to show that

$$l_1 \leq u^* \leq L_1 \quad \text{and} \quad l_2 \leq v^* \leq L_2. \tag{3.15}$$

In order to compute the Brouwer degree, let us consider the homotopy

$$H_\mu(z) = \mu QN(z) + (1-\mu)F(z), \quad \mu \in [0,1],$$

where

$$F(z) = \begin{pmatrix} \bar{a} - \bar{b}\exp\{u\} - \dfrac{1}{\omega}\displaystyle\sum_{k=0}^{\omega-1} \dfrac{c(k)\exp\{v\}}{m(k)\exp\{v\}+\exp\{u\}} \\ -\bar{d} + \dfrac{1}{\omega}\displaystyle\sum_{k=0}^{\omega-1} \dfrac{f(k)\exp\{u\}}{m(k)\exp\{v\}+\exp\{u\}} \end{pmatrix}.$$

From (3.15), it follows that $0 \notin H_\mu(\partial\Omega \cap \operatorname{Ker} L)$ for $\mu \in [0,1]$. Moreover, by [11, Lemma 3.3] the algebraic equation $F(z) = 0$ has a unique solution in $\mathbb{R}^2$, so by the invariance property of homotopy and taking

$$J : \operatorname{Im} Q \to \operatorname{Ker} L, \quad J\left(\begin{pmatrix} u \\ v \end{pmatrix}, \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}_{m=1}^{q}\right) = \begin{pmatrix} u \\ v \end{pmatrix},$$

we obtain

$$\begin{aligned} \deg\{JQN, \Omega \cap \operatorname{Ker} L, (0,0)^\top\} &= \deg\{QN, \Omega \cap \operatorname{Ker} L, (0,0)^\top\} \\ &= \deg\{F, \Omega \cap \operatorname{Ker} L, (0,0)^\top\} \\ &= \frac{\bar{b}}{\omega}\sum_{k=0}^{\omega-1} \frac{f(k)m(k)\exp\{u\}\exp\{v\}}{(m(k)\exp\{v\}+\exp\{u\})^2} \neq 0. \end{aligned}$$

We have proved that Ω satisfies all the requirements of Lemma 2.1; then $Lz = Nz$ has at least one solution in $\operatorname{Dom} L \cap \overline{\Omega}$, so the system (3.2) has at least one ω-periodic solution in $\operatorname{Dom} L \cap \overline{\Omega}$, say $(u^*(k), v^*(k))^\top$. Let $x^*(k) = \exp\{u^*(k)\}$ and $y^*(k) = \exp\{v^*(k)\}$, so $(u^*(k), v^*(k))^\top$ is an ω-periodic solution of the system (1.3) with strictly positive components. This completes the proof. ☐

4. Numerical simulation

In this section we present some numerical examples that show the effectiveness of the presented theoretical result. The results could be broadly interpreted in many areas; however, for the sake of generality we just show parameters that satisfy the hypotheses of Theorem 3.1. We will start considering the coefficient functions of the system (1.3) as follows:

$$\begin{aligned} a(k) &= 0.5(3 + 0.84\sin(k\pi/4)), & b(k) &= 0.3(0.8 + 0.5\cos(k\pi/4)), \\ c(k) &= 0.8(1 + 0.7\sin(k\pi/4)), & d(k) &= 0.5(0.9 + 0.6\sin(k\pi/4)), \\ f(k) &= 1.2(1 + 0.8\cos(k\pi/4)), & m(k) &= 3.5(0.5 + 0.3\cos(k\pi/4)); \end{aligned} \tag{4.1}$$

in this case the functions are 8-periodic (i.e., $\omega = 8$).

