REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 65, No. 1, 2023, Pages 79–101
Published online: April 18, 2023
https://doi.org/10.33044/revuma.2133

GLOBAL ATTRACTORS IN THE PARAMETRIZED HÉNON–DEVANEY MAP

BLADISMIR LEAL AND SERGIO MUÑOZ

ABSTRACT. Given two positive real numbers a and c, we consider the (two-parameter) family of nonlinear mappings

$$F_{a,c}(x,y) = \left(ax + \frac{1}{y},\, cy - \frac{c}{y} - acx\right).$$

$F_{1,1}$ is the classical Hénon–Devaney map. For a large region of parameters, we exhibit an invariant non-bounded closed set with fractal structure which is a global attractor. Our approach leads to an in-depth understanding of the Hénon–Devaney map and its perturbations.

1. Introduction

Hénon [2] introduced the nonlinear mapping $H : \mathbb{R}^2 \setminus \{y = 0\} \to \mathbb{R}^2 \setminus \{y = -x\}$ defined by

$$H(x,y) = \left(x + \frac{1}{y},\, y - \frac{1}{y} - x\right)$$

as an asymptotic form of the equations of motion of the restricted three-body problem. Devaney [1] showed that this map is topologically conjugated to the baker transformation hence transitive with dense periodic orbits. For this reason the above map H is known as the *Hénon–Devaney map*. More recently, Lenarduzzi [7] constructed a semi-conjugacy to a subshift of finite type and extended such a coding to a more general class of maps that can be seen as a map in a square with a fixed discontinuity. Leal and Muñoz [4] generalized to a large class of homeomorphisms similar to the Hénon–Devaney map, which are transitive in the whole plane.

Throughout this article, given $a, c \in \mathbb{R}$, $a > 0$, $c > 0$, we consider the (two-parameter) family

$$F_{a,c}(x,y) = \left(ax + \frac{1}{y},\, cy - \frac{c}{y} - acx\right)$$

2020 *Mathematics Subject Classification.* 37E30, 37C10, 37N05.

Key words and phrases. Parametrized Hénon–Devaney map, non-bounded attractors, invariant non-compact sets, Cantor sets.

Work funded by project PYTAUTO1910-2019-ICB0018 of Universidad Técnica de Manabí.

of diffeomorphisms with curves of singularities

$$\gamma = \{(x,y) : y = 0\} \quad \text{and} \quad \delta = \{(x,y) : y = -cx\},$$

whose inverse is given by

$$F_{a,c}^{-1}(x,y) = \left(\frac{x}{a} - \frac{c}{acx + ay}, \frac{cx + y}{c} \right).$$

Note that $F_{1,1}$ is the classical Hénon–Devaney map. For a large region of parameters, we exhibit an *invariant non-bounded closed set* with *fractal* structure which is a *global attractor*. Let us state our results in a precise way.

We will write F instead of $F_{a,c}$. By a *long curve* α we mean a simple curve in $\mathbb{R}^2$ dividing $\mathbb{R}^2$ into two open unbounded regions (denoted by R_α^+ and R_α^- throughout) having the same border α; that is, $R_\alpha^+ \cup \alpha \cup R_\alpha^- = \mathbb{R}^2$, $R_\alpha^+ \cap R_\alpha^- = \emptyset$ and $\overline{R_\alpha^+} \cap \overline{R_\alpha^-} = \alpha$. Note that, in particular, the singularities $\gamma = \{y = 0\}$ and $\delta = \{y = -cx\}$ are long curves. We consider $R_\gamma^+ = \{(x,y) : y > 0\}$, $R_\gamma^- = \{(x,y) : y < 0\}$, $R_\delta^+ = \{(x,y) : y > -cx\}$ and $R_\delta^- = \{(x,y) : y < -cx\}$. In general, given α a long curve which is the graph of a monotone function (that is, $\alpha = \{(x, f(x)) : x \in I\}$, where $f : I \subset \mathbb{R} \to \mathbb{R}$ is an increasing or decreasing function defined at some interval), we choose $R_\alpha^+ = \{(x,y) : x \in I \text{ and } y > f(x)\}$; therefore, $R_\alpha^- = \{(x,y) : x \in I \text{ and } y < f(x)\}$. Note that, given two disjoint long curves α, β which are graphs of decreasing functions with $\alpha \subset R_\beta^-$, the intersection $\overline{R_\alpha^+} \cap \overline{R_\beta^-}$ is a non-bounded closed region limited by α and β.

Main Theorem. *Suppose $0 < c < a < 1$. Then the following statements hold.*

(1) There are two simple curves $L^+ \subset R_\gamma^+ \cap R_\delta^+$ and $L^- \subset R_\gamma^- \cap R_\delta^-$ satisfying the following conditions:
  - *(a) $F(L^+)$ and $F(L^-)$ are long curves contained in R_δ^+ and R_δ^-, respectively, which are graphs of strictly decreasing functions intersecting γ transversely at single points.*
  - *(b) $F(L^+) \cap L^+ = L^+$ and $F(L^-) \cap L^- = L^-$.*

(2) Consider $\overline{R} := \overline{R_{F(L^-)}}^{+} \cap \overline{R_{F(L^+)}}^{-}$ and

$$\overline{\Gamma} := \overline{R} \cap \bigcap_{n=1}^{\infty} \overbrace{F\big(\cdots F\big(F}^{n\ times}\big(\overline{R} \setminus \overbrace{\gamma\big) \setminus \gamma\big) \setminus \cdots \setminus \gamma}^{n\ times}\big).$$

  - *(a) $F\big(\overline{R} \setminus \gamma\big) \subset \overline{R}$ and $\overline{\Gamma}$ is a non-bounded closed set with $F\big(\overline{\Gamma} \setminus \gamma\big) = \overline{\Gamma}$.*
  - *(b) If $p \in \mathbb{R}^2$ and $\{F^n(p)\}_{n \geq 0}$ is defined, then $\omega(p) \neq \emptyset$ and $\omega(p) \subset \overline{\Gamma}$.*

(3) Let r be any long curve which is the graph of a strictly increasing function contained in $\mathbb{R}^2 \setminus \gamma$. Then $r \cap \overline{\Gamma}$ is a Cantor set.

(4) Consider the region $\left\{(a,c) : 0 < c < a < 1 \text{ and } a^2 < c < \sqrt{a^3}\right\}$ and let P be the union of all successive pre-images from γ, that is

$$P = \gamma \cup \bigcup_{n=1}^{\infty} \overbrace{F^{-1}\big(\cdots F^{-1}\big(F^{-1}}^{n\ times}\big(\gamma \setminus \overbrace{\delta\big) \setminus \delta\big) \setminus \cdots \setminus \delta}^{n\ times}\big).$$

(a) *If s is a long curve which is the graph of a strictly decreasing function contained in $\mathbb{R}^2 \setminus \delta$, then $P \cap s$ is dense in s.*
(b) *$\overline{\Gamma}$ is a set formed by long curves which are graphs of strictly decreasing functions; in particular, $P \cap \overline{\Gamma}$ is dense in $\overline{\Gamma}$.*

In [6] it is shown that, in the region of parameters $\{(a,c) : 0 < a < c < 1\}$, there exists a nonbounded region such that its image is contained inside itself, inducing an attractor which is not compact and, via the projection to $\{x = 0\}$ along a stable manifold, it is induced a one-dimensional dynamics whose return map to a compact interval is expanding. In this paper we exhibit a non-compact global attractor in another region of parameters $\{(a,c) : 0 < c < a < 1\}$. We show the existence of invariant curves to give a detailed topological description of the attractor and highlight its fractal appearance. To achieve our goal, we use expansive properties that do not depend on differentiability, which suggests the existence of the same type of non-compact fractals in a large class of plane homeomorphisms with discontinuity curves.

Following [6], there is an important relationship between $F_{a,c}$ and the (one-dimensional) Boole-like expanding maps, via the projection to $\{x = 0\}$ along a C^1 foliation. Similarly, Hénon–Devaney-like maps, as considered in [4], are (two-dimensional) topological versions of expansive (one-dimensional) alternating systems [8, 3] projecting along a continuous foliation.

In a recent work [5] the authors consider another parametrized family, namely $F_{a,b,c}(x,y) = \left(ax + \dfrac{b}{y}, cy - \dfrac{b}{y} - ax\right)$ and parameters $0 < c < 1 < a$, $b > 0$, and $ac > 1$, showing two-dimensional (compact) Cantor invariant sets with two hyperbolic fixed points. In light of all the results of the references cited in [5] and in the present article, the movement of the parameters produces transitive invariant pieces that pass from compact to non-compact sets and, in the non-compact case, the dynamics pass from transitive fractal pieces to being transitive in the entire plane.

We wonder about the dynamics of $F_{a,c}$ and $F_{a,b,c}$ for all parameters a, b, c in $\mathbb{R}$. Since $F_{1,1}$ is the limit of $F_{a,c}$ (when $a, c \to 1$), we believe that the ergodic behaviors of $F_{a,c}$ can be transferred to the Hénon–Devaney map. This is an interesting and tough open problem. We wonder about the existence of persistent C^1-foliations for all parameters.

2. Basic facts and notations

We will write $F^n(x_0, y_0) = (x_n, y_n)$, $n \in \mathbb{Z}$, whenever such iterate is defined. If $y_0 > 0$ then $y_1 = cy_0 - \dfrac{c}{y_0} - acx_0 = cy_0 - cx_1 > -cx_1$, and thus $F\left(R_\gamma^+\right) = R_\delta^+$; similarly, $F\left(R_\gamma^-\right) = R_\delta^-$ (see Figure 1a). We denote by $F_- : R_\gamma^- \to R_\delta^-$ (resp., $F_+ : R_\gamma^+ \to R_\delta^+$) the restriction of F to R_γ^- (resp., to R_γ^+). If $A \subset \mathbb{R}^2$, then $F_\pm(A)$, $F_\pm^{-1}(A)$ mean $F_\pm\left(A \cap R_\gamma^\pm\right)$, $F_\pm^{-1}\left(A \cap R_\delta^\pm\right)$.

Claim 2.1. *F sends graphs of decreasing functions contained in R^+_γ onto graphs of strictly decreasing functions contained in R^+_δ. Also, F sends graphs of decreasing functions contained in R^-_γ onto graphs of strictly decreasing functions contained in R^-_δ. The same conclusions hold for vertical straight lines (see Figure 1b).*

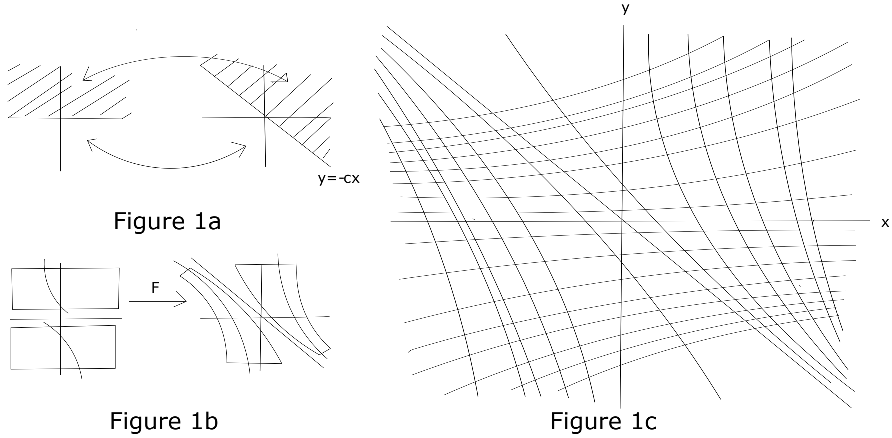

FIGURE 1.

