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TRIVIAL EXTENSIONS OF MONOMIAL ALGEBRAS

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Abstract. We describe the ideal of relations for the trivial extension $T(\Lambda)$ of a finite-dimensional monomial algebra Λ. When Λ is, moreover, a gentle algebra, we solve the converse problem: given an algebra B, determine whether B is the trivial extension of a gentle algebra. We characterize such algebras B through properties of the cycles of their quiver, and show how to obtain all gentle algebras Λ such that $T(\Lambda) \cong B$. We prove that indecomposable trivial extensions of gentle algebras coincide with Brauer graph algebras with multiplicity one in all vertices in the associated Brauer graph, result proven by S. Schroll.

1. Introduction

Let Λ be a finite-dimensional k-algebra (associative, with identity) over an algebraically closed field k. Consider the *trivial extension algebra* $T(\Lambda) = \Lambda \ltimes D(\Lambda)$ of Λ by the Λ-bimodule $D(\Lambda) = \operatorname{Hom}_k(\Lambda, k)$, that is, $T(\Lambda) = \Lambda \oplus D(\Lambda)$ as k-vector space and the multiplication in $T(\Lambda)$ is given by $(a, f)(b, g) = (ab, ag + fb)$ for $a, b \in \Lambda$ and $f, g \in D(\Lambda)$.

The ordinary quiver of the trivial extension $T(\Lambda)$ of a finite-dimensional algebra $\Lambda = kQ_\Lambda/I_\Lambda$ was described by Fernández and Platzeck [3], where also the relations of such trivial extension are given under the assumption that any oriented cycle in the ordinary quiver of Λ is zero in Λ.

In this work we will describe the relations for $T(\Lambda)$ when Λ is monomial. These algebras form a broad family containing, among others, string algebras and gentle algebras.

If Λ is monomial, the ordinary quiver of $T(\Lambda)$ is obtained from Q_Λ by adding t arrows, where t is the number of paths of kQ_Λ which are maximal nonzero in Λ. For each of these maximal paths we add an arrow in the opposite direction. In this way, we obtain an oriented cycle, which we call *elementary*. We prove that each

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nonzero path γ of the quiver of $T(\Lambda)$ is contained in an elementery cycle C such that $C = \gamma\mu$ for some path μ, and we say that μ is a *supplement* of γ. The ideal of relations for $T(\Lambda)$ is generated by

(i) the paths not contained in an elementary cycle, and
(ii) the elements $\mu - \mu'$, where μ, μ' are different paths from $kQ_{T(\Lambda)}$ with a common supplement γ in elementary cycles C and C', respectively.

When Λ is gentle, elementary cycles of $kQ_{T(\Lambda)}$ do not overlap (that is, have no common arrows). Thus the description of $I_{T(\Lambda)}$ is easier, because the generators described in (ii) can be replaced by the elements $C - C'$, where C and C' are elementary cycles starting at the same vertex of $Q_{T(\Lambda)}$.

Let Λ be a gentle algebra. Then the bound quiver of $B = T(\Lambda)$ satisfies the following properties:

(T_1) Any permutation of a maximal cycle is a maximal cycle.
(T_2) Any path u of kQ_B nonzero in B is contained in a maximal cycle of kQ_B, which is unique up to permutations if u is nontrivial.
(T_3) There are at most two different cycles from j to j of kQ_B maximal nonzero in B for any vertex j of $(Q_B)_0$. If there are two such cycles, they are equal in B.
(T_4) If $\alpha_s \cdots \alpha_1 : j \to j$ is a nonzero cycle of kQ_B which is not maximal, then $\widetilde{\alpha_1\alpha_s} = 0$ in B and the dimension of the endomorphism ring of the projective associated to the vertex j is four.

We prove that these properties characterize trivial extensions of gentle algebras. Moreover, we show how to find all gentle algebras Λ such that $T(\Lambda) \simeq B$, as stated in the following result.

Theorem. *Let $B = kQ_B/I_B$ be a finite-dimensional algebra satisfying* (T_1), (T_2), (T_3) *and* (T_4).

(i) *Let Q be the quiver obtained from Q_B by eliminating exactly one arrow of each cycle of Q_B maximal in B, and let $I = kQ \cap I_B$. Then $\Lambda = kQ/I$ is a gentle algebra and $T(\Lambda) \cong B$.*
(ii) *If Λ is a gentle algebra such that $T(\Lambda) \cong B$, then $\Lambda = kQ/I$, with Q and I as in* (i).

Finally, we prove that trivial extensions of indecomposable gentle algebras coincide with Brauer graph algebras with multiplicity one in all vertices in the associated Brauer graph, result proven by S. Schroll in [4] with a different approach. To prove this result we show that an indecomposable algebra B is a Brauer graph algebra with multiplicity one in all vertices in the associated Brauer graph if and only if its cycles satisfy the properties (T_1), (T_2), (T_3) and (T_4) which characterize trivial extensions of gentle algebras.

2. Preliminaries

Throughout this paper k will denote an algebraically closed field. The algebras considered are finite-dimensional k-algebras which we will also assume to be basic and indecomposable. Thus, for an algebra Λ, we have that $\Lambda \simeq kQ_\Lambda/I_\Lambda$, where Q_Λ is a finite connected quiver and the ideal I_Λ is admissible. Given an element x of kQ_Λ, we will denote by $\overline{x}$ the corresponding element of kQ_Λ/I_Λ.

If Q is a quiver, we will denote by Q_0 the set of vertices, and by Q_1 the set of arrows between vertices. For each arrow α, $s(\alpha)$ and $e(\alpha)$ will denote the start and end vertices of α, respectively. For each $i \in Q_0$, S_i will be the simple Λ-module associated to i, and P_i and I_i will denote the projective cover and injective envelope of S_i, respectively. Thus, if e_i is the trivial path of kQ_Λ corresponding to the vertex i of Q_Λ, then $P_i = \Lambda\overline{e_i}$.

We recall now the description of the ordinary quiver for $T(\Lambda)$ given in [3]. Let Λ be an algebra with ordinary quiver Q_Λ, and let $p_1, \ldots, p_t$ be elements in kQ_Λ such that $\mathcal{M} = \{\overline{p_1}, \ldots, \overline{p_t}\}$ is a k-basis for the socle $\operatorname{soc}_{\Lambda^e}\Lambda$ of Λ considered as a module over the enveloping algebra Λ^e, where each p_i is a linear combination of paths with the same origin $s(p_i)$ and the same endpoint $e(p_i)$. Then the ordinary quiver of $T(\Lambda)$ is given by

(i) $(Q_{T(\Lambda)})_0 = (Q_\Lambda)_0$
(ii) $(Q_{T(\Lambda)})_1 = (Q_\Lambda)_1 \cup \{\beta_{p_1}, \ldots, \beta_{p_t}\}$, where β_{p_i} is an arrow from $e(p_i)$ to $s(p_i)$ for each $i = 1, \ldots, t$.

The notion of elementary cycle, given in [3], is essential in the description of the relations for Λ. We recall the definition now. Let $p_{t+1}, \ldots, p_r$ be paths of $Q_{T(\Lambda)}$ such that $\mathcal{B} = \{\overline{p_1}, \ldots, \overline{p_t}, \overline{p_{t+1}}, \ldots, \overline{p_r}\}$ is a k-basis of Λ, and let $\mathcal{B}^* = \{\overline{p_1}^*, \ldots, \overline{p_r}^*\}$ denote the dual basis. Following [3], we say that an oriented cycle C of $kQ_{T(\Lambda)}$ is *elementary* if it is of the form $C = \alpha_j \cdots \alpha_1 \beta_p \alpha_m \cdots \alpha_{j+1}$, where $\alpha_1, \ldots, \alpha_m \in (Q_\Lambda)_1$, $p \in \mathcal{M}$ and $\overline{p}^*(\overline{\alpha_m \cdots \alpha_1}) \neq 0$.

It follows from the definition that cyclic permutations $\sigma_j \cdots \sigma_1 \sigma_m \cdots \sigma_{j+1}$ of elementary cycles $\sigma_m \cdots \sigma_1$ are elementary cycles. When we refer to a permutation of a cycle we will always mean a cyclic permutation of it.

We will say that a path γ in kQ_Λ is *maximal* in Λ if $0 \neq \overline{\gamma}$ in Λ, and $\overline{\alpha\gamma} = 0$, $\overline{\gamma\alpha} = 0$ for any arrow α of Q_Λ.

We will also need the notion of supplement given in [3]. Let q be a path in an elementary cycle C. If $q = C$ we say that the *supplement* of q in C is the trivial path $e_{s(q)}$. Otherwise the *supplement* of q in C is the path consisting of the remaining arrows of C. More precisely, if $C = \mu_s \cdots \mu_1$ is an elementary cycle, with $\mu_1, \ldots, \mu_s \in (Q_\Lambda)_1$, and $q = \mu_{j+r} \cdots \mu_j$ is a subpath of C, then the supplement of q in C is $\mu_{j-1} \cdots \mu_1 \mu_s \cdots \mu_{j+r+1}$.

We consider, as in [3], the morphism of k-algebras $\Phi : kQ_{T(\Lambda)} \to T(\Lambda)$ defined by

$$\Phi(e_i) = (\overline{e_i}, 0) \quad \text{for } i = 1, \ldots, n,$$
$$\Phi(\alpha) = (\overline{\alpha}, 0),\ \Phi(\beta_p) = (0, \overline{p}^*) \quad \text{for every } \alpha \in (Q_\Lambda)_1,\ p \in \mathcal{M}.$$

Then Φ is surjective and so we can identify $T(\Lambda)$ with $kQ_{T(\Lambda)}/\operatorname{Ker}\Phi$. Thus, the class $\overline{x}$ of an element $x \in kQ_{T(\Lambda)}$ is nonzero in $T(\Lambda)$ if and only if $\Phi(x) \neq 0$. Any path in $kQ_{T(\Lambda)}$ containing at least two arrows β_p is zero in $T(\Lambda)$. Moreover, a path in kQ_Λ which is not zero in Λ is always contained in an elementary cycle (see [3, Remark 3.3]) and is therefore not maximal in $T(\Lambda)$. Thus maximal paths in $T(\Lambda)$ contain exactly one arrow β_p.

Associated with Φ are the compositions $\varphi_1 = \pi_1\Phi : kQ_{T(\Lambda)} \to \Lambda$ and $\varphi_2 = \pi_2\Phi : kQ_{T(\Lambda)} \to D(\Lambda)$, where π_1, π_2 are the projections induced by the decomposition $T(\Lambda) = \Lambda \oplus D(\Lambda)$.

