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BILINEAR DIFFERENTIAL OPERATORS AND $\mathfrak{osp}(1|2)$-RELATIVE COHOMOLOGY ON $\mathbb{R}^{1|1}$

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Abstract. We consider the $1|1$-dimensional real superspace $\mathbb{R}^{1|1}$ endowed with its standard contact structure defined by the 1-form α. The conformal Lie superalgebra $\mathcal{K}(1)$ acts on $\mathbb{R}^{1|1}$ as the Lie superalgebra of contact vector fields; it contains the Möbius superalgebra $\mathfrak{osp}(1|2)$. We classify $\mathfrak{osp}(1|2)$-invariant superskew-symmetric binary differential operators from $\mathcal{K}(1)\wedge\mathcal{K}(1)$ to $\mathfrak{D}_{\lambda,\mu;\nu}$ vanishing on $\mathfrak{osp}(1|2)$, where $\mathfrak{D}_{\lambda,\mu;\nu}$ is the superspace of bilinear differential operators between the superspaces of weighted densities. This result allows us to compute the second differential $\mathfrak{osp}(1|2)$-relative cohomology of $\mathcal{K}(1)$ with coefficients in $\mathfrak{D}_{\lambda,\mu;\nu}$.

1. Introduction

The space of weighted densities with weight λ (or λ-densities) on $\mathbb{R}$, denoted by

$$\mathcal{F}_\lambda = \left\{ f(dx)^\lambda \mid f \in C^\infty(\mathbb{R}) \right\}, \quad \lambda \in \mathbb{R},$$

is the space of sections of the line bundle $(T^*\mathbb{R})^{\otimes^\lambda}$ for positive integer λ. The Lie algebra $\mathrm{Vect}(\mathbb{R})$ of vector fields $X_F = F\frac{d}{dx}$ on $\mathbb{R}$, where $F \in C^\infty(\mathbb{R})$, acts by the *Lie derivative*. Alternatively, this action can be written as

$$X_F \cdot (f dx^\lambda) = L^\lambda_{X_F}(f)(dx)^\lambda, \qquad \text{with } L^\lambda_{X_F}(f) = Ff' + \lambda F'f,$$

where f' and F' are, respectively, $\frac{df}{dx}$ and $\frac{dF}{dx}$. For $(\lambda,\mu,\nu) \in \mathbb{R}^3$, each bilinear differential operator A from $C^\infty(\mathbb{R}) \otimes C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$ gives thus rise to a morphism from $\mathcal{F}_\lambda \otimes \mathcal{F}_\mu$ to $\mathcal{F}_\nu$ defined by $f dx^\lambda \otimes g dx^\mu \mapsto A(f\otimes g)dx^\nu$. The Lie algebra $\mathrm{Vect}(\mathbb{R})$ acts on the space $\mathrm{D}_{\lambda,\mu;\nu}$ of these differential operators by

$$X_F \cdot A = L^\nu_{X_F} \circ A - A \circ L^{(\lambda,\mu)}_{X_F},$$

where $L^{(\lambda,\mu)}_{X_F}$ is the Lie derivative on $\mathcal{F}_\lambda \otimes \mathcal{F}_\mu$ defined by the Leibniz rule

$$L^{(\lambda,\mu)}_{X_F}(f\otimes g) = L^\lambda_{X_F}(f)\otimes g + f \otimes L^\mu_{X_F}(g).$$

If we restrict ourselves to the Lie subalgebra of $\mathrm{Vect}(\mathbb{R})$ generated by $\left\{\frac{d}{dx}, x\frac{d}{dx}, x^2\frac{d}{dx}\right\}$, isomorphic to $\mathfrak{sl}(2)$, we get a family of infinite-dimensional $\mathfrak{sl}(2)$-modules,

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still denoted by $\mathrm{D}_{\lambda,\mu;\nu}$. Bouarroudj [6] computed $\mathrm{H}^2_{\mathrm{diff}}\left(\mathrm{Vect}(\mathbb{R}),\mathfrak{sl}(2);\mathrm{D}_{\lambda,\mu}\right)$, where $\mathrm{H}^i_{\mathrm{diff}}$ denotes the differential cohomology; that is, only cochains given by differential operators are considered. These spaces appear naturally in the problem of describing the $\mathfrak{sl}(2)$-trivial deformations of the $\mathrm{Vect}(\mathbb{R})$-module $\mathcal{S}_{\mu-\lambda}=\bigoplus_{k=0}^{\infty}\mathcal{F}_{\mu-\lambda-k}$, the space of symbols of differential operators (for example, see [1, 12]).

In this paper we study the simplest super analog of the problem solved in [6], namely, we consider the superspace $\mathbb{R}^{1|1}$ equipped with the contact structure determined by a 1-form α, and the Lie superalgebra $\mathcal{K}(1)$ of contact vector fields on $\mathbb{R}^{1|1}$. We introduce the $\mathcal{K}(1)$-module $\mathfrak{F}_\lambda$ of λ-densities on $\mathbb{R}^{1|1}$ and the $\mathcal{K}(1)$-module of bilinear differential operators, $\mathfrak{D}_{\lambda,\mu;\nu}=\mathrm{Hom}_{\mathrm{diff}}(\mathfrak{F}_\lambda\otimes\mathfrak{F}_\mu,\mathfrak{F}_\nu)$, which are super analogues of the spaces $\mathcal{F}_\lambda$ and $\mathrm{D}_{\lambda,\mu;\nu}$, respectively. The Lie superalgebra $\mathfrak{osp}(1|2)$, a super analogue of $\mathfrak{sl}(2)$, can be realized as a subalgebra of $\mathcal{K}(1)$. We classify all $\mathfrak{osp}(1|2)$-invariant bilinear differential operators from $\mathcal{K}(1)$ to $\mathfrak{D}_{\lambda,\mu;\nu}$. We use the result to compute $\mathrm{H}^2_{\mathrm{diff}}\left(\mathcal{K}(1),\mathfrak{osp}(1|2);\mathfrak{D}_{\lambda,\mu;\nu}\right)$. We show that nonzero cohomology $\mathrm{H}^2_{\mathrm{diff}}\left(\mathcal{K}(1),\mathfrak{osp}(1|2);\mathfrak{D}_{\lambda,\mu;\nu}\right)$ only appears for resonant values of weights that satisfy $\nu-\mu-\lambda\in\frac{1}{2}\mathbb{N}+3$. These spaces allow us to classify the nontrivial projectively invariant extensions of the Lie superalgebra $\mathcal{K}(1)$ by the module $\mathfrak{D}_{\lambda,\mu;\nu}$.

2. Definitions and notations

Recall that the superalgebra $C^\infty(\mathbb{R}^{1|1})$ of smooth function on the superspace $\mathbb{R}^{1|1}$ consists of elements of the form

$$F(x,\theta)=f_0(x)+f_1(x)\theta,$$

where $f_0,f_1\in C^\infty(\mathbb{R})$, and where x is the even variable and θ is the odd variable ($\theta^2=0$). Let $|F|$ be the parity of a homogeneous function F. Let

$$\mathrm{Vect}(\mathbb{R}^{1|1})=\left\{F_0\partial_x+F_1\partial_\theta \mid F_i\in C^\infty(\mathbb{R}^{1|1})\right\},$$

where $\partial_\theta=\frac{\partial}{\partial\theta}$ and $\partial_x=\frac{\partial}{\partial x}$. Let $\mathcal{K}(1)$ be the Lie superalgebra of contact vector fields on $\mathbb{R}^{1|1}$:

$$\mathcal{K}(1)=\left\{X\in\mathrm{Vect}(\mathbb{R}^{1|1}) \mid \text{there exists } F\in C^\infty(\mathbb{R}^{1|1}) \text{ such that } \mathfrak{L}_X(\alpha)=F\alpha\right\},$$

where $\mathfrak{L}_X$ is the Lie derivative along the vector field X and

$$\alpha=dx+\theta d\theta.$$

Any contact vector field on $\mathbb{R}^{1|1}$ can be expressed as

$$X_F=F\partial_x-\frac{1}{2}(-1)^{|F|}\overline{\eta}(F)\overline{\eta},$$

where $F\in C^\infty(\mathbb{R}^{1|1})$ and $\overline{\eta}=\partial_\theta-\theta\partial_x$. The contact bracket is defined by $[X_F,\,X_G]=X_{\{F,\,G\}}$:

$$\{F,G\}=FG'-F'G-\frac{1}{2}(-1)^{|F|}\overline{\eta}(F)\cdot\overline{\eta}(G).$$

The orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$ can be realized as a subalgebra of $\mathcal{K}(1)$:

$$\mathfrak{osp}(1|2)=\mathrm{Span}(X_1,\,X_x,\,X_{x^2},\,X_{x\theta},\,X_\theta).$$

The space of even elements is isomorphic to $\mathfrak{sl}(2)$, while the space of odd elements is two-dimensional:

$$(\mathfrak{osp}(1|1))_{\bar{1}} = \mathrm{Span}(X_{x\theta},\, X_\theta).$$

We define the space of λ-densities as

$$\mathfrak{F}_\lambda = \left\{ F(x,\theta)\alpha^\lambda \mid F(x,\theta) \in C^\infty(\mathbb{R}^{1|1}) \right\}.$$

As a vector space, $\mathfrak{F}_\lambda$ is isomorphic to $C^\infty(\mathbb{R}^{1|1})$, but the Lie derivative of the density $G\alpha^\lambda$ along the vector field X_F in $\mathcal{K}(1)$ is now

$$\mathfrak{L}_{X_F}(G\alpha^\lambda) = \mathfrak{L}^\lambda_{X_F}(G)\alpha^\lambda, \qquad \text{with } \mathfrak{L}^\lambda_{X_F} = X_F + \lambda F',\ \lambda \in \mathbb{R}. \tag{2.1}$$