Example 4.1. Let us pick

$$\begin{aligned} I_{1m}(x(k_m), y(k_m)) &= 0.8(1 + \sin(k_m\pi/4))x(k_m), \\ I_{2m}(x(k_m), y(k_m)) &= -0.33(1 + \sin(k_m\pi/4))y(k_m), \end{aligned}$$

with $k_m = 4m$. In this case, it can be deduced that conditions (H1)–(H4) hold with $q = 2$, $M = 1.6$ and $l = 0.66$. Since $\overline{a} - \overline{c/m} = 0.92840 > 0$ and $\omega(\overline{f} - \overline{d}) + q\ln(1-l) = 3.8424 > 0$, by Theorem 3.1 the system (1.3) has an 8-periodic solution. This is shown in Figure 1.

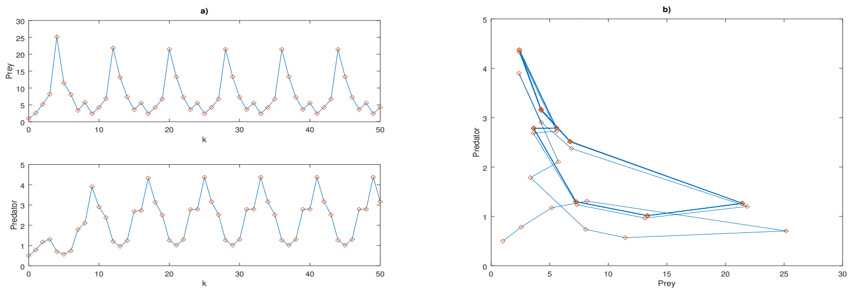

Figure 1. (a) Solution of the system (1.3) on [0,50]. (b) Phase portrait of (1.3) on [0,100].

Example 4.2. Let us pick

$$I_{1m}(x(k_m), y(k_m)) = (1 + \cos(k_m\pi/4))x(k_m),$$
$$I_{2m}(x(k_m), y(k_m)) = -0.1(1 + \sin(k_m\pi/4))y(k_m),$$

with $k_m = 4m$. In this case, it can be deduced that conditions (H1)–(H4) hold with $q = 2$, $M = 2$ and $l = 0.2$. Since $\overline{a} - \overline{c/m} = 0.92840 > 0$ and $\omega(\overline{f} - \overline{d}) + q\ln(1-l) = 5.5537 > 0$, by Theorem 3.1 the system (1.3) has an 8-periodic solution. This is shown in Figure 2.

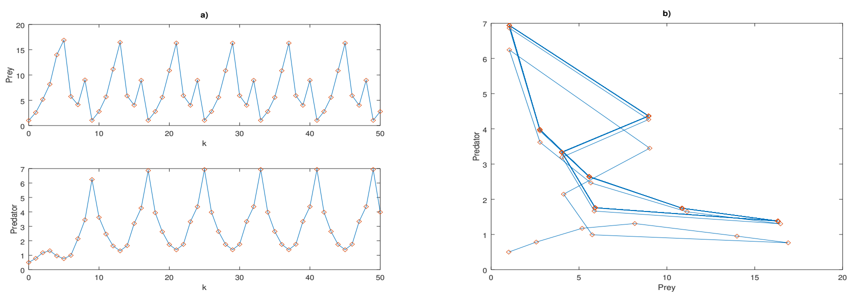

Figure 2. (a) Solution of the system (1.3) on [0,50]. (b) Phase portrait of (1.3) on [0,100].

In the case when the system (1.3) does not have impulses (i.e., $x(k_m^+) = x(k_m)$ and $y(k_m^+) = y(k_m)$) the parameters (4.1) satisfy the conditions of [11, Theorem 3.1], that is, $\overline{f} > \overline{d}$ and $\overline{a} > \overline{c/m}$; therefore the system (1.3) without impulses has a positive periodic solution (see Figure 3).

