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C-GROUPS AND MIXED C-GROUPS OF BOUNDED LINEAR OPERATORS ON NON-ARCHIMEDEAN BANACH SPACES

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Abstract. We introduce and study C-groups and mixed C-groups of bounded linear operators on non-archimedean Banach spaces. Our main result extends some existing theorems on this topic. In contrast with the classical setting, the parameter of a given C-group (or mixed C-group) belongs to a clopen ball Ω_r of the ground field $\mathbb{K}$. As an illustration, we discuss the solvability of some homogeneous p-adic differential equations for C-groups and inhomogeneous p-adic differential equations for mixed C-groups when $\alpha = -1$. Examples are given to support our work.

1. Introduction and preliminaries

In the classical setting, the theory of one-parameter semigroups (or groups) of linear operators on Banach spaces started in the first half of the last century, acquired its core in 1948 with the Hille–Yosida generation theorem [8], and thanks to the efforts of many different schools, the theory reached a certain state of perfection, which is well represented in the monograph by A. Pazy [8]. Recently, the situation is characterized by manifold applications of this theory, not only to traditional areas such as partial differential equations or stochastic processes. Groups have become important tools for integro-differential equations and functional differential equations, in quantum mechanics or in infinite-dimensional control theory. Semigroup methods are also applied with great success to concrete equations arising, e.g., in population dynamics or transport theory. However, semigroup theory is in competition with alternative approaches in all of these fields, and the relevant functional-analytic toolbox now presents a highly diversified picture.

In non-archimedean operator theory, T. Diagana [2] introduced the concept of C_0-groups of bounded linear operators on free non-archimedean Banach space. Also, in [5], A. El Amrani, A. Blali, J. Ettayb and M. Babahmed introduced and studied the notions of C-groups and cosine family of bounded linear operators on non-archimedean Banach space. As an application of C-groups of linear operators is the p-adic abstract Cauchy problem for differential equations on a non-archimedean

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Banach space X given by

$$ACP(A;x)\begin{cases}\dfrac{du(t)}{dt}=Au(t), & t\in\Omega_r,\\ u(0)=Cx,\end{cases}$$

where $A\colon D(A)\to X$ is a linear operator and C is an invertible operator with $x\in D(A)$. By Remark 1.13 below, $ACP(A;x)$ has a solution.

Throughout this paper, X is a non-archimedean (n.a.) Banach space over a (n.a.) non-trivially complete valued field $\mathbb{K}$ with valuation $|\cdot|$, $B(X)$ denotes the set of all bounded linear operators from X into X, $\mathbb{Q}_p$ is the field of p-adic numbers ($p\geq 2$ being a prime) equipped with p-adic valuation $|\cdot|_p$, $\mathbb{Z}_p$ denotes the ring of p-adic integers (the ring of p-adic integers $\mathbb{Z}_p$ is the unit ball of $\mathbb{Q}_p$). For more details and related issues, we refer to [2, 3]. We denote the completion of algebraic closure of $\mathbb{Q}_p$ under the p-adic absolute value $|\cdot|_p$ by $\mathbb{C}_p$ (see [6]). Given $r>0$, Ω_r is the clopen ball of $\mathbb{K}$ centred at 0 with radius r, i.e., $\Omega_r=\{t\in\mathbb{K}:|t|<r\}$.

In the non-archimedean context, a family $(T(t))_{t\in\Omega_r}\subseteq B(X)$ is called a group of bounded linear operators if $T(0)=I$ and for all $t,s\in\Omega_r$, $T(t+s)=T(t)T(s)$, where I is the unit operator of X. For more details, we refer to [2, 5].

Assume that $\mathbb{K}=\mathbb{Q}_p$ and let $A\in B(X)$ be such that $\|A\|<p^{\frac{-1}{p-1}}$; then the function defined for all $t\in\Omega_{p^{\frac{-1}{p-1}}}$ by $f(t)=\left(\sum_{n=0}^{\infty}\dfrac{(tA)^n}{n!}\right)x$ is the solution of the homogeneous p-adic differential equation given by

$$\frac{du}{dt}=Au,\quad u(0)=x,\qquad \text{for a fixed } x\in X.$$

The aim of this paper is to introduce and study the notions of C-groups and mixed C-groups on non-archimedean Banach spaces over $\mathbb{K}$. In contrast with the classical setting, the parameter of a given C-group (or mixed C-group) belongs to a clopen ball Ω_r of the ground field $\mathbb{K}$. As an illustration, we will discuss the solvability of some homogeneous p-adic differential equations for C-groups (Remark 1.13) and inhomogeneous p-adic differential equations for mixed C-groups (Remark 3.18) when $\alpha=-1$.

Definition 1.1 ([3, Definition 2.1])**.** Let X be a vector space over $\mathbb{K}$. A non-negative real valued function $\|\cdot\|\colon X\to\mathbb{R}_+$ is called a *non-archimedean norm* if:

(i) for all $x\in X$, $\|x\|=0$ if and only if $x=0$;
(ii) for any $x\in X$ and $\lambda\in\mathbb{K}$, $\|\lambda x\|=|\lambda|\|x\|$;
(iii) for any $x,y\in X$, $\|x+y\|\leq\max(\|x\|,\|y\|)$.

Property (iii) in Definition 1.1 is referred to as the ultrametric or strong triangle inequality.

Definition 1.2 ([3, Definition 2.2])**.** A *non-archimedean normed space* is a pair $(X;\|\cdot\|)$ where X is a vector space over $\mathbb{K}$ and $\|\cdot\|$ is a non-archimedean norm on X.

Definition 1.3 ([2, Definition 2.2])**.** A *non-archimedean Banach space* is a vector space endowed with a non-archimedean norm, which is complete.

For more details on non-archimedean Banach spaces and related issues, see for example [3].

Proposition 1.4 ([3, Proposition 2.16])**.**

(1) *A closed subspace of a non-archimedean Banach space is a non-archimedean Banach space;*

(2) *The direct sum of two non-archimedean Banach spaces is a non-archimedean Banach space.*

Example 1.5 ([3, Example 2.20])**.** Let $c_0(\mathbb{K})$ denote the set of all sequences $(x_i)_{i\in\mathbb{N}}$ in $\mathbb{K}$ such that $\lim\limits_{i\to\infty} x_i = 0$. Then, $c_0(\mathbb{K})$ is a vector space over $\mathbb{K}$ and

$$\|(x_i)_{i\in\mathbb{N}}\| = \sup_{i\in\mathbb{N}} |x_i|$$

is a non-archimedean norm for which $(c_0(\mathbb{K}), \|\cdot\|)$ a non-archimedean Banach space.

In this section, we define and discuss properties of non-archimedean Banach spaces that have bases.

