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ON CONFORMAL GEOMETRY OF FOUR-DIMENSIONAL GENERALIZED SYMMETRIC SPACES

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Abstract. We study conformal geometry on an important class of four-dimensional (pseudo-)Riemannian manifolds: generalized symmetric spaces. This leads to the general description of conformally Einstein metrics on the spaces under consideration. Finally, the class of oscillator Lie groups is studied for the conformally Einstein property.

1. Introduction

Most geometric objects that exist on a manifold are directly associated with the Riemann curvature tensor, making the study of manifolds with specific curvature tensor properties of great interest to geometers. In certain aspects, the spaces with the simplest form of the curvature tensor are homogeneous manifolds. Due to the straightforward structure of the Riemann curvature tensor, any property valid at one point of a homogeneous manifold can easily be extended to any other point. In other words, studying a homogeneous manifold is akin to studying a single point, allowing for algebraic interpretations of different objects on the manifold. Although homogeneous manifolds have been previously examined in the context of Riemannian manifolds (e.g., [25]), recent research has primarily focused on pseudo-Riemannian geometry. Three-dimensional homogeneous Lorentzian manifolds were studied in [3], while the four-dimensional case due to the signature of the invariant metric was considered in [8], [15]. Special homogeneous pseudo-Riemannian manifolds were also studied in several cases. Three-dimensional Lorentzian manifolds with recurrent curvature were studied in [19] and [10]. Non-reductive homogeneous manifolds of dimension four were studied in [17], [9] and [7]. The oscillator group of dimension four was also considered in several works, e.g. [11] and [4]. One of the most interesting classes of homogeneous pseudo-Riemannian manifolds with several geometric traits are generalized symmetric spaces. These spaces were studied from different points of view after their classification in [14].

Conformal geometry is an interesting topic in many aspects. One says (M, g) and $(M, \tilde{g})$ are *conformally equivalent* if there exists a conformal transformation

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between them. In other words, a locally defined smooth function φ (which is angle preserving) exists such that $\tilde{g} = \varphi^{-2}g$. In this case, the set of metrics $\tilde{g}$ is called the conformal class of g. Different geometric properties of a space could be also studied through its conformal class. For example, a space which is conformally equivalent to a flat (resp., symmetric, Einstein) space is called *conformally flat* (resp., *conformally symmetric, conformally Einstein*). It is well known that the Weyl conformal tensor W is preserved under a conformal transformation but neither the connection nor the curvature tensor will remain invariant. Since the Ricci tensor is not preserved by conformal transformations, it is natural to study the situation of being conformally Einstein. Brinkmann studied this problem and obtained the following necessary and sufficient condition:

$$(n-2)\operatorname{Hes}_\varphi + \varphi\varrho = \frac{1}{n}\{(n-2)\Delta\varphi + \varphi r\}g, \tag{1.1}$$

where $\operatorname{Hes}_\varphi = \nabla d\varphi$ is the Hessian of φ, and ϱ and r respectively denote the Ricci tensor and the scalar curvature of g [1]. Conformal transformations, which map an Einstein manifold to another one were also studied later in [2]. Conformal transformations are trivial in dimension two and equivalent to conformal flatness in dimension three, thus dimension four is the first non-trivial case. Conformally Einstein non-reductive homogeneous manifolds of dimension four were studied in [13]. Conformally Einstein semi-direct extensions of the Heisenberg groups were studied in [12]. The conformal geometry of a special class of four-dimensional product surfaces with non-zero scalar curvature was studied in [22] and [23]. Although Gover and Nurowski presented in [20] some tensorial obstructions as well as non-degeneracy conditions of the Weyl tensor of a metric to be conformally Einstein, it is still an interesting problem to study this property on different spaces by tensorial equations.

This paper is organized in the following way. In Section 2, some basic preliminaries about the subject are given to make the context self-contained and more readable. Section 3 is devoted to the study of the Bach tensor on the spaces under consideration. In Section 4, we completely classify conformally Einstein generalized symmetric spaces of dimension four. Extensions of three-dimensional generalized symmetric spaces to dimension four are contained in Section 5; finally, the conformal geometry of oscillator Lie groups is studied in Section 6.

2. Preliminaries

In order to keep the paper self-contained, we recall some basic material which will be used throughout.

2.1. Generalized symmetric spaces of dimension four.

Let x be an arbitrary point on the connected pseudo-Riemannian manifold (M, g). A *symmetry* at x is defined as an isometry s_x of M which keeps x as an isolated fixed point. On a symmetric space (M, g), each point x admits a symmetry s_x reversing geodesics passing through the point. A. J. Ledger generalized this property and defined a *regular s-structure* as a family $\{s_x : x \in M\}$ of symmetries of (M, g) satisfying

$$s_x \circ s_y = s_z \circ s_x, \qquad z = s_x(y),$$

for all points x, y of M. The smallest integer $k \geq 2$ such that $(s_x)^k = \mathrm{id}_M$ for all $x \in M$ is called the *order* of an s-structure (k may be infinity).

Let (M, g) be a connected pseudo-Riemannian manifold. We call (M, g) a *generalized symmetric space* if it admits a regular s-structure. Since s-structures on (M, g) are not unique, the order of a generalized symmetric space is defined as the infimum of all integers $k \geq 2$ such that M admits a regular s-structure of order k.

Generalized symmetric spaces of dimension four were studied by J. Černý and O. Kowalski, and they classified these spaces both algebraically and in explicit coordinates, as summarized in the following result.