A differential operator on $\mathbb{R}^{1|1}$ is an operator on $C^\infty(\mathbb{R}^{1|1})$ of the form

$$A = \sum_{k=0}^{M} \sum_{\varepsilon} a_{k,\varepsilon}(x,\theta)\partial_x^k \partial_\theta^\varepsilon, \qquad \varepsilon = 0, 1,\ M \in \mathbb{N}.$$

Of course any differential operator defines a linear mapping $F\alpha^\lambda \mapsto A(F)\alpha^\mu$ from $\mathfrak{F}_\lambda$ to $\mathfrak{F}_\mu$ for any λ, $\mu \in \mathbb{R}$, thus the space of differential operators becomes a $\mathcal{K}(1)$-module denoted by $\mathfrak{D}_{\lambda,\mu}$ for the natural action

$$X_F \cdot A = \mathfrak{L}^\mu_{X_F} \circ A - (-1)^{|A||F|} A \circ \mathfrak{L}^\lambda_{X_F}.$$

Similarly, we consider a family of $\mathcal{K}(1)$-modules on the space $\mathfrak{D}_{\lambda,\mu;\nu}$ of bilinear differential operators $A : \mathfrak{F}_\lambda \otimes \mathfrak{F}_\mu \to \mathfrak{F}_\nu$ with the $\mathcal{K}(1)$-action

$$X_F \cdot A = \mathfrak{L}^\nu_{X_F} \circ A - (-1)^{|A||F|} A \circ \mathfrak{L}^{(\lambda,\mu)}_{X_F},$$

where $\mathfrak{L}^{(\lambda,\mu)}_{X_F}$ is the Lie derivative on $\mathfrak{F}_\lambda \otimes \mathfrak{F}_\mu$ defined by the Leibniz rule

$$\mathfrak{L}^{(\lambda,\mu)}_{X_F}(H \otimes G) = \mathfrak{L}^\lambda_{X_F}(H) \otimes G + (-1)^{|F||H|} H \otimes \mathfrak{L}^\mu_{X_F}(G).$$

Since $\overline{\eta}^2 = -\partial_x$ and $\partial_\theta = \overline{\eta} - \theta\overline{\eta}^2$, any differential operator $A \in \mathfrak{D}_{\lambda,\mu}$ can be expressed in the form

$$A(F\alpha^\lambda) = \sum_{i=0}^{\ell} a_i\, \overline{\eta}^i(F)\alpha^\mu, \tag{2.2}$$

where the coefficients $a_i \in C^\infty(\mathbb{R}^{1|1})$ and $\ell \in \mathbb{N}$.

3. The $\mathfrak{osp}(1|2)$-relative cohomology of $\mathcal{K}(1)$ acting on $\mathfrak{D}_{\lambda,\mu;\nu}$

Let us first recall some fundamental concepts from cohomology theory (see, e.g., [8, 9, 10]).

3.1. **Lie superalgebra cohomology.** Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra acting on a superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. (If $\mathfrak{h}$ is omitted, it is assumed to be $\{0\}$.) The space of $\mathfrak{h}$-relative n-cochains of $\mathfrak{g}$ with values in V is

$$C^n(\mathfrak{g}, \mathfrak{h}; V) := \mathrm{Hom}_{\mathfrak{h}}(\Lambda^n(\mathfrak{g}/\mathfrak{h}); V).$$

The *coboundary operator* $\delta_n : C^n(\mathfrak{g}, \mathfrak{h}; V) \to C^{n+1}(\mathfrak{g}, \mathfrak{h}; V)$ is an even map satisfying $\delta_n \circ \delta_{n-1} = 0$ (see, for instance, [10]): for $\phi \in C^n(\mathfrak{g}, \mathfrak{h}; V)$,

$$\begin{aligned}(\delta_n\phi)(g_0, \ldots, g_n) &= \sum_{i=0}^{n} (-1)^i (-1)^{|g_i|(|\phi|+|g_0|+\cdots+|g_{i-1}|)} g_i \phi(g_0, \ldots, \hat{\imath}, \ldots, g_n) \\ &\quad + \sum_{0\le i<j\le n} (-1)^{i+j} (-1)^{|g_i|(|g_0|+\cdots+|g_{i-1}|)} (-1)^{|g_j|(|g_0|+\cdots+\hat{\imath}+\cdots+|g_{j-1}|)} \\ &\qquad\qquad \times \phi([g_i, g_j], g_0, \ldots, \hat{\imath}, \ldots, \hat{\jmath}, \ldots, g_n).\end{aligned}$$

The kernel of δ_n, denoted by $Z^n(\mathfrak{g}, \mathfrak{h}; V)$, is the space of $\mathfrak{h}$-relative n-*cocycles*; among them, the elements in the range of δ_{n-1} are called $\mathfrak{h}$-relative n-*coboundaries*. We denote by $B^n(\mathfrak{g}, \mathfrak{h}; V)$ the space of n-coboundaries.

By definition, the n-th $\mathfrak{h}$-relative cohomology space is the quotient space

$$\mathrm{H}^n(\mathfrak{g}, \mathfrak{h}; V) = Z^n(\mathfrak{g}, \mathfrak{h}; V)/B^n(\mathfrak{g}, \mathfrak{h}; V).$$

We can also define a $\mathfrak{g}$-action π on $C^n(\mathfrak{g}, V)$ by setting, for any $g \in \mathfrak{g}$,

$$\begin{aligned}&(\pi(g)\phi)(g_1, \ldots, g_n) \\ &\quad = g\phi(g_1, \ldots, g_n) - \sum_{i=1}^{n} (-1)^{|g|(|\phi|+|g_1|+\cdots+|g_{i-1}|)} g_i \phi(g_1, \ldots, [g, g_i], \ldots, g_n),\end{aligned}$$

and a contraction operator $\iota(g)$ from C^n to C^{n-1} by

$$(\iota(g)\phi)(g_1, \ldots, g_{n-1}) = (-1)^{|g||\phi|} \phi(g, g_1, \ldots, g_{n-1}).$$

A direct computation gives the classical formula

$$\pi(g)\phi = (\delta_{n-1} \circ \iota(g) + \iota(g) \circ \delta_n)\phi,$$

and thus $\delta_n(\pi(g)\phi) = \pi(g)(\delta_n\phi)$; that is, δ_n is a $\mathfrak{g}$-map. Note that $C^n(\mathfrak{g}, \mathfrak{h}; V)$ may be viewed as the subspace of $C^n(\mathfrak{g}, V)$ annihilated by both $\iota(\mathfrak{h})$ and $\pi(\mathfrak{h})$. We will only need the formula of δ_n (which will be simply denoted by δ) in degrees 0, 1 and 2: for $v \in C^0(\mathfrak{g}, \mathfrak{h}; V) = V^{\mathfrak{h}}$, $\delta v(g) := (-1)^{|g||v|} g \cdot v$, where

$$V^{\mathfrak{h}} = \{v \in V \mid h \cdot v = 0 \text{ for all } h \in \mathfrak{h}\}.$$

3.2. $\mathfrak{osp}(1|2)$**-invariant binary differential operators.** The following steps to compute the cohomology have intensively been used in [2, 4, 5, 6, 7, 11]. First, we classify $\mathfrak{osp}(1|2)$-invariant differential operators, then we isolate among them those that are 2-cocycles. To do that, we need the following lemma.

Lemma 3.1. *Any 2-cocycle vanishing on the subalgebra $\mathfrak{osp}(1|2)$ of $\mathcal{K}(1)$ is $\mathfrak{osp}(1|2)$-invariant.*

Proof. For $X \in \mathfrak{osp}(1|2)$, the 2-cocycle condition reads:

$$c([X,Y],Z) - (-1)^{|Y||Z|}c([X,Z],Y) = (-1)^{|X||c|}\mathfrak{L}_X^{\lambda,\mu}c(Y,Z)$$

for every $Y, Z \in \mathcal{K}(1)$. This relation is nothing but the $\mathfrak{osp}(1|2)$-invariance property of the bilinear map c. □

As our 2-cocycles vanish on $\mathfrak{osp}(1|2)$, we will investigate $\mathfrak{osp}(1|2)$-invariant superskew-symmetric binary differential operators that vanish on $\mathfrak{osp}(1|2)$. Our first main result is the following theorem.