Definition 1.6 ([2, Definition 2.5])**.** A non-archimedean Banach space $(X, \|\cdot\|)$ over a non-archimedean valued field (complete) $(K, |\ |)$ is said to be a *free non-archimedean Banach space* if there exists a family $(x_i)_{i\in I}$ of elements of X, indexed by a set I, such that each element $x \in X$ can be written uniquely as a pointwise convergent series defined by $x = \sum_{i\in I} \lambda_i x_i$, and $\|x\| = \sup_{i\in I} |\lambda_i|\|x_i\|$. The family $(x_i)_{i\in I}$ is then called an *orthogonal basis* for X. If, for all $i \in I$, $\|x_i\| = 1$, then $(x_i)_{i\in I}$ is called an *orthonormal basis* of X. For more details of orthogonality and the concepts of bases in the non-archimedean case, we refer to [9, 10].

The treatment of those non-archimedean Banach spaces in the general case can be found in [4] and in the unpublished manuscript "Geometry of the p-adic Hilbert Spaces" (1999) by B. Diarra. Moreover, X is a free non-archimedean Banach space over $\mathbb{K}$ if and only if X is isometrically isomorphic to $c_0(I, u)$ for a certain index set I and an application $u\colon I \to \mathbb{R}_+^*$. By [9, Theorem 2.58] $c_0(I)$ is of countable type if and only if I is countable. For more details we refer to [9, 10]. In this work, the basis of the free n.a. Banach spaces considered is countable, $I = \mathbb{N}$.

Definition 1.7 ([3])**.** Let $(X, \|\cdot\|)$ be a non-archimedean Banach space. The non-archimedean Banach space $(B(X), \|\cdot\|)$ is the collection of all bounded linear operators from X into itself equipped with the operator norm defined by:

$$\text{for all } A \in B(X), \quad \|A\| = \sup_{x\in X\setminus\{0\}} \frac{\|A(x)\|}{\|x\|}.$$

For more details of non-archimedean linear operator theory, we refer to [2, 3, 4] and to Diarra's unpublished manuscript cited above.

Throughout this paper, $B(X)$ is equipped with the norm of Definition 1.7 and, for all $r > 0$, $\Omega_r^* = \Omega_r\setminus\{0\}$ denotes the clopen ball of center 0 with radius r deprived of zero.

Definition 1.8 ([3, Definition 3.1]). Let $r > 0$ be a real number chosen such that $(T(t))_{t\in\Omega_r}$ are well defined. A one-parameter family $(T(t))_{t\in\Omega_r}$ of bounded linear operators from X into X is a *group* of bounded linear operators on X if

(i) $T(0) = I$, where I is the unit operator of X;
(ii) for all $t, s \in \Omega_r$, $T(t+s) = T(t)T(s)$.

The group $(T(t))_{t\in\Omega_r}$ will be called *of class* C_0 or *strongly continuous* if the following condition holds:

$$\text{For each } x \in X, \quad \lim_{t\to 0} \|T(t)x - x\| = 0.$$

A group of bounded linear operators $(T(t))_{t\in\Omega_r}$ is *uniformly continuous* if and only if $\lim_{t\to 0} \|T(t) - I\| = 0$. The linear operator A defined on

$$D(A) = \left\{ x \in X : \lim_{t\to 0} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

by

$$Ax = \lim_{t\to 0} \frac{T(t)x - x}{t}, \quad \text{for each } x \in D(A),$$

is called the *infinitesimal generator* of the group $(T(t))_{t\in\Omega_r}$.

We introduce the following definition.

Definition 1.9 ([5, Definition 2.21]). Let $r > 0$ and let $C \in B(X)$ be invertible. A one-parameter family $(T(t))_{t\in\Omega_r}$ of bounded linear operators from X into X is called a *C-group* if the following conditions hold:

(i) $T(0) = C$;
(ii) for all $t, s \in \Omega_r$, $CT(t+s) = T(t)T(s)$;
(iii) for all $x \in X$, $T(\cdot)x \colon \Omega_r \to X$ is continuous.

The linear operator A defined on

$$D(A) = \left\{ x \in X : \lim_{t\to 0} \frac{T(t)x - Cx}{t} \text{ exists} \right\}$$

by

$$Ax = C^{-1} \lim_{t\to 0} \frac{T(t)x - Cx}{t}, \quad \text{for each } x \in D(A),$$

is called the *infinitesimal generator* of the C-group $(T(t))_{t\in\Omega_r}$.

Remark 1.10 ([5, Definition 2.21]). Let $(T(t))_{t\in\Omega_r}$ be a C_0-group of infinitesimal generator A, and let $C \in B(X)$ be invertible such that for all $t \in \Omega_r$, $CT(t) = T(t)C$. Define for each $t \in \Omega_r$ the family of linear operators $S(t) = T(t)C$. Then $(S(t))_{t\in\Omega_r}$ is a C-group of infinitesimal generator A. In this sense, Definition 1.9 generalizes Definition 1.8 of C_0-group.

We begin with the following theorem.

Theorem 1.11 ([5, Theorem 2.23]). *Let $(T(t))_{t\in\Omega_r}$ be a C-group satisfying the following condition: there exists $M > 0$ such that for each $t \in \Omega_r$, $\|T(t)\| \leq M$. Let A be its infinitesimal generator. Then, for every $x \in D(A)$ and $t \in \Omega_r$, $T(t)x \in D(A)$. Furthermore,*

$$\frac{dT(t)}{dt}x = AT(t)x = T(t)Ax.$$

Remark 1.12. Let X be a free non-archimedean Banach space, let $(T(t))_{t\in\Omega_r}$ be a C-group of linear operators of infinitesimal generator A on X. From [2, Remark 3.5], A may or may not be a bounded linear operator on X.

These are generalizations of C_0-groups, which can be applied directly to the many p-adic differential and integral equations that may be modeled as a p-adic abstract Cauchy problem on a non-archimedean Banach space. Thanks to Theorem 1.11, we have:

Remark 1.13. One of the consequences of Theorem 1.11 is that the function $v(t) = T(t)x$, $t \in \Omega_r$, for some $x \in D(A)$, is the solution to the homogeneous p-adic differential equation given by

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t), \quad t \in \Omega_r, \\ u(0) = Cx, \end{cases}$$

where $A\colon D(A) \subset X \to X$ is the infinitesimal generator of the C-group $(T(t))_{t\in\Omega_r}$ and $u\colon \Omega_r \to D(A)$ is an X-valued function.

2. Main results

We begin with the following example.