Theorem 2.1 ([14])**.** *Let (M, g) be a proper, simply connected generalized symmetric space of dimension $n = 4$; in this case, (M, g) is of order 3 or infinity. All these spaces are indecomposable and belong (up to an isometry) to one of the following four types:*

Type A. *The underlying homogeneous space is G/H, where*

$$G = \begin{pmatrix} a & b & u \\ c & d & v \\ 0 & 0 & 1 \end{pmatrix}, \qquad H = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $ad - bc = 1$. In local coordinates, (M, g) is the space $R^4(x, y, u, v)$ with the pseudo-Riemannian metric

$$\begin{aligned} g = \pm\big[(-x + \sqrt{1 + x^2 + y^2})du^2 + (x + \sqrt{1 + x^2 + y^2})dv^2 - 2y^2 dudv\big] \\ + \lambda\big[(1 + y^2)dx^2 + (1 + x^2)dy^2 - 2xydxdy\big]/(1 + x^2 + y^2), \end{aligned}$$

where $\lambda \neq 0$ is a real constant. The order is $k = 3$ and the possible signatures are $(4, 0), (0, 4), (2, 2)$. The typical symmetry of order 3 at the initial point $(0, 0, 0, 0)$ is the transformation

$$\begin{aligned} u' &= -(1/2)u - (\sqrt{3}/2)v, \qquad & v' &= -(\sqrt{3}/2)u - (1/2)v, \\ x' &= -(1/2)x + (\sqrt{3}/2)y, \qquad & y' &= -(\sqrt{3}/2)x - (1/2). \end{aligned}$$

Type B. *The underlying homogeneous space is G/H, where*

$$G = \begin{pmatrix} e^{-(x+y)} & 0 & 0 & a \\ 0 & e^x & 0 & b \\ 0 & 0 & e^y & c \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 & 0 & -w \\ 0 & 1 & 0 & -2w \\ 0 & 0 & 1 & 2w \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In local coordinates (M, g) is the space $R^4(x, y, u, v)$ with the pseudo-Riemannian metric

$$g = \lambda(dx^2 + dy^2 + dxdy) + e^{-y}(2dx + dy)dv + e^{-x}(dx + 2dy)du,$$

where λ is a real constant. The order is $k = 3$ and the signature is $(2, 2)$. The typical symmetry of order 3 at the initial point $(0, 0, 0, 0)$ is the transformation

$$u' = -ue^{(y-x)} - v, \quad v' = ue^{-(y+2x)}, \quad x' = y, \quad y' = -(x + y).$$

Type C. *The underlying homogeneous space is the matrix group*

$$G = \begin{pmatrix} e^{-t} & 0 & 0 & x \\ 0 & e^t & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In local coordinates, (M, g) is the space $R^4(x, y, z, t)$ with the pseudo-Riemannian metric

$$g = \pm\big(e^{2t}dx^2 + e^{-2t}dy^2\big) + dzdt.$$

The possible signatures are $(1, 3), (3, 1)$. Following [16], these spaces of type C are indeed symmetric. So, since generalized *symmetric (non-symmetric) spaces are the subject of our study, we eliminate this class in the forthcoming sections.*

Type D. *The underlying homogeneous space is G/H, where*

$$G = \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix}, \qquad H = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $ad - bc = 1$. (M, g) is the space $R^4(x, y, u, v)$ with the pseudo-Riemannian metric

$$\begin{aligned} g = {} & (\sinh(2u) - \cosh(2u)\sin(2v))dx^2 + (\sinh(2u) + \cosh(2u)\sin(2v))dy^2 \\ & - 2\cosh(2u)\cos(2v)dxdy + \lambda(du^2 - \cosh^2(2u)dv^2), \end{aligned}$$

where $\lambda \neq 0$ is a real constant. The order is infinite and the signature is $(2, 2)$. The typical symmetry at the initial point $(0, 0, 0, 0)$ is induced by the automorphism of G of the form

$$a' = a, \quad b' = (1/\alpha^2)b, \quad c' = \alpha^2 c, \quad d' = d, \quad x' = (1/\alpha)x, \quad y' = \alpha y,$$

where $\alpha \neq 0, \pm 1$.

Any generalized symmetric pseudo-Riemannian space is homogeneous. Moreover, it admits at least one structure of reductive homogeneous space with an invariant metric [14]. With regard to the four-dimensional examples, such a reductive decomposition corresponds to their realizations as coset spaces G/H listed in Theorem 2.1 above. Generalized symmetric spaces of dimension four have been the subject of several studies. For example, geometric structures and Ricci solitons were considered in [6] and [5], respectively.

2.2. **Conformally Einstein manifolds.** The study of conformally Einstein manifolds in pseudo-Riemannian spaces is an interesting topic from the perspectives of both geometry and mathematical physics. By applying local coordinates, the conformally Einstein equation (1.1) gives rise to a system of PDEs which is overdetermined and, obviously, very hard to handle in general. The equation is trivial in dimension two and equivalent to conformal flatness for three-dimensional manifolds, so the first remarkable solutions may occur starting from dimension four.

Since any conformally Einstein manifold is Bach flat, it is natural to consider the Bach tensor first. If we denote the Weyl conformal tensor by W and the Ricci tensor by ϱ, the *Bach tensor* on (M^n, g) is given by

$$\mathcal{B} = \operatorname{div}_1 \operatorname{div}_4 W + \frac{n-3}{n-2} W[\varrho],$$

where, with respect to a pseudo-orthonormal basis $\{e_i\}$, with $\varepsilon_i = g(e_i, e_i)$, the tensor $W[\varrho]$ is the Ricci contraction of W defined by

$$W[\varrho](X,Y) = \sum_{i,j} \varepsilon_i \varepsilon_j W(e_i, X, Y, e_j)\varrho(e_i, e_j).$$

One can determine the components of the Bach tensor by using the *Cotton tensor*. The $(0,2)$-tensor field

$$\mathcal{S} = \varrho - \frac{r}{2(n-1)} g \tag{2.1}$$

is a symmetric tensor called the *Schouten tensor*, where r is the scalar curvature. The condition for the Schouten tensor to be a Codazzi tensor (i.e., for its covariant derivative to be totally symmetric) is considered using the Cotton tensor which is defined by the following components:

$$\mathcal{C}_{ijk} = (\nabla_i \mathcal{S})_{jk} - (\nabla_j \mathcal{S})_{ik}. \tag{2.2}$$

Now, the Bach tensor can be computed by the following components:

$$\mathcal{B}_{ij} = \frac{1}{n-2}\left\{ \sum_{k,l=1}^{n} g^{kl}(\nabla_l \mathcal{C}) + \sum_{k,l=1}^{n} \left(\varrho_{kl} \sum_{s,t=1}^{n} g^{ks} g^{lt} W_{isjt} \right) \right\}, \tag{2.3}$$

where W_{isjt} are the components of the Weyl conformal tensor calculated by the identity

$$\begin{aligned} W_{ijhk} = R_{ijhk} &- \frac{1}{n-2}(g_{ih}\varrho_{jk} - g_{jh}\varrho_{ik} - g_{ik}\varrho_{jh} + g_{jk}\varrho_{ih}) \\ &+ \frac{r}{(n-1)(n-2)}(g_{ih}g_{jk} - g_{jh}g_{ik}). \end{aligned}$$

Since any conformally Einstein manifold is necessarily Bach flat, for Bach-flat manifolds of dimension four the conformally Einstein equation

$$2 \operatorname{Hes}_\varphi + \varphi\varrho = \frac{1}{4}\{2\Delta\varphi + \varphi r\} g \tag{2.4}$$

must be carried out to determine conformally Einstein manifolds.