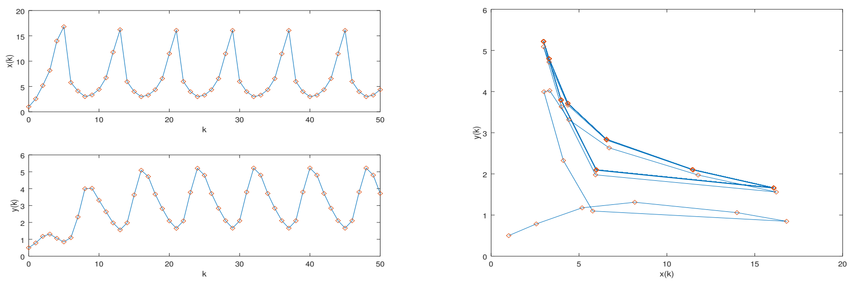

Figure 3. (a) Solution of the system (1.3) without impulses on [0,50]. (b) Phase portrait of (1.3) without impulses on [0,100].

5. Conclusion

In the present paper, we introduce an extension of the discrete ratio-dependent predator-prey model presented by Fan and Wang in [11] (when $x(k_m^+) = x(k_m)$ and $y(k_m^+) = y(k_m)$), where they give sufficient conditions for the existence of periodic solutions when the parameters for the system are periodic. Specifically, we considered the inclusion of impulses in the system and studied their influence on its dynamical properties. We show that the theorem proposed in [11] is contained in Theorem 3.1 when $\overline{f} > \overline{d}$ and $\overline{a} > \overline{c/m}$. On the other hand, our results show that periodic impulsive effects do not destroy the existence of positive periodic solutions of the model without impulses. There still remain many interesting mathematical questions that need to be studied for the system (1.3). For example, the stability of periodic solutions and the existence and stability of almost periodic solutions; or the effect of the inclusion of locality in the system, in the form of a network on local interaction. The generality of our results can be applied to many contexts and areas, where the notion of competition is present and where periodic (or semi-periodic) interventions could be justified. All these questions will be analyzed in a forthcoming paper.

References

[1] R. P. Agarwal, *Difference Equations and Inequalities*, second edition, Monographs and Textbooks in Pure and Applied Mathematics, 228, Marcel Dekker, New York, 2000. MR 1740241.

[2] R. P. Agarwal and P. J. Y. Wong, *Advanced Topics in Difference Equations*, Mathematics and its Applications, 404, Kluwer Academic Publishers, Dordrecht, 1997. MR 1447437.

[3] J. Aguirre, D. Papo, and J. M. Buldú, Successful strategies for competing networks, *Nature Phys.* **9** (2013), 230–234.

[4] H. R. Akcakaya, R. Arditi, and L. R. Ginzburg, Ratio-dependent predation: an abstraction that works, *Ecology* **76** (1995), no. 3, 995–1004.

[5] R. Arditi and L. R. Ginzburg, Coupling in predator-prey dynamics: ratio-dependence, *J. Theor. Biol.* **139** (1989), no. 3, 311–326.

[6] D. Baĭnov and P. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, Pitman Monographs and Surveys in Pure and Applied Mathematics, 66, Longman Scientific & Technical, Harlow, 1993. MR 1266625.

[7] E. Beretta and Y. Kuang, Global analyses in some delayed ratio-dependent predator-prey systems, *Nonlinear Anal.* **32** (1998), no. 3, 381–408. MR 1610586.

[8] A. Cardillo, C. Reyes, F. Naranjo, and J. Gómez, Evolutionary vaccination dilemma in complex networks, *Phys. Rev. E* **88** (2013), 032803.

[9] Y. Chen and Z. Zhou, Stable periodic solution of a discrete periodic Lotka–Volterra competition system, *J. Math. Anal. Appl.* **277** (2003), no. 1, 358–366. MR 1954481.

[10] X. Ding and J. Jiang, Periodicity in a generalized semi-ratio-dependent predator-prey system with time delays and impulses, *J. Math. Anal. Appl.* **360** (2009), no. 1, 223–234. MR 2548378.

[11] M. Fan and K. Wang, Periodic solutions of a discrete time nonautonomous ratio-dependent predator-prey system, *Math. Comput. Modelling* **35** (2002), no. 9-10, 951–961. MR 1910673.

[12] M. Fan, Q. Wang, and X. Zou, Dynamics of a non-autonomous ratio-dependent predator-prey system, *Proc. Roy. Soc. Edinburgh Sect. A* **133** (2003), no. 1, 97–118. MR 1960049.