Proof. Let l be the graph of a decreasing function contained in R^+_γ. Take two different arbitrary points (x_0, y_0), (x'_0, y'_0) in l such that $x'_0 < x_0$; then $y_0 \leq y'_0$ (this is possible because l is the graph of a decreasing function). Since $l \subset R^+_\gamma$, we have that $0 < y_0 \leq y'_0$; thus, since $a > 0$ and $c > 0$,

$$x'_1 = ax'_0 + \frac{1}{y'_0} < x_1 = ax_0 + \frac{1}{y_0},$$
$$y_1 = cy_0 - \frac{c}{y_0} - acx_0 < y'_1 = cy'_0 - \frac{c}{y'_0} - acx'_0.$$

Consequently, $F(l)$ is the graph of a strictly decreasing function. Also, since F sends R^+_γ onto R^+_δ, we conclude that $F(l) \subset R^+_\delta$. If $x'_0 = x_0$ and $y_0 < y'_0$ then the same conclusions hold. The second part of Claim 2.1 is completely analogous. □

Claim 2.2. *F^{-1} sends graphs of increasing functions contained in R^+_δ onto graphs of strictly increasing functions contained in R^+_γ. Quite similarly, F^{-1} sends graphs of increasing functions contained in R^-_δ onto graphs of strictly increasing functions contained in R^-_γ. The same conclusions hold for vertical straight lines.*

Proof. Let l be the graph of an increasing function contained in R^+_δ. Take two different arbitrary points (x_0, y_0), (x'_0, y'_0) in l such that $x_0 < x'_0$; then $y_0 \leq y'_0$ (this is possible because l is the graph of an increasing function). Since $l \subset R^+_\delta$, we have that $y_0 + cx_0 > 0$ and $y'_0 + cx'_0 > 0$; thus, since $a > 0$ and $c > 0$,

$$x_{-1} = \frac{x_0}{a} - \frac{c}{acx_0 + ay_0} < x'_{-1} = \frac{x'_0}{a} - \frac{c}{acx'_0 + ay'_0},$$
$$y_{-1} = \frac{cx_0 + y_0}{c} < y'_{-1} = \frac{cx'_0 + y'_0}{c}.$$

Consequently, $F^{-1}(l)$ is the graph of a strictly increasing function. Also, since F^{-1} sends R^+_δ onto R^+_γ, we conclude that $F^{-1}(l) \subset R^+_\gamma$. If $x_0 = x'_0$ and $y_0 < y'_0$, then the same conclusions hold. The second part of Claim 2.2 is completely analogous. □

From Claim 2.2 the pre-image of $R^+_\delta \cap \gamma = \{(x, 0) : x > 0\}$ (resp., of $R^-_\delta \cap \gamma = \{(x, 0) : x < 0\}$) is the graph of a strictly increasing function contained in R^+_γ (resp., in R^-_γ). Also,

$$x_{-1} = \frac{x}{a} - \frac{c}{acx + ay} \to -\infty \text{ when } y = 0,\, x \to 0^+$$
$$\text{and} \quad y_{-1} = \frac{cx + y}{c} \to 0^+ \text{ when } y = 0,\ x \to 0^+,$$
$$x_{-1} \to +\infty \text{ when } y = 0,\, x \to 0^- \quad \text{and} \quad y_{-1} \to 0^- \text{ when } y = 0,\, x \to 0^-,$$
$$x_{-1} \to +\infty \text{ when } y = 0,\, x \to +\infty \quad \text{and} \quad y_{-1} \to +\infty \text{ when } y = 0,\, x \to +\infty,$$
$$x_{-1} \to -\infty \text{ when } y = 0,\, x \to -\infty \quad \text{and} \quad y_{-1} \to -\infty \text{ when } y = 0,\, x \to -\infty.$$

Consequently, $F^{-1}_+(\{(x, 0) : x > 0\}) =: y_+$ and $F^{-1}_-(\{(x, 0) : x < 0\}) =: y_-$ are long curves, which intersect δ transversely at single points. Similarly, using Claim 2.2 and the formula for F^{-1}, the four curves $F^{-1}_+(y_+) =: y_{++}$, $F^{-1}_-(y_+) =: y_{+-}$, $F^{-1}_+(y_-) =: y_{-+}$, and $F^{-1}_-(y_-) =: y_{--}$ are contained in R^+_γ, R^-_γ, R^+_γ, and R^-_γ, respectively, and they are graphs of strictly increasing functions. Note that these curves have asymptotic behaviors similar to the previous $y_\pm$ and that they intersect δ transversely at single points.

From Claim 2.1 the image of $R^+_\gamma \cap \delta = \{(x, y) : y = -cx,\, y > 0\}$ is the graph of a strictly decreasing function contained in R^+_δ (similarly, the image of $R^-_\gamma \cap \delta = \{(x, y) : y = -cx,\, y < 0\}$ is the graph of a strictly decreasing function contained in R^-_δ). Also,

$$x_1 = ax + \frac{1}{y} \to +\infty \text{ when } x \to 0^-,\, y \to 0^+$$
$$\text{and } y_1 = cy - \frac{c}{y} - acx \to -\infty \text{ when } x \to 0^-,\ y \to 0^+,$$
$$x_1 \to -\infty \text{ when } x \to 0^+,\, y \to 0^- \text{ and } y_1 \to +\infty \text{ when } x \to 0^+,\, y \to 0^-,$$
$$x_1 \to -\infty \text{ when } x \to -\infty,\, y \to +\infty \text{ and } y_1 \to +\infty \text{ when } x \to -\infty,\, y \to +\infty,$$
$$x_1 \to +\infty \text{ when } x \to +\infty,\, y \to -\infty \text{ and } y_1 \to -\infty \text{ when } x \to +\infty,\, y \to -\infty.$$

Consequently, the curves defined by

$$F_+(\{(x, y) : y = -cx,\, y > 0\}) =: x_+ \quad \text{and} \quad F_-(\{(x, y) : y = -cx,\, y < 0\}) =: x_-$$

are long curves, which intersect γ, $y_\pm$, and $y_{\pm\pm}$ transversely at single points. Similarly, using Claim 2.1 and the formula for F, the four curves $F_+(x_+) =: x_{++}$, $F_-(x_+) =: x_{+-}$, $F_+(x_-) =: x_{-+}$, and $F_-(x_-) =: x_{--}$ are contained in R_δ^+, R_δ^-, R_δ^+, and R_δ^-, respectively, and they are graphs of strictly decreasing functions. Note that these curves have asymptotic behaviors similar to the previous $x_\pm$ and that they intersect γ, $y_\pm$, and $y_{\pm\pm}$ transversely at single points.

Using an inductive process, from Claims 2.2, 2.1 and using the formulas for F and F^{-1}, given n and m natural numbers $(n, m \geq 0)$ and two finite sequences $(a_0\, a_1\, \cdots\, a_n)$ and $(b_0\, b_1\, \cdots\, b_m)$ of symbols $\{+,-\}$, we obtain that the curves

$$\begin{aligned} y_{a_0 a_1 \cdots a_n} &:= \left(F_{a_n}^{-1} \circ F_{a_{n-1}}^{-1} \circ \cdots \circ F_{a_0}^{-1}\right)(\gamma) = F_{a_n}^{-1}\left(y_{a_0 \cdots a_{n-1}}\right) \qquad \left(\subset R_\gamma^{a_n}\right), \\ x_{b_0 b_1 \cdots b_m} &:= \left(F_{b_m} \circ F_{b_{m-1}} \circ \cdots \circ F_{b_0}\right)(\delta) = F_{b_m}\left(x_{b_0 \cdots b_{m-1}}\right) \qquad \left(\subset R_\delta^{b_m}\right) \end{aligned} \tag{2.1}$$

are two long curves intersecting transversely at a single point. In fact, the curve $y_{a_0 a_1 \cdots a_n}$ is the graph of a strictly increasing function intersecting δ transversely at a single point. The curve $x_{b_0 b_1 \cdots b_m}$ is the graph of a strictly decreasing function intersecting γ transversely at a single point.

In conclusion, from Claims 2.2, 2.1 and using the formulas for F and F^{-1}, the set of curves (as above in (2.1)), say $\mathcal{P}$ (resp., $\mathcal{I}$), of successive pre-images with respect to γ (resp., of successive images with respect to δ) are sets of long curves which intersect transversely at single points. The curves in $\mathcal{P}$ are graphs of increasing functions and the curves in $\mathcal{I}$ are graphs of decreasing functions (see Figure 1c). We call the curves in $\mathcal{P} \cup \mathcal{I}$ the *guidelines*. Let us distinguish special classes of guidelines: the external guidelines $y_0^\pm = \gamma$, $x_0^\pm = \delta$, and

$$\begin{aligned} \left\{y_n^-\right\} &= \left\{y_{(-)^{n+1}}\right\}_{n\geq 0} & \left\{y_n^+\right\} &= \left\{y_{(+)^{n+1}}\right\}_{n\geq 0} \\ \left\{x_n^-\right\} &= \left\{x_{(-)^{n+1}}\right\}_{n\geq 0} & \left\{x_n^+\right\} &= \left\{x_{(+)^{n+1}}\right\}_{n\geq 0}, \end{aligned}$$

where $(j)^p = \overbrace{jjj\cdots jjj}^{p \text{ times}}$, $j \in \{-,+\}$; and the internal guidelines $u_0^\pm = y_\pm$, $v_0^\pm = x_\pm$, and

$$\begin{aligned} \left\{u_n^-\right\} &= \left\{y_{(+)^n -}\right\}_{n\geq 1} & \left\{u_n^+\right\} &= \left\{y_{(-)^n +}\right\}_{n\geq 1} \\ \left\{v_n^-\right\} &= \left\{x_{(+)^n -}\right\}_{n\geq 1} & \left\{v_n^+\right\} &= \left\{x_{(-)^n +}\right\}_{n\geq 1}. \end{aligned}$$

3. The behavior of forward orbits

Consider the lamination of R_γ^+ by vertical straight lines $\{x = x_0\}_{x_0\in\mathbb{R}}$. For each $x_0 \in \mathbb{R}$, consider the sequences $\left\{z_n^\pm(x_0) = y_n^\pm \cap \{x = x_0\}\right\}_{n\geq 0}$, where $y_n^\pm$ are external guidelines (see Figure 2a).