3. Study of the Bach tensor

We study four classes of generalized symmetric spaces of dimension four separately. For simplicity, in computations of the Bach tensor we apply the algebraic motivation and the results are summarized in the following theorem.

Theorem 3.1. *Let (M, g) be a generalized symmetric space of dimension four. (M, g) is Bach flat if and only if it is of type B.*

Proof. The proof is based on a case-by-case study of generalized symmetric spaces of dimension four, according to the classification which appeared in [14].

Type A: Pseudo-Riemannian case. Let $(M = G/H, g)$ be a four-dimensional generalized symmetric space of type $\mathbf{A}$, where g is an invariant metric of neutral signature $(2, 2)$. We show that (M, g) is never Bach flat. Following [14], the Lie algebra $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ admits a basis $\{u_1, u_2, u_3, u_4, h_1\}$, where $\{u_1, u_2, u_3, u_4\}$ and $\{h_1\}$ are bases of $\mathfrak{m}$ and $\mathfrak{h}$, respectively, such that (reversing the metric when needed [24]) the Lie bracket on $\mathfrak{g}$ and the scalar product on $\mathfrak{m}$ are completely determined by

$[\,,\,]$	u_1	u_2	u_3	u_4	h_1
u_1	0	0	$-\delta u_1$	δu_2	u_2
u_2	0	0	δu_2	δu_1	$-u_1$
u_3	δu_1	$-\delta u_2$	0	$-2\delta^2 h_1$	$-2u_4$
u_4	$-\delta u_2$	$-\delta u_1$	$2\delta^2 h_1$	0	$2u_3$
h_1	$-u_2$	u_1	$2u_4$	$-2u_3$	0

where $\delta > 0$ is a real constant, and

$$g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

By setting $\Lambda[i] = \Lambda_{u_i}$, a direct calculation yields that we can describe the Levi-Civita connection as follows:

$$\Lambda[1] = \begin{pmatrix} 0 & 0 & -\delta & 0 \\ 0 & 0 & 0 & \delta \\ -\frac{\delta}{2} & 0 & 0 & 0 \\ 0 & \frac{\delta}{2} & 0 & 0 \end{pmatrix}, \qquad \Lambda[2] = \begin{pmatrix} 0 & 0 & 0 & \delta \\ 0 & 0 & \delta & 0 \\ 0 & \frac{\delta}{2} & 0 & 0 \\ \frac{\delta}{2} & 0 & 0 & 0 \end{pmatrix},$$

and $\Lambda[3] = \Lambda[4] = 0$. Applying the equation $R_{ij} := R(u_i, u_j) = [\Lambda_{u_i}, \Lambda u_j] - \Lambda_{[u_i,u_j]}$, one can calculate the non-zero components of the curvature tensor as follows:

$$R_{12} = \begin{pmatrix} 0 & -\delta^2 & 0 & 0 \\ \delta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta^2 \\ 0 & 0 & \delta^2 & 0 \end{pmatrix}, \qquad R_{13} = \begin{pmatrix} 0 & 0 & -\delta^2 & 0 \\ 0 & 0 & 0 & \delta^2 \\ -\frac{\delta^2}{2} & 0 & 0 & 0 \\ 0 & \frac{\delta^2}{2} & 0 & 0 \end{pmatrix},$$

$$R_{14} = R_{23} = \begin{pmatrix} 0 & 0 & 0 & -\delta^2 \\ 0 & 0 & -\delta^2 & 0 \\ 0 & -\frac{\delta^2}{2} & 0 & 0 \\ -\frac{\delta^2}{2} & 0 & 0 & 0 \end{pmatrix}, \quad R_{24} = \begin{pmatrix} 0 & 0 & \delta^2 & 0 \\ 0 & 0 & 0 & -\delta^2 \\ \frac{\delta^2}{2} & 0 & 0 & 0 \\ 0 & -\frac{\delta^2}{2} & 0 & 0 \end{pmatrix},$$

$$R_{34} = \begin{pmatrix} 0 & 2\delta^2 & 0 & 0 \\ -\delta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4\delta^2 \\ 0 & 0 & 4\delta^2 & 0 \end{pmatrix}.$$

By contraction on the first and third indices of the curvature tensor, we get the Ricci tensor as

$$(\varrho_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -6\delta^2 & 0 \\ 0 & 0 & 0 & -6\delta^2 \end{pmatrix},$$

so (M, g) is never Einstein in this case. By (2.1), the Schouten tensor is determined to be equal to the Ricci tensor.

The non-zero components of the Cotton tensor (up to symmetries) are

$$\mathcal{C}_{131} = -3\delta^3, \qquad \mathcal{C}_{142} = 3\delta^3, \qquad \mathcal{C}_{232} = 3\delta^3, \qquad \mathcal{C}_{241} = 3\delta^3,$$

and the non-zero components of the Weyl tensor are

$$W_{1234} = 2\delta^2, \qquad W_{1324} = \delta^2, \qquad W_{1423} = -\delta^2.$$

Then, using (2.3), it follows that the Bach tensor is

$$\mathcal{B} = \begin{pmatrix} -\frac{3}{2}\delta^4 & 0 & 0 & 0 \\ 0 & -\frac{3}{2}\delta^4 & 0 & 0 \\ 0 & 0 & -3\delta^4 & 0 \\ 0 & 0 & 0 & -3\delta^4 \end{pmatrix}.$$

So, (M, g) is Bach flat if and only if $\delta = 0$, which is impossible.