Notice that an elementary cycle $C = \alpha_j \cdots \alpha_1\beta_p\alpha_m \cdots \alpha_{j+1} : e \to e$ is nonzero in $T(\Lambda)$. In fact, $\Phi(C) = (0, \varphi_2(C))$, and using the structure of $D(\Lambda)$ as a Λ-bimodule we obtain

$$\begin{aligned}\varphi_2(C)(e) &= \varphi_2(\alpha_j \cdots \alpha_1\beta_p\alpha_m \cdots \alpha_{j+1})(\overline{e}) \\ &= \overline{p}^*(\overline{\alpha_m \cdots \alpha_{j+1}}\,\overline{e}\,\overline{\alpha_j \cdots \alpha_1}) = \overline{p}^*(\overline{\alpha_m \cdots \alpha_1}) \neq 0.\end{aligned}$$

3. Trivial extensions of monomial algebras

From now on we will assume that Λ is a monomial algebra. That is, $\Lambda \simeq kQ_\Lambda/I_\Lambda$, where I_Λ is an admissible ideal generated by paths. In this case, the set of classes of maximal paths is a basis for $\operatorname{soc}_{\Lambda^e}\Lambda$. We will always assume that $p_1, \dots, p_t$ in the chosen basis $\mathcal{M} = \{\overline{p_1}, \dots, \overline{p_t}\}$ of $\operatorname{soc}_{\Lambda^e}\Lambda$ are maximal paths. The extension $\mathcal{B}$ of $\mathcal{M}$ to a basis of Λ considered above is then the set of classes of paths in kQ_Λ that are nonzero in Λ. Notice that this set is a basis for Λ because Λ is a monomial algebra.

In this section we will find generators for the ideal of relations of the trivial extension of a monomial algebra. We do it by adapting the approach followed in [3] to our case.

In order to describe the relations for $T(\Lambda)$ we have to find generators for $\operatorname{Ker}\Phi$. The next proposition gives a description of the elementary cycles of $T(\Lambda)$.

Proposition 3.1. *Let $\Lambda = kQ_\Lambda/I_\Lambda$ be a monomial algebra and C an oriented cycle of $kQ_{T(\Lambda)}$. Then the following conditions are equivalent:*

(i) *C is an elementary cycle.*
(ii) *C is a cyclic permutation of the cycle $p\beta_p$ for some $p \in \mathcal{M}$.*
(iii) *C is maximal in $T(\Lambda)$.*

Proof. (i) → (ii) Let C be an elementary cycle of $kQ_{T(\Lambda)}$, $C = \alpha_j \cdots \alpha_1\beta_p\alpha_n \cdots \alpha_{j+1}$, with $\alpha_1, \dots, \alpha_n \in (Q_A)_1$, $p \in \mathcal{M}$. Since $\mathcal{B}$ is generated by all the paths that are nonzero in Λ and $\overline{p}^*(\overline{\alpha_n \cdots \alpha_1}) \neq 0$, it follows that $\alpha_n \cdots \alpha_1 = p$. Thus C is a permutation of the cycle $p\beta_p$.

(ii) → (i) Let $p \in \mathcal{M}$. Since $\overline{p}^*(\overline{p}) = 1$ the cycle $p\beta_p$ is elementary by definition, and so is any permutation C of $p\beta_p$.

(i) → (iii) Let $C = \alpha_j \cdots \alpha_1\beta_p\alpha_n \cdots \alpha_{j+1}$ be an elementary cycle, with $\alpha_n \cdots \alpha_1 = p \in \mathcal{M}$. We know that $\overline{C} \neq 0$ in $T(\Lambda)$. Suppose C is not maximal. Then there is an arrow $\alpha \in Q_{T(\Lambda)}$ such that $\overline{\alpha}\overline{C} \neq 0$ or $\overline{C}\overline{\alpha} \neq 0$. We assume that $\overline{\alpha}\overline{C} \neq 0$.

Thus $\Phi(\alpha\alpha_j\cdots\alpha_1\beta_p\alpha_n\cdots\alpha_{j+1})\neq(0,0)$. Therefore,

$$\Phi(\alpha\alpha_j\cdots\alpha_1)\Phi(\beta_p)\Phi(\alpha_n\cdots\alpha_{j+1})=(\overline{\alpha\alpha_j\cdots\alpha_1},0)(0,\overline{p}^*)(\overline{\alpha_n\cdots\alpha_{j+1}},0)\neq(0,0).$$

That is, $(0,\overline{\alpha\alpha_j\cdots\alpha_1}\,\overline{p}^*\,\overline{\alpha_n\cdots\alpha_{j+1}})\neq(0,0)$. Thus in $D(\Lambda)$ we have that $\overline{\alpha\alpha_j\cdots\alpha_1}\,\overline{p}^*\,\overline{\alpha_n\cdots\alpha_{j+1}}\neq 0$. So there is a path q of kQ_Λ such that $\overline{\alpha\alpha_j\cdots\alpha_1}\,\overline{p}^*\,\overline{\alpha_n\cdots\alpha_{j+1}}(\overline{q})\neq 0$. By definition of the structure of $D(\Lambda)$ as a Λ-bimodule we get $\overline{\alpha\alpha_j\cdots\alpha_1}\,\overline{p}^*\,\overline{\alpha_n\cdots\alpha_{j+1}}(\overline{q})=\overline{p}^*(\overline{\alpha_n\cdots\alpha_{j+1}}\,\overline{q}\,\overline{\alpha\alpha_j\cdots\alpha_1})\neq 0$, which contradicts that $\overline{p}^*(\overline{p'})=0$ for any path $p'\neq p$ of kQ_Λ. In a similar way we prove that $\overline{C}\overline{\alpha}\neq 0$ in $T(\Lambda)$ leads to a contradiction. Therefore C is maximal.

(iii) $\to$ (i) Suppose that C is a cycle of $Q_{T(\Lambda)}$ maximal in $T(\Lambda)$, $C: e\to e$. As observed above, C contains exactly one arrow β_p. Then $C=\alpha_j\cdots\alpha_1\beta_p\alpha_n\cdots\alpha_{j+1}$ and $\Phi(C)=(0,\varphi_2(C))\neq 0$. So, there is a path γ in Q_Λ such that $\varphi_2(C)(\overline{\gamma})\neq 0$. Then $\varphi_2(C\gamma)(\overline{e})=\varphi_2(C)(\overline{\gamma})\neq 0$, so $\varphi_2(C\gamma)\neq 0$. That is, $0\neq\varphi_2(C\,\gamma)=\overline{C}\overline{\gamma}$ in $T(\Lambda)$. Since C is maximal in $T(\Lambda)$ it follows that γ is a trivial path, that is, $\gamma=e$. Thus $\varphi_2(C)(\overline{e})\neq 0$. Since $\varphi_2(C)(\overline{e})=\overline{p^*}(\overline{\alpha_n\cdots\alpha_{j+1}}\,\overline{e}\,\overline{\alpha_j\cdots\alpha_1})$ it follows that $\overline{p}^*(\overline{\alpha_n\cdots\alpha_1})\neq 0$. The paths $p,\alpha_n\cdots\alpha_1$ belong to the chosen basis $\mathcal{B}$ of Λ, thus $\alpha_n\cdots\alpha_1=p$, and this proves that the cycle C is elementary. □

We will briefly say that "a path has a supplement" to mean that it has a supplement in some elementary cycle. If $\{x_i\}_{i\in I}$ is a family of elements in an algebra, we will denote by $(x_i)_{i\in I}$ the two-sided ideal generated by them.

As a consequence of the preceding proposition we obtain the following result.

Corollary 3.2. *Let $\Lambda=kQ_\Lambda/I_\Lambda$ be a monomial algebra. Then:*

(i) *Every arrow β_p of $Q_{T(\Lambda)}$ is contained in a single elementary cycle, up to permutations for any $p\in\mathcal{M}$.*

(ii) *If a path $\gamma\in(\beta_p)_{p\in\mathcal{M}}$ has supplements μ, μ' then $\mu=\mu'$.*

Proof. (i) The only elementary cycles containing β_p are the permutations of $C=p\beta_p$.

(ii) Let $\gamma\in(\beta_p)_{p\in\mathbb{M}}$ be a path of $kQ_{T(\Lambda)}$ with supplements μ, μ'. Then $\gamma=\delta\beta_p\rho$, with δ,ρ paths of kQ_Λ, and let C be the elementary cycle containing β_p, which is unique up to permutations. Then any supplement of γ is a supplement in C, and is the path consisting of the remaining arrows in C. So $\mu=\mu'$. □

The next technical lemma will be used in what follows.

Lemma 3.3. *Let $\Lambda=kQ_\Lambda/I_\Lambda$ be a monomial algebra.*

(i) *Let q and u be paths in $kQ_{T(\Lambda)}$. If $\varphi_2(q)(\overline{u})\neq 0$, then $\varphi_2(q)(\overline{u})=1$, and u is a supplement of q.*

(ii) *If $C: e\to e$ is an elementary cycle, then $\varphi_2(C)(e)\neq 0$.*

Proof. (i) Suppose that $\varphi_2(q)(\overline{u})\neq 0$. Then $q=\gamma\beta_p\delta$, with γ and δ paths of kQ_Λ. So $0\neq\varphi_2(q)(\overline{u})=\varphi_2(\gamma\beta_p\delta)(\overline{u})=\big((\overline{\gamma},0)(0,\overline{p}^*)(\overline{\delta},0)\big)(\overline{u})=\big((0,\overline{\gamma}\,\overline{p}^*)(\overline{\delta},0)\big)(\overline{u})=(0,\overline{\gamma}\,\overline{p}^*\overline{\delta})(\overline{u})=\overline{p}^*(\overline{\delta u\gamma})$. Thus $\overline{p}^*(\overline{\delta u\gamma})\neq 0$. Then u is a supplement of q in the elementary cycle $\gamma\beta_p\delta u$.

This proves (i), and (ii) follows directly from the definition of φ_2. □

The next proposition is a first step to describe the ideal of relations of the trivial extension of a monomial algebra.

Proposition 3.4. *Let $\Lambda = kQ_\Lambda/I_\Lambda$ be a monomial algebra. Let Φ be the morphism defined above. For each $j \in (Q_{T(\Lambda)})_0$, let Y_j be the ideal of $kQ_{T(\Lambda)}$ generated by*

(i) *oriented cycles from j to j which are not contained in an elementary cycle,*
(ii) *all the elements $C - C'$, where C and C' are elementary cycles with origin j.*

Then $Y_j \subseteq \operatorname{Ker}\Phi \cap e_j kQ_{T(\Lambda)} e_j$.

Proof. It suffices to prove that Φ vanishes in the generators of Yj.

Suppose v is a generator of Y_j as considered in (i), that is, a path from j to j not contained in an elementary cycle. As we observed after the definition of Φ, paths of kQ_Λ are contained in elementary cycles, so v contains one arrow β_p, with $p \in \mathcal{M}$. Thus $\Phi(v) = (0, \varphi_2(v))$.

If v has two or more arrows β_p then $\varphi_2(v) = 0$, so $v \in \operatorname{Ker}\Phi \cap e_j kQ_{T(\Lambda)} e_j$. Otherwise, $v = \gamma\beta_p\delta$, with γ and δ paths of kQ_Λ, and $\varphi_2(v) = \overline{\gamma}\,\overline{p}^*\overline{\delta}$. Suppose $\varphi_2(v) \neq 0$. Then there is a path u of kQ_Λ such that $\varphi_2(v)(\overline{u}) \neq 0$. That is, $\overline{p}^*(\overline{\delta u \gamma}) \neq 0$ and therefore $\gamma\beta_p\delta u$ is an elementary cycle containing $v = \gamma\beta_p\delta$. This contradicts the hypothesis that v is not contained in an elementary cycle. Therefore $\varphi_2(v) = 0$. So $v \in \operatorname{Ker}\Phi \cap e_j kQ_{T(\Lambda)} e_j$.