Theorem 3.2. *The space of superskew-symmetric bilinear differential operators $\mathcal{K}(1)\wedge\mathcal{K}(1) \to \mathfrak{D}_{\lambda,\mu;\nu}$ which are $\mathfrak{osp}(1|2)$-invariant and vanish on $\mathfrak{osp}(1|2)$ is purely even if $\nu-\mu-\lambda$ is integer and is purely odd if $\nu-\mu-\lambda$ is semi-integer; moreover, this space is:*

(i) $(p-2)(4p-9)$-*dimensional if* $(\nu-\mu-\lambda) = 2p-2$ *and* $p \geq 3$;
(ii) $(4p^2-13p+11)$-*dimensional if* $(\nu-\mu-\lambda) = 2p-1$ *and* $p \geq 2$;
(iii) $(p-2)(4p-7)$-*dimensional if* $(\nu-\mu-\lambda) = 2p-\frac{3}{2}$ *and* $p \geq 3$;
(iv) $(4p^2-11p+8)$-*dimensional if* $(\nu-\mu-\lambda) = 2p-\frac{1}{2}$ *and* $p \geq 2$;
(v) *zero-dimensional otherwise.*

Proof. First, it is easy to see that, for the adjoint action, the Lie superalgebra $\mathcal{K}(1)$ is isomorphic to $\mathfrak{F}_{-1}$. So, any such a differential operator can be considered as a 4-ary differential operator $c : \mathfrak{F}_{-1}\otimes\mathfrak{F}_{-1}\otimes\mathfrak{F}_\lambda\otimes\mathfrak{F}_\mu \to \mathfrak{F}_\nu$. Thus, by (2.2), we can see that the operator c has the form

$$c(X_F, X_G, \phi, \psi) = \sum_{\substack{\varepsilon=(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)\\ 0\leq k_1,k_2,k_3,k_4\leq M}} c_\varepsilon^{k_1,k_2,k_3,k_4}(x,\theta,|F|,|G|,|\phi|,|\psi|) \times \overline{\eta}^{\varepsilon_1}(F^{(k_1)})\overline{\eta}^{\varepsilon_2}(G^{(k_2)})\overline{\eta}^{\varepsilon_3}(\phi^{(k_3)})\overline{\eta}^{\varepsilon_4}(\psi^{(k_4)}),$$

where $\varepsilon_i = 0,1$, $M \in \mathbb{N}$ and (with $F^{(k_i)}$ denoting the k_i-th derivative of F by x)

$$\overline{\eta}^{\varepsilon_i}(F^{(k_i)}) = \begin{cases} \overline{\eta}(F^{(k_i)}) & \text{if } \varepsilon_i = 1,\\ F^{(k_i)} & \text{otherwise.}\end{cases}$$

Second, observe that, since the operator c vanishes on $\mathfrak{osp}(1|2)$ (that is, it vanishes when just one argument is from $\mathfrak{osp}(1|2)$), we have $c_\varepsilon^{k_1,k_2,k_3,k_4} = 0$ for $\varepsilon_1 + k_1 \leq 2$ or $\varepsilon_2 + k_2 \leq 2$. The invariance property of c with respect to $X_H \in \mathfrak{osp}(1|2)$ reads:

$$\mathfrak{L}_{X_H}^{\lambda,\mu;\nu}c(X_F,X_G,\phi,\psi) - (-1)^{|c||H|}c([X_H,X_F],X_G,\phi,\psi) - (-1)^{|H|(|c|+|F|)}c(X_F,[X_H,X_G],\phi,\psi) = 0. \quad (3.1)$$

By direct computation using (3.1) together with (2.1) and the graded Leibniz formula

$$\overline{\eta}^j \circ F = \sum_{i=0}^{j} \binom{j}{i}_s (-1)^{|F|(j-i)} \overline{\eta}^i(F) \overline{\eta}^{j-i},$$

with

$$\binom{j}{i}_s = \begin{cases} \binom{[\frac{j}{2}]}{[\frac{i}{2}]} & \text{if } i \text{ is even or } j \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases}$$

we easily check that the invariant property of c with respect to the vector field X_x yields

$$\frac{d}{dx} c_\varepsilon^{k_1,k_2,k_3,k_4} = 0 \qquad \text{and} \qquad \overline{\eta}(c_\varepsilon^{k_1,k_2,k_3,k_4}) = 0.$$

Therefore, the coefficients $c_\varepsilon^{k_1,k_2,k_3,k_4}$ are functions of $|F|$ and $|G|$. We also get

$$\|\varepsilon\| + 2\sum_{i=1}^{4} k_i = 2(\nu - \mu - \lambda) + 4, \quad \text{where } \|\varepsilon\| = \sum_{i=1}^{4} \varepsilon_i.$$

So, the parameters λ, μ, and ν must satisfy $2(\nu - \mu - \lambda) + 4 = n$, where $n \in \mathbb{N}$. The corresponding operator can be expressed as

$$c(X_F, X_G, \phi, \psi) = \sum_{\varepsilon, k_1, k_2, k_3} c_\varepsilon^{k_1,k_2,k_3,n}(|F|, |G|, |\phi|, |\psi|)\, A_\varepsilon^{k_1,k_2,k_3,n}(F, G, \phi, \psi), \tag{3.2}$$

where $\varepsilon_1 + k_1 \geq 3$, $\varepsilon_2 + k_2 \geq 3$, and

$$A_\varepsilon^{k_1,k_2,k_3,n}(F, G, \phi, \psi) = \overline{\eta}^{\varepsilon_1}(F^{(k_1)}) \overline{\eta}^{\varepsilon_2}(G^{(k_2)}) \overline{\eta}^{\varepsilon_3}(\phi^{(k_3)}) \overline{\eta}^{\varepsilon_4}(\psi^{(\frac{1}{2}(n - \|\varepsilon\|) - k_1 - k_2 - k_3)}).$$

We easily check that the operator c is homogeneous: c is even or odd according to whether n is even or odd. Moreover, the superskew-symmetric condition

$$c(X_F, X_G, \phi, \psi) = -(-1)^{|F||G|} c(X_G, X_F, \phi, \psi)$$

leads to the following relation:

$$c_\varepsilon^{k_1,k_2,k_3,n}(|F|, |G|, |\phi|, |\psi|) = -(-1)^{\varepsilon_2 \cdot |F| + \varepsilon_1 \cdot |G| + \varepsilon_1 . \varepsilon_2} c_{\varepsilon_2, \varepsilon_1, \varepsilon_3, \varepsilon_4}^{k_2,k_1,k_3,n}(|G|, |F|, |\phi|, |\psi|). \tag{3.3}$$

Second, we consider the invariance property with respect to X_{x^2} and $X_{x\theta}$. According to the parity of n, we distinguish two cases.

The case where n is even.

In this case, the invariance property of c with respect to $X_{x\theta}$ reads:

$$\mathfrak{L}_{X_{x\theta}}^{\lambda,\mu;\nu} c(X_F, X_G, \phi, \psi) - c([X_{x\theta}, X_F], X_G, \phi, \psi) - (-1)^{|F|} c(X_F, [X_{x\theta}, X_G], \phi, \psi) = 0.$$

Collecting the terms in $x\theta A_\varepsilon^{k_1,k_2,k_3,n}(F, G, \phi, \psi)$, we get

$$\begin{aligned} c_\varepsilon^{k_1,k_2,k_3,n}(|F|, |G|, |\phi|, |\psi|) &= (-1)^{\varepsilon_1} c_\varepsilon^{k_1,k_2,k_3,n}(|F| + 1, |G|, |\phi|, |\psi|) \\ &= (-1)^{\varepsilon_1 + \varepsilon_2} c_\varepsilon^{k_1,k_2,k_3,n}(|F|, |G| + 1, |\phi|, |\psi|) \\ &= (-1)^{\varepsilon_1 + \varepsilon_2 + \varepsilon_3} c_\varepsilon^{k_1,k_2,k_3,n}(|F|, |G|, |\phi| + 1, |\psi|). \end{aligned} \tag{3.4}$$

According to formulae (3.3) and (3.4), we deduce that $c_{0,0,0,0}^{k_1,k_1,k_3,n} = c_{0,0,1,1}^{k_1,k_1,k_3,n} = 0$. The invariance property of c with respect to X_{x^2} reads:

$$\mathfrak{L}_{X_{x^2}}^{\lambda,\mu;\nu} c(X_F, X_G, \phi, \psi) - c([X_{x^2}, X_F], X_G, \phi, \psi) - c(X_F, [X_{x^2}, X_G], \phi, \psi) = 0.$$

Collecting the terms in $\theta A_{\varepsilon}^{k_1,k_2,k_3,n}(F,G)$, we get with the help of (3.3) the following conditions:

$$\begin{aligned}
&\bullet\ \Lambda^n_{k_1,k_2,k_3-2\mu+1} c_{0,0,1,1}^{k_1,k_2,k_3,n} - (k_1-2) c_{1,0,1,0}^{k_1,k_2,k_3,n} - (-1)^{|F|}(k_2-2) c_{0,1,1,0}^{k_1,k_2,k_3,n} \\
&+ (-1)^{|F|+|G|}(k_3-2) c_{0,0,0,0}^{k_1,k_2,k_3+1,n} = 0, \quad k_1+k_2+k_3 \le \frac{n-2}{2} \text{ and } k_1 > k_2 \ge 3; \\
&\bullet\ \Lambda^n_{k_1,k_2,k_3+1} c_{1,0,1,0}^{k_1,k_2,k_3,n} + (k_1+1) c_{0,0,1,1}^{k_1+1,k_2,k_3,n} + (-1)^{|F|}(k_2-2) c_{1,1,1,1}^{k_1,k_2,k_3,n} \\
&- (-1)^{|F|+|G|}(k_3+1) c_{1,0,0,1}^{k_1,k_2,k_3+1,n} = 0, \quad k_1+k_2+k_3 \le \frac{n-4}{2} \text{ and } k_1 \ge 2, k_2 \ge 3; \\
&\bullet\ \Lambda^n_{k_1,k_2,k_3-2\mu+2} c_{1,1,1,1}^{k_1,k_2,k_3,n} + (k_1+1) c_{0,1,1,0}^{k_1+1,k_2,k_3,n} - (-1)^{|F|}(k_2+1) c_{1,0,1,0}^{k_1,k_2+1,k_3,n} \\
&- (-1)^{|F|+|G|}(k_3+1) c_{1,1,0,0}^{k_1,k_2,k_3+1,n} = 0, \quad k_1+k_2+k_3 \le \frac{n-4}{2} \text{ and } k_1 \ge k_2 \ge 2; \\
&\bullet\ \Lambda^n_{k_1,k_2,k_3-2\mu+1} c_{0,1,0,1}^{k_1,k_2,k_3,n} - (k_1-2) c_{1,1,0,0}^{k_1,k_2,k_3,n} + (-1)^{|F|}(k_2+1) c_{0,0,0,0}^{k_1,k_2+1,k_3,n} \\
&+ (-1)^{|F|+|G|}(2\lambda+k_3) c_{0,1,1,0}^{k_1,k_2,k_3,n} = 0, \quad k_1+k_2+k_3 \le \frac{n-2}{2} \text{ and } k_1 \ge 3, k_2 \ge 2; \\
&\bullet\ \Lambda^n_{k_1,k_2,k_3} c_{0,0,0,0}^{k_1,k_2,k_3,n} - (k_1-2) c_{1,0,0,1}^{k_1,k_2,k_3,n} - (-1)^{|F|}(k_2-2) c_{0,1,0,1}^{k_1,k_2,k_3,n} \\
&- (-1)^{|F|+|G|}(2\lambda+k_3) c_{0,0,1,1}^{k_1,k_2,k_3,n} = 0, \quad k_1+k_2+k_3 \le \frac{n-2}{2} \text{ and } k_1 > k_2 \ge 3; \\
&\bullet\ \Lambda^n_{k_1,k_2,k_3+1} c_{1,1,0,0}^{k_1,k_2,k_3,n} + (k_1+1) c_{0,1,0,1}^{k_1+1,k_2,k_3,n} - (-1)^{|F|}(k_2+1) c_{1,0,0,1}^{k_1,k_2+1,k_3,n} \\
&- (-1)^{|F|+|G|}(2\lambda+k_3) c_{1,1,1,1}^{k_1,k_2,k_3,n} = 0, \quad k_1+k_2+k_3 \le \frac{n-4}{2} \text{ and } k_1 \ge k_2 \ge 2,
\end{aligned} \tag{3.5}$$