Example 2.1. Let X be a non-archimedean Banach space over $\mathbb{C}_p$. Let $A, C \in B(X)$ be such that C is invertible, $AC = CA$ and $\|A\| < r$ with $r = p^{\frac{-1}{p-1}}$. Then for all $t \in \Omega_r$, $T(t) = Ce^{tA}$ (in particular, if $C = (I - A)^{-1}$) is a C-group of bounded linear operators on X. In fact:

(i) $T(0) = C$.
(ii) For all $t, s \in \Omega_r$, $T(t)T(s) = Ce^{tA}Ce^{sA} = C^2e^{(t+s)A} = CT(s+t)$.
(iii) It is easy to check that for all $x \in X$, $T(\cdot)x\colon \Omega_r \to X$ is continuous.

Proposition 2.2. *Let X be a non-archimedean Banach space, let $(T(t))_{t\in\Omega_r}$ be a C_1-group of infinitesimal generator A on X and let $C_2 \in B(X)$ be invertible such that for all $t \in \Omega_r$, $C_2T(t) = T(t)C_2$. Then $(C_2T(t))_{t\in\Omega_r}$ is a C_1C_2-group on X.*

Proof. Setting, for all $t \in \Omega_r$, $S(t) = C_2T(t)$, let us show that $(S(t))_{t\in\Omega_r}$ is a C_1C_2-group on X. In fact:

(i) $S(0) = C_2T(0) = C_1C_2$.

(ii) For all $s, t \in \Omega_r$,

$$\begin{aligned} S(s)S(t) &= C_2T(s)C_2T(t) \\ &= T(s)T(t)C_2^2 \\ &= C_1T(s+t)C_2^2 \\ &= C_1C_2^2T(s+t) \\ &= C_1C_2S(s+t). \end{aligned}$$

(iii) It is easy to check that for all $x \in X$, $S(\cdot)x\colon \Omega_r \to X$ is continuous.

Thus, $(S(t))_{t\in\Omega_r}$ is a C_1C_2-group on X. □

We have the following example.

Example 2.3. Assume that $\mathbb{K} = \mathbb{Q}_p$ and $r = p^{\frac{-1}{p-1}}$. Let X be a free non-archimedean Banach space over $\mathbb{Q}_p$ and let $(e_i)_{i\in\mathbb{N}}$ be an orthogonal base of X. Define, for each $t \in \Omega_r$ and $x \in X$ such that $x = \sum_{i\in\mathbb{N}} x_ie_i$,

$$T(t)x = \sum_{i\in\mathbb{N}} (1-\mu_i)e^{t\mu_i}x_ie_i,$$

where $(\mu_i)_{i\in\mathbb{N}} \subset \Omega_r$. It is easy to check that the family $(T(t))_{t\in\Omega_r}$ is well defined on X, and we have the following proposition.

Proposition 2.4. *The family $(T(t))_{t\in\Omega_r}$ of bounded linear operators given above is a C-group of bounded linear operators, whose infinitesimal generator is the bounded diagonal operator A defined by $Ax = \sum_{i\in\mathbb{N}} \mu_ix_ie_i$ for each $x = \sum_{i\in\mathbb{N}} x_ie_i \in X$.*

Proof. Let X be a free non-archimedean Banach space over $\mathbb{Q}_p$ and let $(e_i)_{i\in\mathbb{N}}$ be an orthogonal base of X. Define for each $t \in \Omega_r$, $i \in \mathbb{N}$,

$$T(t)e_i = (1-\mu_i)e^{t\mu_i}e_i \stackrel{\text{def}}{=} \left(\sum_{n\in\mathbb{N}} \frac{(1-\mu_i)\mu_i^n t^n}{n!}\right)e_i,$$

where $(\mu_i)_{i\in\mathbb{N}} \subset \Omega_r$. Since for all $i \in \mathbb{N}$, $t\mu_i \in \Omega_r$, we have for all $t \in \Omega_r$ and $x \in X$, $\|T(t)x\| \leq \sup_{i\in\mathbb{N}} \left|(1-\mu_i)e^{t\mu_i}\right|_p \|x\| < \infty$; then, for all $t \in \Omega_r$, $\|T(t)\|$ is finite. Hence the family $(T(t))_{t\in\Omega_r}$ is well defined on X. Furthermore:

(i) $T(0) = I - A$ (since A is a diagonal operator on X, we have $\|A\| = \sup_{i\in\mathbb{N}} |\mu_i|$, thus $\|A\| < r < 1$, and we have that $I - A$ is invertible).

(ii) For all $t, s \in \Omega_r$,

$$\begin{aligned} T(t)T(s) &= (I-A)e^{tA}(I-A)e^{sA} \\ &= (I-A)(I-A)e^{(t+s)A} \\ &= (I-A)T(t+s). \end{aligned}$$

(iii) It is easy to check that for all $x \in X$, $S(\cdot)x\colon \Omega_r \to X$ is continuous on Ω_r.

Thus $(T(t))_{t\in\Omega_r}$ is a C-group of bounded linear operators on X where $C = I - A$. Let B be the infinitesimal generator of $(T(t))_{t\in\Omega_r}$. It remains to show that $A = B$. Let us show that $D(B) = X(= D(A))$. Clearly, for each $t \in \Omega_r^*$ and $i \in \mathbb{N}$,

$$\frac{T(t)e_i - Ce_i}{t} = C\left(\frac{e^{t\mu_i} - 1}{t}\right)e_i.$$

Thus, for all $t \in \Omega_r^*$ and all $i \in \mathbb{N}$,

$$C^{-1}\left(\frac{T(t)e_i - Ce_i}{t}\right) = \left(\frac{e^{t\mu_i} - 1}{t}\right)e_i.$$

Hence, for all $x = \sum_{i\in\mathbb{N}} x_i e_i \in X$ and $t \in \Omega_r^*$,

$$|x_i|_p\left\|C^{-1}\frac{T(t)e_i - Ce_i}{t}\right\| \le \frac{|x_i|_p\|e_i\|}{|t|_p} \to 0 \quad \text{as } i \to \infty.$$

Thus,

$$D(B) = \left\{x = (x_i)_{i\in\mathbb{N}} : \lim_{i\to\infty} |x_i|_p\left\|C^{-1}\left(\frac{T(t)e_i - Ce_i}{t}\right)\right\| = 0\right\}.$$

To complete the proof, it suffices to prove that

$$\text{for all } i \in \mathbb{N}, \quad \lim_{t\to 0}\left\|Ae_i - C^{-1}\left(\frac{T(t)e_i - Ce_i}{t}\right)\right\| = 0.$$

The latter is actually obvious since $\lim_{t\to 0}\left(\frac{e^{t\mu_i} - 1}{t}\right) = \mu_i$, and hence $A = B$ is the infinitesimal generator of the C-group $(T(t))_{t\in\Omega_r}$. □

We introduce the following definition.

