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BLOW-UP OF POSITIVE-INITIAL-ENERGY SOLUTIONS FOR NONLINEARLY DAMPED SEMILINEAR WAVE EQUATIONS

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Abstract. We consider a class of semilinear wave equations with both strongly and nonlinear weakly damped terms,

$$u_{tt} - \Delta u - \omega \Delta u_t + \mu |u_t|^{m-2} u_t = |u|^{p-2} u,$$

associated with initial and Dirichlet boundary conditions. Under certain conditions, we show that any solution with arbitrarily high positive initial energy blows up in finite time if $m < p$. Furthermore, we obtain a lower bound for the blow-up time.

1. Introduction

In this contribution, we study the blow-up of solutions of the following initial boundary value problem of a semilinear wave equation:

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \mu |u_t|^{m-2} u_t = |u|^{p-2} u, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here, Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Additionally, we assume that

$$u_0 \in H_0^1(\Omega), \quad u_1 \in L^2(\Omega), \tag{1.2}$$

and ω, μ, m and p are positive constants, with

$$\begin{cases} 2 < p \le \frac{2n}{n-2}, & \text{for } n \ge 3, \\ 2 < p < \infty, & \text{for } n = 2. \end{cases} \tag{1.3}$$

The linear strong damping term $-\omega \Delta u_t$ appears in models describing Kelvin–Voigt materials that exhibit both elastic and viscous properties, while the nonlinear frictional damping term $\mu |u_t|^{m-2} u_t$ usually models external friction forces such as air resistance acting on the vibrating structures.

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In the absence of strong damping ($\omega = 0$), the equation in (1.1) reduces to the nonlinearly damped wave equation

$$u_{tt} - \Delta u + \mu |u_t|^{m-2} u_t = |u|^{p-2} u. \tag{1.4}$$

Eq. (1.4) was first studied by Levine [8] in the case of linear weak damping ($m = 2$). By using the concavity method, he showed that solutions with negative initial energy blow up in finite time. Later, for the case $m > 2$, by using a different method, Georgiev and Todorova [4] established a global existence result for Eq. (1.4) if $m \geq p$ and finite time blow-up if $p > m$ and the initial energy is sufficiently negative.

In the presence of the strong damping term, i.e. $\omega > 0$, Gazzola and Squassina [3] studied (1.4) for $m = 2$. They gave a necessary and sufficient condition for blow-up if $E(0) < d$, where d is the depth of the potential well. Recently, Yang and Xu [16] gave a sufficient condition for blow-up if $E(0) > d$. In the case of $\omega > 0$ and $m > 2$, Yu [17] gave a necessary and sufficient condition for blow-up when $E(0) < d$. Boukhatem and Benabderrahmane [2] extended the previous work to a semilinear hyperbolic equation for a uniformly elliptic operator with nonlinear damping and source terms. For results of the same nature, we refer the reader to [1, 6, 5, 9, 12, 14, 15, 18] and the references therein.

In related work, Messaoudi [11] considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau) \Delta u(\tau)\, d\tau + |u_t|^{m-2} u_t = |u|^{p-2} u. \tag{1.5}$$

He proved, for $m < p$, that solutions with $E(0) < d$ blow up in finite time. Later, by using a modified concavity method, Kafini and Messaoudi [7] established a blow-up result for (1.5) when the damping term is linear. When $m > 2$, by introducing a new technique, Song [13] obtained a finite time blow-up result for solutions of (1.5) with arbitrarily high initial energy.

In this paper, motivated by the above-cited works, we give sufficient conditions for the finite time blow-up of solutions of (1.1) in both cases: $E(0) < 0$ and $E(0) > 0$. Furthermore, we give a lower bound for the blow-up time.