Type A: Riemannian case. Let $(M = G/H, g)$ be a four-dimensional generalized symmetric space of type **A**, where the invariant metric g is of signature $(4, 0)$ or $(0, 4)$. Following [14], the Lie algebra $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ admits a basis $\{u_1, u_2, u_3, u_4, h_1\}$, where $\{u_1, u_2, u_3, u_4\}$ and $\{h_1\}$ are bases of $\mathfrak{m}$ and $\mathfrak{h}$, respectively, such that (reversing the metric if needed) the Lie bracket on $\mathfrak{g}$ and the scalar product on $\mathfrak{m}$ are completely determined by

[,]	u_1	u_2	u_3	u_4	h_1
u_1	0	0	$-u_1$	u_2	u_2
u_2	0	0	u_2	u_1	$-u_1$
u_3	u_1	$-u_2$	0	$-2h_1$	$-2u_4$
u_4	$-u_2$	$-u_1$	$2h_1$	0	$2u_3$
h_1	$-u_2$	u_1	$2u_4$	$-2u_3$	0

and

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2}{\rho} & 0 \\ 0 & 0 & 0 & \frac{2}{\rho} \end{pmatrix},$$

where $\rho \neq 0$ is a real constant. Direct calculations yield that we can describe the Levi-Civita connection as follows:

$$\Lambda[1] = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\varrho}{2} & 0 & 0 & 0 \\ 0 & -\frac{\varrho}{2} & 0 & 0 \end{pmatrix}, \qquad \Lambda[2] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{\varrho}{2} & 0 & 0 \\ -\frac{\varrho}{2} & 0 & 0 & 0 \end{pmatrix},$$

and $\Lambda[3] = \Lambda[4] = 0$. The components of the curvature tensor are deduced as

$$R_{12} = \begin{pmatrix} 0 & \rho & 0 & 0 \\ -\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho \\ 0 & 0 & -\rho & 0 \end{pmatrix}, \quad R_{13} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\varrho}{2} & 0 & 0 & 0 \\ 0 & -\frac{\varrho}{2} & 0 & 0 \end{pmatrix},$$

$$R_{14} = R_{23} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{\varrho}{2} & 0 & 0 \\ \frac{\varrho}{2} & 0 & 0 & 0 \end{pmatrix}, \quad R_{24} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -\frac{\varrho}{2} & 0 & 0 & 0 \\ 0 & \frac{\varrho}{2} & 0 & 0 \end{pmatrix},$$

$$R_{34} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 4 & 0 \end{pmatrix},$$

and the Ricci tensor is calculated immediately as

$$(\varrho_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix}.$$

Obviously, (M, g) is never Einstein. By (2.1), the Schouten tensor is determined to be equal to the Ricci tensor. Then, the non-zero components of the Cotton tensor (up to symmetries) are given by

$$\mathcal{C}_{131} = 3\rho, \qquad \mathcal{C}_{142} = -3\rho, \qquad \mathcal{C}_{232} = -3\rho, \qquad \mathcal{C}_{241} = -3\rho,$$

and the non-zero components of the Weyl tensor (up to symmetries) are

$$W_{1234} = 2, \qquad W_{1324} = 1, \qquad W_{1423} = -1.$$

Then, using (2.3), it follows that the Bach tensor is

$$(\mathcal{B}_{ij}) = \begin{pmatrix} -\frac{3}{2}\rho^2 & 0 & 0 & 0 \\ 0 & -\frac{3}{2}\rho^2 & 0 & 0 \\ 0 & 0 & 3\rho & 0 \\ 0 & 0 & 0 & 3\rho \end{pmatrix}.$$

So, (M, g) is Bach flat if and only if $\rho = 0$, which is impossible.

Type B. Let (M, g) be a four-dimensional generalized symmetric space of type **B**. In this case, we have $(M = G/H, g)$ and the Lie algebra $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ is spanned by $\{u_1, u_2, u_3, u_4, h_1\}$, where $\{u_1, u_2, u_3, u_4\}$ and $\{h_1\}$ span $\mathfrak{m}$ and $\mathfrak{h}$, respectively. The Lie brackets on $\mathfrak{g}$ and the scalar product on $\mathfrak{m}$ are given by the following relations:

$[\,,\,]$	u_1	u_2	u_3	u_4	h_1
u_1	0	0	$-u_1$	$\varepsilon h_1 + u_2$	0
u_2	0	0	$-\varepsilon h_1 + u_2$	u_1	0
u_3	u_1	$\varepsilon h_1 - u_2$	0	0	$2u_2$
u_4	$-\varepsilon h_1 - u_2$	$-u_1$	0	0	$-2u_1$
h_1	0	0	$-2u_2$	$2u_1$	0

where $\varepsilon = \pm 1$, and

$$g = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 2\lambda & 0 \\ 0 & -1 & 0 & 2\lambda \end{pmatrix},$$

where λ is an arbitrary real constant (see [14]).

By direct calculations with respect to the basis $\{u_i\}$, we get $\Lambda[1] = \Lambda[2] = 0$ and

$$\Lambda[3] = \begin{pmatrix} 1 & 0 & -2\lambda & 0 \\ 0 & -1 & 0 & 2\lambda \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \Lambda[4] = \begin{pmatrix} 0 & -1 & 0 & 2\lambda \\ -1 & 0 & 2\lambda & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then, the curvature tensor is obtained by the following non-zero components:

$$R_{14} = -R_{23} = \begin{pmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad R_{34} = \begin{pmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix},$$

and immediately the Ricci tensor is

$$(\varrho_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}.$$

We conclude that (M, g) is never Einstein. By (2.1), the Schouten tensor is determined to be equal to the Ricci tensor. The non-zero components of the Weyl tensor (up to symmetries) are given by

$$W_{3434} = 4\lambda.$$

Also, the Cotton tensor which is calculated by (2.2) will vanish identically. Then, using (2.3), it follows that the Bach tensor vanishes too. So, (M, g) is always Bach flat in this case.