Definition 2.5. Let $(T(t))_{t\in\Omega_r}$ be a C-group of bounded linear operators on X. $(T(t))_{t\in\Omega_r}$ is said to be a *uniformly continuous* C-group on X if

$$\lim_{t\to 0}\|T(t) - C\| = 0.$$

Theorem 2.6. *Let X be a non-archimedean Banach space over $\mathbb{Q}_p$ and let $A \in B(X)$ be such that $\|A\| < r\left(= p^{\frac{-1}{p-1}}\right)$. Then A is the infinitesimal generator of an uniformly continuous C-group of bounded linear operators $(T(t))_{t\in\Omega_r}$.*

Proof. Suppose that A is a bounded linear operator on X with $\|A\| < r\left(= p^{\frac{-1}{p-1}}\right)$ and set, for all $t \in \Omega_r$,

$$T(t) = (I - A)e^{tA} \stackrel{\text{def}}{=} \sum_{n\in\mathbb{N}} \frac{(I - A)(tA)^n}{n!}.$$

Clearly, this series converges in norm and defines a family of bounded linear operators on X by $|t|\|A\| < r$. Furthermore:

(i) $T(0) = I - A$ (since $\|A\| < r < 1$, $I - A$ is invertible).
(ii) The same as in Proposition 2.4.
(iii) It is easy to check that for all $x \in X$, $S(\cdot)x\colon \Omega_r \to X$ is continuous.

Thus $(T(t))_{t\in\Omega_r}$ is a C-group of bounded linear operators on X where $C = I - A$. For all $t \in \Omega_r$,

$$\begin{aligned}\|T(t) - C\| &= \|(I - A)(e^{tA} - 1)\|\\ &\leq \|I - A\|\|e^{tA} - I\|\\ &\leq \|e^{tA} - I\|.\end{aligned}$$

Hence

$$\lim_{t\to 0} \|T(t) - C\| = 0.$$

For all $t \in \Omega_r^*$,

$$\frac{T(t) - C}{t} = C\Big(\frac{e^{tA} - I}{t}\Big).$$

Thus, for all $t \in \Omega_r^*$,

$$C^{-1}\Big(\frac{T(t) - C}{t}\Big) = \Big(\frac{e^{tA} - I}{t}\Big) = \sum_{n=0}^{\infty} \frac{t^n A^{n+1}}{(n+1)!}.$$

Hence, for all $t \in \Omega_r^*$,

$$\begin{aligned}\Big\|C^{-1}\Big(\frac{T(t) - C}{t}\Big) - A\Big\| &= \Big\|\frac{e^{tA} - I}{t} - A\Big\|\\ &\leq |t|\|A\|\|\xi_t\|,\end{aligned}$$

where $\xi_t = \sum_{n=0}^{\infty} \frac{t^n A^{n+1}}{(n+2)!}$ converges. Consequently, $\lim_{t\to 0} \Big\|C^{-1}\Big(\frac{T(t) - C}{t}\Big) - A\Big\| = 0$. Then, $(T(t))_{t\in\Omega_r}$ given above is an uniformly continuous C-group of bounded linear operators of infinitesimal generator A. □

We introduce the following definition.

Definition 2.7 ([5, Definition 2.16])**.** Let X and Y be two non-archimedean Banach spaces over a non-archimedean valued field $\mathbb{K}$. For all $T \in B(X)$ and $S \in B(Y)$, the operator $T \oplus S$ is defined on the Banach space $X \oplus Y = \{(x, y) : x \in X,\ y \in Y\} = \{x \oplus y : x \in X,\ y \in Y\}$ endowed with the n.a. norm $\|x \oplus y\| = \max(\|x\|, \|y\|)$, by

$$\text{for all } x \oplus y \in X \oplus Y, \quad (T \oplus S)(x \oplus y) = Tx \oplus Sy = (Tx, Sy).$$

Proposition 2.8. *For all invertible operators A and B in $B(X)$, $A \oplus B$ is invertible on $X \oplus X$. Furthermore, its inverse, denoted by $(A \oplus B)^{-1}$, satisfies*

$$(A \oplus B)^{-1} = A^{-1} \oplus B^{-1}.$$

Proof. Let $A, B \in B(X)$ be two invertible operators; then $A^{-1}A = AA^{-1} = I$ and $B^{-1}B = BB^{-1} = I$. Therefore for all $(x \oplus y) \in X \oplus X$, we have

$$\begin{aligned}\big(A^{-1} \oplus B^{-1}\big)(A \oplus B)(x \oplus y) &= \big(A^{-1} \oplus B^{-1}\big)(Ax \oplus By)\\ &= \big(A^{-1}Ax\big) \oplus \big(B^{-1}By\big)\\ &= x \oplus y\end{aligned}$$

and

$$\begin{aligned}(A\oplus B)\left(A^{-1}\oplus B^{-1}\right)(x\oplus y) &= (A\oplus B)\left(A^{-1}x\oplus B^{-1}y\right)\\ &= \left(AA^{-1}x\oplus BB^{-1}x\right)\\ &= x\oplus y.\end{aligned}$$

Thus, for all $x\oplus y\in X\oplus X$,

$$(A\oplus B)\left(A^{-1}\oplus B^{-1}\right)(x\oplus y) = \left(A^{-1}\oplus B^{-1}\right)(A\oplus B)(x\oplus y) = x\oplus y.$$

Then, $A\oplus B$ is invertible on $X\oplus X$ and its inverse is $(A\oplus B)^{-1} = A^{-1}\oplus B^{-1}$. □

Example 2.9. Let $A, B\in B(X)$ be such that $\max(\|A\|,\|B\|) < r\left(= p^{\frac{-1}{p-1}}\right)$. Set, for all $t\in\Omega_r$,

$$T(t) = e^{tA}\oplus e^{tB}.$$

It is easy to see that for all $t\in\Omega_r$, $T(t)$ is invertible and, for all $t\in\Omega_r$, $T(t)^{-1} = T(-t)$.

We begin with the following theorem.

Theorem 2.10. *Let $(T(t))_{t\in\Omega_r}$ be a C-group of infinitesimal generator A on X. Set, for all $t\in\Omega_r$, $S(t) = T(t)\oplus I$. Then we have:*

(i) *$(S(t))_{t\in\Omega_r}$ is a $C\oplus I$-group on $X\oplus X$.*
(ii) *The infinitesimal generator of $(S(t))_{t\in\Omega_r}$ is the operator T defined on $D(T) = D(A)\oplus X$ by $T(x\oplus y) = Ax\oplus 0$, for all $x\in D(A)$, $y\in X$.*

Proof. (i) Since $(T(t))_{t\in\Omega_r}$ is a C-group of infinitesimal generator A on X,

$$S(0) = T(0)\oplus I = C\oplus I.$$

Let $x\oplus y\in X\oplus X$ and $t, s\in\Omega_r$. We have

$$\begin{aligned}(C\oplus I)S(t+s)(x\oplus y) &= (C\oplus I)T(t+s)(x)\oplus y\\ &= CT(t+s)(x)\oplus y\\ &= T(t)T(s)(x)\oplus y\\ &= (T(t)\oplus I)(T(s)(x)\oplus y)\\ &= S(t)\left((T(s)\oplus I)(x\oplus y)\right)\\ &= S(t)S(s)(x\oplus y).\end{aligned}$$

On the other hand,

$$\begin{aligned}\lim_{t\to 0}\|S(t)(x\oplus y)-(C\oplus I)(x\oplus y)\| &= \lim_{t\to 0}\|(T(t)x-Cx)\oplus 0\|\\ &= \lim_{t\to 0}\max(\|T(t)x-Cx\|, 0)\\ &= \lim_{t\to 0}\|T(t)x-Cx\|\\ &= 0.\end{aligned}$$

Therefore $(S(t))_{t\in\Omega_r}$ is a $C\oplus I$-group on $X\oplus X$.