2. Preliminaries

We denote by $\|\cdot\|_p$ the $L^p(\Omega)$ norm ($2 \leq p < \infty$), and by $(\cdot,\cdot)$ the L^2 inner product. The notation $\langle \cdot,\cdot \rangle$ is used in this paper to denote the duality paring between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. We introduce the energy functional

$$E(t) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u_t\|_2^2 - \frac{1}{p}\|u\|_p^p.$$

By simple calculation we have

$$E'(t) = -\mu \|u_t\|_m^m - \omega \|\nabla u_t\|_2^2 \leq 0,$$

which implies that

$$E(t) \leq E(0) \quad \forall t \geq 0,$$

and

$$-E'(t) \geq \mu \|u_t\|_m^m, \quad -E'(t) \geq \omega \|\nabla u_t\|_2^2. \tag{2.1}$$

Definition 2.1. By *solution of problem* (1.1) *over* $[0,T]$ we mean a function

$$u \in C\big([0,T], H_0^1(\Omega)\big) \cap C^1\big([0,T], L^2(\Omega)\big) \cap C^2\big([0,T], H^{-1}(\Omega)\big),$$

with $u_t \in L^m\big([0,T], H_0^1(\Omega)\big)$, such that $u(0) = u_0$, $u_t(0) = u_1$ and

$$\langle u_{tt}(t), \eta\rangle + \int_\Omega \nabla u(t)\nabla\eta\, dx + \omega\int_\Omega \nabla u_t(t)\nabla\eta\, dx + \mu\int_\Omega |u(t)|^{m-2}u_t(t)\eta\, dx = \int_\Omega |u|^{p-2}u\eta\, dx$$

for all $\eta \in H_0^1(\Omega)$ and a.e. $t \in [0,T]$.

Theorem 2.2 ([17])**.** *Assume that conditions* (1.2) *and* (1.3) *hold. Then the problem* (1.1) *admits a unique local solution defined on* $[0,T]$*. Moreover, if*

$$T_{\max} := \sup\{T > 0 : u = u(t) \text{ exists on } [0,T]\} < \infty,$$

then

$$\lim_{t\to T_{\max}} \|u(t)\|_q = \infty \quad \text{for all } q \geq 1 \text{ such that } q > \frac{n(p-2)}{2}.$$

3. Blow-up with negative initial energy

In this section we show that the solution of (1.1) blows up in finite time if $m < p$ and $E(0) < 0$. To prove the main result in this section, we define $H(t) := -E(t)$ and we use the following lemma. For the proof, see [10].

Lemma 3.1. *Suppose* (1.3) *holds. Then we have*

$$\|u\|_p^s \leq C\left[|H(t)| + \|u_t\|_2^2 + \|u\|_p^p\right]$$

for any $u \in H_0^1(\Omega)$ *and* $2 \leq s \leq p$.

Theorem 3.2. *Suppose* (1.2) *and* (1.3) *hold. Assume further that* $p > m \geq 2$ *and* $E(0) < 0$*. Then the solution of the problem* (1.1) *blows up in finite time.*

Proof. To obtain a contradiction, we suppose that the solution of (1.1) is global; then, for every fixed $T > 0$, there exists a constant $K > 0$ such that

$$\max\left\{\|\nabla u\|_2^2, \|u_t\|_2^2, \|u\|_p^p\right\} \leq K \qquad \forall t \in [0,T]. \tag{3.1}$$

We have $H'(t) = -E'(t) \geq 0$, which together with $E(0) < 0$ shows that

$$0 < H(0) \leq H(t) \leq \frac{1}{p}\|u\|_p^p; \tag{3.2}$$

furthermore,

$$\mu\|u_t\|_m^m \leq H'(t). \tag{3.3}$$

We now define an auxiliary function

$$L(t) := H^{1-\alpha}(t) + \varepsilon\,(u_t, u) + \varepsilon\frac{\omega}{2}\|\nabla u\|_2^2,$$

for ε small (to be chosen later) and

$$0 < \alpha \leq \min\left\{\frac{p-2}{2p}, \frac{p-m}{p(m-1)}\right\} < \frac{1}{2}.$$

By taking the derivative of $L(t)$ we obtain

$$L'(t) = (1-\alpha)H'(t)H^{-\alpha}(t) + \varepsilon\langle u_{tt}, u\rangle + \varepsilon\|u_t\|_2^2 + \varepsilon\omega(\nabla u_t, \nabla u). \tag{3.4}$$

Using (1.1), the equation (3.4) takes the form

$$\begin{aligned} L'(t) = (1-\alpha)H'(t)H^{-\alpha}(t) - \varepsilon\|\nabla u\|_2^2 + \varepsilon\|u_t\|_2^2 \\ + \varepsilon\|u\|_p^p - \varepsilon\mu(|u_t|^{m-2}u_t, u). \end{aligned} \tag{3.5}$$