where $\Lambda^n_{k_1,k_2,k_3} = (-1)^{|F|+|G|+|\phi|}(\frac{n}{2} - k_1 - k_2 - k_3)$. For each n and any λ, we can see, with the help of Maple, that the system (3.5) is linearly independent. Now according to formula (3.3), we can see that all the coefficients $c_{\varepsilon}^{k_1,k_2,k_3,n}$ can be expressed in terms of

$$\begin{cases}
c_{1,0,1,0}^{k_1,k_2,k_3,n}, & k_1 \ge 2 \text{ and } k_2 \ge 3; \\
c_{1,0,0,1}^{k_1,k_2,k_3,n}, & k_1 \ge 2 \text{ and } k_2 \ge 3; \\
c_{0,0,0,0}^{k_1,k_2,k_3,n}, & k_1 > k_2 \ge 3; \\
c_{0,0,1,1}^{k_1,k_2,k_3,n}, & k_1 > k_2 \ge 3; \\
c_{1,1,0,0}^{k_1,k_2,k_3,n}, & k_1 \ge k_2 \ge 2; \\
c_{1,1,1,1}^{k_1,k_2,k_3,n}, & k_1 \ge k_2 \ge 2.
\end{cases} \tag{3.6}$$

So, we deduce that the dimension of the space of solutions is equal to

$$\#(\text{the coefficients } c_{\varepsilon}^{k_1,k_2,k_3,n} \text{ given by (3.6)}) - \#\left(\text{equations given by (3.5)}\right).$$

We will need the following lemma.

Lemma 3.3.

(1) *For* $n = 4p$*, the number of the coefficients* $c_{\varepsilon}^{k_1,k_2,k_3,n}$ *given by* (3.6) *is* $\frac{1}{48}(4n^3 - 93n^2 + 758n - 335)$ *and the number of equations given by* (3.5) *is* $\frac{1}{48}(4n^3 - 109n^2 + 1090n - 3472)$*. Moreover, for generic* λ *and* μ*, the space of* $\mathfrak{osp}(1|2)$*-invariant operators is spanned by* (3.7) *(see below).*
(2) *For* $n = 4p + 2$*, the number of the coefficients* $c_{\varepsilon}^{k_1,k_2,k_3,n}$ *given by* (3.6) *is* $\frac{1}{12}(n^3 - 24n^2 + 194n - 528)$ *and the number of equations given by* (3.5) *is* $\frac{1}{12}(n^3 - 27n^2 + 245n - 750)$*. Moreover, for generic* λ *and* μ*, the space of* $\mathfrak{osp}(1|2)$*-invariant operators is spanned by* (3.8) *(see below).*

Proof. First, we can see, by a direct computation, that the number of the coefficients $c_{\varepsilon}^{k_1,k_2,k_3,n}$ given by (3.6) and the number of equations given by (3.5) are as in Lemma 3.3 for $n = 4p$ and $4p + 2$.

Second, for $k_1 + k_2 + k_3 \leq \frac{n}{2} - 1$ with $k_1 > k_2 \geq 3$ and for generic λ, μ, it follows from the first (resp., fifth) equation in (3.5) that the coefficients $c_{0,0,1,1}^{k_1,k_2,k_3,n}$ $\big($resp., $c_{0,0,0,0}^{k_1,k_2,k_3,n}$ with $k_1+k_2+k_3 \leq \frac{n}{2}-1\big)$ are determined in terms of $c_{1,0,1,0}^{k_1,k_2,k_3,n}$ and $c_{0,0,0,0}^{k_1,k_2,k_3,n}$ $\big($resp., $c_{1,0,0,1}^{k_1,k_2,k_3,n}$ and $c_{0,0,0,0}^{k_1,k_2,\frac{n}{2}-k_1-k_2,n}\big)$. Moreover, for $k_1+k_2+k_3 \leq \frac{n}{2}-1$ with $k_1 \geq 3$, $k_2 \geq 2$ and for generic λ, μ, it follows from the fourth equation in (3.5) that the coefficients $c_{1,0,0,1}^{k_2,k_1,k_3,n}$ can be expressed in terms of $c_{0,0,0,0}^{k_1,k_2,\frac{n}{2}-k_1-k_2,n}$, $c_{1,0,1,0}^{k_2,k_1,k_3,n}$ and $c_{1,1,0,0}^{k_1,k_2,k_3,n}$. Furthermore, for $k_1 + k_2 + k_3 \leq \frac{n}{2} - 2$ with $k_1 \geq k_2 \geq 2$ and for generic λ, μ, it follows from the third (resp., sixth) equation in (3.5) that the coefficients $c_{1,1,1,1}^{k_1,k_2,k_3,n}$ $\big($resp., $c_{1,1,0,0}^{k_1,k_2,k_3,n}\big)$ are determined in terms of $c_{1,0,1,0}^{k_1,k_2+1,k_3,n}$ and $c_{1,1,0,0}^{k_1,k_2,k_3+1,n}$ $\big($resp., $c_{0,0,0,0}^{k_1,k_2,\frac{n}{2}-k_1-k_2,n}$, $c_{1,0,1,0}^{k_2,k_1,k_3,n}$, and $c_{1,1,0,0}^{k_1,k_2,\frac{n}{2}-k_1-k_2-1,n}\big)$. Finally, for $k_1 + k_2 + k_3 \leq \frac{n}{2} - 2$ with $k_1 \geq 2$, $k_2 \geq 3$, it follows from the second equation in (3.5) that the coefficients $c_{1,0,1,0}^{k_1,k_2,k_3,n}$ can be expressed in terms of

$$c_{0,0,0,0}^{k_1,k_2,\frac{n}{2}-k_1-k_2,n}, \quad c_{1,0,1,0}^{k_1,k_2,\frac{n}{2}-k_1-k_2-1,n}, \quad \text{and} \quad c_{1,1,0,0}^{k_1,k_2,\frac{n}{2}-k_1-k_2-1,n}.$$

Thus, we deduce, for generic λ and μ, that the space of $\mathfrak{osp}(1|2)$-invariant operators has the following structure:

(i) For $n=4p$, it is $\frac{1}{4}(n-8)(n-9)$-dimensional and spanned by

$$\begin{array}{ll}
c^{2,2,2p-5,n}_{1,1,0,0}, & c^{4,3,2p-7,n}_{0,0,0,0}, \\
c^{3,2,2p-6,n}_{1,1,0,0},\ c^{3,3,2p-7,n}_{1,1,0,0}, & c^{5,3,2p-8,n}_{0,0,0,0},\ c^{5,4,2p-9,n}_{0,0,0,0}, \\
c^{4,2,2p-7,n}_{1,1,0,0},\ c^{4,3,2p-8,n}_{1,1,0,0},\ c^{4,4,2p-9,n}_{1,1,0,0}, & c^{6,3,2p-9,n}_{0,0,0,0},\ c^{6,4,2p-10,n}_{0,0,0,0},\ c^{6,5,2p-11,n}_{0,0,0,0}, \\
\vdots & \vdots \\
c^{2p-4,2,1,n}_{1,1,0,0},\ c^{2p-4,3,0,n}_{1,1,0,0}, & c^{2p-4,3,1,n}_{0,0,0,0},\ c^{2p-4,4,0,n}_{0,0,0,0}, \\
c^{2p-3,2,0,n}_{1,1,0,0}, & c^{2p-3,3,0,n}_{0,0,0,0},
\end{array}$$

and

$$\begin{array}{l}
c^{2,3,2p-6,n}_{1,0,1,0},\ c^{2,4,2p-7,n}_{1,0,1,0}\cdots,\ c^{2,2p-3,0,n}_{1,0,1,0}, \\
c^{3,3,2p-7,n}_{1,0,1,0},\ c^{3,4,2p-8,n}_{1,0,1,0}\cdots,\ c^{3,2p-4,0,n}_{1,0,1,0}, \\
\vdots \\
c^{2p-5,3,1,n}_{1,0,1,0},\ c^{2p-5,4,0,n}_{1,0,1,0}, \\
c^{2p-4,3,0,n}_{1,0,1,0}.
\end{array} \tag{3.7}$$