Lemma 3.1. *Suppose that $0 < c \leq a$. If $x_0 > 0$ then $z_n^+(x_0)$ does not converge. Similarly, if $x_0 < 0$ then $z_n^-(x_0)$ does not converge.*

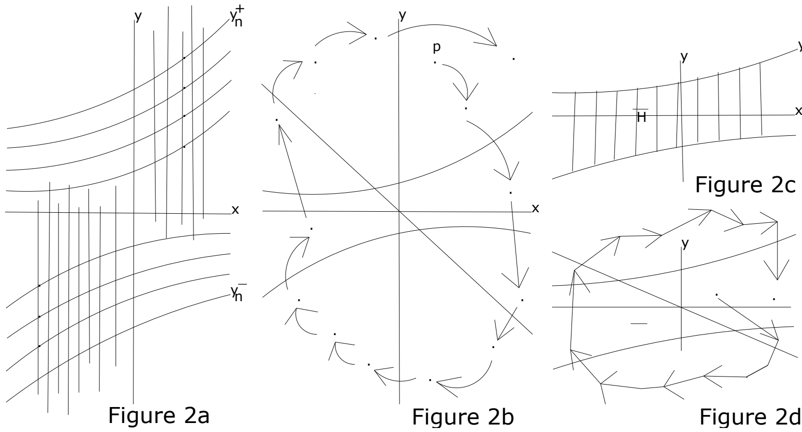

FIGURE 2.

Proof. Fix $x_0 > 0$. For each $n \in \mathbb{N}$, write (x_0, z_n) instead of $z_n^+(x_0)$ (this notation is consistent since, for all $n \in \mathbb{N}$, the first component of $z_n^+(x_0)$ is x_0). Computing (inductively) the orbit of order n of the point (x_0, z_n) we obtain

$$F^n(x_0, z_n) = \left(a^n x_0 + \frac{a^{n-1}}{z_n} + \sum_{j=0}^{n-2} \frac{a^{n-2-j}}{y_{j+1}},\, c^n z_n - \sum_{j=0}^{n-1} c^{n-j} x_{j+1} \right).$$

Since after n iterates $F^n(x_0, z_n)$ belongs to the x axis, we have

$$c^n z_n - c^n x_1 - c^{n-1} x_2 - \cdots - c x_n = 0;$$

therefore, solving for z_n, it follows that

$$\begin{aligned} z_n = &\left(a x_0 + \frac{1}{z_n} \right) + \frac{1}{c} \left(a^2 x_0 + \frac{a}{z_n} + \frac{1}{y_1} \right) + \cdots \\ &+ \frac{1}{c^{n-1}} \left(a^n x_0 + \frac{a^{n-1}}{z_n} + \sum_{j=0}^{n-2} \frac{a^{n-2-j}}{y_{j+1}} \right); \end{aligned}$$

consequently,

$$z_n \geq a x_0 + \frac{a^2 x_0}{c} + \cdots + \frac{a^n x_0}{c^{n-1}} = c x_0 \sum_{k=1}^{n} \left(\frac{a}{c} \right)^k. \tag{3.1}$$

Observe from our assumptions that $\frac{a}{c} \geq 1$; therefore, since $z_n > 0$ and $y_i > 0$ (for all $i \in \{1, \ldots, n-1\}$), the series on the right of formula (3.1) diverges to $+\infty$, proving the first part of Lemma 3.1. Since $-F(x, y) = F(-x, -y)$, the second part of Lemma 3.1 follows similarly. $\square$

Lemma 3.2. *Suppose that $0 < c \leq a < 1$. Given $x_0 \leq 0$ and $y_0 > 0$, there exists a natural number n such that $x_n \geq 0$ or $y_n \leq 0$. Analogously, given $x_0 \geq 0$ and $y_0 < 0$, there exists a natural number n such that $x_n \leq 0$ or $y_n \geq 0$.*

Proof. If $x_0 = 0$ and $y_0 > 0$ then $x_1 > 0$. Fix $x_0 < 0$ and $y_0 > 0$ and suppose (by contradiction) that $x_n < 0$ and $y_n > 0$ for all $n \in \mathbb{N}$. Under this assumption we will then have that

$$\begin{aligned}
x_1 &= ax_0 + \frac{1}{y_0} > x_0 \\
y_1 &= cy_0 - cx_1 < cy_0 - cx_0 < y_0 - x_0 \\
cy_1 &< c\,(y_0 - x_0) < y_0 - x_0 \\
x_2 &= ax_1 + \frac{1}{y_1} > x_1 \\
y_2 &= cy_1 - cx_2 < y_0 - x_0 - cx_0 < y_0 - 2x_0.
\end{aligned}$$

Then, it is shown inductively that $y_n < y_0 - nx_0$ for all $n \in \mathbb{N}$. On the other hand,

$$x_2 = ax_1 + \frac{1}{y_1} > x_1 + \frac{1}{y_1} = ax_0 + \frac{1}{y_0} + \frac{1}{y_1} > x_0 + \frac{1}{y_0} + \frac{1}{y_0 - x_0}.$$

Again inductively it can be shown that, for all $n \in \mathbb{N}$,

$$x_n > x_0 + \frac{1}{y_0} + \sum_{k=1}^{n-1} \frac{1}{y_0 - kx_0}.$$

Consequently, since the series $\sum_{k=1}^{\infty} \frac{b}{y_0 - kx_0}$ diverges to $+\infty$ (use, for example, the integral criterion), we have that there exists $N \in \mathbb{N}$ such that $x_N > 0$. This contradiction completes the proof of the first part of Lemma 3.2. The second part follows similarly, since $F(x, y) = -F(-x, -y)$. $\square$

Note that both Lemmas 3.1 and 3.2 prove that, for every $x_0 \in \mathbb{R}$, the sequences $z_n^{\pm}(x_0)$ do not converge. Also, note that there is a geometric order of the external guidelines $\{y_n^{\pm}\}$ in the entire plane; since the inferior component $R^-_{y^+_{n+1}}$ strictly contains the inferior component $R^-_{y^+_n}$ and $F^n\left(y_n^+\right) \subset \gamma$, by Lemmas 3.1 and 3.2, given any point $p \in R^+_\gamma$, there is an n such that p goes to R^-_γ after n iterates. Those are the main ingredients proving that every point $p \in \mathbb{R}^2$ whose forward orbit $\{F^n(p)\}_{n \geq 0}$ is defined goes from the region R^+_γ to R^-_γ and from R^-_γ to R^+_γ infinitely many times (see Figure 2b). If p is a successive pre-image of some point in γ, then its orbit ends in γ, as follows.

Proposition 3.3. *Suppose that $0 < c \leq a < 1$ and let $p = (x_0, y_0)$ be an arbitrary point. If $y_0 > 0$ then there exists $N \geq 1$ such that $y_1 > 0, \ldots, y_{N-1} > 0$, and $y_N \leq 0$. Similarly, if $y_0 < 0$ then there exists $M \geq 1$ such that $y_1 < 0, \ldots, y_{M-1} < 0$, and $y_M \geq 0$.*

Proof. We will only prove the first part of the Proposition, since the second part is similar. Let us fix $p = (x_0, y_0)$ with $y_0 > 0$. Suppose that $x_0 > 0$. Let N be the natural number such that p is between the guidelines y_N^+ and y_{N-1}^+. Such

N exists due to Lemma 3.1 and it satisfies the required inequalities $y_1 > 0, \dots, y_{N-1} > 0$, and $y_N \leq 0$ (note that $y_N = 0$ whenever $p_0 \in y_N^+$). If $x_0 = 0$, then $x_1 = ax_0 + \dfrac{1}{y_0} = \dfrac{1}{y_0} > 0$ and we use the previous case. If $x_0 < 0$, the proof follows from Lemma 3.2 and the previous cases. □

Let $\overline{H} := \overline{R^+_{y_0^-}} \cap \overline{R^-_{y_0^+}}$ be the non-bounded closed region limited by y_0^+ and y_0^- (see Figure 2c). Then

$$\mathbb{R}^2 \setminus \gamma = \bigcup_{n=1}^{\infty} \overbrace{F^{-1}\big(\cdots F^{-1}\big(F^{-1}}^{n \text{ times}}\big(\overline{H} \setminus \overbrace{\delta\big) \setminus \delta\big) \setminus \cdots \setminus \delta}^{n \text{ times}}\big). \tag{3.2}$$

Indeed, for every $n \in \mathbb{Z}^+$ we have:

$$\begin{aligned}
u_{n+1}^- &\to F_-\left(u_{n+1}^-\right) = y_n^+ \cap R_\delta^- && \left(\text{or}\ \ F_-^{-1}\left(y_n^+ \cap R_\delta^-\right) = u_{n+1}^-\right)\\
y_n^+ &\to F_+\left(y_n^+\right) = y_{n-1}^+ \cap R_\delta^+ && \left(\text{or}\ \ F_+^{-1}\left(y_{n-1}^+ \cap R_\delta^+\right) = y_n^+\right)\\
&\vdots\\
y_1^+ &\to F_+\left(y_1^+\right) = y_0^+ \cap R_\delta^+ && \left(\text{or}\ \ F_+^{-1}\left(y_0^+ \cap R_\delta^+\right) = y_1^+\right)\\
y_0^+ &\to F_+\left(y_0^+\right) = \gamma \cap R_\delta^+ && \left(\text{or}\ \ F_+^{-1}\left(\gamma \cap R_\delta^+\right) = y_0^+\right)\\
\\
u_{n+1}^+ &\to F_+\left(u_{n+1}^+\right) = y_n^- \cap R_\delta^+ && \left(\text{or}\ \ F_+^{-1}\left(y_n^- \cap R_\delta^+\right) = u_{n+1}^+\right)\\
y_n^- &\to F_-\left(y_n^-\right) = y_{n-1}^- \cap R_\delta^- && \left(\text{or}\ \ F_-^{-1}\left(y_{n-1}^- \cap R_\delta^-\right) = y_n^-\right)\\
&\vdots\\
y_1^- &\to F_-\left(y_1^-\right) = y_0^- \cap R_\delta^- && \left(\text{or}\ \ F_-^{-1}\left(y_0^- \cap R_\delta^-\right) = y_1^-\right)\\
y_0^- &\to F_-\left(y_0^-\right) = \gamma \cap R_\delta^- && \left(\text{or}\ \ F_-^{-1}\left(\gamma \cap R_\delta^-\right) = y_0^-\right).
\end{aligned}$$

Given curves l and m, denote by $[l, m]$ the region enclosed by l and m (including both l and m); then

$$\begin{aligned}
\left[y_n^-, u_n^-\right] \cup \left[u_n^+, y_n^+\right] &= F^{-1}\left(\overline{H} \setminus \delta\right) \cup F^{-1}\left(F^{-1}\left(\overline{H} \setminus \delta\right) \setminus \delta\right)\\
&\qquad \cdots \cup \overbrace{F^{-1}\big(\cdots F^{-1}\big(F^{-1}}^{n \text{ times}}\big(\overline{H} \setminus \overbrace{\delta\big) \setminus \delta\big) \setminus \cdots \setminus \delta}^{n \text{ times}}\big) \quad \left(n \in \mathbb{Z}^+\right).
\end{aligned}$$

As said above, for every $x_0 \in \mathbb{R}$, the sequences $z_n^\pm(x_0) = y_n^\pm \cap \{x = x_0\}$ do not converge. These facts imply that $y_n^+ \cap \{x = x_0\} \to +\infty$, $y_n^- \cap \{x = x_0\} \to -\infty$, $u_n^+ \cap \{x = x_0\} \to (x_0, 0)$, and $u_n^- \cap \{x = x_0\} \to (x_0, 0)$, which proves (3.2). Consequently, we have the following remark.