To estimate the last term in the right-hand side of (3.5), we use Young's inequality

$$(|u_t|^{m-2}u_t, u) \leq \frac{\delta^m}{m}\|u\|_m^m + \frac{m-1}{m}\delta^{m/(1-m)}\|u_t\|_m^m \qquad \forall \delta > 0. \tag{3.6}$$

By taking

$$\delta = \left[kH^{-\alpha}(t)\right]^{\frac{1-m}{m}},$$

for a large constant k to be chosen later, (3.6) becomes

$$(|u_t|^{m-2}u_t, u) \leq \frac{1}{m}\left[kH^{-\alpha}(t)\right]^{1-m}\|u\|_m^m + \frac{m-1}{m}kH^{-\alpha}\|u_t\|_m^m. \tag{3.7}$$

Combining (3.5) and (3.7), and using (3.3), yields

$$\begin{aligned} L'(t) \geq \left[1-\alpha-\varepsilon k\frac{m-1}{m}\right]H'(t)H^{-\alpha}(t) - \varepsilon\|\nabla u\|_2^2 + \varepsilon\|u\|_p^p \\ + \varepsilon\|u_t\|_2^2 - \frac{\varepsilon\mu}{m}\left[kH^{-\alpha}(t)\right]^{1-m}\|u\|_m^m. \end{aligned} \tag{3.8}$$

By using (3.2), we obtain

$$H^{\alpha(m-1)}(t)\|u\|_m^m \leq p^{-\alpha(m-1)}\|u\|_p^{p\alpha(m-1)}\|u\|_m^m,$$

and hence by the inequality

$$\|u\|_m \leq C\|u\|_p,$$

we get

$$H^{\alpha(m-1)}(t)\|u\|_m^m \leq Cp^{-\alpha(m-1)}\|u\|_p^{p\alpha(m-1)+m}. \tag{3.9}$$

Thus, by (3.9) and Lemma 3.1, for $s = p\alpha(m-1)+m \leq p$, we obtain

$$H^{\alpha(m-1)}(t)\|u\|_m^m \leq Cp^{-\alpha(m-1)}\left[H(t) + \|u_t\|_2^2 + \|u\|_p^p\right].$$

Therefore, in view of the last inequality, (3.8) becomes

$$\begin{aligned} L'(t) \geq \left[1-\alpha-\varepsilon k\frac{m-1}{m}\right]H'(t)H^{-\alpha}(t) + \frac{\varepsilon}{2}(p-2)\|\nabla u\|_2^2 \\ + \frac{\varepsilon}{2}(p+2)\|u_t\|_2^2 + \varepsilon\left\{pH(t) - \lambda k^{1-m}\left[H(t) + \|u_t\|_2^2 + \|u\|_p^p\right]\right\}, \end{aligned} \tag{3.10}$$

where

$$\lambda = Cp^{-\alpha(m-1)}\frac{\mu}{m}.$$

Writing $p = (p+2)/2 + (p-2)/2$ in (3.10) yields

$$L'(t) \geq \gamma_1 H'(t)H^{-\alpha}(t) + \gamma_2 H(t) + \gamma_3\|u_t\|_2^2 + \gamma_4\|u\|_p^p + \gamma_5\|\nabla u\|_2^2, \tag{3.11}$$

where

$$\gamma_1 = 1 - \alpha - \varepsilon k \frac{m-1}{m}, \qquad \gamma_2 = \varepsilon \left(\frac{p+2}{2} - \lambda k^{1-m} \right),$$
$$\gamma_3 = \varepsilon \left(\frac{p+6}{4} - \lambda k^{1-m} \right), \quad \gamma_4 = \varepsilon \left(\frac{p-2}{2p} - \lambda k^{1-m} \right),$$
$$\gamma_5 = \frac{\varepsilon}{4}(p-2) > 0.$$