(ii) For $n=4p+2$, it is $\frac{1}{4}(n^2-17n+74)$-dimensional and spanned by

$$\begin{array}{ll}
c^{2,2,2p-4,n}_{1,1,0,0}, & c^{4,3,2p-6,n}_{0,0,0,0}, \\
c^{3,2,2p-5,n}_{1,1,0,0},\ c^{3,3,2p-6,n}_{1,1,0,0}, & c^{5,3,2p-7,n}_{0,0,0,0},\ c^{5,4,2p-8,n}_{0,0,0,0}, \\
c^{4,2,2p-6,n}_{1,1,0,0},\ c^{4,3,2p-7,n}_{1,1,0,0},\ c^{4,4,2p-8,n}_{1,1,0,0}, & c^{6,3,2p-8,n}_{0,0,0,0},\ c^{6,4,2p-9,n}_{0,0,0,0},\ c^{6,5,2p-10,n}_{0,0,0,0}, \\
\vdots & \vdots \\
c^{2p-3,2,1,n}_{1,1,0,0},\ c^{2p-3,3,0,n}_{1,1,0,0}, & c^{2p-3,3,1,n}_{0,0,0,0},\ c^{2p-3,4,0,n}_{0,0,0,0}, \\
c^{2p-2,2,0,n}_{1,1,0,0}, & c^{2p-2,3,0,n}_{0,0,0,0},
\end{array}$$

and

$$\begin{array}{l}
c^{2,3,2p-5,n}_{1,0,1,0},\ c^{2,4,2p-6,n}_{1,0,1,0}\cdots,\ c^{2,2p-2,0,n}_{1,0,1,0}, \\
c^{3,3,2p-6,n}_{1,0,1,0},\ c^{3,4,2p-7,n}_{1,0,1,0}\cdots,\ c^{3,2p-3,0,n}_{1,0,1,0}, \\
\vdots \\
c^{2p-4,3,1,n}_{1,0,1,0},\ c^{2p-4,4,0,n}_{1,0,1,0}, \\
c^{2p-3,3,0,n}_{1,0,1,0}.
\end{array} \tag{3.8}$$

□

The case where n is odd.

In this case, the invariance property of c with respect to $X_{x\theta}$ reads:

$$\mathfrak{L}^{\lambda,\mu;\nu}_{X_{x\theta}} c(X_F, X_G, \phi, \psi) - c([X_{x\theta}, X_F], X_G, \phi, \psi) - (-1)^{|F|} c(X_F, [X_{x\theta}, X_G], \phi, \psi) = 0.$$

Collecting the terms in $x\theta A_\varepsilon^{k_1,k_2,k_3,n}(F,G,\phi,\psi)$, we get

$$\begin{aligned} c_\varepsilon^{k_1,k_2,k_3,n}(|F|,|G|,|\phi|,|\psi|) &= (-1)^{\varepsilon_2+\varepsilon_3+\varepsilon_4} c_\varepsilon^{k_1,k_2,k_3,n}(|F|+1,|G|,|\phi|,|\psi|) \\ &= (-1)^{\varepsilon_3+\varepsilon_4} c_\varepsilon^{k_1,k_2,k_3,n}(|F|,|G|+1,|\phi|,|\psi|) \\ &= (-1)^{\varepsilon_4} c_\varepsilon^{k_1,k_2,k_3,n}(|F|,|G|,|\phi|+1,|\psi|). \end{aligned} \tag{3.9}$$

According to formulae (3.3) and (3.9), we deduce that $c_{0,0,1,0}^{k_1,k_1,k_3,n} = c_{0,0,0,1}^{k_1,k_1,k_3,n} = 0$. The invariance property of c with respect to X_{x^2} reads:

$$\mathfrak{L}_{X_{x^2}}^{\lambda,\mu;\nu} c(X_F,X_G,\phi,\psi) - c([X_{x^2},X_F],X_G,\phi,\psi) - c(X_F,[X_{x^2},X_G],\phi,\psi) = 0.$$

Collecting the terms in $\theta A_\varepsilon^{k_1,k_2,k_3,n}(F,G)$, we get with the help of (3.3) the following conditions:

$$\begin{aligned}
&\bullet\ \Lambda^n_{k_1,k_2,k_3+\frac12} c_{0,0,1,0}^{k_1,k_2,k_3,n} - (k_1-2) c_{1,0,1,1}^{k_1,k_2,k_3,n} - (-1)^{|F|}(k_2-2) c_{0,1,1,1}^{k_1,k_2,k_3,n} \\
&+ (-1)^{|F|+|G|}(k_3+1) c_{0,0,0,1}^{k_1,k_2,k_3+1,n} = 0, \quad k_1+k_2+k_3 \le \frac{n-3}{2} \text{ and } k_1 > k_2 \ge 3; \\
&\bullet\ \Lambda^n_{k_1,k_2,k_3-2\mu+\frac32} c_{0,1,1,1}^{k_1,k_2,k_3,n} + (k_1-2) c_{1,1,1,0}^{k_1,k_2,k_3,n} - (-1)^{|F|}(k_2+1) c_{0,0,1,0}^{k_1,k_2+1,k_3,n} \\
&+ (-1)^{|F|+|G|}(k_3+1) c_{0,1,0,0}^{k_1,k_2,k_3+1,n} = 0, \quad k_1+k_2+k_3 \le \frac{n-3}{2} \text{ and } k_1 \ge 3, k_2 \ge 2; \\
&\bullet\ \Lambda^n_{k_1,k_2,k_3+\frac32} c_{1,1,1,0}^{k_1,k_2,k_3,n} - (k_1+1) c_{0,1,1,1}^{k_1+1,k_2,k_3,n} + (-1)^{|F|}(k_2+1) c_{1,0,1,1}^{k_1,k_2+1,k_3,n} \\
&- (-1)^{|F|+|G|}(k_3+1) c_{1,1,0,1}^{k_1,k_2,k_3+1,n} = 0, \quad k_1+k_2+k_3 \le \frac{n-5}{2} \text{ and } k_1 \ge k_2 \ge 2; \\
&\bullet\ \Lambda^n_{k_1,k_2,k_3-2\mu+\frac12} c_{0,0,0,1}^{k_1,k_2,k_3,n} + (k_1-2) c_{1,0,0,0}^{k_1,k_2,k_3,n} + (-1)^{|F|}(k_2-2) c_{0,1,0,0}^{k_1,k_2,k_3,n} \\
&+ (-1)^{|F|+|G|}(2\lambda+k_3) c_{0,0,1,0}^{k_1,k_2,k_3,n} = 0, \quad k_1+k_2+k_3 \le \frac{n-1}{2} \text{ and } k_1 > k_2 \ge 3; \\
&\bullet\ \Lambda^n_{k_1,k_2,k_3-2\mu+\frac32} c_{1,1,0,1}^{k_1,k_2,k_3,n} - (k_1+1) c_{0,1,0,0}^{k_1+1,k_2,k_3,n} + (-1)^{|F|}(k_2+1) c_{1,0,0,0}^{k_1,k_2+1,k_3,n} \\
&+ (-1)^{|F|+|G|}(2\lambda+k_3) c_{1,1,1,0}^{k_1,k_2,k_3,n} = 0, \quad k_1+k_2+k_3 \le \frac{n-3}{2} \text{ and } k_1 \ge k_2 \ge 2; \\
&\bullet\ \Lambda^n_{k_1,k_2,k_3+\frac12} c_{0,1,0,0}^{k_1,k_2,k_3,n} + (k_1-2) c_{1,1,0,1}^{k_1,k_2,k_3,n} - (-1)^{|F|}(k_2+1) c_{0,0,0,1}^{k_1,k_2+1,k_3,n} \\
&- (-1)^{|F|+|G|}(2\lambda+k_3) c_{0,1,1,1}^{k_1,k_2,k_3,n} = 0, \quad k_1+k_2+k_3 \le \frac{n-3}{2} \text{ and } k_1 \ge 3, k_2 \ge 2,
\end{aligned} \tag{3.10}$$

where $\Lambda^n_{k_1,k_2,k_3} = (-1)^{|F|+|G|+|\phi|}(\frac{n}{2} - k_1 - k_2 - k_3)$. For each n and any λ, we can see, with the help of Maple, that the system (3.10) is linearly independent. Now according to formulae (3.3), we can see that all the coefficients $c_{\varepsilon}^{k_1,k_2,k_3,n}$ can be expressed in terms of

$$\begin{cases} c_{1,0,0,0}^{k_1,k_2,k_3,n}, & k_1 \geq 2 \text{ and } k_2 \geq 3; \\ c_{1,0,1,1}^{k_1,k_2,k_3,n}, & k_1 \geq 2 \text{ and } k_2 \geq 3; \\ c_{0,0,1,0}^{k_1,k_2,k_3,n}, & k_1 > k_2 \geq 3; \\ c_{0,0,0,1}^{k_1,k_2,k_3,n}, & k_1 > k_2 \geq 3; \\ c_{1,1,1,0}^{k_1,k_2,k_3,n}, & k_1 \geq k_2 \geq 2; \\ c_{1,1,0,1}^{k_1,k_2,k_3,n}, & k_1 \geq k_2 \geq 2. \end{cases} \tag{3.11}$$

So, we deduce that the dimension of the space of solutions is equal to

$$\#(\text{the coefficients } c_{\varepsilon}^{k_1,k_2,k_3,n} \text{ given by (3.11)}) - \#(\text{equations given by (3.10)}).$$

We will need the following lemma.

Lemma 3.4.