Type D. Let $(M = G/H, g)$ denote a generalized symmetric space of type **D**. According to the classification in [14], for the Lie algebra $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ of the Lie group G, there exists a basis $\{u_1, u_2, u_3, u_4, h_1\}$, with $\{u_1, u_2, u_3, u_4\}$ and $\{h_1\}$ bases of $\mathfrak{m}$ and $\mathfrak{h}$, respectively, such that

$[\,,\,]$	u_1	u_2	u_3	u_4	h_1
u_1	0	0	0	$-u_2$	u_1
u_2	0	0	$-u_1$	0	$-u_2$
u_3	0	u_1	0	$-h_1$	$2u_3$
u_4	u_2	0	h_1	0	$-2u_4$
h_1	$-u_1$	u_2	$-2u_3$	$2u_4$	0

and the invariant metric is

$$g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & \lambda & 0 \end{pmatrix},$$

where $\lambda \neq 0$ is a real constant. With respect to $\{u_i\}$, we deduce that

$$\Lambda[1] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \frac{1}{\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Lambda[2] = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 & 0 \end{pmatrix},$$

$\Lambda[3] = \Lambda[4] = 0$, and the curvature tensor is completely determined by the following non-zero components:

$$R_{12} = \begin{pmatrix} \frac{1}{\lambda} & 0 & 0 & 0 \\ 0 & -\frac{1}{\lambda} & 0 & 0 \\ 0 & 0 & -\frac{1}{\lambda} & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda} \end{pmatrix}, \quad R_{14} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 & 0 \end{pmatrix},$$

$$R_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \frac{1}{\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_{34} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Then, the Ricci tensor is immediately calculated as

$$(\varrho_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & -3 & 0 \end{pmatrix},$$

from which we conclude that (M, g) is never Einstein. By (2.1), the Schouten tensor is determined to be equal to the Ricci tensor. Then, the non-zero components of the Cotton tensor (up to symmetries) are given by

$$\mathcal{C}_{141} = \mathcal{C}_{232} = \frac{3}{\lambda},$$

and the Weyl tensor is determined by the following non-zero components:

$$W_{1234} = 1, \qquad W_{1324} = \frac{1}{2}, \qquad W_{1423} = -\frac{1}{2}.$$

Then, using (2.3), it follows that the Bach tensor is

$$(\mathcal{B}_{ij}) = \begin{pmatrix} 0 & -\frac{3}{2\lambda^2} & 0 & 0 \\ -\frac{3}{2\lambda^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{2\lambda} \\ 0 & 0 & \frac{3}{2\lambda} & 0 \end{pmatrix}.$$

Clearly, $\mathcal{B} \neq 0$, and so (M, g) is never Bach flat in this case. □

4. Conformally Einstein generalized symmetric spaces

Let φ be a solution of the conformally Einstein equation (2.4). Then, $\sigma = -2\ln(\varphi)$ is a function that vanishes the $(0,3)$-tensor field $\mathfrak{C} := \mathcal{C} - W(\cdot,\cdot,\cdot,\nabla\sigma)$; in fact, by [21, Proposition 4.1], $\mathfrak{C} = 0$ and $\mathcal{B} = 0$ are necessary conditions for any solution of (2.4). These conditions are also sufficient to be conformally Einstein if (M, g) is weakly generic, in the sense that its Weyl tensor W does not define an injective map from TM to $\otimes^3 TM$.

As we observed in the previous section, the only Bach-flat case was **B** (Theorem 3.1), so the only case that remains to be considered for non-trivial conformally Einstein solutions is the case **B** of generalized symmetric spaces of dimension four.

Theorem 4.1. *Let (M,g) be a generalized symmetric space of dimension four. In this case, (M,g) is non-trivially conformally Einstein if and only if it is of type* ***B*** *with the multiplying function*

$$\varphi(x,y) = c_1e^x + c_2e^{-(x+y)} + c_3e^y,$$

in local coordinates (x,y,u,v) on (M,g), where c_1,c_2,c_3 are arbitrary real constants.

Proof. As mentioned above, the only case through four-dimensional generalized symmetric spaces which we must study for non-trivial conformally Einstein examples is the case **B**.

Let (M,g) be a generalized symmetric space of class **B**. In this case, (M,g) is the space $R^4(x,y,u,v)$ with the pseudo-Riemannian metric

$$g = \lambda(dx^2 + dy^2 + dxdy) + e^{-y}(2dx + dy)dv + e^{-x}(dx + 2dy)du,$$

where λ is an arbitrary real constant. The order is $k = 3$ and the signature is $(2,2)$. By direct calculations, the non-zero components of the Levi-Civita connection are

$$\begin{aligned}
\nabla_{\partial_x}\partial_x &= \frac{1}{3}\partial_x - \frac{2}{3}\partial_y + \frac{2}{3}\lambda e^x\partial_u - \frac{1}{3}\lambda e^y\partial_v,\\
\nabla_{\partial_x}\partial_y &= -\frac{1}{3}\partial_x - \frac{1}{3}\partial_y + \frac{1}{3}\lambda e^x\partial_u + \frac{1}{3}\lambda e^y\partial_v,\\
\nabla_{\partial_x}\partial_u &= -\frac{2}{3}\partial_u + \frac{1}{3}e^{y-x}\partial_v,\\
\nabla_{\partial_x}\partial_v &= \frac{2}{3}e^{x-y}\partial_u - \frac{1}{3}\partial_v,\\
\nabla_{\partial_y}\partial_y &= -\frac{2}{3}\partial_x + \frac{1}{3}\partial_y - \frac{1}{3}\lambda e^x\partial_u + \frac{2}{3}\lambda e^y\partial_v,\\
\nabla_{\partial_y}\partial_u &= -\frac{1}{3}\partial_u + \frac{2}{3}e^{y-x}\partial_v,\\
\nabla_{\partial_y}\partial_v &= \frac{1}{3}e^{x-y}\partial_u - \frac{2}{3}\partial_v,
\end{aligned}$$

where $\partial_x := \frac{\partial}{\partial x}$, $\partial_y := \frac{\partial}{\partial y}$, $\partial_u := \frac{\partial}{\partial u}$, $\partial_v := \frac{\partial}{\partial v}$ are the coordinate vector fields. The curvature tensor is completely determined by the following components:

$$\begin{aligned}
R(\partial_x,\partial_y) &= -\frac{1}{3}\partial_x\otimes dx - \frac{2}{3}\partial_x\otimes dy + \frac{2}{3}\partial_y\otimes dx + \frac{1}{3}\partial_y\otimes dy + \frac{1}{3}\partial_u\otimes du\\
&\quad + \frac{2}{3}e^{x-y}\partial_u\otimes dv - \frac{2}{3}e^{y-x}\partial_v\otimes du - \frac{1}{3}\partial_v\otimes dv,\\
R(\partial_x,\partial_u) &= \frac{2}{3}\partial_u\otimes dx + \frac{1}{3}\partial_u\otimes dy - \frac{1}{3}e^{y-x}\partial_v\otimes dx - \frac{2}{3}e^{y-x}\partial_v\otimes dy,\\
R(\partial_y,\partial_u) &= -\frac{2}{3}e^{x-y}\partial_u\otimes dx - \frac{1}{3}e^{x-y}\partial_u\otimes dy + \frac{1}{3}\partial_v\otimes dx + \frac{2}{3}\partial_v\otimes dy;
\end{aligned}$$

then, the Ricci tensor is computed as

$$\varrho = -\frac{4}{3}dx\otimes dx - \frac{2}{3}(dx\otimes dy + dy\otimes dx) - \frac{4}{3}dy\otimes dy.$$