We choose now k large enough such that the coefficients γ_i, for $2 \leq i \leq 4$, are positive. Once k is fixed, we choose ε small enough such that

$$\gamma_1 > 0 \quad \text{and} \quad L(0) > 0.$$

Hence, the inequality (3.11) becomes

$$L'(t) \geq A \left[H(t) + \|u_t\|_2^2 + \|u\|_p^p \right] \tag{3.12}$$

for some constant $A > 0$. Consequently, we have

$$L(t) \geq L(0) > 0 \qquad \text{for all } t \geq 0.$$

Next, by using Hölder's and Young's inequalities, we obtain

$$\begin{aligned} |(u_t, u)|^{1/(1-\alpha)} &\leq \|u_t\|_2^{1/(1-\alpha)} \|u\|_2^{1/(1-\alpha)} \\ &\leq C \left[\|u_t\|_2^{s/(1-\alpha)} + \|u\|_2^{r/(1-\alpha)} \right] \end{aligned}$$

for $1/s + 1/r = 1$. We take $s = 2(1-\alpha)$, which gives

$$\frac{s}{(1-\alpha)} = 2 \quad \text{and} \quad \frac{r}{(1-\alpha)} = \frac{2}{(1-2\alpha)} \leq p.$$

Therefore, by using Lemma 3.1, we obtain

$$|(u_t, u)|^{1/(1-\alpha)} \leq C \left[H(t) + \|u_t\|_2^2 + \|u\|_p^p \right]. \tag{3.13}$$

From (3.1) and (3.2), we have

$$\|\nabla u\|^{2/(1-\alpha)} \leq K^{1/(1-\alpha)} \leq K^{1/(1-\alpha)} \frac{H(t)}{H(0)}. \tag{3.14}$$

So, by using Jensen's inequality, we get

$$L(t)^{1/(1-\alpha)} \leq C \left[H(t) + |(u_t, u)|^{1/(1-\alpha)} + \|\nabla u\|^{2/(1-\alpha)} \right],$$

and by combining it with (3.13) and (3.14), we deduce

$$L(t)^{1/(1-\alpha)}) \leq B \left[H(t) + \|u_t\|_2^2 + \|u\|_p^p \right]. \tag{3.15}$$

From the inequalities (3.12) and (3.15), we finally obtain the differential inequality

$$L'(t) \geq D L(t)^{1/(1-\alpha)} \tag{3.16}$$

for some $D > 0$. A simple integration of (3.16) over $(0, t)$ immediately yields

$$L(t) \geq \left[L^{-\alpha/(1-\alpha)}(0) - \frac{\alpha D}{1-\alpha} t \right]^{1-1/\alpha}, \tag{3.17}$$

which shows that the functional $L(t)$ blows up in finite time. □

Remark 3.3. From (3.17) we obtain the following upper bound of the blow-up time:

$$T_{\max} \leq \frac{1-\alpha}{\alpha D} \left[L(0)\right]^{1-1/\alpha}.$$

4. Blow-up with positive initial energy

In this section, we consider the blow-up of solutions of the problem (1.1) when $E(0) > 0$. To prove the main theorem of this paper, we employ the following lemma.

Lemma 4.1. *If* $2 < m < p$ *then*

$$\frac{1}{m}\|u\|_m^m \leq \frac{s}{2}\|u\|_2^2 + \frac{1-s}{p}\|u\|_p^p, \qquad \text{where } s = \frac{p-m}{p-2}.$$

Proof. By the convexity of the function u^x/x for $u \geq 0$ and $x > 0$. □

Theorem 4.2. *Suppose* (1.2) *and* (1.3) *hold. Assume further that* $p > m \geq 2$. *If the solution of* (1.1) *satisfies*

$$(u_t(0), u(0)) > ME(0) > 0 \tag{4.1}$$

for some $M > 0$ *to be specified later in the proof, then* $u(t)$ *blows up in finite time.*

Proof. Assume, towards a contradiction, that $u(t)$ is a global solution of (1.1). Setting $F(t) := \frac{1}{2}\|u(t)\|_2^2$, it follows from (1.1) that

$$F''(t) = \|u_t(t)\|_2^2 + \|u(t)\|_p^p - \|\nabla u(t)\|_2^2 - \omega(\nabla u_t, \nabla u) - \mu(|u_t|^{m-2}u_t, u). \tag{4.2}$$