Suppose now that v is a generator of Y_j of the type (ii), that is $v = C - C'$, where C and C' are elementary cycles with origin j. Then $\Phi(v) = (0, \varphi_2(v))$ and $\varphi_2(v) = \varphi_2(C) - \varphi_2(C')$.

Let $u \in kQ_\Lambda$. We will prove that $\varphi_2(v)(\overline{u}) = 0$.

If $u \neq e_j$, then u is not a supplement for C, and thus $\varphi_2(C)(\overline{u}) = \varphi_2(C')(\overline{u}) = 0$, by Lemma 3.3 (i), so $\varphi_2(v)(\overline{u}) = 0$. On the other hand, $\varphi_2(C)(\overline{e_j}) = 1$ and $\varphi_2(C')(\overline{e_j}) = 1$, by Lemma 3.3 (ii). Thus $\varphi_2(v)(\overline{e_j}) = 1 - 1 = 0$.

Thus, $\varphi_2(v) = 0$, that is, $v \in \operatorname{Ker}\Phi \cap e_j kQ_{T(\Lambda)} e_j$. □

Now we state the main result of this section.

Theorem 3.5. *Let $\Lambda = kQ_\Lambda/I_\Lambda$ be a monomial algebra. Let I' be the ideal in $kQ_{T(\Lambda)}$ generated by*

(i) *the paths not contained in an elementary cycle, and*
(ii) *the elements $\mu - \mu'$, where μ, μ' are different paths from $kQ_{T(A)}$ whith a common supplement γ in elementary cycles C and C', respectively.*

Then I' is admissible and $I' = \operatorname{Ker}\Phi$. That is, $T(\Lambda) \simeq kQ_{T(\Lambda)}/I'$.

Before proving this theorem, we will make some useful observations about the ideal I' defined in its statement.

Remark 3.6.

(a) $I_\Lambda \subseteq I'$, because if q is a path of I_Λ, then $\overline{q} = 0$ in Λ and so q is not contained in an elementary cycle, because elementary cycles are nonzero in $T(\Lambda)$. Thus $I_\Lambda \subseteq I' \cap kQ_\Lambda$. Conversely, a path in kQ_Λ which is not zero in Λ is contained in an elementary cycle, as observed above. Thus $I_\Lambda = I' \cap kQ_\Lambda$.

(b) If $\mu - \mu'$ is a generator of I' of type (ii) in Theorem 3.5, then $\mu, \mu' \in (\beta_p)_{p\in\mathcal{M}}$. In fact, since they are different paths with a common supplement γ it follows from Corollary 3.2 (ii) that $\gamma \in kQ_\Lambda$. Thus any supplement of γ has an arrow β_p.

(c) Suppose q in $kQ_{T(\Lambda)}$ is not in I'. Then q has a supplement in some elementary cycle. In fact, q is contained in an elementary cycle C, by the definition of I'. Thus q has a supplement in C.

Now we will prove Theorem 3.5

Proof. We must prove that $I' = \operatorname{Ker}\Phi$, where Φ is the morphism defined above.

We will prove first that $I' \subseteq \operatorname{Ker}\Phi$. For this, we prove that the generators given in I' belong to $\operatorname{Ker}\Phi$.

Case 1. The generator is a path q in I' not contained in an elementary cycle. Then:

- If $q \in kQ_\Lambda$, then $q \in kQ_\Lambda \cap I' = I_\Lambda$, by Remark 3.6 (a). Then $\overline{q} = 0$ and thus $\Phi(q) = (0, 0)$.
- If $q \in (\beta_p)_{p\in\mathcal{M}}$, then $q = \gamma\beta_p\delta$, with $p \in \mathcal{M}$, where γ and δ are paths of kQ_Λ. Thus $\Phi(q) = (0, \varphi_2(q))$.

Suppose that $\varphi_2(q) \neq 0$. Then there is a path u of kQ_Λ such that $\varphi_2(q)(\overline{u}) \neq 0$. Thus $\overline{p}^*(\overline{\delta u\gamma}) \neq 0$. So u is a supplement of q in the elementary cycle $\gamma\beta_p\delta u$. This is a contradiction because we are assuming that q is not contained in an elementary cycle. Thus $\varphi_2(q) = 0$ and then $q \in \operatorname{Ker}\Phi$.

Case 2. Suppose v is a generator of I' of the form $v = \mu - \mu'$, where μ, μ' are different paths of $kQ_{T(\Lambda)}$ from i to j with a common supplement γ in elementary cycles C and C', respectively. Then $C = \gamma\,\mu$ and $C' = \gamma\,\mu'$, and we have that $\gamma \in kQ_\Lambda$, because $\mu, \mu' \in (\beta_p)_{p\in\mathcal{M}}$ by Remark 3.6 (b). Since $v \in (\beta_p)_{p\in\mathcal{M}}$ we know that $\Phi(v) = (0, \varphi_2(v))$. We will prove that $\varphi_2(v) = 0$.

The product $\gamma\, v = \gamma\mu - \gamma\mu' = C - C'$ is in the ideal Y_j defined in Proposition 3.4. We proved in the same proposition that $Y_j \subseteq \operatorname{Ker}\Phi$, so $\gamma v \in \operatorname{Ker}\Phi$ and therefore $\varphi_2(\gamma\, v) = 0$. Then $0 = \varphi_2(\gamma\, v)(\overline{e_i}) = \varphi_2(v)(\overline{e_i\gamma}) = \varphi_2(v)(\overline{\gamma})$. Thus $\varphi_2(v)(\overline{\gamma}) = 0$.

Suppose that there is a path $q \in kQ_\Lambda$ such that $\varphi_2(v)(\overline{q}) \neq 0$. Then $\varphi_2(v)(\overline{q}) = \varphi_2(\mu - \mu')(\overline{q}) = \varphi_2(\mu)(\overline{q}) - \varphi_2(\mu')(\overline{q}) \neq 0$. So either $\varphi_2(\mu)(\overline{q})$ or $\varphi_2(\mu')(\overline{q})$ is not zero.

Without loss of generality we may assume that $\varphi_2(\mu)(\overline{q}) \neq 0$. Then q is a supplement of μ by Lemma 3.3, and therefore $q = \gamma$, because paths in $(\beta_p)_{p\in\mathcal{M}}$ have a unique supplement, by Corollary 3.2 (ii). So $\varphi_2(v)(\overline{\gamma}) = \varphi_2(v)(\overline{q}) \neq 0$, which contradicts that $\varphi_2(v)(\overline{\gamma}) = 0$. This proves that $\varphi_2(v) = 0$, as desired.

The preceding case-by-case analysis proves that $I' \subseteq \operatorname{Ker}\Phi$.

Let $\pi : kQ_{T(\Lambda)} \to kQ_{T(\Lambda)}/I'$ be the canonical epimorphism and denote $\pi(y) = \widetilde{y}$.

Since $I' \subseteq \operatorname{Ker}\Phi$, the epimorphism $\Phi : kQ_{T(\Lambda)} \twoheadrightarrow kQ_{T(\Lambda)}/\operatorname{Ker}\Phi = T(\Lambda)$ induces an epimorphism $\overline{\Phi} : kQ_{T(\Lambda)}/I' \twoheadrightarrow kQ_{T(\Lambda)}/\operatorname{Ker}\Phi = T(\Lambda)$ such that $\overline{\Phi} \circ \pi = \Phi$. To prove the equality $I' = \operatorname{Ker}\Phi$ it is enough to prove that $\dim_k kQ_{T(\Lambda)}/I' = \dim_k T(\Lambda) = 2\dim_k \Lambda$.

The inclusion of Λ in $T(\Lambda)$ factors through $kQ_{T(\Lambda)}/I'$ because $I_\Lambda \subseteq I'$. So, the morphism $\iota : \Lambda \to kQ_{T(\Lambda)}/I'$ induced by the inclusion of kQ_Λ en $kQ_{T(\Lambda)}$ is a monomorphism.

Thus we have the following commutative diagram:

$$
\begin{array}{ccc}
kQ_\Lambda & \hookrightarrow & kQ_{T(\Lambda)} \\
\downarrow & & \downarrow \pi \\
\Lambda = kQ_\Lambda/I_\Lambda & \xrightarrow{\ \iota\ } & kQ_{T(\Lambda)}/I' \\
 & \searrow & \downarrow \overline{\Phi} \\
 & & T(\Lambda) = kQ_{T(\Lambda)}/\operatorname{Ker}\Phi
\end{array}
$$

with $\overline{\Phi} \circ \pi = \Phi$.

We know that $kQ_{T(\Lambda)} = kQ_\Lambda + (\beta_p)_{p\in\mathcal{M}}$. Therefore $e_j kQ_{T(\Lambda)} e_i = e_j kQ_\Lambda e_i + e_j(\beta_p)_{p\in\mathcal{M}} e_i$ for each $i, j \in (Q_{T(\Lambda)})_0$. We consider in $kQ_{T(\Lambda)}/I'$ the subspaces $\mathbb{P}_{ij} = \pi(e_j kQ_\Lambda e_i)$ and $\mathbb{F}_{ij} = \pi(e_j(\beta_p)_{p\in\mathcal{M}} e_i)$. Then $\mathbb{P}_{ij} = \iota(e_j \Lambda e_i) \simeq e_j \Lambda e_i$. So $\sum_{i,j} \dim_k \mathbb{P}_{ij} = \dim_k \Lambda$. We will show that $\dim_k(\mathbb{P}_{ij}) \geq \dim_k(\mathbb{F}_{ji})$.

We start by proving that $\mathbb{F}_{ji} \neq 0$ if and only if $\mathbb{P}_{ij} \neq 0$. In fact, if $\mathbb{F}_{ji} \neq 0$, there is a path q of $e_i(\beta_p)_{p\in\mathcal{M}} e_j$ which is not in I'. So q has a supplement γ by Remark 3.6 (c). Then γ is a path of kQ_Λ, because q is in $(\beta_p)_{p\in\mathcal{M}}$, and γ does not belong to I'. Then $0 \neq \widetilde{\gamma} \in \mathbb{P}_{ij}$ and so $\mathbb{P}_{ij} \neq 0$. Conversely, if $\mathbb{P}_{ij} \neq 0$ there is a path q in kQ_Λ such that q does not belong to I'. Then q has a supplement γ, again by Remark 3.6 (c), and γ is a path of $e_i(\beta_p)_{p\in\mathcal{M}} e_j$ which does not belong to I'. Then $0 \neq \widetilde{\gamma} \in \mathbb{F}_{ji}$ and so $\mathbb{F}_{ji} \neq 0$.