[13] H. I. Freedman and R. M. Mathsen, Persistence in predator-prey systems with ratio-dependent predator influence, *Bull. Math. Biol.* **55** (1993), no. 4, 817–827.

[14] H. I. Freedman, *Deterministic Mathematical Models in Population Ecology*, Monographs and Textbooks in Pure and Applied Mathematics, 57, Marcel Dekker, New York, 1980. MR 0586941.

[15] R. E. Gaines and J. L. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Lecture Notes in Mathematics, 568, Springer-Verlag, Berlin, 1977. MR 0637067.

[16] S.-B. Hsu, T.-W. Hwang, and Y. Kuang, Global analysis of the Michaelis-Menten-type ratio-dependent predator-prey system, *J. Math. Biol.* **42** (2001), no. 6, 489–506. MR 1845589.

[17] H.-F. Huo and W.-T. Li, Stable periodic solution of the discrete periodic Leslie-Gower predator-prey model, *Math. Comput. Modelling* **40** (2004), no. 3-4, 261–269. MR 2091059.

[18] C. Jost, O. Arino, and R. Arditi, About deterministic extinction in ratio-dependent predator-prey models, *Bull. Math. Biol.* **61** (1999), 19–32.

[19] B. Karrer and M. E. J. Newman, Competing epidemics on complex networks, *Phys. Rev. E* **84** (2011), 036106.

[20] Y. Kuang, Rich dynamics of Gause-type ratio-dependent predator-prey system, in *Differential Equations with Applications to Biology (Halifax, NS, 1997)*, 325–337, Fields Inst. Commun., 21, Amer. Math. Soc., Providence, RI, 1999. MR 1662624.

[21] Y. Kuang and E. Beretta, Global qualitative analysis of a ratio-dependent predator-prey system, *J. Math. Biol.* **36** (1998), no. 4, 389–406. MR 1624192.

[22] Z. Li and F. Chen, Almost periodic solutions of a discrete almost periodic logistic equation, *Math. Comput. Modelling* **50** (2009), no. 1-2, 254–259. MR 2542607.

[23] M. Lizana and J. J. Marín V., Pattern formation in a reaction-diffusion ratio-dependent predator-prey model, *Miskolc Math. Notes* **6** (2005), no. 2, 201–216. MR 2199165.

[24] J. D. Murray, *Mathematical Biology*, Biomathematics, 19, Springer-Verlag, Berlin, 1989. MR 1007836.

[25] C. Poletto, S. Meloni, A. Van Metre, V. Colizza, Y. Moreno, and S. Vespignani, Characterising two-pathogen competition in spatially structured environments, *Sci. Rep.* **5** (2015), 7895.

[26] S. Shen and P. Weng, Positive periodic solution of a discrete predator-prey patch-system, *Acta Math. Appl. Sin. Engl. Ser.* **24** (2008), no. 4, 627–642. MR 2448362.

[27] I. Stamova and G. Stamov, *Applied Impulsive Mathematical Models*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, 2016. MR 3496133.

[28] Q. Wang, Y. Zhang, Z. Wang, M. Ding, and H. Zhang, Periodicity and attractivity of a ratio-dependent Leslie system with impulses, *J. Math. Anal. Appl.* **376** (2011), no. 1, 212–220. MR 2745401.

[29] D. Xiao and S. Ruan, Global dynamics of a ratio-dependent predator-prey system, *J. Math. Biol.* **43** (2001), no. 3, 268–290. MR 1868217.

[30] J. Zhang and J. Wang, Periodic solutions for discrete predator-prey systems with the Beddington–DeAngelis functional response, *Appl. Math. Lett.* **19** (2006), no. 12, 1361–1366. MR 2264191.

[31] Z. Zhou and X. Zou, Stable periodic solutions in a discrete periodic logistic equation, *Appl. Math. Lett.* **16** (2003), no. 2, 165–171. MR 1962311.

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