Remark 3.4. The *first return map* $F_{\overline{H}} : \overline{H} \setminus \gamma \to \overline{H}$, defined by $F_{\overline{H}}(p) = F^{n(p)}(p)$, where $n(p) = \min\left\{n \geq 1 : F^n(p) \in \overline{H}\right\}$, is well defined (see Figure 2d). Given $p \in \overline{H} \setminus \gamma$, we call $n(p)$ the *first return time* of p (for forward iterates) to $\overline{H}$.

4. Right and left invariant curves

Consider the lamination of R^+_γ (resp., of R^-_γ) by horizontal straight lines $\{y = y_0\}_{y_0>0}$ (resp., $\{y = y_0\}_{y_0<0}$). For each $y_0 > 0$, consider the sequence $\big\{w^+_n(y_0) = x^+_n \cap \{y = y_0\}\big\}_{n\geq 0}$ (resp., $\big\{w^-_n(y_0) = x^-_n \cap \{y = y_0\}\big\}_{n\geq 0}$, for each $y_0 < 0$), where $x^\pm_n$ are external guidelines (see Figure 3a).

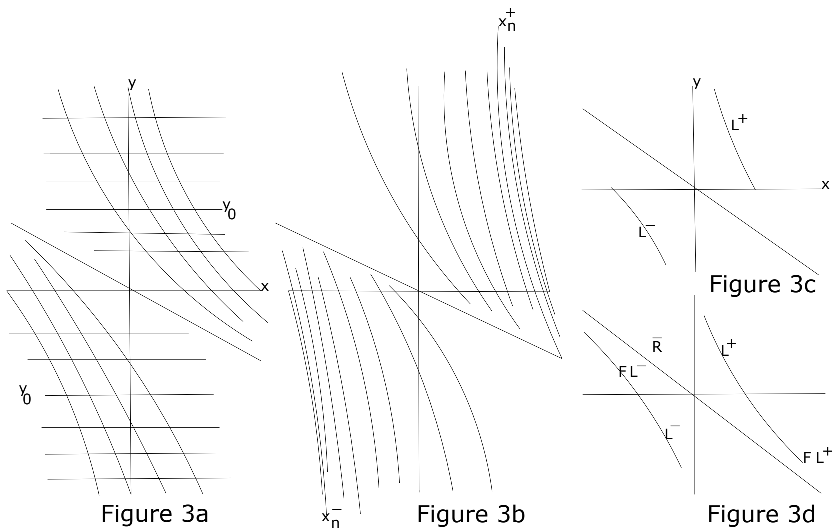

Figure 3.

Lemma 4.1. *For parameters $0 < c < 1$, if $y_0 > 0$ then $\{w^+_n(y_0)\}$ is convergent. Similarly, if $y_0 < 0$ then $\{w^-_n(y_0)\}$ is convergent.*

Proof. Fix $y_0 > 0$. For each $n \in \mathbb{N}$, let us write (w_n, y_0) instead of $w^+_n(y_0)$ (this notation is consistent since, for all $n \in \mathbb{N}$, the second component of $w^+_n(y_0)$ is y_0). First we note that $\frac{-y_0}{c} \leq w_n$ and $w_n \leq w_{n+1}$ for all $n \in \mathbb{N}$. Let us limit $w^+_n(y_0)$ on the right. Indeed, computing the orbit of order n (with respect to F^{-1}) of the point $(w_n, y_0) = (x_0, y_0)$ we obtain

$$
\begin{aligned}
F^{-1}(w_n, y_0) &= \left(\frac{w_n}{a} - \frac{1}{ay_{-1}},\, y_{-1}\right) = (x_{-1}, y_{-1})\\
F^{-2}(w_n, y_0) &= \left(\frac{w_n}{a^2} - \frac{1}{a^2y_{-1}} - \frac{1}{ay_{-2}},\, y_{-2}\right) = (x_{-2}, y_{-2})\\
&\ \ \vdots\\
F^{-n}(w_n, y_0) &= \left(\frac{w_n}{a^n} - \frac{1}{a^ny_{-1}} - \frac{1}{a^{n-1}y_{-2}} - \cdots - \frac{1}{ay_{-n}},\, y_{-n}\right) = (x_{-n}, y_{-n})\,.
\end{aligned}
$$

After n iterates, $F^{-n}(w_n, y_0)$ belongs to $\delta = \{y = -cx\}$, that is,

$$y_{-n} = -\frac{cw_n}{a^n} + \frac{c}{a^n y_{-1}} + \frac{c}{a^{n-1} y_{-2}} + \cdots + \frac{c}{a y_{-n}}.$$

Similarly, after computing the n-th iterate of (x_{-n}, y_{-n}) (but now with respect to F), it follows that

$$y_0 = c^n y_{-n} - c^n x_{-n+1} - c^{n-1} x_{-n+2} - \cdots - c^2 x_{-1} - cw_n.$$

After the respective substitutions, we get

$$\begin{aligned}
w_n &= -\frac{c^n}{a^n} w_n + \frac{c^n}{a^n y_{-1}} + \frac{c^n}{a^{n-1} y_{-2}} + \cdots + \frac{c^n}{a y_{-n}} - c^{n-1} x_{-n+1} - \cdots - c x_{-1} - \frac{y_0}{c} \\
&= -\frac{c^n}{a^n} w_n + \frac{c^n}{a^n y_{-1}} + \frac{c^n}{a^{n-1} y_{-2}} + \cdots + \cdots + \frac{c^n}{a^2 y_{-n+1}} + \frac{c^n}{a y_{-n}} \\
&\quad - \frac{c^{n-1}}{a^{n-1}} w_n + \frac{c^{n-1}}{a^{n-1} y_{-1}} + \frac{c^{n-1}}{a^{n-2} y_{-2}} + \cdots + \frac{c^{n-1}}{a y_{-n+1}} \\
&\quad \vdots \\
&\quad - \frac{c^2}{a^2} w_n + \frac{c^2}{a^2 y_{-1}} + \frac{c^2}{a y_{-2}} \\
&\quad - \frac{c}{a} w_n + \frac{c}{a y_{-1}} - \frac{y_0}{c}.
\end{aligned}$$

Then, solving for w_n and regrouping, we obtain

$$\begin{aligned}
w_n \sum_{k=0}^{n} \left(\frac{c}{a}\right)^k &= \frac{1}{y_{-1}} \sum_{k=1}^{n} \left(\frac{c}{a}\right)^k + \frac{c}{y_{-2}} \sum_{k=1}^{n-1} \left(\frac{c}{a}\right)^k + \cdots \\
&\quad + \frac{c^{n-2}}{y_{-n+1}} \sum_{k=1}^{2} \left(\frac{c}{a}\right)^k + \frac{c^{n-1}}{y_{-n}} \frac{c}{a} - \frac{y_0}{c}.
\end{aligned}$$

(Note that, for large values of y_0, w_n $(= x_0)$ can be negative). Let p be the first in $\{0, 1, \ldots, n\}$ such that $x_{-p} < 0$. Therefore

$$x_0 \geq 0, \quad x_{-1} \geq 0, \quad \ldots, \quad x_{-p+1} \geq 0, \quad x_{-p} < 0.$$

We stress that such p exists (since the n-th iterate of (w_n, y_0), with respect to F^{-1}, belongs to the straight line $\{y = -cx\}$). Also:

(1) For all $i \in \{1, \ldots, p\}$, $y_{-i} = \dfrac{cx_{-i+1} + y_{-i+1}}{c} = x_{-i+1} + \dfrac{y_{-i+1}}{c} > \dfrac{y_{-i+1}}{c} > y_{-i+1}$ (since $0 < c < 1$); in particular, $y_{-i} > y_0 > 0$ and $0 < 1/y_{-i} < 1/y_0$ for all $i \in \{1, \ldots, p\}$.

(2) For all $i \in \{p+1, \ldots, n\}$, $x_{-i} = \dfrac{x_{-i+1}}{a} - \dfrac{c}{acx_{-i+1} + ay_{-i+1}} < \dfrac{x_{-i+1}}{a} < 0$ (since $y_{-i+1} > -cx_{-i+1}$).

Let K be the intersection between x_1^+ and $\{(0, y) : y > 0\}$, that is,

$$F(-K, K) = \left(-aK + \frac{1}{K},\, cK + acK - \frac{c}{K}\right) = \left(0,\, cK + acK - \frac{c}{K}\right),$$

so $-aK+\dfrac{1}{K}=0$ and therefore $K=\sqrt{\dfrac{1}{a}}$. Following item (2) above and the definitions of the x_n^+'s, we have $y_{-i}>K>0$ for all $i\in\{p+1,\dots,n\}$, so $0<1/y_{-i}<1/K$ for all $i\in\{p+1,\dots,n\}$. Finally,

$$w_n\le \frac{\sum_{k=1}^{n}\left(\frac{c}{a}\right)^k}{\sum_{k=0}^{n}\left(\frac{c}{a}\right)^k}\frac{c^0}{y_0}+\frac{\sum_{k=1}^{n-1}\left(\frac{c}{a}\right)^k}{\sum_{k=0}^{n}\left(\frac{c}{a}\right)^k}\frac{c^1}{y_0}+\cdots+\frac{\sum_{k=1}^{n-(p-1)}\left(\frac{c}{a}\right)^k}{\sum_{k=0}^{n}\left(\frac{c}{a}\right)^k}\frac{c^{p-1}}{y_0}+\frac{\sum_{k=1}^{n-p}\left(\frac{c}{a}\right)^k}{\sum_{k=0}^{n}\left(\frac{c}{a}\right)^k}\frac{c^p}{K}$$
$$+\cdots+\frac{\sum_{k=1}^{2}\left(\frac{c}{a}\right)^k}{\sum_{k=0}^{n}\left(\frac{c}{a}\right)^k}\frac{c^{n-2}}{K}+\frac{\left(\frac{c}{a}\right)}{\sum_{k=0}^{n}\left(\frac{c}{a}\right)^k}\frac{c^{n-1}}{K}\le\frac{1}{M}\sum_{k=0}^{n-1}c^k,\quad M=\min\{y_0,K\}.$$

Since $0<c<1$, the series on the right is convergent. This completes the proof of the first part of the lemma. The second part is completely analogous. $\square$

Define the curves $L^+:\big\{(0,y):y>0\big\}\to\mathbb{R}^2$ and $L^-:\big\{(0,y):y<0\big\}\to\mathbb{R}^2$ as follows (see Figures 3b and 3c):

$$L^+(y_0)=\lim_{n\to+\infty}w_n^+(y_0),\quad y_0>0,$$
$$L^-(y_0)=\lim_{n\to+\infty}w_n^-(y_0),\quad y_0<0.$$

Set $L^+=L^+\big(\big\{(0,y):y>0\big\}\big)$ and $L^-=L^-\big(\big\{(0,y):y<0\big\}\big)$. We call L^+ (resp., L^-) the *right invariant curve* (resp., the *left invariant curve*) induced by F.