Therefore we assume that both, $\mathbb{F}_{ji}$ and $\mathbb{P}_{ij}$ are nonzero and we choose paths $\mu_1, \dots, \mu_f \in (\beta_p)_{p\in\mathcal{M}}$ such that $\{\widetilde{\mu_1}, \dots, \widetilde{\mu_f}\}$ is a basis of $\mathbb{F}_{ji}$. Then $\mu_t \notin I'$ if $t \in \{1, \dots, f\}$. So, μ_t has a supplement in an elementary cycle $C_t = \mu_t\, \gamma_t$ for all $t \in \{1, \dots, f\}$ by Remark 3.6 (c). The paths $\gamma_1, \dots, \gamma_f$ belong to kQ_Λ. We will prove that that $\widetilde{\gamma_1}, \dots, \widetilde{\gamma_f}$ are linearly independent in $kQ_{T(\Lambda)}/I'$. Assume, on the contrary, that $\widetilde{\gamma_1}, \dots, \widetilde{\gamma_f}$ are linearly dependent in $kQ_{T(\Lambda)}/I'$. Then $\overline{\gamma_1}, \dots, \overline{\gamma_f}$ are linearly dependent in Λ, because ι is a monomorphism. Since $\overline{\gamma_t}$ is not zero in Λ for $t \in \{1, \dots, f\}$, we have that $\gamma_1, \dots, \gamma_f$ are not pairwise different, because Λ is monomial, and therefore classes in Λ of pairwise different nonzero paths are linearly independent. Therefore two of the paths $\gamma_1, \dots, \gamma_f$ are equal, let's say $\gamma_1 = \gamma_2$. Then $C_1 = \mu_1\gamma_1$ and $C_2 = \mu_2\gamma_1$, so the elementary cycles C_1 and C_2 contain the common path γ_1, and this proves that $\mu_2 - \mu_1$ is an element of I'. Then $\widetilde{\mu_2} = \widetilde{\mu_1}$, which contradicts the fact that $\widetilde{\mu_1}, \widetilde{\mu_2}$ are elements of a basis of $\mathbb{F}_{ji}$. This contradiction shows that $\widetilde{\gamma_1}, \dots, \widetilde{\gamma_f}$ are linearly independent in $kQ_{T(\Lambda)}/I'$. So $\dim_k(\mathbb{P}_{ij}) \geq \dim_k(\mathbb{F}_{ji})$.

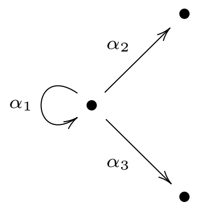
Therefore

$$
\begin{aligned}
\dim_k kQ_{T(\Lambda)}/I' &\leq \sum_{i,j} (\dim_k \mathbb{P}_{ij} + \dim_k \mathbb{F}_{ij}) \leq \sum_{i,j} (\dim_k \mathbb{P}_{ij} + \dim_k \mathbb{P}_{ji}) \\
&= 2 \dim_k \Lambda = \dim_k T(\Lambda).
\end{aligned}
$$

Thus $\dim_k kQ_{T(\Lambda)}/I' \leq \dim_k T(\Lambda)$, and this proves that the surjective morphism $\overline{\Phi} : kQ_{T(\Lambda)}/I' \twoheadrightarrow T(\Lambda)$ is an isomorphism, which ends the proof of the theorem. □

Example 3.7. Let $\Lambda = kQ_\Lambda/I_\Lambda$, where

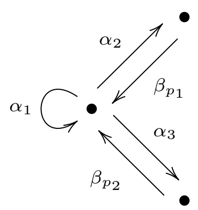
Q_Λ:



with relation $(\alpha_1)^2 = 0$.

The maximal paths are $p_1 = \alpha_2\alpha_1$ and $p_2 = \alpha_3\alpha_1$, which induce in $Q_{T(\Lambda)}$ elementary cycles $C_1 = \beta_{p_1}\alpha_2\alpha_1$ and its permutations $C_2 = \alpha_1\beta_{p_1}\alpha_2$, $C_3 = \alpha_2\alpha_1\beta_{p_1}$, and $C_4 = \beta_{p_2}\alpha_3\alpha_1$ and its permutations $C_5 = \alpha_1\beta_{p_2}\alpha_3$, $C_6 = \alpha_3\alpha_1\beta_{p_2}$. Then $T(\Lambda)$ is given by

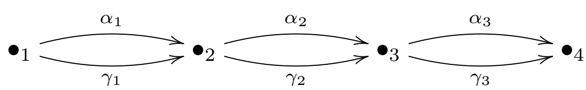
$Q_{T(\Lambda)}$:



with relations
$(\alpha_1)^2$, $\alpha_3\beta_{p_1}$, $\alpha_2\beta_{p_2}$,
$\alpha_3\alpha_1\beta_{p_1}$, $\alpha_2\alpha_1\beta_{p_2}$,
$\alpha_1 C_1$, $\alpha_2 C_2$, $\beta_{p_1} C_3$,
$\alpha_1 C_4$, $\alpha_3 C_5$, $\beta_{p_2} C_6$,
$\beta_{p_1}\alpha_2 - \beta_{p_2}\alpha_3$.

Example 3.8. Let $\Lambda = kQ_\Lambda/I_\Lambda$, where

Q_Λ:

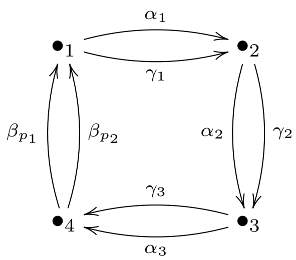


with relations
$\alpha_{i+1}\,\alpha_i$, $\gamma_{i+1}\,\gamma_i$ for $i = 1, 2$.

The maximal paths are $p_1 = \alpha_3\gamma_2\alpha_1$ and $p_2 = \gamma_3\alpha_2\gamma_1$, which induce in $Q_{T(\Lambda)}$ elementary cycles $C_1 = \beta_{p_1}\alpha_3\gamma_2\alpha_1$ and its permutations, $C_2 = \alpha_1\beta_{p_1}\alpha_3\gamma_2$, $C_3 = \gamma_2\alpha_1\beta_{p_1}\alpha_3$, $C_4 = \alpha_3\gamma_2\alpha_1\beta_{p_1}$ and $C_5 = \beta_{p_2}\gamma_3\alpha_2\gamma_1$ and its permutations, $C_6 = \gamma_1\beta_{p_2}\gamma_3\alpha_2$, $C_7 = \alpha_2\gamma_1\beta_{p_2}\gamma_3$, and $C_8 = \gamma_3\alpha_2\gamma_1\beta_{p_2}$.

Then $T(\Lambda)$ is given by

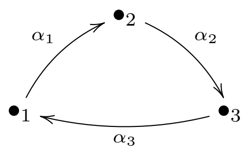
$Q_{T(\Lambda)}$:



with relations
$\alpha_{i+1}\,\alpha_i,\ \gamma_{i+1}\,\gamma_i$ for $i=1,2$,
$\alpha_1\,C_1,\ \gamma_2\,C_2,\ \alpha_3\,C_3,\ \beta_{p_1}\,C_4$,
$\gamma_1\,C_5,\ \alpha_2\,C_6,\ \gamma_3\,C_7,\ \beta_{p_2}\,C_8$,
$\alpha_1\beta_{p_2},\ \beta_{p_2}\alpha_3,\ \beta_{p_1}\,\gamma_3,\ \gamma_1\,\beta_{p_1}$,
$C_1-C_5,\ C_3-C_7,\ C_2-C_6,\ C_4-C_8$.

Example 3.9. Let $\Lambda = kQ_\Lambda/I_\Lambda$, where

Q_Λ:

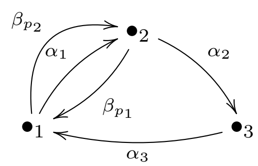


with relations $\alpha_2\,\alpha_1,\ \alpha_1\,\alpha_3$.

The maximal paths are $p_1=\alpha_1$ and $p_2=\alpha_3\alpha_2$, which induce in $Q_{T(A)}$ the elementary cycles $C_1=\beta_{p_1}\alpha_1$ and its permutation, $C_2=\alpha_1\beta_{p_1}$, $C_3=\beta_{p_2}\alpha_3\alpha_2$ and its permutations, $C_4=\alpha_2\beta_{p_2}\alpha_3$ and $C_5=\alpha_3\alpha_2\beta_{p_2}$.

Then $T(\Lambda)$ is given by

$Q_{T(A)}$:



with relations
$\alpha_2\,\alpha_1,\ \alpha_1\,\alpha_3,\ \beta_{p_2}\beta_{p_1},\ \beta_{p_1}\beta_{p_2}$
$\alpha_1C_1,\ \beta_{p_1}C_2,\ \alpha_2C_3,\ \beta_{p_2}C_5,\ \alpha_3C_4$,
$C_1-C_5,\ C_2-C_3$.

4. The gentle case

In this section we study the trivial extension $T(\Lambda)$ in the particular case when Λ is a gentle algebra. We will prove that the given description of the relations of the trivial extension of a monomial algebra can be formulated in a simple way when the monomial algebra is gentle. Also, we will see that it is possible to determine when an algebra $B=kQ_B/I_B$, given by its quiver and relations, is the trivial extension of a gentle algebra. Moreover, we characterize such algebras B using properties of their cycles and show how to find all the gentle algebras Λ such that $T(\Lambda) \cong B$.

We recall from [1] that an algebra Λ is called *gentle* if it is Morita equivalent to kQ/I, where:

(G_1) I is generated by paths of length two.
(G_2) Each vertex of Q is the beginning and the target of at most two arrows.
(G_3) For each arrow α of Q there exists at most one arrow β such that $\alpha\beta \in I$, and there exists at most one arrow γ such that $\gamma\alpha \in I$.
(G_4) For each arrow α of Q there exists at most one arrow δ such that $\alpha\delta \notin I$ and there exists at most one arrow ϵ such that $\epsilon\alpha \notin I$.

It follows from the definition of a gentle algebra that every arrow is contained in a unique maximal path, at most two maximal paths begin at the same given vertex, and at most two maximal paths end at a given vertex. Thus, different maximal paths can not have common arrows.

In the next proposition we prove properties of the maximal cycles of the trivial extension of a gentle algebra, which will be very useful in what follows.

Proposition 4.1. *Let $\Lambda = kQ_\Lambda/I_\Lambda$ be a gentle algebra. Then the trivial extension $T(\Lambda)$ satisfies the following properties:*

(i) *Any nontrivial path of $kQ_{T(\Lambda)}$ which is nonzero in $T(\Lambda)$ is contained in a maximal cycle, unique up to permutations.*
(ii) *If C_1 and C_2 are two maximal cycles from j to j, then either C_1 is a permutation of C_2, or C_1 and C_2 do not overlap (i.e., they do not have common arrows) and $\widetilde{C_1} = \widetilde{C_2}$ in $T(\Lambda)$.*
(iii) *In $kQ_{T(\Lambda)}$ there are at most two different cycles from j to j which are maximal in $T(\Lambda)$ for all j in $\{1, \ldots, n\}$.*

Proof. (i) Gentle algebras are monomial, so we know from Theorem 3.5 that every nontrivial path γ of $kQ_{T(\Lambda)}$ which is nonzero in $T(\Lambda)$ is contained in an elementary cycle C. On the other hand, we know from Proposition 3.1 that elementary cycles coincide with maximal cycles in $T(\Lambda)$, and are permutations of cycles of the form $p_h\beta_{p_h}$, with $p_h \in \mathcal{M}$. Then the uniqueness up to permutations of the maximal cycle C containing γ is clear when γ contains an arrow β_p. Otherwise such uniqueness follows from the fact that every nontrivial path of kQ_Λ is contained in a unique maximal path, because Λ is a gentle algebra.