(1) *For $n = 4p + 1$, the number of the coefficients $c_{\varepsilon}^{k_1,k_2,k_3,n}$ given by* (3.11) *is* $\frac{1}{12}(n^3 - 24n^2 + 194n - 531)$ *and the number of equations given by* (3.10) *is* $\frac{1}{12}(n^3 - 27n^2 + 245n - 747)$. *Moreover, for generic λ and μ, the space of* $\mathfrak{osp}(1|2)$*-invariant operators is spanned by* (3.12) *(see below).*
(2) *For $n = 4p + 3$, the number of the coefficients $c_{\varepsilon}^{k_1,k_2,k_3,n}$ given by* (3.11) *is* $\frac{1}{12}(n^3 - 24n^2 + 194n - 525)$ *and the number of equations given by* (3.10) *is* $\frac{1}{12}(n^3 - 27n^2 + 245n - 747)$. *Moreover, for generic λ and μ, the space of* $\mathfrak{osp}(1|2)$*-invariant operators is spanned by* (3.13) *(see below).*

Proof. First, we can see, by a direct computation, that the number of the coefficients $c_{\varepsilon}^{k_1,k_2,k_3,n}$ given by (3.11) and the number of equations given by (3.10) are as in Lemma 3.4 for $n = 4p+1$ and $4p+3$. Moreover, in a similar way as in the proof of Lemma 3.3, we deduce, for generic λ and μ, that the space of $\mathfrak{osp}(1|2)$-invariant operators has the following structure:

(i) For $n = 4p+1$, it is $\frac{1}{4}(n-8)(n-9)$-dimensional and spanned by

$$
\begin{array}{ll}
c^{2,2,2p-5,n}_{1,1,1,0}, & c^{4,3,2p-7,n}_{0,0,1,0}, \\
c^{3,2,2p-6,n}_{1,1,1,0},\ c^{3,3,2p-7,n}_{1,1,1,0}, & c^{5,3,2p-8,n}_{0,0,1,0},\ c^{5,4,2p-9,n}_{0,0,1,0}, \\
c^{4,2,2p-7,n}_{1,1,1,0},\ c^{4,3,2p-8,n}_{1,1,1,0},\ c^{4,4,2p-9,n}_{1,1,1,0}, & c^{6,3,2p-9,n}_{0,0,1,0},\ c^{6,4,2p-10,n}_{0,0,1,0},\ c^{6,5,2p-11,n}_{0,0,1,0}, \\
\vdots & \vdots \\
c^{2p-4,2,1,n}_{1,1,1,0},\ c^{2p-4,3,0,n}_{1,1,1,0}, & c^{2p-4,3,1,n}_{0,0,1,0},\ c^{2p-4,4,0,n}_{0,0,1,0}, \\
c^{2p-3,2,0,n}_{1,1,1,0}, & c^{2p-3,3,0,n}_{0,0,1,0},
\end{array}
$$

and

$$
\begin{array}{l}
c^{2,3,2p-5,n}_{1,0,0,0},\ c^{2,4,2p-6,n}_{1,0,0,0} \cdots ,\ c^{2,2p-2,0,n}_{1,0,0,0}, \\
c^{3,3,2p-6,n}_{1,0,0,0},\ c^{3,4,2p-7,n}_{1,0,0,0} \cdots ,\ c^{3,2p-3,0,n}_{1,0,0,0}, \\
\vdots \\
c^{2p-4,3,1,n}_{1,0,0,0},\ c^{2p-4,4,0,n}_{1,0,0,0}, \\
c^{2p-3,3,0,n}_{1,0,0,0}.
\end{array}
\tag{3.12}
$$

(ii) For $n = 4p+3$, it is $\frac{1}{4}(n^2-17n+74)$-dimensional and spanned by

$$
\begin{array}{ll}
c^{2,2,2p-4,n}_{1,1,1,0}, & c^{4,3,2p-6,n}_{0,0,1,0}, \\
c^{3,2,2p-5,n}_{1,1,1,0},\ c^{3,3,2p-6,n}_{1,1,1,0}, & c^{5,3,2p-7,n}_{0,0,1,0},\ c^{5,4,2p-8,n}_{0,0,1,0}, \\
c^{4,2,2p-6,n}_{1,1,1,0},\ c^{4,3,2p-7,n}_{1,1,1,0},\ c^{4,4,2p-8,n}_{1,1,1,0}, & c^{6,3,2p-8,n}_{0,0,1,0},\ c^{6,4,2p-9,n}_{0,0,1,0},\ c^{6,5,2p-10,n}_{0,0,1,0}, \\
\vdots & \vdots \\
c^{2p-3,2,1,n}_{1,1,1,0},\ c^{2p-3,3,0,n}_{1,1,1,0}, & c^{2p-3,3,1,n}_{0,0,1,0},\ c^{2p-3,4,0,n}_{0,0,1,0}, \\
c^{2p-2,2,0,n}_{1,1,1,0}, & c^{2p-2,3,0,n}_{0,0,1,0},
\end{array}
$$

and

$$
\begin{array}{l}
c^{2,3,2p-4,n}_{1,0,0,0},\ c^{2,4,2p-5,n}_{1,0,0,0} \cdots ,\ c^{2,2p-1,0,n}_{1,0,0,0}, \\
c^{3,3,2p-5,n}_{1,0,0,0},\ c^{3,4,2p-6,n}_{1,0,0,0} \cdots ,\ c^{3,2p-2,0,n}_{1,0,0,0}, \\
\vdots \\
c^{2p-3,3,1,n}_{1,0,0,0},\ c^{2p-3,4,0,n}_{1,0,0,0}, \\
c^{2p-2,3,0,n}_{1,0,0,0}.
\end{array}
\tag{3.13}
$$

□

Now, using Lemma 3.3 and Lemma 3.4, we easily check that Theorem 3.2 is proved. □

3.3. **The $\mathfrak{osp}(1|2)$-relative cohomology of $\mathcal{K}(1)$.** In this subsection, we will compute the second differential $\mathfrak{osp}(1|2)$-relative cohomology spaces $\mathrm{H}^2_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu;\nu})$. Our second main result is the following:

Theorem 3.5. *For $\nu-\mu-\lambda \leq \frac{9}{2}$, the space $\mathrm{H}^2_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu;\nu})$ has the following structure:*

(i) *If $\nu-\mu-\lambda=3$, then*

$$\mathrm{H}^2_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu;\nu}) \simeq \begin{cases} \mathbb{R} & \text{if } (\lambda,\mu) \in \{(0,0),(0,-\frac{5}{2}),(-\frac{5}{2},0)\}, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *If $\nu-\mu-\lambda=\frac{7}{2}$, then*

$$\mathrm{H}^2_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu;\nu}) \simeq \begin{cases} \mathbb{R} & \text{if } (\lambda,\mu) \in \left\{ \begin{array}{c} (0,0),(\frac{-3}{2},0), \\ (-\frac{5}{4},0),(0,-\frac{5}{4}) \end{array} \right\}, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) *If $\nu-\mu-\lambda=4$, then*

$$\mathrm{H}^2_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu;\nu}) \simeq \begin{cases} \mathbb{R} & \text{if } (\lambda,\mu) \in \left\{ \begin{array}{c} (0,-2),(0,-\frac{1}{2}), \\ (-1,0),(-\frac{1}{2},0) \end{array} \right\}, \\ 0 & \text{otherwise.} \end{cases}$$

(iv) *If $\nu-\mu-\lambda=\frac{9}{2}$, then*

$$\mathrm{H}^2_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu;\nu}) \simeq \begin{cases} \mathbb{R} & \text{if } (\lambda,\mu) \in \{(-2,0),(-\frac{5}{2},0)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.6. $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu;\nu})$ has been computed in [3].

The proof of Theorem 3.5 will be the subject of subsection 3.5. In fact, we first need the description of $\mathfrak{osp}(1|2)$-invariant trilinear operators, from $\mathfrak{F}_{-1}\otimes\mathfrak{F}_\lambda\otimes\mathfrak{F}_\mu$ to $\mathfrak{F}_{\lambda+\mu+k-1}$.

3.4. **$\mathfrak{osp}(1|2)$-invariant trilinear differential operators.**

Proposition 3.7 ([3]). *The space of trilinear differential operators $T:\mathcal{K}(1)\otimes\mathfrak{F}_\lambda\otimes\mathfrak{F}_\mu \to \mathfrak{F}_{\lambda+\mu+k-1}$ which are $\mathfrak{osp}(1|2)$-invariant and vanish on $\mathfrak{osp}(1|2)$ is purely even if $\nu-\mu-\lambda$ is integer and is purely odd if $\nu-\mu-\lambda$ is semi-integer; moreover, it is:*

(i) *$2(\nu-\mu-\lambda-1)$-dimensional if $2(\nu-\mu-\lambda)\in\mathbb{N}+3$, generated by*

$$c_{1,0,0}^{\frac{k-1}{2},0,0},\ c_{1,0,0}^{\frac{k-3}{2},1,0},\ c_{1,0,0}^{\frac{k-5}{2},2,0},\ \dots,\ c_{1,0,0}^{2,\frac{k-5}{2},0},$$
$$c_{1,1,1}^{\frac{k-3}{2},0,0},\ c_{1,1,1}^{\frac{k-5}{2},1,0},\ c_{1,1,1}^{\frac{k-7}{2},2,0},\ \dots,\ c_{1,1,1}^{2,\frac{k-7}{2},0} \quad \text{if } \nu-\mu-\lambda \text{ is semi-integer;}$$

and

$$c_{1,1,0}^{\frac{k}{2}-1,0,0},\ c_{1,1,0}^{\frac{k}{2}-2,0,1},\ c_{1,1,0}^{\frac{k}{2}-3,0,2},\ \dots,\ c_{1,1,0}^{2,0,\frac{k}{2}-3},$$
$$c_{1,0,1}^{\frac{k}{2}-1,0,0},\ c_{1,0,1}^{\frac{k}{2}-2,1,0},\ c_{1,0,1}^{\frac{k}{2}-3,2,0},\ \dots,\ c_{1,0,1}^{2,\frac{k}{2}-3,0} \quad \text{if } \nu-\mu-\lambda \text{ is integer.}$$

(ii) *zero-dimensional otherwise.*

In order to prove Theorem 3.5, we will study properties of the coboundaries.