(ii) Let $x \in D(A)$ and $y \in X$. We have

$$\begin{aligned}\lim_{t\to 0}\frac{S(t)(x\oplus y)-(C\oplus I)\,(x\oplus y)}{t} &= \lim_{t\to 0}\frac{(T\,(t)\,(x)-Cx)\oplus 0}{t}\\ &= CAx\oplus 0=(C\oplus I)(Ax\oplus 0).\end{aligned}$$

Thus, for all $x \in D(A)$ and $y \in X$ we have

$$(C\oplus I)^{-1}\left(\lim_{t\to 0}\frac{S(t)(x\oplus y)-(C\oplus I)(x\oplus y)}{t}\right)=Ax\oplus 0.$$

Then $D(T) = D(A) \oplus X$ and $T(x \oplus y) = A(x) \oplus 0$, for all $x \in D(A)$ and for all $y \in X$. □

Theorem 2.11. *Let $(A(t))_{t\in\Omega_r}$ and $(B(t))_{t\in\Omega_r}$ be, respectively, a C_1-group on X of infinitesimal generator A and a C_2-group on X of infinitesimal generator B. We set, for all $t \in \Omega_r$, $T(t) = A(t) \oplus B(t)$. Then*

(i) *$(T(t))_{t\in\Omega_r}$ is a $C_1 \oplus C_2$-group on $X \oplus X$.*
(ii) *The infinitesimal generator of $(T(t))_{t\in\Omega_r}$ is the operator T defined on $D(T) = D(A) \oplus D(B)$ by $T(x \oplus y) = Ax \oplus By$ for all $(x, y) \in X^2$.*

Proof. (i) Let $x \oplus y \in X \oplus X$. Since $(A(t))_{t\in\Omega_r}$ and $(B(t))_{t\in\Omega_r}$ are, respectively, a C_1-group and a C_2-group on X, we have

$$T(0)(x\oplus y)=A(0)x\oplus B(0)y=C_1x\oplus C_2y=(C_1\oplus C_2)(x\oplus y).$$

Hence $T(0) = C_1 \oplus C_2$. We also have, for all $(t, s) \in \Omega_r^2$,

$$\begin{aligned}(C_1\oplus C_2)T(t+s)(x\oplus y) &= (C_1\oplus C_2)(A(t+s)x\oplus B(t+s)y)\\ &= C_1A(t+s)x\oplus C_2B(t+s)y\\ &= A(t)A(s)x\oplus B(t)B(s)y\\ &= (A(t)\oplus B(t))(A(s)x\oplus B(s)y)\\ &= T(t)(A(s)\oplus B(s)(x\oplus y))\\ &= T(t)T(s)(x\oplus y).\end{aligned}$$

Then, $(C_1 \oplus C_2)T(t + s) = T(t)T(s)$. On the other hand,

$$\begin{aligned}\lim_{t\to 0}\|T(t)(x\oplus y)-(C_1\oplus C_2)\,(x\oplus y)\,\| &= \lim_{t\to 0}\|A(t)x\oplus B(t)y-C_1x\oplus C_2y\|\\ &= \lim_{t\to 0}\|(A(t)x-C_1x)\oplus(B(t)y-C_2y)\|\\ &= \lim_{t\to 0}\max(\|A(t)x-C_1x\|,\|B(t)y-C_2y\|)\\ &= 0.\end{aligned}$$

Therefore, $(T(t))_{t\in\Omega_r}$ is a $C_1 \oplus C_2$-group on $X \oplus X$.

(ii) Let $x \in D(A)$ and $y \in D(B)$. We have

$$\begin{aligned}\lim_{t\to 0}\frac{T(t)(x\oplus y)-(C_1\oplus C_2)(x\oplus y)}{t} &= \lim_{t\to 0}\frac{(A(t)x-C_1x)\oplus(B(t)y-C_2y)}{t}\\ &= C_1Ax\oplus C_2By\\ &= (C_1\oplus C_2)(Ax\oplus By).\end{aligned}$$

Thus, for all $x \in D(A)$, $y \in D(B)$ we have

$$(C_1 \oplus C_2)^{-1}\left(\lim_{t\to 0}\frac{T(t)(x\oplus y)-(C_1\oplus C_2)(x\oplus y)}{t}\right) = Ax \oplus By.$$

Consequently, $D(T) = D(A) \oplus D(B)$ and $T(x \oplus y) = Ax \oplus By$. □

3. Mixed C-groups of bounded linear operators on some non-archimedean Banach spaces

We introduce the following definition.

Definition 3.1. Let $r > 0$ be a real number and let $C \in B(X)$ be invertible. A family $(T(t))_{t\in\Omega_r}$ of bounded linear operators is said to satisfy a *p-adic H-C-generalized Cauchy equation of bounded linear operator on* X if

$$\text{for all } t, s \in \Omega_r, \quad CT(s+t) = H(T(s), T(t)),$$

where $H\colon B(X) \times B(X) \to B(X)$ is a function.

Remark 3.2. If $H\big(T(s), T(t)\big) = T(s)T(t)$ with $T(0) = C$, then $(T(t))_{t\in\Omega_r}$ satisfies the first condition of C-groups of bounded linear operators on X.

Definition 3.3. Let $r > 0$ be a real number and let $C \in B(X)$ be invertible. A family $(S(t))_{t\in\Omega_r}$ of bounded linear operators will be called an *H-C-group* or a *generalized C-group* of bounded linear operators on X if

(i) $S(0) = C$;

(ii) there is a C-group $(T(t))_{t\in\Omega_r}$ of bounded linear operators and $D \in B(X)$ such that for all $t, s \in \Omega_r$,

$$\begin{aligned} CS(s+t) &= H(S(s), S(t)) \\ &= S(s)S(t) + D(S(s) - T(s))(S(t) - T(t)); \end{aligned}$$

(iii) for each $x \in X$, $S(\cdot)x\colon \Omega_r \to S(t)x$ is continuous on Ω_r.