(ii) If C_1 and C_2 are two maximal cycles from j to j, then they have the common supplement e_j. So the path $C_1 - C_2 \in I'$ because it satisfies condition (ii) in Proposition 3.5. Thus $\widetilde{C_1} = \widetilde{C_2}$ in $T(\Lambda)$. The fact that C_1 and C_2 do not overlap is a direct consequence of (i).

(iii) Suppose that there are three cycles C_1, C_2, C_3 from j to j in $kQ_{T(\Lambda)}$ maximal in $T(\Lambda)$. Then each C_i has exactly one arrow β_{p_i}, with $p_i \in \mathcal{M}$, and the length of C_i is at least 2. So we write $C_1 = \delta_1\, q_1\, \gamma_1$, $C_2 = \delta_2\, q_2\, \gamma_2$ and $C_3 = \delta_3\, q_3\, \gamma_3$, where γ_i, δ_i are arrows of $Q_{T(\Lambda)}$ and q_i is a path of $kQ_{T(\Lambda)}$ for $i \in \{1, 2, 3\}$.

By (ii) we have that the cycles C_1, C_2, C_3 have no common arrow. Moreover, γ_1, γ_2, γ_3 are not all arrows of Q_Λ because Λ is gentle. If none of them is an arrow of Q_Λ then δ_1, δ_2, δ_3 lie in Q_Λ, and this can not happen because Λ is gentle. Assume two of the arrows γ_i, say γ_1 and γ_2, are arrows of Q_Λ. Then $\gamma_3 = \beta_{p_3}$, and therefore δ_3 is an arrow of Q_Λ. Thus one of the paths $\gamma_1\,\delta_3$, $\gamma_2\,\delta_3$ determines a nonzero class in Λ, because Λ is a gentle algebra. Then such a path is contained in a maximal cycle C'. The maximal cycles C' and C_3 have the common arrow δ_3, so $C' = C_3$, because maximal cycles do not overlap. This is a contradiction because γ_1, γ_2 are not arrows of C_3. This proves that only one arrow γ_i is an arrow of Q_Λ. In the same way one proves that only one arrow δ_i is an arrow of Q_Λ. We may assume that the arrow γ_i in Q_Λ is γ_1. Then $\gamma_2 = \beta_{p_2}$ and $\gamma_3 = \beta_{p_3}$. So δ_2, δ_3 are arrows of Q_Λ, and this contradiction shows that there are at most two maximal cycles from j to j. □

The above result shows that the description of the ideal of relations for $T(\Lambda)$ given in Theorem 3.5 can be simplified in the gentle case, as we state in the following theorem.

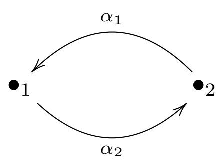
Theorem 4.2. *Let $\Lambda = kQ_\Lambda/I_\Lambda$ be a gentle algebra. Let I' be the ideal in $kQ_{T(\Lambda)}$ generated by*

(i) *the paths not contained in elementary cycles, and*
(ii) *the elements $C - C'$, where C and C' are elementary cycles starting at the same vertex of $Q_{T(\Lambda)}$.*

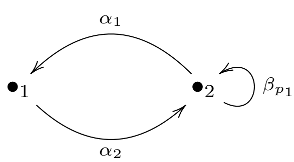
Then I' is admissible and $I' = \operatorname{Ker}\Phi$. That is, $T(\Lambda) \simeq kQ_{T(\Lambda)}/I'$.

We illustrate the previous theorem with an example.

Example 4.3. Let Λ be given the quiver



with the relation $\alpha_1\,\alpha_2 = 0$. Then the unique maximal path is $p_1 = \alpha_2\alpha_1$. So $Q_{T(\Lambda)}$ is the quiver



The elementary cycles are $C_1 = \alpha_2\alpha_1\beta_{p_1}$ and its permutations, $C_2 = \beta_{p_1}\alpha_2\alpha_1$ and $C_3 = \alpha_1\beta_{p_1}\alpha_2$. The relations are

$$\alpha_1\,\alpha_2 = 0, \quad \beta_{p_1}^2 = 0, \quad \beta_{p_1}\alpha_2\,\alpha_1\beta_{p_1} = 0,$$
$$\alpha_1\beta_{p_1}\alpha_2\,\alpha_1 = 0, \quad \alpha_2\,\alpha_1\beta_{p_1}\alpha_2 = 0, \quad \alpha_2\,\alpha_1\beta_{p_1} = \beta_{p_1}\alpha_2\alpha_1.$$

Our next goal is to give a characterization of the trivial extension of a gentle algebra through the description of its cycles. We will see that maximal cycles play a fundamental role. This will require some preliminary lemmas.

Lemma 4.4. *If $\Lambda = kQ_\Lambda/I_\Lambda$ is an algebra such that I_Λ is generated by paths of length two and $C = \alpha_s \cdots \alpha_1$ is a cycle with origin j and nonzero in Λ, then $\overline{\alpha_1\,\alpha_s} = 0$.*

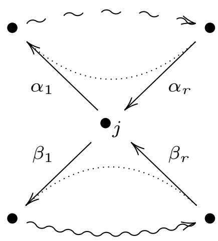
Proof. We have that $\overline{\alpha_s \cdots \alpha_1} \neq 0$ because C is not zero. Suppose $\overline{\alpha_1\,\alpha_s} \neq 0$. Then $\overline{\alpha_1\,C} = \overline{\alpha_1\alpha_s \cdots \alpha_1} \neq 0$ since I_Λ is generated by paths of length two. For the same reason $\overline{\alpha_1\,C^k} \neq 0$ for all k greater than one. This contradicts that A is of finite dimension. Thus $\overline{\alpha_1\,\alpha_s} = 0$. □

Observe that the previous lemma holds for gentle algebras.

Lemma 4.5. *Let $\Lambda = kQ_\Lambda/I_\Lambda$ be a gentle algebra. Then*

(i) *There is at most one cycle of kQ_Λ with origin j and nonzero in Λ for any vertex j of Q_Λ.*
(ii) *There are at most two elementary cycles of $kQ_{T(\Lambda)}$ with origin j in the same permutation class.*

Proof. (i) Assume that there are two nonzero cycles C_1, C_2 of kQ_Λ with origin j and nonzero in Λ. Let $C_1 = \alpha_r \cdots \alpha_1$, $C_2 = \beta_s \cdots \beta_1$, with $\alpha_1, \ldots, \alpha_r$, $\beta_1, \ldots, \beta_s$ arrows of Q_Λ. Since Λ is gentle and C_1, C_2 are nonzero in Λ we have that $\alpha_1 \neq \beta_1$ and $\alpha_r \neq \beta_s$. By Lemma 4.4 we know that $\overline{\alpha_1\alpha_r} = 0$ and $\overline{\beta_1\beta_s} = 0$. Then, $\overline{\beta_1\alpha_r} \neq 0$ and $\overline{\alpha_1\beta_s} \neq 0$ because Λ is gentle. Therefore $(C_1C_2)^t \neq 0$ for all t greater than zero, because Λ is generated by paths of length two. This contradicts that Λ is a finite-dimensional algebra.



Thus (i) holds, and (ii) is a direct consequence of Proposition 4.1 □

Lemma 4.6. *Let $\Lambda = kQ_\Lambda/I_\Lambda$ be a gentle algebra. If $\dim_k \operatorname{End}_{T(\Lambda)}(P_j) > 2$ for some indecomposable projective $T(\Lambda)$-module P_j, then there is a cycle of kQ_Λ with origin j and nonzero in Λ.*

Proof. We know from Proposition 4.1 (ii) that elementary cycles with the same origin are equal in $T(\Lambda)$. Thus the hypothesis $\dim_k \operatorname{End}_{T(\Lambda)}(P_j) > 2$ implies that there is a cycle γ from j to j which is not elementary and is nonzero in $T(\Lambda)$. Then there is an elementary cycle C with origin j containing γ, so $C = \delta\gamma$, with δ not trivial. Then δ is a cycle with origin j and is nonzero in $T(\Lambda)$ because elementary cycles do not vanish in $T(\Lambda)$. On the other hand, a permutation of C has the form

$p\,\beta_p$ with $p \in \mathcal{M}$, by Proposition 3.1. The arrow β_p is either in δ or in γ. In the first case the cycle γ is in kQ_Λ, and otherwise the cycle δ is in kQ_Λ. This ends the proof of the lemma because both cycles are nonzero in $T(\Lambda)$. □

The previous lemmas are helpful to determine $\dim_k \operatorname{End}_{T(\Lambda)}(P_j)$ when Λ is a gentle algebra with a cycle in j nonzero in Λ, as we show in the next proposition.

Proposition 4.7. *Let $\Lambda = kQ_\Lambda/I_\Lambda$ be a gentle algebra with a cycle γ with origin j nonzero in Λ. Then there is a cycle δ with origin j in $kQ_{T(\Lambda)}$ nonzero in $T(\Lambda)$, $\delta \neq \gamma$, such that $\widetilde{e_j}$, $\widetilde{\gamma}$, $\widetilde{\delta}$, $\widetilde{\gamma\,\delta} = \widetilde{\delta\,\gamma}$ form a basis of $\operatorname{End}_{T(\Lambda)} P_j$.*

Proof. Suppose γ is a cycle with origin j of kQ_Λ nonzero in Λ, $\gamma = \alpha_s \cdots \alpha_1$, where α_i are arrows of Q_Λ for all $i \in \{1, \dots, s\}$. We know that there is an elementary cycle C of $kQ_{T(\Lambda)}$ with origin j containing γ which is unique up to permutations, by Proposition 4.1 (i). Let $\delta = \epsilon_r \cdots \epsilon_1$ be the supplement of γ in C, where $\epsilon_1, \dots, \epsilon_r$ are arrows of $Q_{T(\Lambda)}$. Then δ is a cycle with origin j and is nonzero in $T(\Lambda)$, $\delta \neq \gamma$, and C is a permutation of $\delta\gamma$. From the fact, proven in Lemma 4.5 that there is at most one cycle of Q_Λ starting at j, we can conclude that either $C = \delta\gamma$ or $C = \gamma\delta$. We may assume $C = \gamma\,\delta$.

We have that $\widetilde{\alpha_1\,\alpha_s} = 0$ from Lemma 4.4. Moreover, $\widetilde{\epsilon_1\,\epsilon_r} = 0$ because the path $\epsilon_1\,\epsilon_r$ is not contained in any elementary cycle. Moreover, the paths $\widetilde{\epsilon_1\,\alpha_s}$ and $\widetilde{\alpha_1\,\epsilon_r}$ are nonzero in $kQ_{T(\Lambda)}$ because any permutation of a maximal cycle is a maximal cycle.