Remark 4.2. Let $\mathcal{F}^+$ (resp., $\mathcal{F}^-$) be any lamination of R_γ^+ (resp., of R_γ^-) by long curves intersecting transversely, at single points, the sequence of curves $\{x_n^+\}$ (resp., $\{x_n^-\}$). Then, given $l_+\in\mathcal{F}^+$ (resp., $l_-\in\mathcal{F}^-$), the sequence $\{l_+\cap x_n^+\}_{n\ge0}$ (resp., $\{l_-\cap x_n^+\}_{n\ge0}$) converges to some point in L^+ (resp., in L^-).

Proof of item (1) of the Main Theorem. Due to our construction, $L^+\subset R_\delta^+\cap R_\gamma^+$ and $L^-\subset R_\delta^-\cap R_\gamma^-$. Also, both $L^\pm$ are limits of the sequences $\{x_n^\pm\}$. From Claim 2.1 the guidelines $\{x_n^\pm\}$ are graphs of (strictly) decreasing continuous functions, and following the formula for F we have that $\overline{R^+_{x^+_{n+1}}}\subset R^+_{x^+_n}$ and $\overline{R^-_{x^-_{n+1}}}\subset R^-_{x^-_n}$. Since both $L^\pm$ are uniform limits (on compact regions) of the sequences $\{x_n^\pm\}$, $L^\pm$ are both graphs of decreasing continuous functions. From Claim 2.1 and using the formula for F, it follows that $F(L^+)$ and $F(L^-)$ are both strictly decreasing long curves, intersecting γ at single points, and they are contained respectively in R_δ^+ and R_δ^- (see Figure 3d). Note that the region $R^-_{F(L^+)}\cap R_\delta^+$ (resp., $R^+_{F(L^-)}\cap R_\delta^-$) contains all external guidelines $\{x_n^+\}$ (resp., $\{x_n^-\}$). Let us prove that $F(L^+)\cap R_\gamma^+=L^+$; the other case is analogous. Take $q\in F(L^+)\cap R_\gamma^+$; then $q=F(p)\in R_\gamma^+$ for some $p\in L^+$. Take $y_0>0$ with $p_n=w_n^+(y_0)\subset\{y=y_0\}$ and

$p_n \to p$. Since F is continuous in R^+_γ, we have $F(p_n) \to F(p) = q$, so $q \in R^+_\gamma$ is a limit point of a sequence in the leaf $F(\{y = y_0\})$ (which intersects transversely, at single points, the sequence of curves $\{x^+_n\}$). Thus, from Remark 4.2, $q \in L^+$. Similar arguments prove that $L^+ \subset F(L^+) \cap R^+_\gamma$, using the continuity of F^{-1}. □

5. The global attractor induced by the invariant curves

Let $\overline{R} = \overline{R^+_{F(L^-)}} \cap \overline{R^-_{F(L^+)}}$ be the non-bounded closed region (cylinder) limited by $F(L^-)$ and $F(L^+)$ (see Figure 3d above). Observe that all guidelines with respect to δ are contained in R. Similarly to Equation (2.1) above, let us consider the successive images from $L^\pm$. Indeed, given $n \in \mathbb{Z}^+$ $(n \geq 0)$,

$$\begin{aligned} L^+_{a_0a_1\cdots a_n} &:= \left(F_{a_n} \circ F_{a_{n-1}} \circ \cdots \circ F_{a_0}\right)\left(L^+\right), \quad a_j \in \{-,+\},\ 0 \leq j \leq n,\ a_0 \neq -, \\ L^-_{a_0a_1\cdots a_n} &:= \left(F_{a_n} \circ F_{a_{n-1}} \circ \cdots \circ F_{a_0}\right)\left(L^-\right), \quad a_j \in \{-,+\},\ 0 \leq j \leq n,\ a_0 \neq + \end{aligned} \tag{5.1}$$

Note that $L^+_{+++\cdots+} = F(L^+)$ and $L^-_{---\cdots-} = F(L^-)$ for any string of $+$'s and $-$'s (this follows from the invariant property in item (1) of the Main Theorem). From Claim 2.1 all curves induced by $L^\pm$ (given in (5.1)) are graphs of (strictly) decreasing functions and, due to the formula for F, they are in fact long curves contained in $\overline{R}$. Also, from Claim 2.2, all curves induced by $L^\pm$ intersect transversely, at single points, the set of guidelines with respect to γ.

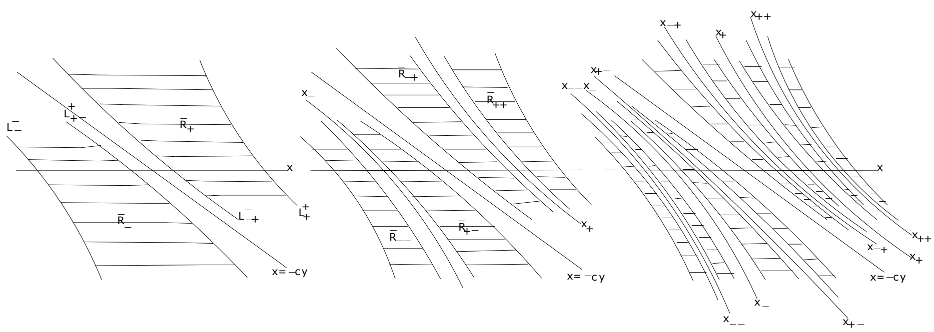

Figure 4.

Note that

$$F\left(\overline{R} \setminus \gamma\right) = \overline{R}_- \cup \overline{R}_+,$$

where $\overline{R}_+ = F_+\left(\overline{R}\right)$ and $\overline{R}_- = F_-\left(\overline{R}\right)$. Therefore, $\overline{R}_+$ is the non-bounded region enclosed by L^+_+ and L^-_{-+}, and so $\overline{R}_+ \subset R^+_\delta \cap \overline{R}$. Similarly, $\overline{R}_-$ is the non-bounded region enclosed by L^+_{+-} and L^-_-, and so $\overline{R}_- \subset R^-_\delta \cap \overline{R}$. Thus $F\left(\overline{R} \setminus \gamma\right) = \overline{R}_- \cup \overline{R}_+ \subset \overline{R}$, as required in item (2) of the Main Theorem. Similarly, note that

$$F\left\{F\left(\overline{R} \setminus \gamma\right) \setminus \gamma\right\} = F\left\{\left(\overline{R}_- \cup \overline{R}_+\right) \setminus \gamma\right\} = \overline{R}_{--} \cup \overline{R}_{+-} \cup \overline{R}_{-+} \cup \overline{R}_{++},$$

where $\overline{R}_{++} = F_+\left(\overline{R}_+\right)$, $\overline{R}_{-+} = F_+\left(\overline{R}_-\right)$, $\overline{R}_{+-} = F_-\left(\overline{R}_+\right)$, and $\overline{R}_{--} = F_-\left(\overline{R}_-\right)$. Therefore, $\overline{R}_{++}$ is the non-bounded region enclosed by $L^+_{+++} = L^+_+$ and L^-_{-++}, and so $\overline{R}_{++} \subset \overline{R}_{+}$; $\overline{R}_{-+}$ is the non-bounded region enclosed by L^{+}_{+-+} and L^{-}_{-+}, and so $\overline{R}_{-+} \cap \overline{R}_{++} = \emptyset$ and $\overline{R}_{-+} \subset \overline{R}_{+}$; $\overline{R}_{+-}$ is the non-bounded region enclosed by L^{+}_{+-} and L^{-}_{-+-}, and so $\overline{R}_{+-} \subset \overline{R}_{-}$; $\overline{R}_{--}$ is the non-bounded region enclosed by L^{+}_{+--} and $L^{-}_{--} = L^{-}_{-}$, and so $\overline{R}_{--} \cap \overline{R}_{+-} = \emptyset$ and $\overline{R}_{--} \subset \overline{R}_{-}$. Thus $F\left\{F\left(\overline{R} \setminus \gamma\right) \setminus \gamma\right\} = \overline{R}_{--} \cup \overline{R}_{+-} \cup \overline{R}_{-+} \cup \overline{R}_{++} \subset \overline{R}_{-} \cup \overline{R}_{+} \subset \overline{R}$. Similarly,

$$\begin{aligned} F\left(F\left(F\left(\overline{R} \setminus \gamma\right) \setminus \gamma\right) \setminus \gamma\right) = {} & \overline{R}_{---} \cup \overline{R}_{+--} \cup \overline{R}_{-+-} \cup \overline{R}_{++-} \\ & \cup \overline{R}_{--+} \cup \overline{R}_{+-+} \cup \overline{R}_{-++} \cup \overline{R}_{+++}, \end{aligned}$$

$$\begin{aligned} &\overline{R}_{+++} = F_{+}\left(\overline{R}_{++}\right) \subset \overline{R}_{++}, && \overline{R}_{-++} = F_{+}\left(\overline{R}_{-+}\right) \subset \overline{R}_{++}, \\ &\overline{R}_{+-+} = F_{+}\left(\overline{R}_{+-}\right) \subset \overline{R}_{-+}, && \overline{R}_{--+} = F_{+}\left(\overline{R}_{--}\right) \subset \overline{R}_{-+}, \\ &\overline{R}_{++-} = F_{-}\left(\overline{R}_{++}\right) \subset \overline{R}_{+-}, && \overline{R}_{-+-} = F_{-}\left(\overline{R}_{-+}\right) \subset \overline{R}_{+-}, \\ &\overline{R}_{+--} = F_{-}\left(\overline{R}_{+-}\right) \subset \overline{R}_{--}, && \overline{R}_{---} = F_{-}\left(\overline{R}_{--}\right) \subset \overline{R}_{--}. \end{aligned}$$

The 2^3 sets $\overline{R}_{ijk}$ $(i, j, k \in \{+, -\})$ are closed and non-bounded regions which are pairwise disjoint (see Figure 4), and so on. We consider the closed set

$$\begin{aligned} \overline{\Gamma} := {} & \overline{R} \cap \bigcap_{n=1}^{\infty} \overbrace{F\big(\cdots F\big(F}^{n \text{ times}}\left(\overline{R} \setminus \overbrace{\gamma\right) \setminus \gamma\big) \setminus \cdots \setminus \gamma}^{n \text{ times}}\big) \\ = {} & \overline{R} \;\cap\; \left(\overline{R}_{-} \cup \overline{R}_{+}\right) \;\cap\; \left(\overline{R}_{--} \cup \overline{R}_{+-} \cup \overline{R}_{-+} \cup \overline{R}_{++}\right) \;\cap\; \cdots . \end{aligned}$$

Similarly to Equation (3.2), we have

$$\overline{R} \setminus \gamma = \overline{R} \cap \bigcup_{n=1}^{\infty} \overbrace{F^{-1}\big(\cdots F^{-1}\big(F^{-1}}^{n \text{ times}}\left(\overline{H} \setminus \overbrace{\delta\right) \setminus \delta\big) \setminus \cdots \setminus \delta}^{n \text{ times}}\big).$$

Remark 5.1. As in Remark 3.4, following the results in Sections 3, 4 and the present section, the first return map $F_{\overline{H} \cap \overline{R}} : \left(\overline{H} \cap \overline{R}\right) \setminus \gamma \to \overline{H} \cap \overline{R}$, defined by $F_{\overline{H} \cap \overline{R}}(p) = F^{n(p)}\left(p\right)$, where $n(p) = \min\left\{n \geq 1 : F^{n}(p) \in \overline{H} \cap \overline{R}\right\}$, is well defined. Given $p \in \overline{H} \cap \overline{R} \setminus \gamma$, we call $n(p)$ the *first return time* of p to $\overline{H} \cap \overline{R}$. We stress that $\overline{H} \cap \overline{R}$ is a bounded set.