The linear operator A defined on

$$D(A) = \left\{x \in X : \lim_{t\to 0}\frac{S(t)x - Cx}{t} \text{ exists}\right\}$$

by

$$Ax = C^{-1}\lim_{t\to 0}\frac{S(t)x - Cx}{t}, \quad \text{for each } x \in D(A),$$

is called the *infinitesimal generator* of the H-C-group $(S(t))_{t\in\Omega_r}$.

Remark 3.4. Let $(S(t))_{t\in\Omega_r}$ be a generalized C-group on X. If $D = 0$, then $(S(t))_{t\in\Omega_r}$ is a C-group of bounded linear operators on X.

Question 3.5. Can you characterise the infinitesimal generator of an H-C-group of linear operators on an infinite dimensional non-archimedean Banach space?

Definition 3.6. Let $r > 0$ be a real number and let $C \in B(X)$ be invertible. A family $(S(t))_{t\in\Omega_r}$ is said to be a *mixed C-group* of bounded linear operators on X if

(i) $S(0) = C$;
(ii) there is a C-group $(T(t))_{t\in\Omega_r}$ of bounded linear operators and $\alpha \in \mathbb{K}$ such that for all $s, t \in \Omega_r$,

$$\begin{aligned} CS(s+t) &= H(S(s), S(t)) \\ &= S(s)S(t) + \alpha(S(s) - T(s))(S(t) - T(t)); \end{aligned}$$

(iii) for each $x \in X$, $S(\cdot)x\colon \Omega_r \to S(t)x$ is continuous on Ω_r.

The linear operator A defined on

$$D(A) = \left\{ x \in X : \lim_{t\to 0} \frac{S(t)x - Cx}{t} \text{ exists} \right\}$$

by

$$Ax = C^{-1} \lim_{t\to 0} \frac{S(t)x - Cx}{t}, \quad \text{for each } x \in D(A),$$

is called the *infinitesimal generator* of the mixed C-group $(S(t))_{t\in\Omega_r}$.

Remark 3.7. Let $(S(t))_{t\in\Omega_r}$ be a mixed C-group on X. If $\alpha = 0$, then $(S(t))_{t\in\Omega_r}$ is a C-group of bounded linear operators on X.

We have the following example.

Example 3.8. Let $r = p^{\frac{-1}{p-1}}$. Suppose that X is a non-archimedean Banach space over $\mathbb{Q}_p$, and $A, C \in B(X)$ are such that C is invertible, $AC = CA$ and $\|A\| < r$. Set, for all $t \in \Omega_r$,

$$S(t) = Ce^{tA} + tACe^{tA}.$$

Then one can see that with $D = -I$, $(S(t))_{t\in\Omega_r}$ is an H-C group where for all $t \in \Omega_r$, $T(t) = Ce^{tA}$. In this case, for all $t, s \in \Omega_r$, $S(s)S(t) = S(t)S(s)$.

In fact, for $D = -I$, we have for all $t, s \in \Omega_r$, $CS(s+t) = C^2e^{(s+t)A} + (s+t)AC^2e^{(s+t)A}$ and

$$\begin{aligned} S(s)S(t) &= \big(Ce^{sA} + sCAe^{sA}\big)\big(Ce^{tA} + tACe^{tA}\big) \\ &= C^2e^{(s+t)A} + tAC^2e^{(s+t)A} + sAC^2e^{(s+t)A} + st(AC)^2e^{(s+t)A} \\ &= C^2e^{(s+t)A} + (s+t)AC^2e^{(s+t)A} + st(AC)^2e^{(s+t)A} \end{aligned}$$

and

$$(S(s) - T(s))(S(t) - T(t)) = st(AC)^2e^{(t+s)A}.$$

Hence,

$$\begin{aligned} S(s)S(t) - (S(s) - T(s))(S(t) - T(t)) &= C^2e^{(s+t)A} + (s+t)AC^2e^{(s+t)A} \\ &= CS(s+t). \end{aligned}$$

Conditions (i) and (iii) of Definition 3.3 are easy to verify, so $(S(t))_{t\in\Omega_r}$ is an H-C-group.

The following proposition gives a condition under which an H-C-group family commute.

Proposition 3.9. *Let $(S(t))_{t\in\Omega_r}$ be an H-C-group family on X. If $I+D$ is injective and for all $t,s\in\Omega_r$, $T(s)S(t)=S(t)T(s)$, then for all $t,s\in\Omega_r$, $S(s)S(t)=S(t)S(s)$.*

Proof. Assume that $I+D$ is injective and for all $t,s\in\Omega_r$, $T(s)S(t)=S(t)T(s)$; then for all $t,s\in\Omega_r$,

$$\begin{aligned} S(t)S(s) + D\big(S(t)-T(t)\big)\big(S(s)-T(s)\big) \\ &= CS(t+s) \\ &= CS(s+t) \\ &= S(s)S(t) + D\big(S(s)-T(s)\big)\big(S(t)-T(t)\big). \end{aligned}$$

Thus, $(I+D)(S(t)S(s)-S(s)S(t))=0$. Then for all $t,s\in\Omega_r$, $S(s)S(t)=S(t)S(s)$. □

Theorem 3.10. *Let $(S(t))_{t\in\Omega_r}$ be an H-C-group family of infinitesimal generator A on X, with $(T(t))_{t\in\Omega_r}$ a C-group of infinitesimal generator A_0, such that there are M_1, $M_2>0$ such that for all $t,s\in\Omega_r$, $\|S(t)\|\le M_1$, $\|T(t)\|\le M_2$, $T(s)S(t)=S(t)T(s)$ and $S(t)S(s)=S(s)S(t)$. If $x\in D(A)$, then for all $t\in\Omega_r$,*

$$S(t)x,\, T(t)x \in D(A) \quad \text{and} \quad AS(t)x = S(t)Ax.$$

Furthermore, for any $x\in D(A_0)$,

$$S(t)x,\, T(t)x\in D(A_0) \quad \text{and} \quad A_0S(t)x = S(t)A_0x, A_0T(t)x = T(t)A_0x.$$

Proof. Let $x\in D(A)$, $s\in\Omega_r^*$ and $t\in\Omega_r$. From the boundedness of the $(S(t))_{t\in\Omega_r}$, it is easy to see that

$$C^{-1}\left(\frac{S(s)S(t)x-CS(t)x}{s}\right) = S(t)C^{-1}\left(\frac{S(s)x-Cx}{s}\right) \to S(t)Ax \quad \text{as } s\to 0.$$

Consequently, for all $t\in\Omega_r$, $S(t)Ax\in D(A)$ and $AS(t)x=S(t)Ax$.