We claim that the only cycles with origin j and nonzero in $T(\Lambda)$ are γ, δ, $C = \gamma\,\delta$ and $C' = \delta\,\gamma$. In fact, we know that $C = \alpha_s \cdots \alpha_1\epsilon_r \cdots \epsilon_1$ and $\widetilde{\alpha_1\,\alpha_s} = 0$. Then $\epsilon_r \neq \alpha_s$ and $\epsilon_1 \neq \alpha_1$. Suppose that there is another cycle of $kQ_{T(\Lambda)}$ with origin j and nonzero in $T(\Lambda)$. Such a cycle is contained in an elementary cycle $C'' = \rho_t \cdots \rho_1$ starting at j. Then either C'' is a permutation of C, or C and C'' do not have common arrows, by Proposition 4.1 (ii). We also know that C'' has a permutation of the form $p'\,\beta_{p'}$ with $p' \in \mathcal{M}$. On the other hand, we know that the permutation class of C contains at most two cycles, by Lemma 4.5 (ii), so C and C'' have no common arrow. We consider two cases:

Case 1. γ is maximal in Λ. Then $\delta = \beta_p$. Since C'' is an elementary cycle, it contains only one arrow β'_p, so $\rho_1 \neq \beta_{p'}$ or $\rho_t \neq \beta_{p'}$. Suppose $\rho_1 \neq \beta_{p'}$. Then $\rho_1\alpha_s \neq 0$ because Λ is gentle. This contradicts the maximality of γ. Analogously, if $\rho_t \neq \beta_{p'}$ we have a contradiction.

Case 2. γ is not maximal in Λ. So $\epsilon_1 \neq \beta_p$ or $\epsilon_r \neq \beta_p$. Suppose $\epsilon_1 \neq \beta_p$. Then $\rho_1 = \beta_{p'}$ because Λ is gentle. So $\rho_t \neq \beta_{p'}$ and $\alpha_1\rho_t \neq 0$ again because Λ is gentle. Therefore $\alpha_1\rho_t$ is contained in a maximal cycle C'' and so α_1 is contained in two different cycles, which is a contradiction.

Therefore the only cycles with origin j and nonzero in $T(\Lambda)$ are γ, δ, $C = \gamma\,\delta$ and its permutation $\delta\,\gamma$. So $\operatorname{End}_{T(A)} P_j$ is generated by $\widetilde{e_j}$, $\widetilde{\gamma}$, $\widetilde{\delta}$, $\widetilde{\gamma\,\delta} = \widetilde{\delta\,\gamma}$. □

Summarizing the previous results we can state important properties of the cycles of the trivial extension of a gentle algebra, which we list in the following proposition.

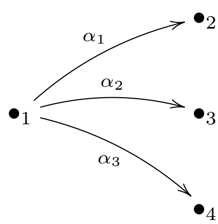
Proposition 4.8. *Let Λ be a gentle algebra. Then the bound quiver of $B = T(\Lambda)$ satisfies the following properties:*

- (T_1) *Any permutation of a maximal cycle is a maximal cycle.*
- (T_2) *Any path u of kQ_B which is nonzero in B is contained in a maximal cycle of kQ_B, which is unique up to permutations if u is nontrivial.*
- (T_3) *There are at most two different cycles with origin j in kQ_B which are maximal and nonzero in B for any vertex j of $(Q_B)_0$. If there are two such cycles, they are equal in B.*
- (T_4) *If $\alpha_s \dots \alpha_1 : j \to j$ is a cycle of kQ_B which is nonzero and not maximal in B, then $\widetilde{\alpha_1 \alpha_s} = 0$ in B and $\dim_k \operatorname{End}_K(P_j) = 4$.*

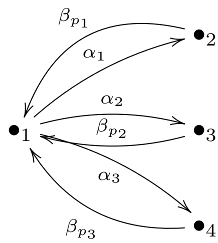
Proof. Property (T_1) holds because maximal cycles coincide with elementary cycles, by Proposition 3.1. Properties (T_2) and (T_3) follow directly from Proposition 4.1, and property (T_4) is a consequence of Proposition 4.7. □

Notice that (T_1) holds for any monomial algebra as proven in Proposition 3.1. However, (T_2), (T_3) and (T_4) do not hold in general for monomial algebras. In fact, in Example 3.7 the arrow α_1 belongs to the elementary cycles C_1 and C_4, so (T_2) does not hold in this case.

On the other hand, let Λ be the hereditary algebra



Then $T(\Lambda)$ has quiver



where the elementary cycles are $C_1 = \beta_{p_1}\alpha_1$, $C_2 = \beta_{p_2}\alpha_2$, $C_3 = \beta_{p_3}\alpha_3$ and their permutations. Then the vertex 1 is the origin of C_1, C_2 and C_3, so (T_3) does not hold.

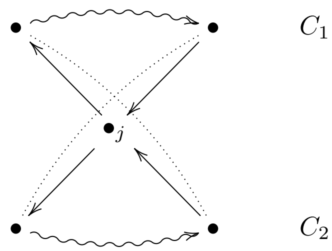
Finally, if Λ is given by the quiver  with relation $\alpha^4 = 0$, then $T(\Lambda)$ does not satisfy (T_4).

Remark 4.9. Let $B = kQ_B/I_B$ be an algebra satisfying (T_1), (T_2), (T_3) and (T_4). Then (T_2) implies that B is not semisimple, and any arrow of Q_B is contained in a nonzero cycle.

Let j be a vertex of Q_B. Then:

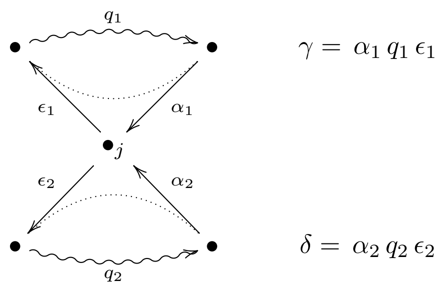
(a) If there is only one cycle C in kQ_B with origin j and nonzero in B then C is maximal by (T_2), and there is a unique arrow starting at j.

(b) If there are exactly two different cycles C_1 and C_2 with origin j nonzero in B then these are maximal in B from (T_4), since otherwise we would have $\dim_k \mathrm{End}_B(P) = 4$. This situation is illustrated in the following diagram:



where $\overline{C_1} = \overline{C_2}$, and $\dim_k \mathrm{End}_B(P_j) = 2$.

(c) If there are more than two different cycles with origin j and nonzero in B, then we know from (T_3) that one of them is not maximal. So $\dim_k \mathrm{End}_B(P_j) = 4$ from (T_4), and kQ_B contains four cycles with origin j. Two of them are maximal and are in the same permutation class. The situation is locally as follows:



where γ and δ nonmaximal cycles. The cycles of kQ_B maximal in B are $C_1 = \gamma\,\delta$, $C_2 = \delta\,\gamma$, and they are equal in B.

In any case, at most two arrows start at j and at most two arrows end at j.

Remark 4.10.

(a) An algebra $B = kQ_B/I_B$ that satisfies properties (T_1), (T_2), (T_3) and (T_4) is uniquely determined by its quiver Q_B and by the cycles of kQ_B maximal in B. That is, if the algebras A, B satisfy (T_1), (T_2), (T_3) and (T_4), then they are isomorphic if and only if they have the same quiver and the same maximal cycles.

(b) Assume B is an algebra satisfying (T_1) and (T_2). Let $C_1, \ldots, C_t$ be a complete set of representatives of the equivalence classes under permutations of nonzero cycles of Q_B maximal in B, and let α_i be an arrow in C_i for $i = 1, \ldots, t$. Then α_i is not an arrow of C_j for $i \neq j$, because (T_2) holds. Thus we can always consider a quiver obtained from Q_B by eliminating exactly one arrow of each cycle of Q_B maximal in B.

For an algebra B satisfying properties (T_1), (T_2), (T_3) and (T_4), it is possible to construct a gentle algebra whose trivial extension is isomorphic to B. The following theorem shows this fact and describes all such gentle algebras.

Theorem 4.11. *Let $B = kQ_B/I_B$ be a finite-dimensional algebra satisfying (T_1), (T_2), (T_3) and (T_4).*

(i) *Let Q be the quiver obtained from Q_B by eliminating exactly one arrow of each cycle of Q_B maximal in B, and let $I = kQ \cap I_B$. Then $\Lambda = kQ/I$ is a gentle algebra and $T(\Lambda) \cong B$.*

(ii) *If Λ is a gentle algebra such that $T(\Lambda) \cong B$, then $\Lambda = kQ/I$, with Q and I as in* (i).

Proof. (i) Let Q be the quiver obtained from Q_B by eliminating exactly one arrow of each cycle of Q_B maximal in B. We will prove all the conditions that must be satisfied so that $\Lambda = kQ/I$ is gentle.

$(G1)$ I is generated by paths of length two. Indeed:

Assume that there is a relation $\overline{k_1 q_1} + \overline{k_2 q_2} + \ldots + \overline{k_r q_r} = 0$, where $0 \neq k_i \in k$ for all $i \in \{1, \ldots, r\}$ and all the q_i are different paths of kQ_Λ nonzero in Λ. We choose such a sum with r minimum. If one of the q_i's, say q_1, is not maximal in Λ, then it can be extended to a maximal nonzero path $q' q_1 q$ and we replace the original relation by $\overline{k_1 q' q_1 q} + \overline{k_2 q' q_2 q} + \ldots + \overline{k_r q' q_r q} = 0$. So we may assume that all the q_i's are maximal paths in Λ. From the construction of Q we know that there is an arrow $\epsilon \in Q_B$ such that $q_1 \epsilon$ is a maximal cycle of B, and $\epsilon \notin Q$. Then $\overline{k_1 q_1 \epsilon} + \overline{k_2 q_2 \epsilon} + \ldots + \overline{k_r q_r \epsilon} = 0$ in B with the first summand nonzero. Then there is another summand which is nonzero. Let's say $\overline{q_2 \epsilon} \neq 0$. Next we prove that the path $q_2 \epsilon$ is maximal in B. Since property (T_2) holds, this path can be completed to a maximal cycle C. On the other hand, by (T_1) we know that permutations of maximal cycles are maximal cycles, so we may assume that $C = g q_2 \epsilon$, where g is a path in kQ_B. Since ϵ is not an arrow of Q, we have that $g\, q_2$ is in kQ. Since $g\, q_2$ is nonzero and q_2 is maximal in Λ, it follows that the path g is trivial, so $C = q_2 \epsilon$ is a maximal cycle of B. Thus $q_1 \epsilon$, $q_2 \epsilon$ are maximal cycles of B and they contain ϵ. Since (T_2) holds, there is, up to permutations, a unique cycle containing ϵ, so $q_1 \epsilon = q_2 \epsilon$. Thus $q_1 = q_2$, and this is a contradiction because all the q_i's are different paths of kQ. This proves that I is generated by paths.