By using Hölder's and Young's inequalities, we estimate the last two terms in the right-hand side of the previous equation as follows:

$$(\nabla u_t, \nabla u) \leq \eta\|\nabla u\|_2^2 + \frac{1}{4\eta}\|\nabla u_t\|_2^2, \quad \eta > 0,$$
$$(|u_t|^{m-2}u_t, u) \leq \frac{1}{m}\delta^m\|u\|_m^m + \frac{m-1}{m}\delta^{m/(1-m)}\|u_t\|_m^m, \quad \delta > 0.$$

So, by Lemma 4.1, we get

$$\frac{\delta^m}{m}\|u\|_m^m \leq \frac{s}{2}\delta^m\|u\|_2^2 + \frac{1-s}{p}\delta^m\|u\|_p^p.$$

Hence, (4.2) becomes

$$F''(t) \geq \|u_t(t)\|_2^2 - (1+\omega\eta)\|\nabla u(t)\|_2^2 + \left[1 - \frac{\mu(1-s)}{p}\delta^m\right]\|u(t)\|_p^p$$
$$- \frac{\mu s}{2}\delta^m\|u(t)\|_2^2 - \frac{\omega}{4\eta}\|\nabla u_t(t)\|_2^2 - \mu\frac{m-1}{m}\delta^{-\frac{m}{m-1}}\|u_t(t)\|_m^m.$$

Adding and subtracting $p(1-\varepsilon)E(t)$ for $\varepsilon \in (0,1)$ in the right-hand side of the last inequality, and using (2.1) and the Poincaré inequality, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac{dF(t)}{dt} - \left[\frac{1}{4\eta} + \frac{m-1}{m}\delta^{-\frac{m}{m-1}}\right]E(t)\right\} \\
&\quad \geq F''(t) + \frac{\omega}{4\eta}\|\nabla u_t(t)\|_2^2 + \mu\frac{m-1}{m}\delta^{-\frac{m}{m-1}}\|u_t(t)\|_m^m \\
&\quad \geq \|u_t(t)\|_2^2 - (1+\omega\eta)\|\nabla u(t)\|_2^2 \\
&\qquad + \left[1 - \frac{\mu(1-s)}{p}\delta^m\right]\|u(t)\|_p^p - \frac{\mu}{2}\delta^m\|u(t)\|_2^2 \\
&\quad \geq \left[1 + \frac{p}{2}(1-\varepsilon)\right]\|u_t(t)\|_2^2 + \left[\frac{p}{2}(1-\varepsilon) - (1+\omega\eta)\right]\|\nabla u(t)\|_2^2 \\
&\qquad + \left[\varepsilon - \frac{\mu(1-s)}{p}\delta^m\right]\|u(t)\|_p^p - \frac{\mu s}{2}\delta^m\|u(t)\|_2^2 - p(1-\varepsilon)E(t) \\
&\quad \geq \left[1 + \frac{p}{2}(1-\varepsilon)\right]\|u_t(t)\|_2^2 + \left\{\alpha(\varepsilon)B - \frac{\mu s}{2}\delta^m\right\}\|u(t)\|_2^2 \\
&\qquad + \left[\varepsilon - \frac{\mu(1-s)}{p}\delta^m\right]\|u(t)\|_p^p - p(1-\varepsilon)E(t),
\end{aligned}
\tag{4.3}
$$

where

$$\alpha(\varepsilon) = \frac{p}{2}(1-\varepsilon) - (1+\omega\eta)$$

and B is the best constant of the Poincaré inequality

$$\|\nabla u\|_2^2 \geq B\|u\|_2^2.$$

Therefore, taking $\eta = \varepsilon$ and

$$\delta = \left[\frac{p\varepsilon}{\mu(1-s)}\right]^{1/m},$$

setting

$$\gamma_1(\varepsilon) = \frac{1}{4\varepsilon} + \frac{m-1}{m}\left(\frac{1-s}{p\varepsilon}\right)^{-\frac{1}{m-1}}$$

and substituting in (4.3), we arrive at

$$
\begin{aligned}
\frac{d}{dt}\left[F'(t) - \gamma_1(\varepsilon)E(t)\right] &\geq \left[1 + \frac{p}{2}(1-\varepsilon)\right]\|u_t(t)\|_2^2 \\
&\quad + \left[\alpha(\varepsilon)B - \frac{ps}{2(1-s)}\varepsilon\right]\|u(t)\|_2^2 - p(1-\varepsilon)E(t).
\end{aligned}
$$