Proof of item (2) of the Main Theorem. It was noted above that $F\left(\overline{R} \setminus \gamma\right) = \overline{R}_{-} \cup \overline{R}_{+} \subset \overline{R}$. Clearly, $\overline{\Gamma}$ is closed, and since all curves induced by $L^{\pm}$ are long curves contained in $\overline{\Gamma}$ (see (5.1) above), $\overline{\Gamma}$ is non-bounded. Also, since $F\left(\overline{R} \setminus \gamma\right) \subset \overline{R}$, we have

$$\begin{aligned} F\left(\overline{\Gamma}\setminus\gamma\right) &= F\left(\left[\overline{R}\cap\bigcap_{n=1}^{\infty}\overbrace{F\big(\cdots F\big(F}^{n\text{ times}}\big(\overline{R}\setminus\overbrace{\gamma\big)\setminus\gamma\big)\setminus\cdots\setminus\gamma}^{n\text{ times}}\big)\right]\setminus\gamma\right)\\ &= F\left(\left[\overline{R}\setminus\gamma\right]\cap\bigcap_{n=1}^{\infty}\left[\overbrace{F\big(\cdots F\big(F}^{n\text{ times}}\big(\overline{R}\setminus\overbrace{\gamma\big)\setminus\gamma\big)\setminus\cdots\setminus\gamma}^{n\text{ times}}\big)\setminus\gamma\right]\right)\\ &= \bigcap_{n=1}^{\infty}\overbrace{F\big(\cdots F\big(F}^{n\text{ times}}\big(\overline{R}\setminus\overbrace{\gamma\big)\setminus\gamma\big)\setminus\cdots\setminus\gamma}^{n\text{ times}}\big)\\ &= \overline{R}\cap\bigcap_{n=1}^{\infty}\overbrace{F\big(\cdots F\big(F}^{n\text{ times}}\big(\overline{R}\setminus\overbrace{\gamma\big)\setminus\gamma\big)\setminus\cdots\setminus\gamma}^{n\text{ times}}\big)=\overline{\Gamma}. \end{aligned}$$

Let $p\in\mathbb{R}^2$ such that $\{F^n(p)\}_{n\geq 0}$ is defined, that is, no image of p ends in γ. If $p\in\overline{R}$, then, from Remark 5.1, there is a sub-sequence $\{F^{n_k}(p)\}$ contained in $\overline{H}\cap\overline{R}$, which is bounded. Therefore $\{F^{n_k}(p)\}$ has a convergent sub-sequence, and so the ω-limit $\omega\,(p)$ is not empty. Let $p^*\in\omega\,(p)$ and let $\{F^{n_k}(p)\}$ be a sub-sequence of $\{F^n(p)\}$ such that $F^{n_k}(p)\to p^*$ $(k\to\infty)$. Suppose that $p^*\notin\overline{\Gamma}$. Since

$$\overbrace{F\big(\cdots F\big(F}^{n+1\text{ times}}\big(\overline{R}\setminus\overbrace{\gamma\big)\setminus\gamma\big)\setminus\cdots\setminus\gamma}^{n+1\text{ times}}\big)\subset\overbrace{F\big(\cdots F\big(F}^{n\text{ times}}\big(\overline{R}\setminus\overbrace{\gamma\big)\setminus\gamma\big)\setminus\cdots\setminus\gamma}^{n\text{ times}}\big)$$

and $\overline{\Gamma}$ is closed, there is a large enough $N\in\mathbb{N}$ and a small enough neighborhood of $p^*\in U\subset\mathbb{R}^2\setminus\overline{\Gamma}$ such that

$$U\cap\bigcup_{n=N}^{\infty}\overbrace{F\big(\cdots F\big(F}^{n\text{ times}}\big(\overline{R}\setminus\overbrace{\gamma\big)\setminus\gamma\big)\setminus\cdots\setminus\gamma}^{n\text{ times}}\big)=\emptyset. \tag{5.2}$$

Take $n_k\geq N$ such that $F^{n_k}(p)\in U$. On the other hand,

$$F^{n_k}(p)\in\overbrace{F\big(\cdots F\big(F}^{n_k\text{ times}}\big(\overline{R}\setminus\overbrace{\gamma\big)\setminus\gamma\big)\setminus\cdots\setminus\gamma}^{n_k\text{ times}}\big),$$

which is in contradiction with (5.2). Consequently, $\omega\,(p)\subset\overline{\Gamma}$. Suppose that $p\notin\overline{R}$. If $p\in R_\gamma^+\cap R_\delta^+$ then, following Proposition 3.3, take the first n such that $F^n(p)\in R_\gamma^-$. Since $F\left(R_\gamma^+\right)\subset R_\delta^+$, $F^n(p)\in R_\gamma^-\cap R_\delta^+$. Therefore, since $F\left(R_\gamma^-\cap R_\delta^+\right)\subset\overline{R}$, much as before, $\omega(p)\subset\overline{\Gamma}$. The other cases, i.e. $p\in R_\gamma^-\cap R_\delta^-$ or $p\in R_\gamma^-\cap R_\delta^+$ or $p\in R_\gamma^+\cap R_\delta^-$, are similar. Thus the proof is completed. □

6. Fractal structures

In this section we describe the fractal appearance of the attractor built in the previous section. The main idea is to take the region limited by the invariant curves $L^\pm$, which after forward iterations produces long cylinders, in number equal to 2^n, which are removed at successive intersections producing long gaps. This idea suggests the appearance of a Cantor-like set but not bounded. We will prove that the intersection of $\overline{\Gamma}$ with the graph of an increasing function is in fact a classical Cantor set. On the other hand, we will also prove that the intersection of $\overline{\Gamma}$ with the set P (the union of all successive pre-images from γ) is a dense set in $\overline{\Gamma}$.

Let $R = R^+_{F(L^-)} \cap R^-_{F(L^+)}$ be the non-bounded open region (cylinder) limited by $F(L^-)$ and $F(L^+)$. Consider $\Lambda = \Lambda(F, L^-, L^+)$ the maximal invariant set of F in R. Similarly to Section 5,

$$\begin{aligned}\Gamma &:= \bigcap_{n=1}^{\infty} \overbrace{F\big(\cdots F\big(F}^{n \text{ times}}\big(R \setminus \overbrace{\gamma) \setminus \gamma) \setminus \cdots \setminus \gamma}^{n \text{ times}}\big) \\ &= (R_- \cup R_+) \;\cap\; (R_{--} \cup R_{+-} \cup R_{-+} \cup R_{++}) \;\cap\; \cdots ,\end{aligned}$$

where $R_+ = F_+(R)$, $R_- = F_-(R)$, $R_{++} = F_+(R_+)$, $R_{-+} = F_+(R_-)$, $R_{+-} = F_-(R_+)$, and $R_{--} = F_-(R_-)$. And so on successively. On the other hand, $F^{-1}(R \setminus \delta) \cap R = R \setminus \gamma$, $F^{-1}\left\{F^{-1}(R \setminus \delta) \setminus \delta\right\} \cap R = R \setminus (y_- \cup \gamma \cup y_+)$, and so on successively. Thus

$$\begin{aligned}\Lambda &:= \bigcap_{n=1}^{\infty} \overbrace{F^{-1}\big(\cdots F^{-1}\big(F^{-1}}^{n \text{ times}}\big(R \setminus \overbrace{\delta) \setminus \delta) \setminus \cdots \setminus \delta}^{n \text{ times}}\big) \cap R \\ &\quad \cap \bigcap_{n=1}^{\infty} \overbrace{F\big(\cdots F\big(F}^{n \text{ times}}\big(R \setminus \overbrace{\gamma) \setminus \gamma) \setminus \cdots \setminus \gamma}^{n \text{ times}}\big) \\ &= \cdots \cap (R \setminus (y_- \cup \gamma \cup y_+)) \cap (R \setminus \gamma) \cap R \\ &\quad \cap (R_- \cup R_+) \cap (R_{--} \cup R_{+-} \cup R_{-+} \cup R_{++}) \cap \cdots \\ &= \left(R \setminus \bigcup_{\alpha \in \mathcal{P}} \alpha\right) \cap \Gamma.\end{aligned}$$

Remark 6.1. Following our constructions we make the following remarks:

(1) For every point p belonging to some image from L^+ (see (5.1) above), its backward orbit remains in L^+; that is, there exists $N \geq 0$ such that $F^{-n}(p) \in L^+$ for all $n \geq N$. Similarly, the backward orbits of those points belonging to some image from L^- remain in L^-.

(2) Set $A^+ = R^+_{F(L^+)} \cap R^+_\gamma$ (the non-bounded region limited by L^+ and γ) and $A^- = R^-_{F(L^-)} \cap R^-_\gamma$ (the non-bounded region limited by L^- and γ). For those points which do not belong to Λ, their backward orbits end in (the discontinuity) δ or they remain in $A^+ \cup A^-$.

(3) If p is a successive pre-image of some point in γ then its orbit ends in γ. For every point $p \in \Lambda$, its full orbit $\{F^n(p)\}_{n\in\mathbb{Z}}$ is defined.

(4) For those points that belong to $\mathbb{R}^2 \setminus \bigcup_{\alpha\in\mathcal{P}} \alpha$, their forward orbits go from the region $R^+_\gamma \cap R$ to $R^-_\gamma \cap R$ and from $R^-_\gamma \cap R$ to $R^+_\gamma \cap R$ infinitely many times.

(5) For points in Λ, their backward orbits go from $R_\delta^+ \cap R$ to $R_\delta^- \cap R$ and from $R_\delta^- \cap R$ to $R_\delta^+ \cap R$ infinitely many times. Also, the pre-image of $\overline{R}_+ \cap \overline{R_\gamma^-}$ goes to $\overline{H} \cap \overline{R} \cap R_\gamma^+$ and the pre-image of $\overline{R}_- \cap \overline{R_\gamma^+}$ goes to $\overline{H} \cap \overline{R} \cap R_\gamma^-$ (where, as above, $\overline{R}_+ = F_+\left(\overline{R}\right)$ and $\overline{R}_- = F_-\left(\overline{R}\right)$). Consequently, for every point in $p \in \overline{R} \cap \overline{H} \cap \Lambda$ which does not belong to any image from $L^\pm$, there exists a natural number $m(p) \geq 1$ such that $F^{-m(p)}(p) \in \overline{R} \cap \overline{H} \cap \Lambda$. We call $m(p)$ the *first return time* of p (for backward iterates) to $\overline{R} \cap \overline{H} \cap \overline{R}$ (where, as in Section 3, $\overline{H} = \overline{R^+_{y_0^-}} \cap \overline{R^-_{y_0^+}}$: the non-bounded closed region limited by y_0^+ and y_0^-).

The next lemma shows an *expansive-like* property for a large region of parameters. Similarly to [1], given (x_0, y_0) and (x_0', y_0') define $\Delta x_n = x_n' - x_n$ and $\Delta y_n = y_n' - y_n$ for each $n \geq 0$.