Suppose now that $\alpha_r \cdots \alpha_2 \alpha_1$ is an element of I with $r > 2$, where each α_i is an arrow of Q, and $\alpha_r \cdots \alpha_2, \alpha_{r-1} \cdots \alpha_1 \notin I$. Then $\alpha_r \cdots \alpha_2$ and $\alpha_{r-1} \cdots \alpha_1$ are nonzero paths in Λ whith a common arrow α_2. From (T_2) we know that α_2 is contained in a cycle of kQ_B, maximal in B, and such a cycle is unique up to permutations. Thus $\alpha_{r-1} \cdots \alpha_1$ and $\alpha_r \cdots \alpha_2$ are both contained in the same maximal cycle C. We observe first that $\alpha_1 \neq \alpha_2, \cdots \alpha_r$. In fact, if we assume, on the contrary, that $\alpha_1 = \alpha_k$, with $2 \leq k \leq r$, we get that $\alpha_1\alpha_{k-1}\cdots\alpha_2$ is a cycle contained in C, and it is not maximal because any maximal cycle contains one arrow which is not an arrow of Q, and all α_i's are in Q_1. Since (T_4) holds we have that $\overline{\alpha_2\alpha_1} = 0$, contradicting our assumption that $\overline{\alpha_{r-1}\cdots\alpha_1} \neq 0$. Thus $\alpha_r \neq \alpha_2, \cdots \alpha_1$, and therefore $C = \alpha_r \cdots \alpha_2\beta_s\cdots\beta_1$, where $\beta_1, \ldots, \beta_s$ are arrows of Q_B, and one of them is α_1. Then we have $\beta_k = \alpha_1$, with $1 \leq k \leq s$, and $\alpha_1\beta_{k-1}\cdots\beta_1\alpha_r\cdots\alpha_2 = C_1$ is a cycle contained in C, which is not maximal unless $s = k$. Thus, if $s \neq k$, then (T_4) implies that $\overline{\alpha_2\alpha_1} = 0$, contradicting again that $\overline{\alpha_{r-1}\cdots\alpha_1} \neq 0$. So $s = k$ and $C_1 = C$ is a maximal cycle. Since (T_1) holds we obtain that the permutation $\beta_{k-1}\cdots\beta_1\alpha_r\cdots\alpha_2\alpha_1$ of C_1 is also a maximal cycle, contradicting that $\alpha_r\cdots\alpha_2\alpha_1 \in I$. This contradiction proves that I is generated by paths of length two.

$(G2)$ Each vertex of Q is the beginning and the target of at most two arrows. Indeed:

The vertices of Q_B satisfy this property from Remark 4.9. Then the subquiver Q of Q_B inherits this property.

$(G3)$ For each arrow α of Q there exists at most one arrow β such that $\alpha\beta \in I$, and there exists at most one arrow γ such that $\gamma\alpha \in I$, and $(G4)$ For each arrow α of Q there exists at most one arrow δ such that $\alpha\delta \notin I$ and there exists at most one arrow ϵ such that $\epsilon\alpha \notin I$ is satisfied because any arrow α of Q is an arrow of Q_B and so it is a consequence of Remark 4.9.

Further, by the construction of Λ we have that the quiver of the trivial extension $T(\Lambda)$ of Λ is precisely the quiver Q_B of the algebra B and, moreover, the cycles of kQ_B maximal in B coincide with the cycles of $kQ_{T(\Lambda)}$ maximal in $T(\Lambda)$. On the other hand, we know from Proposition 4.8 that $T(\Lambda)$ satisfies the properties (T_1), (T_2), (T_3) and (T_4). Thus $T(\Lambda) \cong B$ by Remark 4.10.

(ii) The quiver Q_Λ is obtained from $Q_{T(\Lambda)}$ by deleting the arrow β_p in each elementary cycle $p\beta_p$. Then (ii) holds because the maximal cycles of B coincide with the elementary cycles of $T(\Lambda) \cong B$. □

The previous theorem and Proposition 4.8 yield the following characterization of trivial extensions of gentle algebras.

Theorem 4.12. *Let B an indecomposable finite-dimensional k-algebra. Then B satisfies (T_1), (T_2), (T_3) and (T_4) if and only if there is a gentle algebra Λ such that $T(\Lambda) \cong B$.*

Given an algebra Λ, the quiver of its trivial extension is obtained from the quiver of Λ by adding certain arrows β_p. When Λ is a monomial algebra, we proved in Proposition 3.1 that elementary cycles coincide with maximal nonzero cycles in $T(\Lambda)$. Since Λ is obtained from $T(\Lambda)$ by deleting the arrows β_p, and there is one arrow β_p in each elementary cycle, it follows that Λ is obtained from $T(\Lambda)$ by deleting exactly one arrow from each maximal cycle, and considering the induced relations. This shows that the construction in Theorem 4.11 of a gentle algebra Λ

whose trivial extension is isomorphic to B gives us in fact all gentle algebras with trivial extension isomorphic to B.

5. Trivial extensions and Brauer graph algebras

In this section we relate Brauer graph algebras with the finite-dimensional algebras satisfying the properties (T_1), (T_2), (T_3) and (T_4) defined in the previous section.

We start with the necessary definitions, which generalize notions from the representation theory of finite groups, following the approach of Benson [2, section 4.18].

A *Brauer graph* is a finite connected graph (possibly with multiple edges and loops) where each vertex is assigned a cyclic ordering of the edges which are incident on it, and an integer greater than zero called the *multiplicity* of the vertex. We will always assume that the multiplicity at each vertex is one. If $j_1, j_2, \dots, j_r, j_1$ is the cyclic ordering of the edges around the vertex u, then $j_1, j_2, \dots, j_r$ is called a *sequence of successors of* j_1 *at the vertex* u.

If a loop j_k is incident on u it occurs twice in any sequence of successors at u, and these occurrences are labeled j_k, $\widehat{j}_k$.

The sequence of successors of j_1 at the vertex u is unique if j_1 is not a loop; otherwise there are two: one starts at j_1, the other at $\widehat{j}_1$.

We recall the definition of the Brauer graph algebra associated to a Brauer graph, which is a generalization of the classical Brauer tree algebra.

Definition 5.1. B_Γ is the *Brauer graph algebra associated to the Brauer graph* Γ if there is a one to one correspondence between the edges j of Γ and the simple modules S_j over B_Γ, and $P_j = P_0(S_j)$ is described by:

$$P_j/\mathrm{rad}(P_j) \cong \mathrm{soc}(P_j) \cong S_j$$
$$\mathrm{rad}(P_j)/\mathrm{soc}(P_j) = U_j \oplus V_j,$$

where U_j, V_j are the uniserial modules at the vertices u, v on which the edge j is incident, defined as follows: Let $j = j_1, j_2, \dots, j_r$ be the sequence of successors of j at the vertex u. Then U_j is the uniserial module with composition factors (from the top) $S_{j_2}, S_{j_3}, \dots, S_{j_r}$ ($U_j = 0$ if $r = 1$). If j is not a loop, V_j corresponds analogously to the sequence of successors of j at the vertex v; otherwise, it corresponds to the sequence of successors at the vertex u starting at $\widehat{j}_1$.

The *quiver* Q_Γ *associated to the Brauer graph* Γ is defined as follows. For each edge j_k of Γ there is a vertex v_{j_k} of Q_Γ. If the edge j_{k+1} of Γ immediately follows the edge j_k in some sequence of successors, there is an arrow $u_{j_k} \to u_{j_{k+1}}$ of Q_Γ.

We observe that Q_Γ coincides with the quiver associated to the algebra B_Γ. We describe next the maximal cycles in B_Γ. For each sequence of successors $j_1, j_2, \dots, j_r$ of j_1 at the vertex u with $r > 1$, there is a cycle $u = u_{j_1} \to u_{j_2} \to \cdots \to u_{j_r} \to u_{j_1}$ which is maximal in B_Γ. If j_1 is not a loop, we will denote this cycle by $C_{j_1,u}$. If j_1 is a loop, we will call $C_{j_1,u}$, $C_{\widehat{j}_1,u}$ the cycles which correspond to the two occurrences of j_1 in the cyclic ordering of the edges around the vertex u.

Then the cycles described are all the maximal cycles in B_Γ, and every arrow is contained in one of them.

From the description of the indecomposable projective modules we can describe the relations for B_Γ, which are of the following three types:

• Relations of type one: $C_{j_1,u} - C_{j_1,v}$ if the edge j_1 of Γ is incident on two different vertices u, $v \in \Gamma$; $C_{j_1,u} - C_{\widehat{j_1,u}}$ if the edge j_1 of Γ is a loop in u.

• Relations of type two: $\alpha\, C_{j_1,u}$, where $u_{j_1} \xrightarrow{\alpha} u_{j_2}$.

• Relations of type three: paths of length two of kQ_Γ which are not subpaths of a cycle $C_{j_1,u}$.

In the next theorem we characterize Brauer graph algebras given by their quiver and relations from the properties of their cycles, using for this the properties (T_1), (T_2), (T_3) and (T_4) given in Section 4.

Theorem 5.2. *Let $B = kQ/I$ be an indecomposable finite-dimensional algebra. Then B satisfies (T_1), (T_2), (T_3) and (T_4) if and only if there is a Brauer graph Γ with multiplicity one in all the vertices such that the associated Brauer graph algebra B_Γ is isomorphic to B.*

Proof. Let B_Γ be a Brauer graph algebra with associated Brauer graph Γ. We will prove that B_Γ satisfies the properties (T_1), (T_2), (T_3) and (T_4). In fact:

(T_1) Any permutation of a maximal cycle in B_Γ is a maximal cycle in B_Γ.

This property is a direct consequence of the description of the maximal cycles of a Brauer graph algebra. In fact, a maximal cycle in B_Γ is of the form $C_{j_1,u}$, and every permutation of the cycle $C_{j_1,u}$ is obtained by considering a cyclic permutation of the sequence of successors $j_1, j_2, \dots, j_r$ of j_1 at the vertex u, which is also a sequence of successors at the same vertex, and therefore gives rise to another cycle of kQ_Γ maximal in B_Γ.

(T_2) Any path u of kQ_B which is nonzero in B is contained in a maximal cycle of kQ_B, unique up to permutations if u is nontrivial.

Since Brauer graph algebras are indecomposable and not semisimple, the trivial paths are not maximal. Suppose $u = u_{j_1} \to u_{j_2} \to \cdots \to u_{j_k}$ is a nontrivial path of kQ_Γ which is nonzero in B_Γ. From the description of Q_Γ we know that the edge j_{t+1} immediately follows the edge j_t in a sequence of successors of j_1 at some vertex u of Γ for $t = 1, \dots, k-1$. Then the path considered is a subpath of the cycle $C_{j_1,u}$, which is maximal in B_Γ, and is the only maximal nonzero cycle containing the path u up to permutations, by construction.

(T_3) There are at most two different cycles with origin j in kQ_B which are maximal nonzero in B_Γ for any vertex j of $(Q_B)_0$. If there are two such cycles, they are equal in B_Γ.

The fact that this property holds in B_Γ follows from the above description of the maximal cycles of B_Γ. In fact, if there are two maximal cycles starting at the same vertex, they are of the form $C_{j_1,u}$ and $C_{j_1,v}$ if j_1 is not a loop, or $C_{j_1,u}$ and $C_{\widehat{j_1},u}$ if j_1 is a loop. In the first case $C_{j_1,u} - C_{j_1,v}$ is a relation of type one and in the second $C_{j_1,u} - C_{\widehat{j_1},u}$ is also a relation of type one. Thus the two cycles are equal in B_Γ.