The Cotton tensor given by (2.2) will vanish and the Weyl tensor is given by

$$\begin{aligned} W = \frac{\lambda}{2}(&dx \otimes dy \otimes dx \otimes dy - dx \otimes dy \otimes dy \otimes dx - dy \otimes dx \otimes dx \otimes dy \\ &+ dy \otimes dx \otimes dy \otimes dx), \end{aligned}$$

which shows that (M, g) is not weakly generic. The case $\lambda = 0$ leads to conformal flatness, so we focus on the cases with $\lambda \neq 0$.

Let $\varphi(x, y, u, v)$ be an arbitrary positive function on M and $\sigma = -2\ln(\varphi)$. Then, by performing a simple calculation the gradient of σ is found to be

$$\begin{aligned} \nabla\sigma = \frac{4}{3\varphi}\big\{ &- (2e^y\partial_v\varphi - e^x\partial_u\varphi)\partial_x \\ &+ (e^y\partial_v\varphi - 2e^x\partial_u\varphi)\partial_y \\ &- (\lambda e^{x+y}\partial_v\varphi - 2\lambda e^{2x}\partial_u\varphi + 2e^x\partial_y\varphi - e^x\partial_x\varphi)\partial_u \\ &+ (2\lambda e^{2y}\partial_v\varphi - \lambda e^{x+y}\partial_u\varphi + e^y\partial_y\varphi - 2e^y\partial_x\varphi)\partial_v \big\}. \end{aligned}$$

Therefore, the only non-zero components of the tensor $\mathfrak{C}$ are as follows (observe that $\mathfrak{C}_{ijk} = -\mathfrak{C}_{jik}$ for all $i, j, k \in \{1, \dots, 4\}$):

$$\begin{cases} \frac{3}{2}\varphi\mathfrak{C}_{121} = \lambda(2e^x\partial_u\varphi - e^y\partial_v\varphi), \\ \frac{3}{2}\varphi\mathfrak{C}_{122} = \lambda(e^x\partial_u\varphi - 2e^y\partial_v\varphi). \end{cases}$$

Since the tensor field $\mathfrak{C}$ of any conformally Einstein manifold necessarily vanishes, we have

$$\begin{cases} \frac{3}{2}\varphi(\mathfrak{C}_{121} - 2\mathfrak{C}_{122}) = 3\lambda e^y\partial_v\varphi = 0, \\ \frac{3}{2}\varphi(2\mathfrak{C}_{121} - \mathfrak{C}_{122}) = 3\lambda e^x\partial_u\varphi = 0; \end{cases}$$

and since $\lambda \neq 0$, the function φ does not depend on the variables u and v.

We now check the conformally Einstein equation (2.4) for some smooth function $\varphi(x, y)$. We set

$$\psi := 2\,\mathrm{Hes}_\varphi + \varphi\varrho - \frac{1}{4}\{2\Delta\varphi + \varphi r\}g;$$

obviously, $\psi_{ij} = -\psi_{ji}$ for all $i, j \in \{1, \dots, 4\}$. By standard calculations, the components of the tensor field ψ are as follows:

$$\psi_{11} = -\frac{4}{3}\varphi + \frac{4}{3}\partial_y\varphi - \frac{2}{3}\partial_x\varphi + 2\partial^2_{xx}\varphi,$$

$$\psi_{12} = -\frac{2}{3}\varphi + \frac{2}{3}\partial_y\varphi + \frac{2}{3}\partial_x\varphi + 2\partial^2_{xy}\varphi,$$

$$\psi_{22} = -\frac{4}{3}\varphi - \frac{2}{3}\partial_y\varphi + \frac{4}{3}\partial_x\varphi + 2\partial^2_{yy}\varphi.$$

Now, we set $\psi = 0$, and from the resulting system of PDEs, we obtain the solution $\varphi(x, y) = c_1e^x + c_2e^{-(x+y)} + c_3e^y$, for some arbitrary real constants c_1, c_2 and c_3, and this finishes the proof. □

5. Extensions of generalized symmetric spaces of dimension three

In this section, we extend three-dimensional generalized symmetric spaces to dimension four by considering direct products of them with $\mathbb{R}$. From the classification in [14], any proper, simply connected generalized symmetric pseudo-Riemannian space (M,g) of dimension $n=3$ is of order 4. Those are indecomposable (the generalized symmetric spaces which are not direct products of any pseudo-Riemannian manifolds of lower dimensions) and described, up to an isometry, as follows: the underlying homogeneous space G/H is the matrix group

$$\begin{pmatrix} e^{-t} & 0 & x \\ 0 & e^{t} & y \\ 0 & 0 & 1 \end{pmatrix},$$

and (M,g) is the space $\mathbb{R}^3(x,y,z)$ with a pseudo-Riemannian metric

$$\bar{g} = \begin{pmatrix} \pm e^{2z} & 0 & 0 \\ 0 & \pm e^{-2z} & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \tag{5.1}$$

where λ is a real constant. Using this classification, the extension of these spaces to generalized symmetric spaces of dimension four gives the following result.