Lemma 3.8. *Let $B : \mathcal{K}(1) \to \mathfrak{D}_{\lambda,\mu;\nu}$ be an operator vanishing on $\mathfrak{osp}(1|2)$. If $\delta(B)$ belongs to $B^2(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu;\nu})$, then B is an $\mathfrak{osp}(1|2)$-invariant trilinear differential operator.*

Proof. For all X, $Y \in \mathcal{K}(1)$, $\phi\alpha^\lambda \in \mathfrak{F}_\lambda$, and $\psi\alpha^\mu \in \mathfrak{F}_\mu$, we have

$$\begin{aligned}\delta(B)(X, Y, \phi, \psi) := (-1)^{|X||B|}\mathfrak{L}_X^{\lambda,\mu;\nu}B(Y, \phi, \psi) - (-1)^{|Y|(|X|+|B|)}\mathfrak{L}_Y^{\lambda,\mu;\nu}B(X, \phi, \psi)\\ - B([X, Y], \phi, \psi).\end{aligned}$$

Since $\delta(B)(X, Y, \phi, \psi) = B(X, \phi, \psi) = 0$ for all $X \in \mathfrak{osp}(1|2)$, we deduce that

$$(-1)^{|X||B|}\mathfrak{L}_X^{\lambda,\mu;\nu}B(Y, \phi, \psi) - B([X, Y], \phi, \psi) = 0.$$

Thus, the operator B is $\mathfrak{osp}(1|2)$-invariant; therefore it coincides with $\mathfrak{osp}(1|2)$-invariant trilinear differential operators. □

Now, clearly, the coboundary $\delta(T)$ has the form

$$\delta(T)(X_F, X_G, \phi, \psi) = \sum_{\varepsilon,k_1,k_2k_3,k_4} \beta_\varepsilon^{k_1,k_2,k_3,n}(|F|, |G|, |\phi|, |\psi|)A_\varepsilon^{k_1,k_2,k_3,n}(F, G, \phi, \psi),$$

where $\varepsilon_i = 0, 1$.

3.5. **Proof of Theorem 3.5.** According to Lemma 3.1, any 2-cocycle of $\mathcal{K}(1)$ with coefficients in $\mathfrak{D}_{\lambda,\mu;\nu}$ vanishing on $\mathfrak{osp}(1|2)$ is $\mathfrak{osp}(1|2)$-invariant. So, by Theorem 3.2, it is identically zero if $\nu - \mu - \lambda < 3$ and expressed as in (3.2) for $\nu - \mu - \lambda \in \frac{1}{2}\mathbb{N} + 3$.

For $\nu - \mu - \lambda \in \frac{1}{2}\mathbb{N} + 3$, the proof of Theorem 3.5 consists in two steps. First, we investigate operators that belong to $Z^2(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu;\nu})$. The 2-cocycle condition imposes conditions on the coefficients $c_\varepsilon^{k_1,k_2,k_3,n}$: we get a linear system for $c_\varepsilon^{k_1,k_2,k_3,n}$. Second, taking into account these conditions, we eliminate all coefficients underlying coboundaries. Gluing these bits of information together we deduce that $\dim \mathrm{H}^2$ is equal to the number of independent coefficients $c_\varepsilon^{k_1,k_2,k_3,n}$ remaining in the expression of the 2-cocycle (3.2).

3.5.1. *The case where $\nu - \mu - \lambda = 3$.* In this case, according to Theorem 3.2, the 2-cocycle (3.2) can be expressed as

$$c(X_F, X_G, \phi, \psi) = c_{1,1,0,0}^{2,2,0,10}\gamma(X_F, X_G, \phi, \psi),$$

where

$$\gamma(X_F, X_G, \phi, \psi) = \overline{\eta}(F'')\overline{\eta}(G'')\phi\psi.$$

Therefore, by a direct computation, we can see that the 2-cocycle condition is always satisfied. Let us study the triviality of this 2-cocycle. According to subsection 3.4, we can see that any coboundary $\delta(B) \in B^2(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu;\nu})$ can be expressed as

$$\delta(B) = \delta(T).$$

A direct computation confirms that the coefficients of $\delta(T)$ are expressed in terms of

$$\beta^{2,2,0,10}_{1,1,0,0} = \mu\big(\mu+\tfrac{5}{2}\big)c^{2,0,1}_{1,0,1} + (-1)^{|G|}\lambda\left(\big(\lambda+\tfrac{5}{2}\big)c^{2,1,0}_{1,1,0} + 2\mu c^{2,0,1}_{1,1,0}\right) + 3(-1)^{|F|}\lambda\mu c^{3,0,0}_{0,1,1}.$$

So, for $(\lambda,\mu) = (0,0), (0,-\frac{5}{2}), (-\frac{5}{2},0)$, clearly the coefficients $c^{2,2,0,10}_{1,1,0,0}$ cannot be eliminated by adding a coboundary because $\beta^{2,2,0,10}_{1,1,0,0}$ is zero. Hence, the cohomology is one-dimensional.

For $(\lambda,\mu) \notin \big\{(0,0), (0,-\frac{5}{2}), (-\frac{5}{2},0)\big\}$, the coefficients $c^{2,2,0,10}_{1,1,0,0}$ can be eliminated by adding a coboundary since $\beta^{2,2,0,10}_{1,1,0,0}$ is nonzero. Hence, the cohomology is zero-dimensional.

3.5.2. *The case where* $\nu-\mu-\lambda = \frac{7}{2}$. In this case, according to Theorem 3.2, the space of solutions is spanned by

$$c^{2,2,0,11}_{1,1,1,0},\ c^{2,3,0,11}_{1,0,0,0}.$$

Therefore, by a direct computation, we can see that the 2-cocycle condition is always satisfied. Let us study the triviality of this 2-cocycle. According to subsection 3.4, we can see that any coboundary $\delta(B) \in B^2(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu;\nu})$ can be expressed as

$$\delta(B) = \delta(T).$$

A direct computation confirms that the coefficients of $\delta(T)$ are expressed in terms of

$$\beta^{2,2,0,11}_{1,1,1,0} = \mu c^{2,1,0}_{1,1,1} - 2\mu c^{2,0,1}_{1,1,1} - \tfrac{3}{2}(-1)^{|F|+|G|}\left(\big(\lambda+\tfrac{3}{2}\big)c^{3,1,0}_{0,1,0} + \mu c^{3,0,1}_{0,1,0}\right),$$

$$\begin{aligned}\beta^{2,3,0,11}_{1,0,0,0} = {} & (-1)^{|F|}\lambda\left(4\mu c^{3,0,1}_{0,1,0} + \big(2\lambda+\tfrac{5}{2}\big)c^{3,1,0}_{0,1,0}\right) + (-1)^{|F|+|G|}\mu\big(2\mu+\tfrac{5}{2}\big)c^{3,0,1}_{0,0,1}\\ & + \tfrac{1}{3}(-1)^{|F|+|G|}\lambda\mu\left((4\lambda+1)c^{2,1,0}_{1,1,1} + (4\mu+1)c^{2,0,1}_{1,1,1}\right).\end{aligned}$$

So, in the same way as before, for $(\lambda,\mu) = (-\frac{3}{2},0)$ (resp., $(\lambda,\mu) = (0,0), (0,-\frac{5}{4})$, $(-\frac{5}{4},0)$), clearly the coefficients $c^{2,2,0,11}_{1,1,1,0}$ (resp., $c^{2,3,0,11}_{1,0,0,0}$) cannot be eliminated by adding a coboundary because $\beta^{2,2,0,11}_{1,1,1,0}$ (resp., $\beta^{2,3,0,11}_{1,0,0,0}$) is zero. Hence, the cohomology is one-dimensional.

For $(\lambda,\mu) \notin \big\{(-\frac{5}{4},0), (0,-\frac{5}{4}), (-\frac{3}{2},0), (0,0)\big\}$, the coefficients $c^{2,2,0,11}_{1,1,1,0}$ and $c^{2,3,0,11}_{1,0,0,0}$ can be eliminated by adding a coboundary since $\beta^{2,2,0,11}_{1,1,1,0}$ and $\beta^{2,3,0,11}_{1,0,0,0}$ are nonzero. Hence, the cohomology is zero-dimensional.