(T_4) If $\alpha_s \dots \alpha_1 : j \to j$ is a nonzero cycle of kQ_{B_Γ} which is not maximal in B_Γ, then $\widetilde{\alpha_1\alpha_s} = 0$ in B_Γ and $\dim_k \mathrm{End}_K(P_j) = 4$.

Let $\gamma : u_{j_1} \to u_{j_2} \to \cdots \to u_{j_{s-1}} \to u_{j_1}$ be a cycle of kQ_Γ nonzero and not maximal in B_Γ. From the above description of the maximal cycles of B_Γ we conclude that the edge j_1 is a loop, there are two sequences of successors $j_1, j_2, \dots, j_{s-1}, \widehat{j}_1, j_{s+1}, \dots, j_r$ and $\widehat{j}_1, j_{s+1}, \dots, j_r, j_1, j_2, \dots, j_{s-1}$, and the maximal cycles with origin u_{j_1} are $C_{j_1,u_{j_1}}$, $C_{\widehat{j_1},u_{j_1}}$. Since the path $u_{j_{s-1}} \overset{\alpha_s}{\to} u_{j_1} \overset{\alpha_1}{\to} u_{j_2}$ is not a subpath of any of these two maximal cycles, it follows that $\alpha_1\alpha_s$ is a relation of type three. Therefore $\overline{\alpha_1\alpha_s} = 0$ in B_Γ.

The cycle $\gamma' : u_{j_1} \to u_{j_{s+1}} \to \cdots \to u_{j_r} \to u_{j_1}$ with origin u_{j_1} is also nonzero and not maximal in B_Γ, since it is properly contained in $C_{\widehat{j_1},u}$. Then, the cycles starting at u_j are γ, γ' and the maximal cycles $\gamma'\gamma = C_{j_1,u_{j_1}}$ and $\gamma\gamma' = C_{\widehat{j_1},u_{j_1}}$. Since $C_{j_1,u_{j_1}} - C_{\widehat{j_1},u_{j_1}}$ is a relation of type one, we have that $\mathrm{End}_{B_\Gamma} P_{j_1}$ is generated by $\overline{e_{j_1}}$, $\overline{\gamma}$, $\overline{\gamma'}$ and $\overline{C_{j_1,u_{j_1}}}$. Thus $\dim_k \mathrm{End}_{B_\Gamma} P_{j_1} = 4$, so (T_4) holds for B_Γ. This ends the proof that Brauer graph algebras satisfy the properties (T_1), (T_2), (T_3) and (T_4).

Now consider $B = kQ/I$ satisfying (T_1), (T_2), (T_3) and (T_4). We will construct a Brauer graph Γ such that $B \simeq B_\Gamma$. The set of maximal cycles of B is not empty because (T_2) holds. Consider the equivalence relation defined in this set by $C \sim C'$ if and only if C is a permutation of C', if C, C' are maximal cycles. Let $\{\overline{C_1}, \dots, \overline{C_s}\}$ be the set of equivalence classes.

We associate to the quiver Q a Brauer graph Γ as follows. The set of vertices of Γ is

$$\{u_{\overline{C}_1}, u_{\overline{C}_2}, \dots, u_{\overline{C}_s}\} \cup \{u_{v_j} : v_j \in Q_0 \text{ and } v_j \text{ is the origin of a unique maximal cycle}\}.$$

The edges a_i of Γ correspond to vertices v_i of Q in the following way: If v_i is the beginning of two different maximal cycles C_1, C_2, the endpoints of a_i are $u_{\overline{C_1}}$, $u_{\overline{C_2}}$. Notice that when $\overline{C_1} = \overline{C_2}$ we obtain a loop. If v_i is the beginning of a unique maximal cycle C' then the endpoints of a_i are $u_{\overline{C'}}$, u_{v_i}, and so a_i is the only edge with u_{v_i} as an endpoint. For each equivalence class $\overline{C_k}$, the edges that are incident on the vertex $u_{\overline{C}_k}$ correspond to the vertices of the maximal cycle C_k, and we define a cyclic ordering in this set of edges as follows. The edge a_{j+1} is the immediate successor of the edge a_j if there is an arrow $v_j \to v_{j+1}$ of Q contained in a maximal cycle of $\overline{C}_k$. This ordering is well defined since any arrow is contained in a unique maximal cycle up to permutations, by (T_2). Finally, we assign the multiplicity one to each vertex of Γ, and Γ is then a Brauer graph. From the Brauer graph Γ we obtain the Brauer graph algebra $B_\Gamma = Q_\Gamma/I_\Gamma$. We know that B_Γ satisfies (T_1), (T_2), (T_3) and (T_4), since we just proved that Brauer graph algebras do.

We will prove that $Q = Q_\Gamma$, and that the cycles of Q maximal in B coincide with the cycles of $Q = Q_\Gamma$ maximal in B_Γ.

We prove next that $Q = Q_\Gamma$. The Brauer graph Γ was constructed in such a way that the vertices of Q are in bijective correspondence with the edges of Γ. On the other hand, these edges are in bijective correspondence with the vertices of Q_Γ, as follows from the definition of Brauer graph algebra associated to a Brauer graph. Then the vertices of Q are in bijective correspondence with those of Q_Γ. We will denote by u_i the vertex of Q_Γ corresponding to the vertex v_i of Q under this bijection. Then a_i is the edge of Γ corresponding to u_i in Q_Γ and to v_i in Q.

Let $v_j \to v_{j+1}$ be an arrow in Q. Then this arrow belongs to a unique maximal cycle C, by (T_2), and the edge a_{j+1} of Γ is an immediate predecessor of a_j in the cyclic ordering of the edges of Γ at the vertex $u_{\overline{C}}$. This means that there is an arrow $u_j \to u_{j+1}$ in Q_Γ. Conversely, if there is an arrow $u_j \to u_{j+1}$ in Q_Γ a similar argument shows that there is an arrow $v_j \to v_{j+1}$ in Q. Thus Q and Q_Γ have the same arrows, so we can identify the quivers Q and Q_Γ.

Finally, we will prove that the cycles of Q maximal in B coincide with the cycles of Q_Γ maximal in B_Γ. In fact, saying that $C_j = v_{j_1} \to v_{j_2} \to \cdots \to v_{j_t}$ is a cycle of kQ maximal in B is equivalent to saying that $a_{j_1}, a_{j_2}, \ldots, a_{j_t}$ are all the edges of Γ with endpoint $u_{\overline{C}_j}$ with the cyclic ordering $a_{j_1} \leq a_{j_2} \leq \cdots \leq a_{j_t}$. This means that $a_{j_1}, a_{j_2}, \ldots, a_{j_t}$ is a sequence of successors of a_{j_1} at the vertex $u_{\overline{C_j}}$, which amounts to say that $C_{j_1, u_{\overline{C}_j}} = u_{j_1} \to u_{j_2} \to \cdots \to u_{j_t}$ is a cycle of Q_Γ maximal in B_Γ. Then cycles of Q maximal in B coincide with cycles of Q_Γ maximal in Γ.

We know from Remark 4.10 that algebras that satisfy the properties (T_1), (T_2), (T_3) and (T_4) with the same quiver and the same maximal cycles are isomorphic, thus $B_\Gamma \cong B$. □

In Theorem 4.12 we proved that properties (T_1), (T_2), (T_3) and (T_4) characterize trivial extensions of gentle algebras. Combining this result with the previous theorem we get the following characterization of trivial extensions of gentle algebras, which was obtained by S. Schroll using a different approach.

Corollary 5.3 ([4, Theorem 1.2 and Corollary 1.4])**.** *Let B be an indecomposable finite-dimensional algebra. Then B is a Brauer graph algebra with multiplicity one in all the vertices of the associated Brauer graph if and only if B is the trivial extension of a gentle algebra.*

We summarize the results of the last two sections in the following theorem.

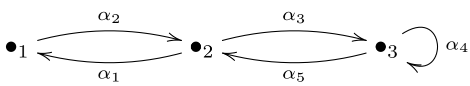
Theorem 5.4. *Let $B = kQ_B/I_B$ be a finite-dimensional indecomposable k-algebra. Then the following conditions are equivalent:*

(i) *B is isomorphic to the trivial extension of a gentle algebra.*
(ii) *B satisfies the properties (T_1), (T_2), (T_3) and (T_4).*
(iii) *B is isomorphic to the Brauer graph algebra B_Γ associated to a Brauer graph Γ with multiplicity one in all the vertices.*

Let us see an example to illustrate this result.

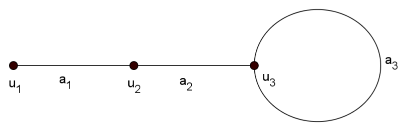
Example 5.5. Let $B = kQ_B/I_B$, where

Q_B:



$$I_B = (\alpha_1\,\alpha_2\,\alpha_1,\ \alpha_2\,\alpha_1\,\alpha_2,\ \alpha_3\,\alpha_2,\ \alpha_1\,\alpha_5,\ \alpha_5\,\alpha_3,\ \alpha_4\,\alpha_4,\ \alpha_3\,\alpha_5\,\alpha_4\,\alpha_3,\ \alpha_4\,\alpha_3\,\alpha_5\,\alpha_4,\ \alpha_5\,\alpha_4\,\alpha_3\,\alpha_5,\ \alpha_2\,\alpha_1 - \alpha_5\,\alpha_4\,\alpha_3,\ \alpha_3\,\alpha_5\,\alpha_4 - \alpha_4\,\alpha_3\,\alpha_5).$$

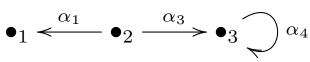
The cycles of kQ_B maximal in B are $C_1 = \alpha_1\,\alpha_2$ and its permutation $\alpha_2\,\alpha_1$, $C_2 = \alpha_5\,\alpha_4\,\alpha_3$ and its permutations $\alpha_3\,\alpha_5\,\alpha_4$ and $\alpha_4\,\alpha_3\,\alpha_5$. Then the algebra B satisfies the properties (T_1), (T_2), (T_3) and (T_4). To describe the Brauer graph Γ associated to Q_B we observe that the vertex 1 is the beginning of a unique maximal cycle. The vertices of Γ are u_1, $u_2 = u_{\overline{C_1}}$ and $u_3 = u_{\overline{C_2}}$. Since Q_B has three vertices, Γ has three edges: a_1 with endpoints u_1 and u_2, a_2 with endpoints u_2 and u_3, and a_3, which is a loop at u_3. Thus Γ is the graph



where we choose the cyclic counterclockwise ordering of the edges around each vertex and we assign multiplicity one to each vertex.

On the other hand, if we delete in Q_B one arrow of each maximal cycle we obtain a gentle algebra Λ such that $T(\Lambda) \simeq B$. For example, when we choose the arrows α_2 and α_5 we obtain $\Lambda = kQ_\Lambda/I_\Lambda$, with

Q_Λ:



$I_\Lambda = (\alpha_4^2)$.

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