Hence, we choose ε small enough such that

$$\alpha(\varepsilon)B - \frac{ps}{2(1-s)}\varepsilon > 0.$$

By using the Schwarz inequality, we have

$$\begin{aligned}2\left[1+\frac{p}{2}(1-\varepsilon)\right]^{1/2}&\left[\alpha(\varepsilon)B-\frac{ps}{2(1-s)}\varepsilon\right]^{1/2}(u_t,u)\\&\leq\left[1+\frac{p}{2}(1-\varepsilon)\right]\|u_t(t)\|_2^2+\left[\alpha(\varepsilon)B-\frac{ps}{2(1-s)}\varepsilon\right]\|u(t)\|_2^2.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\frac{d}{dt}\left[F'(t)-\gamma_1(\varepsilon)E(t)\right]&\geq\beta(\varepsilon)(u_t,u)-p(1-\varepsilon)E(t)\\&=\beta(\varepsilon)\left[F'(t)-\gamma_2(\varepsilon)E(t)\right],\end{aligned}\tag{4.4}$$

where

$$\begin{aligned}\beta(\varepsilon)&=2\left[1+\frac{p}{2}(1-\varepsilon)\right]^{1/2}\left[\alpha(\varepsilon)B-\frac{ps}{2(1-s)}\varepsilon\right]^{1/2},\\\gamma_2(\varepsilon)&=\frac{p(1-\varepsilon)}{\beta(\varepsilon)}.\end{aligned}$$

Since

$$\alpha(\varepsilon)B-\frac{ps}{2(1-s)}\varepsilon\to\begin{cases}\frac{2B}{p-2}>0&\text{as }\varepsilon\to0^+,\\-\frac{B}{1+\omega}-\frac{ps}{2(1-s)}<0&\text{as }\varepsilon\to1^-,\end{cases}$$

there exists $\varepsilon_*\in(0,1)$ such that

$$\beta(\varepsilon_*)=0\quad\text{ and }\quad\beta(\varepsilon)>0\ \ \forall\varepsilon\in(0,\varepsilon_*).$$

Hence, we have

$$\gamma_1(\varepsilon)-\gamma_2(\varepsilon)\to\begin{cases}+\infty&\text{as }\varepsilon\to0^+,\\-\infty&\text{as }\varepsilon\to\varepsilon_*^-.\end{cases}$$

Therefore, there exists $\varepsilon_0\in(0,\varepsilon_*)$ such that $\gamma_1(\varepsilon_0)=\gamma_2(\varepsilon_0)>0$. So, by setting

$$\begin{aligned}L(t)&=F'(t)-\gamma_1(\varepsilon_0)E(t),\\M&=\gamma_1(\varepsilon_0),\end{aligned}$$

and by using (4.1), we obtain

$$\begin{aligned}L(0)&=(u_t(0),u(0))-\gamma_1(\varepsilon_0)E(0)\\&>(u_t(0),u(0))-ME(0)>0.\end{aligned}$$

Moreover, with this choice of ε_0, (4.4) becomes

$$\frac{d}{dt}L(t)\geq\beta(\varepsilon_0)L(t),$$

which gives

$$L(t)\geq L(0)e^{\beta(\varepsilon_0)t}\qquad\forall t\geq0.$$

Since $u(t)$ is global and $E(0)>0$, by Theorem 3.2 we have that $E(t)>0$ for all $t\geq0$. Hence, we arrive at the inequality

$$F'(t)\geq L(0)e^{\beta(\varepsilon_0)t}\qquad\forall t\geq0.$$

By integrating this inequality over $(0,t)$, we get

$$\|u(t)\|_2^2 = 2F(t) \geq 2F(0) + 2\frac{L(0)}{\beta(\varepsilon_0)}\left[e^{\beta(\varepsilon_0)t} - 1\right] \quad \forall t \geq 0. \tag{4.5}$$