From the boundedness of the $(T(t))_{t\in\Omega_r}$, we have

$$C^{-1}\left(\frac{S(s)T(t)x-CT(t)x}{s}\right) = T(t)C^{-1}\left(\frac{S(s)x-Cx}{s}\right) \to T(t)Ax \quad \text{as } s\to 0.$$

Consequently, for all $t\in\Omega_r$, $T(t)x\in D(A)$ and $AT(t)x=T(t)Ax$.

The last part can be proved similarly. □

Set $A_1=(1+\alpha)A-\alpha A_0$, where $\alpha\in\mathbb{K}\backslash\{-1\}$, A_0 is the infinitesimal generator of the C-group $(T(t))_{t\in\Omega_r}$ and A is the infinitesimal generator of a mixed C-group $(S(t))_{t\in\Omega_r}$. We have the following theorem.

Theorem 3.11. *Let X be a non-archimedean Banach space over $\mathbb{K}$. Let $(S(t))_{t\in\Omega_r}$ be a mixed C-group family of bounded linear operators on X with $\alpha\in\mathbb{K}\backslash\{-1\}$. Set, for all $t\in\Omega_r$, $T_1(t)=(1+\alpha)S(t)-\alpha T(t)$. Then $(T_1(t))_{t\in\Omega_r}$ is a C-group of bounded linear operators whose infinitesimal generator is an extension of A_1. Furthermore, for all $x\in X$ and $t\in\Omega_r$,*

$$S(t)x = \frac{1}{1+\alpha}T_1(t)x + \frac{\alpha}{1+\alpha}T(t)x.$$

Proof. Let us see the conditions in Definition 1.9:

(i) Trivially, $T_1(0) = (1+\alpha)S(0) - \alpha T(0) = C$.

(ii) For all $t, s \in \Omega_r$ and $x \in X$, we have

$$CT_1(s+t)x = (1+\alpha)CS(s+t) - \alpha CT(s+t)x,$$

and

$$\begin{aligned}
CT_1(s+t)x &= (1+\alpha)(S(s)S(t) + \alpha(S(s)-T(s))(S(t)-T(t)))x - \alpha T(s)T(t)x \\
&= (1+\alpha)S(s)S(t)x + \alpha(1+\alpha)S(s)S(t)x - \alpha(1+\alpha)S(s)T(t)x \\
&\quad - \alpha(1+\alpha)T(s)S(t)x + \alpha(1+\alpha)T(s)T(t)x - \alpha T(s)T(t)x \\
&= (1+\alpha)^2 S(s)S(t)x - \alpha(1+\alpha)S(s)T(t)x \\
&\quad - \alpha(1+\alpha)T(s)S(t)x + \alpha^2 T(s)T(t)x \\
&= ((1+\alpha)S(s) - \alpha T(s))((1+\alpha)S(t) - \alpha T(t))x \\
&= T_1(s)T_1(t)x.
\end{aligned}$$

(iii) Since $(T(t))_{t\in\Omega_r}$ and $(S(t))_{t\in\Omega_r}$ are continuous on Ω_r, $(T_1(t))_{t\in\Omega_r}$ is also continuous on Ω_r. So, $(T_1(t))_{t\in\Omega_r}$ is a C-group of bounded linear operators on X.

Now we show that an extension of $A_1 = (1+\alpha)A - \alpha A_0$, where $\alpha \in \mathbb{K}\backslash\{-1\}$, is the infinitesimal generator of $(T_1(t))_{t\in\Omega_r}$. Let B be the infinitesimal generator of $(T_1(t))_{t\in\Omega_r}$. By definition of $D(A)$ and $D(A_0)$, for $x \in D(A_1)$ $(= D(A) \cap D(A_0))$ we have

$$\lim_{t\to 0}\left(\frac{S(t)x - Cx}{t}\right) = CAx \quad \text{and} \quad \lim_{t\to 0}\left(\frac{T(t)x - Cx}{t}\right) = CA_0x.$$

Then,

$$\begin{aligned}
\lim_{t\to 0}\left(\frac{T_1(t)x - Cx}{t}\right) &= \lim_{t\to 0}\left(\frac{(1+\alpha)S(t)x - \alpha T(t)x - Cx}{t}\right) \\
&= (1+\alpha)\lim_{t\to 0}\left(\frac{S(t)x - Cx}{t}\right) - \alpha \lim_{t\to 0}\left(\frac{T(t)x - Cx}{t}\right)
\end{aligned}$$

exists in X. It follows that $x \in D(B)$ and $A_1x = Bx$, hence the infinitesimal generator of $(T_1(t))_{t\in\Omega_r}$ is an extension of A_1. ☐

For $\alpha \in \mathbb{K}\backslash\{-1\}$ and $D = \alpha I$, from Proposition 3.9 and Theorem 3.10 we conclude:

Proposition 3.12. *Let X be a non-archimedean Banach space over $\mathbb{K}$ and let $(S(t))_{t\in\Omega_r}$ be a mixed C-group family of bounded linear operators on X with $\alpha \in \mathbb{K}\backslash\{-1\}$ such that for all $t, s \in \Omega_r$, $T(s)S(t) = S(t)T(s)$. Then for all $t, s \in \Omega_r$, $S(s)S(t) = S(t)S(s)$.*

Theorem 3.13. *Let X be a non-archimedean Banach space over $\mathbb{K}$. Let $(S(t))_{t\in\Omega_r}$ be a mixed C-group family of infinitesimal generator A on X, with $(T(t))_{t\in\Omega_r}$ a C-group of infinitesimal generator A_0, and $\alpha \in \mathbb{K}\backslash\{-1\}$ such that there are M_1, $M_2 > 0$ such that for all $t, s \in \Omega_r$, $\|S(t)\| \le M_1$, $\|T(t)\| \le M_2$, $T(s)S(t) = S(t)T(s)$ and $S(s)S(t) = S(t)S(s)$. If $x \in D(A)$, then for all $t \in \Omega_r$, $S(t)x, T(t)x \in D(A)$ and*

$AS(t)x = S(t)Ax$. Furthermore, $S(t)x, T(t)x \in D(A_0)$ and $A_0S(t)x = S(t)A_0x$, $A_0T(t)x = T(t)A_0x$ for any $x \in D(A_0)$.

For $\alpha = -1$, we have the following theorem.