Theorem 5.1. *Let (M,g) be a generalized symmetric space which is a direct product of a generalized symmetric space of dimension three with $\mathbb{R}$. In this case, (M,g) is never Bach flat, and so it is never conformally Einstein.*

Proof. Let $(M=G/H,\bar{g})$ be a three-dimensional generalized symmetric space of dimension three, where $\bar{g}$ is described in local coordinates (x,y,z) as (5.1). We study the extended four-dimensional generalized symmetric space $(M\times\mathbb{R},g)$. Clearly, with respect to the local coordinates (x,y,z,t), g will be

$$g = \begin{pmatrix} \varepsilon e^{2z} & 0 & 0 & 0 \\ 0 & \varepsilon e^{-2z} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix},$$

where ε,δ are ±1 and λ is an arbitrary real constant. The non-zero components of the Levi-Civita connection are

$$\begin{aligned} \nabla_{\partial_x}\partial_z &= \partial_x, \\ \nabla_{\partial_x}\partial_x &= \frac{-\varepsilon}{\lambda}e^{2z}\partial_z, \\ \nabla_{\partial_y}\partial_z &= -\partial_y, \\ \nabla_{\partial_y}\partial_y &= \frac{\varepsilon}{\lambda}e^{-2z}\partial_z. \end{aligned}$$

By direct calculation, the Ricci tensor is found to be

$$(\varrho_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The scalar curvature is $r = \frac{-2}{\lambda}$, and clearly, (M, g) is never Einstein. By (2.1), the components of the Schouten tensor are

$$(\mathcal{S}_{ij}) = \begin{pmatrix} \frac{\varepsilon}{3\lambda}e^{2z} & 0 & 0 & 0 \\ 0 & \frac{\varepsilon}{3\lambda}e^{-2z} & 0 & 0 \\ 0 & 0 & -\frac{5}{3} & 0 \\ 0 & 0 & 0 & \frac{\delta}{3\lambda} \end{pmatrix}.$$

The non-zero components of the Cotton tensor (up to symmetries) are given by

$$\mathcal{C}_{311} = \frac{2\varepsilon}{\lambda}e^{2z}, \qquad \mathcal{C}_{232} = \frac{2\varepsilon}{\lambda}e^{-2z},$$

and the non-zero components of the Weyl tensor are

$$\begin{aligned} W_{1212} &= \frac{2}{3\lambda}, & W_{1313} &= -\frac{\varepsilon}{3}e^{2z}, \\ W_{1414} &= -\frac{\varepsilon\delta}{3\lambda}e^{2z}, & W_{2323} &= -\frac{\varepsilon}{3}e^{-2z}, \\ W_{2424} &= -\frac{\varepsilon\delta}{3\lambda}e^{-2z}, & W_{3434} &= \frac{2\delta}{3}. \end{aligned}$$

Now, using (2.3), it follows that the Bach tensor matrix is

$$(\mathcal{B}_{ij}) = \begin{pmatrix} \frac{2\varepsilon}{3\lambda^2}e^{2z} & 0 & 0 & 0 \\ 0 & \frac{2\varepsilon}{3\lambda^2}e^{-2z} & 0 & 0 \\ 0 & 0 & -\frac{2}{\lambda} & 0 \\ 0 & 0 & 0 & \frac{2\delta}{3\lambda^2} \end{pmatrix}.$$

So, (M, g) is never Bach flat and this finishes the proof. □

6. The oscillator group

The four-dimensional *oscillator algebra* $\mathfrak{g}$ is considered as the matrix subalgebra of $\mathfrak{gl}(4, \mathbb{R})$ with four generators X, Y, P, Q as follows:

$$X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Clearly, the non-vanishing Lie brackets will be

$$[X, Y] = P, \qquad [Q, X] = Y, \qquad [Q, Y] = -X.$$

The corresponding connected simply connected Lie group G, which is a subgroup of $\mathrm{GL}(4, \mathbb{R})$, is called the *(four-dimensional) oscillator group*.

Following [18], we equip G with the 1-parameter family of left-invariant Lorentzian metrics $g_a = \langle\, ,\rangle$, described by nonvanishing products as

$$\langle e_1, e_1\rangle = \langle e_4, e_4\rangle = a, \qquad \langle e_2, e_2\rangle = \langle e_3, e_3\rangle = 1, \qquad \langle e_1, e_4\rangle = \langle e_4, e_1\rangle = 1,$$

for any real constant a with $-1 < a < 1$. The metric is bi-invariant whenever $a = 0$, and in all other cases g_a is only left-invariant [18]. With respect to local coordinates (x_1, x_2, x_3, x_4), one can evaluate the invariant metric g_a explicitly as

$$g_a = adx_1^2 + 2ax_3dx_1dx_2 + (1+ax_3^2)dx_2^2 + dx_3^2 + 2dx_1dx_4 + 2x_3dx_2dx_4 + adx_4^2. \tag{6.1}$$

Direct calculations yield that g_a is locally symmetric if and only if $a = 0$ [18]. In order to study the conformally Einstein property of the oscillator group, we consider the necessary condition of Bach flatness, which yields the following theorem.

Theorem 6.1. *Let (G, g_a) be the oscillator group of dimension four. The following statements are equivalent:*

(i) *(G, g_a) is Bach flat.*
(ii) *(G, g_a) is locally conformally flat.*
(iii) *(G, g_a) is locally symmetric.*
(iv) *g_a is bi-invariant.*

Proof. Starting from (6.1), we can describe the Levi-Civita connection ∇, and then the curvature of (G, g_a), with respect to the basis $\{\partial_i\}$ of coordinate vector fields. Explicitly, the Levi-Civita connection is completely determined by the following possibly non-vanishing components:

$$\nabla_{\partial_1}\partial_2 = -\frac{a}{2}\partial_3, \qquad \nabla_{\partial_1}\partial_3 = -\frac{ax_3}{2}\partial_1 + \frac{a}{2}\partial_2, \quad \nabla_{\partial_2}\partial_2 = -ax_3\partial_3,$$

$$\nabla_{\partial_2}\partial_3 = \frac{1-ax_3^2}{2}\partial_1 + \frac{ax_3}{2}\partial_2, \quad \nabla_{\partial_2}\partial_4 = -\frac{1}{2}\partial_3, \qquad \nabla_{\partial_3}\partial_4 = -\frac{x_3}{2}\partial_1 + \frac{1}{2}\partial_2.$$