On the other hand, by using Hölder's inequality and (2.1), we have

$$\begin{aligned}
\|u(t)\|_2 &\leq \|u(0)\|_2 + \int_0^t \|u_\tau(\tau)\|_2 \, d\tau \\
&\leq \|u(0)\|_2 + C\int_0^t \|u_\tau(\tau)\|_m \, d\tau \\
&\leq \|u(0)\|_2 + Ct^{\frac{m-1}{m}} \int_0^t \|u_\tau(\tau)\|_m^m \, d\tau \\
&\leq \|u(0)\|_2 + Ct^{\frac{m-1}{m}} \int_0^t \frac{-1}{\mu}\frac{dE(\tau)}{d\tau} \, d\tau \\
&\leq \|u(0)\|_2 + Ct^{\frac{m-1}{m}} \left[\frac{E(0)-E(t)}{\mu}\right]^{1/m} \\
&\leq \|u(0)\|_2 + C\left[\frac{E(0)}{\mu}\right]^{1/m} t^{\frac{m-1}{m}},
\end{aligned}$$

which clearly contradicts (4.5). □

5. Lower bound for the blow-up time

In this section, we give a lower bound for the blow-up time $T_{\max}$. To this end, we define

$$G(t) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u_t\|_2^2.$$

Theorem 5.1. *Assume that* (1.2) *and* (1.3) *hold, and let u be the solution of* (1.1), *which blows up at a finite $T_{\max}$. Then*

$$T_{\max} \geq \int_{G(0)}^{+\infty} \left\{A_1 + A_2\sigma^{\beta/2}\right\}^{-1} d\sigma,$$

where β, A_1 and A_2 are positive constants to be determined later in the proof.

Proof. By differentiating $G(t)$ and using (1.1), we obtain

$$\begin{aligned}
G'(t) &= (\nabla u_t, \nabla u) + \langle u_{tt}, u_t\rangle \\
&= (\nabla u_t, \nabla u) + (\Delta u, u_t) + \omega(\Delta u_t, u_t) - \mu\|u_t\|_m^m + \left(|u|^{p-2}u, u_t\right) \\
&= -\omega\|\nabla u_t\|_2^2 - \mu\|u_t\|_m^m + \left(|u|^{p-2}u, u_t\right).
\end{aligned}$$

Thus,

$$G'(t) \leq -\omega\|\nabla u_t\|_2^2 + \left(|u|^{p-1}, |u_t|\right). \tag{5.1}$$

Using Hölder's inequality, Young's inequality and the Sobolev inequality

$$\|v\|_q \leq B_q\|\nabla v\|_q \qquad \forall q \in [1, 2^*], \ \forall v \in H_0^1(\Omega),$$

we get

$$\begin{aligned}\left(|u|^{p-1},|u_t|\right) &\le \|u_t\|_p\|u\|_p^{p-1}\\ &\le \|u_t\|_p^{\alpha}+C_1\|u\|_p^{\beta}\\ &\le B_p^{\alpha}\|\nabla u_t\|_p^{\alpha}+C_2\|\nabla u\|_2^{\beta}\\ &\le A_1+\|\nabla u_t\|_2^2+A_2(G(t))^{\beta/2},\end{aligned}\tag{5.2}$$

where $1<\alpha<2$ is some positive constant, $\beta=\alpha(p-1)/(\alpha-1)$ and

$$\begin{aligned}C_1&=(\alpha-1)\alpha^{-\alpha/(\alpha-1)}, & C_2&=C_1B_p^{\beta},\\ A_1&=(2-\alpha)2^{-2/(2-\alpha)}B_p^{2\alpha/(2-\alpha)}\alpha^{\alpha/(2-\alpha)}, & A_2&=C_2 2^{\beta/2}.\end{aligned}$$

Combining (5.2) and (5.1) gives

$$G'(t)\le A_1+A_2(G(t))^{\beta/2}.\tag{5.3}$$

Finally, integrating inequality (5.3) over $(0,T_{\max})$ we get

$$T_{\max}\ge\int_0^{T_{\max}}\left\{A_1+A_2(G(\tau))^{\beta/2}\right\}^{-1}G'(\tau)\,d\tau,$$

and so

$$T_{\max}\ge\int_{G(0)}^{+\infty}\left\{A_1+A_2\sigma^{\beta/2}\right\}^{-1}\,d\sigma.$$ □

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