Lemma 6.2. *Suppose that* $0 < c < a < 1$ *and let* $p_0 = (x_0, y_0)$ *and* $p_0' = (x_0', y_0')$ *be points in* Λ *which do not belong to any image from* $L^\pm$*. If* $\Delta x_0 > 0$ *and* $\Delta y_0 > 0$*, then there exists* $n \in \mathbb{Z}^+$ *such that* $y_{-n} y'_{-n} < 0$*. (That is, for some* $n \in \mathbb{Z}^+$*, either* $F^{-n}(p_0) \in R_\gamma^+$ *and* $F^{-n}(p_0') \in R_\gamma^-$*, or* $F^{-n}(p_0) \in R_\gamma^-$ *and* $F^{-n}(p_0') \in R_\gamma^+$*).*

Proof. (By contradiction.) Suppose that, for all positive integers n, we have

$$y_{-n} y'_{-n} > 0. \tag{6.1}$$

Note that

$$\begin{aligned}
\Delta y_{-1} &= y'_{-1} - y_{-1} = \frac{(cx_0' + y_0')}{c} - \frac{(cx_0 + y_0)}{c} = \Delta x_0 + \frac{\Delta y_0}{c} > \frac{\Delta y_0}{c} > 0 \\
\Delta x_{-1} &= x'_{-1} - x_{-1} = \frac{x_0'}{a} - \frac{c}{acx_0' + ay_0'} - \frac{x_0}{a} + \frac{c}{acx_0 + ay_0} \\
&= \frac{\Delta x_0}{a} + \frac{1}{ay_{-1}} - \frac{1}{ay'_{-1}} = \frac{\Delta x_0}{a} + \frac{(y'_{-1} - y_{-1})}{ay_{-1}y'_{-1}} \\
&= \frac{\Delta x_0}{a} + \frac{\Delta y_{-1}}{ay_{-1}y'_{-1}} > \frac{\Delta x_0}{a} > 0 \\
\Delta y_{-2} &= y'_{-2} - y_{-2} = \frac{(cx'_{-1} + y'_{-1})}{c} - \frac{(cx_{-1} + y_{-1})}{c} \\
&= \Delta x_{-1} + \frac{\Delta y_{-1}}{c} > \frac{\Delta y_{-1}}{c} > \frac{\Delta y_0}{c^2} > 0 \\
\Delta x_{-2} &= x'_{-2} - x_{-2} = \frac{x'_{-1}}{a} - \frac{c}{acx'_{-1} + ay'_{-1}} - \frac{x_{-1}}{a} + \frac{c}{acx_{-1} + ay_{-1}} \\
&= \frac{\Delta x_{-1}}{a} + \frac{1}{ay_{-2}} - \frac{1}{ay'_{-2}} = \frac{\Delta x_{-1}}{a} + \frac{(y'_{-2} - y_{-2})}{ay_{-2}y'_{-2}} \\
&= \frac{\Delta x_{-1}}{a} + \frac{\Delta y_{-2}}{ay_{-2}y'_{-2}} > \frac{\Delta x_{-1}}{a} > \frac{\Delta x_0}{a^2} > 0.
\end{aligned}$$

In general, we obtain

$$\Delta x_{-n} > \frac{\Delta x_{-n+1}}{a} > \cdots > \frac{\Delta x_{-1}}{a^{n-1}} > \frac{\Delta x_0}{a^n}$$
$$\Delta y_{-n} > \frac{\Delta y_{-n+1}}{c} > \cdots > \frac{\Delta y_{-1}}{c^{n-1}} > \frac{\Delta y_0}{c^n};$$

consequently, $\Delta y_{-n} \to +\infty$ and $\Delta x_{-n} \to +\infty$ (when $n \to \infty$). Following the item 5 of Remark 6.1, at negative iterates, the orbits of p_0 and p_0' go to the bounded region $\overline{H} \cap \overline{R}$. If we suppose that the backward orbits of p_0 and p_0' go to $\overline{H} \cap \overline{R}$ at the same times (in other case, there is nothing to prove) then it is impossible for them to have the same signs at return times. Therefore we obtain a contradiction with (6.1). $\square$

Lemma 6.3. *Suppose that $0 < c < \sqrt{a^3} < a < \sqrt{c} < 1$. Let $p_0 = (x_0, y_0)$ and $p_0' = (x_0', y_0')$ be points in $\mathbb{R}^2$ such that both $\{F^n(p_0)\}_{n\geq 0}$ and $\{F^n(p_0')\}_{n\geq 0}$ are defined. If $\Delta x_0 < 0$ and $\Delta y_0 > 0$ then there exists $n \in \mathbb{N}$ such that $y_n y_n' < 0$. (That is, for some n in $\mathbb{N}$, either $F^n(p_0) \in R_\gamma^+$ and $F^n(p_0') \in R_\gamma^-$, or $F^n(p_0) \in R_\gamma^-$ and $F^n(p_0') \in R_\gamma^+$).*

Proof. (By contradiction.) Suppose that, for all natural numbers n, we have

$$y_n y_n' > 0. \tag{6.2}$$

Take $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$\frac{1}{c\left(\displaystyle\sum_{j=1}^{\infty} a^j\right)^2}\left(\frac{a^3}{c^2}\right)^{n-1} - \frac{2c}{\sqrt{ac}\displaystyle\sum_{j=1}^{\infty} a^j}\left(\frac{a^2}{c}\right)^{n-1}|\overline{H}\cap\overline{R}| + ca^{n-1}|\overline{H}\cap\overline{R}|^2 \geq C > 1 \tag{6.3}$$

for some constant C, where$|\overline{H} \cap \overline{R}|$ is the diameter of $\overline{H} \cap \overline{R}$ (which is well defined since $\overline{H} \cap \overline{R}$ is bounded). Observe that (6.3) follows from our hypotheses $0 < c < \sqrt{a^3} < a < \sqrt{c} < 1$ or, equivalently, $\dfrac{a^3}{c^2} > 1$ and $\dfrac{a^2}{c} < 1$.

Note that

$$\Delta x_1 = x_1' - x_1 = ax_0' + \frac{1}{y_0'} - ax_0 - \frac{1}{y_0} = a\Delta x_0 + \frac{(y_0 - y_0')}{y_0' y_0} = a\Delta x_0 - \frac{\Delta y_0}{y_0' y_0}$$
$$\Delta y_1 = y_1' - y_1 = cy_0' - \frac{c}{y_0'} - acx_0' - cy_0 + \frac{c}{y_0} + acx_0 = c\Delta y_0 - c\Delta x_1$$
$$\Delta x_2 = x_2' - x_2 = ax_1' + \frac{1}{y_1'} - ax_1 - \frac{1}{y_1} = a\Delta x_1 + \frac{(y_1 - y_1')}{y_1' y_1} = a\Delta x_1 - \frac{\Delta y_1}{y_1' y_1}$$
$$\Delta y_2 = y_2' - y_2 = cy_1' - \frac{c}{y_1'} - acx_1' - cy_1 + \frac{c}{y_1} + acx_1 = c\Delta y_1 - c\Delta x_2.$$

In general, we obtain

$$\begin{aligned}
\Delta x_n &= a\Delta x_{n-1} - \frac{\Delta y_{n-1}}{y'_{n-1}y_{n-1}} < 0 \quad \forall n \geq 1\\
&= a^n\Delta x_0 - a^{n-1}\frac{\Delta y_0}{y_0y'_0} - a^{n-2}\frac{\Delta y_1}{y_1y'_1} - \cdots - a\frac{\Delta y_{n-2}}{y_{n-2}y'_{n-2}} - \frac{\Delta y_{n-1}}{y_{n-1}y'_{n-1}}\\
\Delta y_n &= c\Delta y_{n-1} - c\Delta x_n > c^n\Delta y_0 > 0 \quad \forall n \geq 1\\
\Delta y_n &= c^n\Delta y_0 - c^n\Delta x_1 - c^{n-1}\Delta x_2 - \cdots - c\Delta x_n\\
&= c^n\Delta y_0 - c^n\left(a\Delta x_0 - \frac{\Delta y_0}{y'_0y_0}\right) - c^{n-1}\left(a\Delta x_1 - \frac{\Delta y_1}{y'_1y_1}\right)\\
&\qquad - \cdots - c\left(a\Delta x_{n-1} - \frac{\Delta y_{n-1}}{y'_{n-1}y_{n-1}}\right).
\end{aligned}$$

Similarly to item (2) of the Main Theorem, suppose (without losing generality) that $p_0 \in \overline{R}$; in other case, for some n, $F^n(p_0)$ does. Therefore suppose that also $p'_0 \in \overline{R}$ (in other case, there is nothing to prove). Using Remark 5.1, suppose (without losing generality) that p_0 is in the region $\overline{H} \cap \overline{R}$; therefore suppose that also $p'_0 \in \overline{H} \cap \overline{R}$ (in other case, there is nothing to prove). Suppose also that $p_0 \in R^-_{u^+_N} \cap R^+_{u^-_N}$, where $R^-_{u^+_N}$ and $R^+_{u^-_N}$ are the inferior and superior components induced by the guidelines u^+_N and u^-_N (in other case, for some n, $F^n(p_0)$ does). Therefore suppose that also $p'_0 \in R^-_{u^+_N} \cap R^+_{u^-_N}$ (in other case, there is nothing to prove). This last assumption implies that the first return time of p_0 to $\overline{H} \cap \overline{R}$ is at least N, where N is given by (6.3). Suppose that $y_0 < 0$, $y'_0 < 0$ (if $y_0 > 0$, $y'_0 > 0$, the proof is similar). Consequently, $y_1 > 0$, $y'_1 > 0$, and take n to be the first return time of p_0 to $\overline{H} \cap \overline{R}$. Observe from (6.2) that we can suppose that n is also the first return time of p'_0 to $\overline{H} \cap \overline{R}$ (in other case, there is nothing to prove). Note from (6.2) that $y_1 > 0$, $y'_1 > 0$, $\ldots$, $y_n > 0$, $y'_n > 0$, $y_{n+1} < 0$, $y'_{n+1} < 0$. Also, it is crucial to note that, in this case, $x_1 < 0$, $x'_1 < 0$. Therefore

$$\begin{aligned}
y_1 &= cy_0 - cx_1 < -cx_1 = a\frac{c}{a}(-x_1)\\
y_2 &= cy_1 - cx_2 = cy_1 - acx_1 - \frac{c}{y_1} < a\frac{c^2}{a}(-x_1) + \frac{a^3}{c}\frac{c^2}{a^2}(-x_1)\\
&= a^2\frac{c^2}{a^2}(-x_1) + \frac{a^3}{c}\frac{c^2}{a^2}(-x_1) < a^2\frac{c^2}{a^2}(-x_1) + a\frac{c^2}{a^2}(-x_1)\\
&= (a^2+a)\left(\frac{c}{a}\right)^2(-x_1)\\
y_3 &= cy_2 - cx_3 = cy_2 - acx_2 - \frac{c}{y_2} < cy_2 - acx_2 = cy_2 - a^2cx_1 - \frac{c}{y_1}\\
&< cy_2 - a^2cx_1 < (a^2+a)\frac{c^3}{a^2}(-x_1) + \frac{a^5}{c^2}\frac{c^3}{a^3}(-x_1)
\end{aligned}$$

$$
\begin{aligned}
&= a(a^2+a)\frac{c^3}{a^3}(-x_1) + \frac{a^5}{c^2}\frac{c^3}{a^3}(-x_1) < (a^3+a^2)\frac{c^3}{a^3}(-x_1) + a\frac{c^3}{a^3}(-x_1)\\
&= (a^3+a^2+a)\left(\frac{c}{a}\right)^3(-x_1).
\end{aligned}
$$