Then, we can describe the Riemannian curvature tensor R of (G, g_a) with respect to $\{\partial_i\}$. Setting $R_{ij} := R(\partial_i, \partial_j)$, we have

$$R_{12} = \begin{pmatrix} \frac{a^2x_3}{4} & \frac{a^2x_3^2+a}{4} & 0 & \frac{ax_3}{4} \\ -\frac{a^2}{4} & -\frac{a^2x_3}{4} & 0 & -\frac{a}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad R_{13} = \begin{pmatrix} 0 & 0 & \frac{a}{4} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{a^2}{4} & -\frac{a^2x_3}{4} & 0 & -\frac{a}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_{14} = 0, \qquad R_{23} = \begin{pmatrix} 0 & 0 & ax_3 & 0 \\ 0 & 0 & -\frac{3a}{4} & 0 \\ -\frac{a^2x_3}{4} & \frac{3a-a^2x_3^2}{4} & 0 & -\frac{ax_3}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_{24} = \begin{pmatrix} -\frac{ax_3}{4} & -\frac{ax_3^2+1}{4} & 0 & -\frac{x_3}{4} \\ \frac{a}{4} & \frac{ax_3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad R_{34} = \begin{pmatrix} 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{a}{4} & \frac{ax_3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Ricci tensor is then obtained by contraction of the curvature tensor. Hence, with respect to $\{\partial_i\}$, the Ricci tensor is described by the matrix

$$(\varrho_{ij}) = \begin{pmatrix} \frac{1}{2}a^2 & \frac{1}{2}a^2x_3 & 0 & \frac{1}{2}a \\ \frac{1}{2}a^2x_3 & \frac{1}{2}a(ax_3^2-1) & 0 & \frac{1}{2}ax_3 \\ 0 & 0 & -\frac{1}{2}a & 0 \\ \frac{1}{2}a & \frac{1}{2}ax_3 & 0 & \frac{1}{2} \end{pmatrix},$$

so (G, g_a) is never Einstein. By (2.1), the Schouten tensor is calculated as follows:

$$(\mathcal{S}_{ij}) = \begin{pmatrix} \frac{7}{12}a^2 & \frac{7}{12}a^2x_3 & 0 & \frac{7}{12}a \\ \frac{7}{12}a^2x_3 & \frac{7}{12}a^2x_3^2 - \frac{5}{12}a & 0 & \frac{7}{12}ax_3 \\ 0 & 0 & -\frac{5}{12}a & 0 \\ \frac{7}{12}a & \frac{7}{12}ax_3 & 0 & \frac{1}{12}a^2+\frac{1}{2} \end{pmatrix}.$$

The non-zero components of the Cotton tensor (up to symmetries) are

$$\mathcal{C}_{123} = \mathcal{C}_{312} = \frac{1}{2}a^2, \qquad \mathcal{C}_{321} = a^2, \qquad \mathcal{C}_{322} = \frac{3}{2}a^2x_3,$$
$$\mathcal{C}_{324} = a, \qquad \mathcal{C}_{342} = \mathcal{C}_{423} = \frac{1}{2}a.$$

Finally, with respect to $\{\partial_i\}$, the Weyl conformal tensor W is completely determined by the following possibly non-vanishing matrices $W_{ij} := W(\partial_i, \partial_j)$:

$$W_{12} = \begin{pmatrix} \frac{a^2x_3}{6} & \frac{a(1+ax_3^2)}{6} & 0 & \frac{ax_3}{6} \\ -\frac{a^2}{6} & -\frac{a^2x_3}{6} & 0 & -\frac{a}{6} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad W_{13} = \begin{pmatrix} 0 & 0 & \frac{a}{6} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{a^2}{6} & -\frac{a^2x_3}{6} & 0 & -\frac{a}{6} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$W_{14} = \begin{pmatrix} -\frac{a}{3} & -\frac{ax_3}{3} & 0 & -\frac{a^2}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{a^2}{3} & \frac{a^2x_3}{3} & 0 & \frac{a}{3} \end{pmatrix}, \quad W_{23} = \begin{pmatrix} 0 & 0 & \frac{ax_3}{2} & 0 \\ 0 & 0 & -\frac{a}{3} & 0 \\ -\frac{a^2x_3}{6} & \frac{a(2-ax_3^2)}{6} & 0 & -\frac{ax_3}{6} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$W_{24} = \begin{pmatrix} -\frac{ax_3}{2} & -\frac{ax_3^2}{2} & 0 & -\frac{a^2x_3}{2} \\ \frac{a}{6} & \frac{ax_3}{6} & 0 & \frac{a^2}{6} \\ 0 & 0 & 0 & 0 \\ \frac{a^2x_3}{3} & \frac{a(2ax_3^2-1)}{6} & 0 & \frac{ax_3}{3} \end{pmatrix}, \quad W_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{a}{6} & \frac{ax_3}{6} & 0 & \frac{a^2}{6} \\ 0 & 0 & -\frac{a}{6} & 0 \end{pmatrix}.$$

Observe that these equalities yield at once that g_a is locally conformally flat if and only if $a = 0$.

Then, using (2.3), it follows that the Bach tensor is

$$(\mathcal{B}_{ij}) = \begin{pmatrix} -\frac{5}{6}a^3 & -\frac{5}{6}a^3x_3 & 0 & -\frac{5}{6}a^2 \\ -\frac{5}{6}a^3x_3 & -\frac{a^2}{6}(5ax_3^2-3) & 0 & -\frac{5}{6}a^2x_3 \\ 0 & 0 & \frac{1}{2}a^2 & 0 \\ -\frac{5}{6}a^2 & -\frac{5}{6}a^2x_3 & 0 & -\frac{a}{6}(a^2+4) \end{pmatrix}.$$

So, (G, g_a) is Bach flat if and only if $a = 0$, which is equivalent to being locally conformally flat and being locally symmetric. Clearly, the metric g_0 is bi-invariant in this case. □

7. Conclusion

In this paper, we studied conformally Einstein generalized symmetric spaces of dimension four. The mentioned spaces, classified in [14], were deeply studied for different geometric properties in the literature. We proved that generalized symmetric spaces of dimension four which accept non-trivial conformally Einstein metrics belong to the purely pseudo-Riemannian class **B**. We also determined the non-zero function φ in which $\varphi^{-2}g$ is Einstein, where g is an arbitrary invariant metric on a generalized symmetric space of class **B**. We also studied four-dimensional extensions of generalized symmetric spaces of dimension three and showed that these spaces are never Bach flat. As an example of metric Lie groups, we examined the four-dimensional oscillator Lie group and proved that, for these metrics, being conformally Einstein is equivalent to being bi-invariant.

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