3.5.3. *The case where* $\nu-\mu-\lambda = 4$. In this case, according to Theorem 3.2, the space of solutions is spanned by

$$c^{3,2,0,12}_{1,1,0,0},\ c^{2,3,0,12}_{1,0,1,0},\ c^{2,2,1,12}_{1,1,0,0}.$$

Therefore, by a direct computation, we can see that the 2-cocycle condition is always satisfied. Let us study the triviality of this 2-cocycle. According to subsection 3.4, we can see that any coboundary $\delta(B) \in B^2(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu;\nu})$ can be expressed as

$$\delta(B)=\delta(T).$$

A direct computation confirms that the coefficients of $\delta(T)$ are expressed in terms of

$$\begin{aligned}\beta^{2,2,0,12}_{1,1,0,0}&=3(-1)^{|F|}\lambda\left((\mu+\tfrac{3}{2})c^{3,0,1}_{0,1,1}-(\lambda+\tfrac{5}{2})c^{3,1,0}_{0,1,1}\right)+2(-1)^{|G|}\lambda(2\mu+1)c^{2,0,2}_{1,1,0}\\&\qquad+\lambda(2\lambda+5)c^{2,2,0}_{1,0,1}+2(\mu+2)(\mu+\tfrac{1}{2})c^{2,0,2}_{1,0,1},\\ \beta^{2,3,0,12}_{1,0,1,0}&=(-1)^{|F|}\mu\left(c^{3,0,1}_{0,1,1}-(\lambda+\tfrac{1}{2})c^{3,1,0}_{0,1,1}\right)+\tfrac{2}{3}\lambda\mu c^{2,2,0}_{1,0,1}\\&\qquad+\tfrac{1}{6}(-1)^{|G|}\left(2(\lambda+1)(\lambda+\tfrac{1}{2})c^{2,2,0}_{1,1,0}+\mu(2\mu+7)c^{2,0,2}_{1,1,0}\right),\\ \beta^{2,2,0,12}_{1,1,1,1}&=c^{2,0,2}_{1,0,1}-c^{2,2,0}_{1,0,1}-\tfrac{3}{4}(-1)^{|F|}\left((2\lambda+1)c^{3,1,0}_{0,1,1}+(2\mu+1)c^{3,0,1}_{0,1,1}\right).\end{aligned}$$

So, in the same way as before, for $(\lambda,\mu)=(-\frac{1}{2},0),(-1,0)$ (resp., for $(\lambda,\mu)=(0,-\frac{1}{2}),(0,-2)$), clearly the coefficients $c^{2,3,0,12}_{1,0,1,0}$ (resp., $c^{2,2,0,12}_{1,1,0,0}$) cannot be eliminated by adding a coboundary because $\beta^{2,3,0,12}_{1,0,1,0}$ (resp., $\beta^{2,2,0,12}_{1,1,0,0}$) is zero; moreover, the coefficient $c^{2,2,0,12}_{1,1,1,1}$ can be eliminated by adding a coboundary since $\beta^{2,2,0,12}_{1,1,1,1}$ is nonzero. Hence, the cohomology is one-dimensional.

For $(\lambda,\mu)\notin\left\{(-\frac{1}{2},0),(-1,0),(0,-\frac{1}{2}),(0,-2)\right\}$, the coefficients $c^{2,2,0,12}_{1,1,1,1}$, $c^{2,3,0,12}_{1,0,1,0}$, and $c^{2,2,0,12}_{1,1,0,0}$ can be eliminated by adding a coboundary since $\beta^{2,2,0,12}_{1,1,1,1}$, $\beta^{2,3,0,12}_{1,0,1,0}$, and $\beta^{2,2,0,12}_{1,1,0,0}$ are nonzero. Hence, the cohomology is zero-dimensional.

3.5.4. *The case where* $\nu-\mu-\lambda=\frac{9}{2}$. In this case, a straightforward computation shows that the condition of 2-cocycle is equivalent to formulae (3.10) corresponding to $\mathfrak{osp}(1|2)$-invariant operators together with the equation

$$\lambda(-1)^{|F|+|G|}c^{2,2,1,13}_{1,1,1,0}+\mu c^{2,2,0,13}_{1,1,0,1}=0.$$

Thus, we have just proved that the coefficients of every 2-cocycle are expressed in terms of

$$c^{3,2,0,13}_{0,1,1,1},\ c^{2,2,0,13}_{1,1,0,1},\ c^{3,2,0,13}_{1,1,1,0},\ c^{2,2,1,13}_{1,1,1,0}.$$

On the other hand, according to subsection 3.4, we can see that any coboundary $\delta(B)\in B^2(\mathcal{K}(1),\mathfrak{osp}(1|2);\mathfrak{D}_{\lambda,\mu;\nu})$ can be expressed as

$$\delta(B)=\delta(T).$$

A direct computation confirms that the coefficients of $\delta(T)$ are expressed in terms of

$$\begin{aligned}\beta^{3,2,0,13}_{1,1,1,0}&=2(-1)^{|F|+|G|}\mu(\lambda+1)c^{3,2,0}_{0,0,1}+(\lambda+2)(\lambda+\tfrac{5}{2})c^{2,3,0}_{1,0,0}+(-1)^{|F|}\mu(\mu+\tfrac{1}{2})c^{3,0,2}_{0,1,0}\\&\qquad+(-1)^{|G|}\mu\left(\tfrac{1}{3}(4\mu+7)c^{2,0,2}_{1,1,1}+2(\lambda+1)(\lambda+\tfrac{5}{6})c^{2,2,0}_{1,1,1}\right),\end{aligned}$$

$$\beta^{2,2,1,13}_{1,1,1,0} = -3(\lambda-\tfrac{3}{2})c^{2,3,0}_{1,0,0} + (-1)^{|G|}\left((\lambda+1)(4\mu+3)c^{2,2,0}_{1,1,1} - \mu(\mu+\tfrac{5}{2})c^{2,1,1}_{1,1,1}\right)$$
$$- 3(-1)^{|F|+|G|}(\mu-\tfrac{3}{2})c^{3,2,0}_{0,0,1},$$
$$\beta^{3,2,0,13}_{0,1,1,1} = 4c^{4,0,0}_{1,1,1} + \tfrac{1}{2}(-1)^{|F|}c^{3,2,0}_{0,0,1} + \tfrac{1}{2}(-1)^{|F|+|G|}c^{3,0,2}_{0,1,0},$$
$$\beta^{2,2,0,13}_{1,1,0,1} = -9(-1)^{|F|+|G|}c^{3,2,0}_{0,0,1} - 3(-1)^{|F|}\lambda c^{3,0,2}_{0,1,0} - 3(\mu+1)c^{2,1,2}_{1,0,0}$$
$$+ (-1)^{|G|}\left(4\lambda(\mu+1)c^{2,0,2}_{1,1,1} - 6(\lambda+1)c^{2,2,0}_{1,1,1} + \lambda(\lambda+\tfrac{5}{2})c^{2,1,1}_{1,1,1}\right).$$

So, in the same way as before, for $(\lambda,\mu) = (-\frac{5}{2},0),(-2,0)$, clearly the coefficient $c^{3,2,0,13}_{1,1,1,0}$ cannot be eliminated by adding a coboundary because $\beta^{3,2,0,13}_{1,1,1,0}$ is zero; moreover, the coefficients $c^{2,2,1,13}_{1,1,1,0}$, $c^{3,2,0,13}_{0,1,1,1}$, and $c^{2,2,0,13}_{1,1,0,1}$ can be eliminated by adding a coboundary since $\beta^{2,2,1,13}_{1,1,1,0}$, $\beta^{3,2,0,13}_{0,1,1,1}$, and $\beta^{2,2,0,13}_{1,1,0,1}$ are nonzero. Hence, the cohomology is one-dimensional.

For $(\lambda,\mu) \notin \{(-\frac{5}{2},0),(-2,0)\}$, the coefficients $c^{3,2,0,13}_{1,1,1,0}$, $c^{2,2,1,13}_{1,1,1,0}$, $c^{3,2,0,13}_{0,1,1,1}$, and $c^{2,2,0,13}_{1,1,0,1}$ can be eliminated by adding a coboundary since $\beta^{3,2,0,13}_{1,1,1,0}$, $\beta^{2,2,1,13}_{1,1,1,0}$, $\beta^{3,2,0,13}_{0,1,1,1}$, and $\beta^{2,2,0,13}_{1,1,0,1}$ are nonzero. Hence, the cohomology is zero-dimensional.

This completes the proof of Theorem 3.5. □

Conjecture 3.9. *For $\nu-\mu-\lambda \geq 5$, the second differential $\mathfrak{osp}(1|2)$-relative cohomology of $\mathcal{K}(1)$ with coefficients in $\mathfrak{D}_{\lambda,\mu;\nu}$ is trivial.*

3.6. **Extensions of $\mathcal{K}(1)$.** The theory of algebra extensions and their interpretation in terms of cohomology is well known; see, e.g., [9]. The second cohomology space $\mathrm{H}^2(\mathfrak{g},V)$ classifies the nontrivial extensions of the Lie superalgebra $\mathfrak{g}$ by the module V:

$$0 \longrightarrow V \longrightarrow \mathfrak{g}_V \longrightarrow \mathfrak{g} \longrightarrow 0,$$

the Lie structure on $\mathfrak{g}_V = \mathfrak{g}\oplus V$ being given by

$$[(g_1,a),\,(g_2,b)] = ([g_1,g_2],\, g_1.b - g_2.a + c(g_1,g_2)),$$

where c is a 2-cocycle with values in V.

We consider a natural class of "non-central" extensions of $\mathcal{K}(1)$, namely extensions by the module $\mathfrak{D}_{\lambda,\mu;\nu}$ of bilinear differential operators acting on weighted densities. We will be interested in the projectively invariant extensions which are given by projectively invariant 2-cocycles c. The cocycle c in this case represents a nontrivial cohomology class of the second cohomology space $\mathrm{H}^2_{\mathrm{diff}}(\mathcal{K}(1),\mathfrak{osp}(1|2);\mathfrak{D}_{\lambda,\mu;\nu})$. We mention that the same problem was considered in [13, 14]. The result is quite surprising:

Proposition 3.10. *In any of these four cases:*

- $\nu-\mu-\lambda = 3$ *and* $(\lambda,\mu) = (0,0),(0,-\frac{5}{2}),(-\frac{5}{2},0)$,
- $\nu-\mu-\lambda = \frac{7}{2}$ *and* $(\lambda,\mu) = (0,0),(-\frac{3}{2},0),(0,-\frac{5}{4}),(-\frac{5}{4},0)$,
- $\nu-\mu-\lambda = 4$ *and* $(\lambda,\mu) = (0,-2),(0,-\frac{1}{2}),(-\frac{1}{2},0),(-1,0)$,
- $\nu-\mu-\lambda = \frac{9}{2}$ *and* $(\lambda,\mu) = (-2,0),(-\frac{5}{2},0)$,

there exists a unique non-trivial extension of $\mathcal{K}(1)$ by $\mathfrak{D}_{\lambda,\mu;\nu}$.

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