It should be noted that we used the hypothesis $0 < a < \sqrt{c}$. In general, we have

$$
\begin{aligned}
0 < K < y_{n-1} &< (a^{n-1}+\cdots+a^2+a)\left(\frac{c}{a}\right)^{n-1}(-x_1)\\
&\leq \left(\sum_{j=1}^{\infty} a^j\right)\left(\frac{c}{a}\right)^{n-1}\left(-ax_0 - \frac{1}{y_0}\right),
\end{aligned}
$$

where $K = \frac{1}{\sqrt{ac}}$ is the intersection of the guidelines y_0^+ and x_0^+, which occurs in the y-axis; therefore

$$
\begin{aligned}
\frac{1}{|y_0|} &= -\frac{1}{y_0} > \frac{1}{\sqrt{ac}\sum_{j=1}^{\infty} a^j}\frac{a^{n-1}}{c^{n-1}} - |ax_0| > \frac{1}{\sqrt{ac}\sum_{j=1}^{\infty} a^j}\frac{a^{n-1}}{c^{n-1}} - |\overline{H}\cap\overline{R}|\\
\frac{1}{|y_0'|} &= -\frac{1}{y_0'} > \frac{1}{\sqrt{ac}\sum_{j=1}^{\infty} a^j}\frac{a^{n-1}}{c^{n-1}} - |ax_0'| > \frac{1}{\sqrt{ac}\sum_{j=1}^{\infty} a^j}\frac{a^{n-1}}{c^{n-1}} - |\overline{H}\cap\overline{R}|.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\Delta y_n &> -ac\Delta x_{n-1} > caa^{n-2}\frac{\Delta y_0}{y_0' y_0}\\
&> ca^{n-1}\Delta y_0\left(\frac{1}{\sqrt{ac}\sum_{j=1}^{\infty} a^j}\frac{a^{n-1}}{c^{n-1}} - |\overline{H}\cap\overline{R}|\right)\left(\frac{1}{\sqrt{ac}\sum_{j=1}^{\infty} a^j}\frac{a^{n-1}}{c^{n-1}} - |\overline{H}\cap\overline{R}|\right)\\
&= \Delta y_0\left(\frac{1}{a\left(\sum_{j=1}^{\infty} a^j\right)^2}\left(\frac{a^3}{c^2}\right)^{n-1} - \frac{2c}{\sqrt{ac}\sum_{j=1}^{\infty} a^j}\left(\frac{a^2}{c}\right)^{n-1}|\overline{H}\cap\overline{R}| + ca^{n-1}|\overline{H}\cap\overline{R}|^2\right)\\
&\geq C\Delta y_0.
\end{aligned}
$$

Similarly, denoting $n = n_1$, if n_2 is the first return time of $F^{n_1}(p_0) = p_{n_1}$ to $\overline{H}\cap\overline{R}$, then

$$
\Delta y_{n_2} \geq C\Delta y_{n_1} \geq C^2\Delta y_0
$$

and so on successively. Thus, since $C > 1$, we have $\Delta y_{n_j} \to \infty$ $(j \to \infty)$. So, with j sufficiently large, it is impossible for $F^{n_j}(p_0)$ and $F^{n_j}(p_0')$ to have the same sign, and so we obtain a contradiction with (6.2). $\square$

Interchanging the roles of p_0 and p_0', Lemma 6.2 (resp., Lemma 6.3) holds for the case $\Delta x_0 > 0$ and $\Delta y_0 < 0$ (resp., $\Delta x_0 < 0$ and $\Delta y_0 < 0$). In the next lemma, as before, we consider $\mathcal{P}$ and $\mathcal{I}$ the sets of guidelines induced by γ and δ. Also, we consider the set $\mathcal{L}$ of all curves induced by $L^{\pm}$ (as in (5.1) above).

Lemma 6.4. *Suppose that $0 < c < a < 1$ and let r be a long curve contained in R_γ^+ which is the graph of an increasing function. Then the union of all curves belonging to $\mathcal{L}$ is dense in $r \cap \overline{\Gamma}$.*

Proof. Since all curves in $\mathcal{L}$ are long curves which are graphs of decreasing functions, $\mathcal{L}$ intersects r transversely at single points. Suppose that the union of all curves belonging to $\mathcal{L}$ is not dense in $r \cap \overline{\Gamma}$. Then, there are two different points p, p' in r such that the connected curve (p, p'), joining p with p' and contained in r, does not intersect any curve induced by $L^{\pm}$. Therefore (p, p') does not intersect any guideline with respect to δ. So no point in (p, p') is an image from δ; in particular, $(p, p') \subset R_\delta^+$ or $(p, p') \subset R_\delta^-$. Suppose, without losing generality, that neither p nor p' is an image from δ; if not, then take two different points in (p, p') with that property. Since p and p' belong to a graph of an increasing function, the hypotheses of Lemma 6.2 hold. Due to our assumption, for every $n \geq 0$, $n \in \mathbb{Z}^+$, we have that $F^{-n}([p, p'])$ is connected. In other words, for every $n \geq 0$, $n \in \mathbb{Z}^+$, $F^{-n}([p, p']) \subset R_\gamma^+$ or $F^{-n}([p, p']) \subset R_\gamma^-$. On the other hand, following Lemma 6.2 we can take $N \in \mathbb{Z}^+$ such that $F^{-N}(p) \in R_\gamma^-$ and $F^{-N}(p') \in R_\gamma^+$ (or $F^{-N}(p) \in R_\gamma^+$ and $F^{-N}(p') \in R_\gamma^-$). Therefore, since $F^{-N}([p, p'])$ is connected, we obtain the contradictory fact $F^{-N}([p, p']) \cap \gamma \neq \emptyset$. Quite similarly, the same conclusions are obtained for long curves contained in R_γ^- which are graphs of increasing functions. $\square$

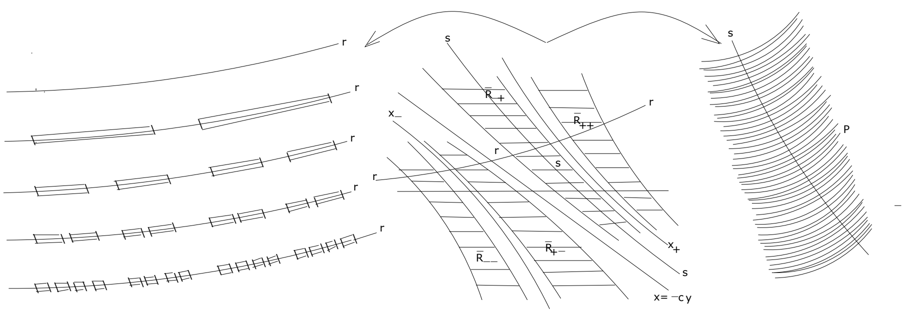

Figure 5.

Proof of item (3) of the Main Theorem. Fix a long curve r contained in R_γ^+ which is the graph of a strictly increasing function. Since $\mathcal{L}$ is a set of long curves which are graphs of decreasing functions, r intersects $\bigcup_{\alpha \in \mathcal{L}} \alpha$ (the union of all curves contained in $\mathcal{L}$) transversely. In particular, r intersects the border of $\overline{R}$ transversely and $r \cap \overline{\Gamma}$ is a bounded set (see Figure 5b). The curve r is a closed set in $\mathbb{R}^2$ and clearly, by its definition, $\overline{\Gamma}$ is also closed, so $r \cap \overline{\Gamma}$ is closed. Therefore $r \cap \overline{\Gamma}$ is a set contained in the connected curve $r \cap \overline{R}$, which is compact. Lemma 6.4 implies both: (1) $r \cap \overline{\Gamma}$ does not have connected curves contained in $r \cap \overline{R}$; and (2) every point in $r \cap \overline{\Gamma}$ is a limit point of a sequence contained in $r \cap \left(\bigcup_{\alpha\in\mathcal{L}} \alpha\right)$ and $r \cap \left(\bigcup_{\alpha\in\mathcal{L}} \alpha\right) \subset r \cap \overline{\Gamma}$. Consequently $r \cap \overline{\Gamma}$ is a Cantor set (see Figure 5a). Similarly, any long curve contained in R_γ^- which is the graph of an increasing function intersects $\overline{\Gamma}$ in a Cantor set. □

Proof of item (4) of the Main Theorem. Γ is the set containing the union of all curves in $\mathcal{L}$ or all curves that are uniform limits of curves in $\mathcal{L}$; since all these curves are graphs of strictly decreasing functions, subitem (b) of item (4) follows. Let s be a long curve contained in R_δ^+ which is the graph of a decreasing function. Since all curves in $\mathcal{P}$ are long curves which are graphs of increasing functions, $\mathcal{P}$ intersects s transversely at single points (see Figure 5c). Suppose that the union of all curves belonging to $\mathcal{P}$ is not dense in s. Then, there are two different points p, p' in s such that the connected curve (p,p'), joining p with p' and contained in s, does not intersect any curve belonging to $\mathcal{P}$. Therefore (p,p') does not intersect any guideline with respect to γ. So no point in (p,p') is an image from γ; in particular, $(p,p') \subset R_\gamma^+$ or $(p,p') \subset R_\gamma^-$. Suppose, without losing generality, that neither p nor p' is an image from γ; if not, then take two different points in (p,p') with that property. Since p and p' belong to a graph of a decreasing function, the hypotheses of Lemma 6.3 hold. Due to our assumption, for every $n \geq 0$, $n \in \mathbb{Z}^+$, we have that $F^n([p,p'])$ is connected. In other words, for every $n \geq 0$, $n \in \mathbb{Z}^+$, $F^n([p,p']) \subset R_\gamma^+$ or $F^n([p,p']) \subset R_\gamma^-$. On the other hand, following Lemma 6.3 we can take $N \in \mathbb{Z}^+$ such that $F^N(p) \in R_\gamma^-$ and $F^N(p') \in R_\gamma^+$ (or $F^N(p) \in R_\gamma^+$ and $F^N(p') \in R_\gamma^-$). Thus, since $F^N([p,p'])$ is connected, we obtain the contradictory fact $F^N([p,p']) \cap \gamma \neq \emptyset$. The same conclusions are obtained for long curves contained in R_δ^- which are graphs of decreasing functions. □

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Bladismir Leal✉
Instituto de Ciencias Básicas, Universidad Técnica de Manabí, Av. José María Urbina, Portoviejo, Ecuador
Facultad de Ingeniería, Universidad Nacional de Chimborazo, Vía Guano km 1.5, Riobamba, Ecuador
bladismir@gmail.com

Sergio Muñoz
Faculdade de Tecnologia, Dpto. de Matemática, Física e Computação, Universidade do Estado do Rio de Janeiro, Resende, Brasil
sergio.munoz@uerj.br

Received: August 14, 2020
Accepted: May 26, 2021