Theorem 3.14. *Let X be a non-archimedean Banach space over $\mathbb{K}$. Let $(S(t))_{t\in\Omega_r}$ be a mixed C-group family of infinitesimal generator A on X, such that there are M_1, $M_2 > 0$ such that for all $t, s \in \Omega_r$, $\|S(t)\| \le M_1$, $\|T(t)\| \le M_2$, $T(s)S(t) = S(t)T(s)$ and $S(s)S(t) = S(t)S(s)$, where $(T(t))_{t\in\Omega_r}$ is a C-group of infinitesimal generator A_0. Then, for all $x \in D(A) \cap D(A_0)$,*

$$\frac{dCS(t)}{dt}x = C(A_0S(t)x + (A - A_0)T(t)x).$$

Proof. Let $x \in D(A) \cap D(A_0)$. Using Theorem 3.10, we have

$$\begin{aligned}
\frac{d}{dt}S(t)Cx &= \lim_{h\to 0}\frac{CS(h+t)x - S(t)Cx}{h}\\
&= \lim_{h\to 0}\frac{S(h)S(t)x + (T(h) - S(h))(S(t) - T(t))x - S(t)Cx}{h}\\
&= \lim_{h\to 0}\frac{T(h)S(t)x - CS(t)x}{h} + \lim_{h\to 0}\frac{S(h)T(t)x - CT(t)x}{h}\\
&\quad - \lim_{h\to 0}\frac{T(h)T(t)x - CT(t)x}{h}\\
&= C\left(A_0S(t)x + AT(t)x - A_0T(t)x\right).
\end{aligned}$$

□

Proposition 3.15. *Let X be a finite dimensional Banach space over $\mathbb{K}$. Let $(T(t))_{t\in\Omega_r}$ be a C-group of infinitesimal generator A_0 on X such that for all $t \in \Omega_r$, $A_0T(t) = T(t)A_0$. Let $A \in B(X)$ be such that for all $t \in \Omega_r$, $AT(t) = T(t)A$. Then, for all $t \in \Omega_r$, $S(t) = T(t) + t(A - A_0)T(t)$ is a mixed C-group of infinitesimal generator A on X with $\alpha = -1$.*

Proof. Since $(T(t))_{t\in\Omega_r}$ is a C-group on X, we have $T(0) = C$, hence $S(0) = T(0) = C$. By assumption, for all $t \in \Omega_r$, $AT(t) = T(t)A$ and $A_0T(t) = T(t)A_0$; then $CA = AC$ and $CA_0 = A_0C$. Let $s, t \in \Omega_r$. We have

$$\begin{aligned}
CS(s+t) &= CT(s+t) + (s+t)(A - A_0)CT(s+t)\\
&= T(s)T(t) + sAT(s)T(t) - sA_0T(s)T(t) + tAT(s)T(t) - tA_0T(s)T(t),
\end{aligned}$$

and

$$\begin{aligned}
S(s)S(t) &- (S(s) - T(s))(S(t) - T(t))\\
&= (T(s) + s(A - A_0)T(s))(T(t) + t(A - A_0)T(t))\\
&\quad - st(A - A_0)T(s)(A - A_0)T(t)\\
&= T(s)T(t) + tT(s)AT(t) - tT(s)A_0T(t)\\
&\quad + sAT(s)T(t) - sA_0T(s)T(t)\\
&\quad + st(A - A_0)T(s)(A - A_0)T(t)\\
&\quad - st(A - A_0)(A - A_0)T(s)T(t)\\
&= CS(s+t).
\end{aligned}$$

Also we have that, for all $x \in X$, $t \in \Omega_r \mapsto T(t)x$ is continuous, and then $t \in \Omega_r \mapsto S(t)x$ is continuous. Consequently, $(S(t))_{t\in\Omega_r}$ is a mixed C-group of bounded linear operators on X with $\alpha = -1$. It is easy to see that A is the infinitesimal generator of $(S(t))_{t\in\Omega_r}$. □

Example 3.16. Assume that $\mathbb{K} = \mathbb{Q}_p$. Let $A, A_0, C \in B(X)$ be such that $\|A_0\| < p^{\frac{1}{1-p}}$, $CA = AC$ and $CA_0 = A_0C$. We consider the family of bounded linear operators on X defined by

$$\text{for all } t \in \Omega_r, \quad S(t) = Ce^{tA_0} + t(A - A_0)Ce^{tA_0}.$$

It is easy to see that for $\alpha = -1$, $(S(t))_{t\in\Omega_r}$ is a mixed C-group of bounded linear operators of infinitesimal generator A on X.

In the following theorem it will be proved that multiplication of an H-C-group and a C-group is an H-C-group if these two families commute.

Theorem 3.17. *Let X be a non-archimedean Banach space over $\mathbb{K}$. Let $(B(t))_{t\in\Omega_r}$ be a commuting strongly continuous H-C-group of infinitesimal generator B on X with the C-group $(T(t))_{t\in\Omega_r}$ of infinitesimal generator A_0 and $D \in B(X)$. Let $(A(t))_{t\in\Omega_r}$ be a C-group of infinitesimal generator A such that for all $t, s \in \Omega_r$, $A(t)D = DA(t)$ and $A(s)B(t) = B(t)A(s)$. Then, for all $t \in \Omega_r$, $V(t) = A(t)B(t)$ is an H-C^2-group of infinitesimal generator $A + B$.*

Proof. Trivially, $V(0) = C^2$. Also for any $s, t \in \Omega_r$,

$$\begin{aligned} C^2V(s+t) &= CA(s+t)CB(s+t) \\ &= A(s)A(t)(B(s)B(t) + D(B(s) - T(s))(B(t) - T(t))) \\ &= V(s)V(t) + D(V(s) - A(s)T(s))(V(t) - A(t)T(t)). \end{aligned}$$

Thus $(V(t))_{t\in\Omega_r}$ is an H-C^2-group which is obviously strongly continuous. Also for any $x \in D(A) \cap D(B)$,

$$\begin{aligned} \lim_{t\to 0} \frac{V(t)x - C^2x}{t} &= \lim_{t\to 0} \frac{A(t)B(t)x - CB(t)x}{t} + \lim_{t\to 0} \frac{CB(t)x - C^2x}{t} \\ &= C^2Ax + C^2Bx. \end{aligned}$$

Then,

$$C^{-2} \lim_{t\to 0} \frac{V(t)x - C^2x}{t} = (A + B)x.$$ □

Remark 3.18. Let X be a non-archimedean Banach space over $\mathbb{K}$. Theorem 3.14 shows that for $\alpha = -1$, if $(S(t))_{t\in\Omega_r}$ is a mixed C-group of infinitesimal generator A, with $(T(t))_{t\in\Omega_r}$ a C-group of infinitesimal generator A_0 such that for all $t, s \in \Omega_r$, $T(s)S(t) = S(t)T(s)$ and $S(s)S(t) = S(t)S(s)$, then $u(t) = S(t)x$ is a solution of the inhomogeneous p-adic differential equation given by

$$\frac{du(t)}{dt} = A_0u(t) + (A - A_0)f(t), \quad t \in \Omega_r,$$

and $u(0) = Cx$, $x \in D(A_0) \cap D(A)$ with $f(t) = T(t)x$.

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