REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 63, No. 2, 2022, Pages 523–530
Published online: December 24, 2022
https://doi.org/10.33044/revuma.2136

ON THE COMMON FIXED POINTS OF DOUBLE SEQUENCES OF TWO-PARAMETER NONEXPANSIVE SEMIGROUPS IN STRICTLY CONVEX BANACH SPACES

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Abstract. We obtain some results about common fixed points of strongly continuous two-parameter semigroups of nonexpansive mappings in strictly convex Banach spaces. Also, by using them we characterize common fixed points of double sequences of such semigroups.

1. Introduction

Let $\mathcal{A}$ be a *strictly convex* Banach space, i.e., for all $x, y \in \mathcal{A}$ with $\|x\| = \|y\| = 1$ and $x \neq y$, we have $\|x + y\| < 2$, and let C be a closed convex subset of $\mathcal{A}$. The mapping T on C is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We will denote the set of all fixed points of T by $F(T)$.

In this paper, for a two-parameter semigroup of nonexpansive mappings on a closed convex subset of a strictly convex Banach space, we obtain some results related to its common fixed points and apply them to characterize common fixed points of double sequences of such semigroups. This is an extension of the one-parameter case considered in [13].

Two-parameter semigroups of operators, which are an extension of the one-parameter case, were studied by E. Hille in 1944 [3], and in 1946 N. Dunford and I. Segal [2] applied them to prove the theorem of Weierstrass. X-parameter semigroups were studied by Hille and Phillips [4]. O. A. Ivanova obtained some other results in n-parameter groups in 1966 [5]. Semigroups of operators have many applications in several areas of applied mathematics, such as theory of random fields [9, 14], statistical mechanics, partial differential equations and time evolution in quantum field theory [10, 11, 12].

2. Main results

In this section, we discuss some basic theorems about common fixed points of a two-parameter semigroup of nonexpansive mappings on a strictly convex subset of a Banach space.

2020 *Mathematics Subject Classification.* 47D03, 47H09, 47H10.
Key words and phrases. Fixed points, nonexpansive semigroup, Banach space.

Definition 2.1. Let C be a closed convex subset of a Banach space $\mathcal{A}$. The family $\{\alpha(s,t) : (s,t) \in \mathbb{R}_+ \times \mathbb{R}_+\}$ is a *two-parameter semigroup of nonexpansive mappings on* C if:

i) for each $(s,t) \in \mathbb{R}_+ \times \mathbb{R}_+$, $\alpha(s,t)$ is a nonexpansive mapping on C;
ii) $\alpha(s+s', t+t') = \alpha(s,t)\alpha(s',t')$ for all $s, s', t, t' \in \mathbb{R}_+$;
iii) for each $x \in C$, the mapping $(s,t) \mapsto \alpha(s,t)x$ from $\mathbb{R}_+ \times \mathbb{R}_+$ into C is strongly continuous.

For more information about properties of two-parameter semigroups, the reader may consult [1, 6, 7, 8].

Theorem 2.2. *Let C be a closed convex subset of a strictly convex Banach space $\mathcal{A}$. Let $a, b > 0$ and let $\{\alpha(s,t) : (s,t) \in [0,a) \times [0,b)\}$ be a family of mappings on C such that for each $(s,t) \in [0,a) \times [0,b)$, $\alpha(s,t)$ is nonexpansive and the mapping $(s,t) \mapsto \alpha(s,t)x$ is weakly continuous on $[a_n, a_{n+1}) \times [b_m, b_{m+1})$ for all $x \in C$ and $m, n \in \mathbb{N}$ for some increasing sequences $\{a_n\}$ and $\{b_m\}$ with $a_1 = b_1 = 0$, $a_n \to a$ and $b_m \to b$. Suppose*

$$\bigcap_{\substack{0 \le s < a \\ 0 \le t < b}} F(\alpha(s,t)) \neq \emptyset,$$

and define a nonexpansive mapping T on C as follows:

$$Tx = \frac{1}{ab} \int_0^a \int_0^b \alpha(s,t)x \, dt \, ds.$$

Then

$$\bigcap_{\substack{0 \le s < a \\ 0 \le t < b}} F(\alpha(s,t)) = F(T).$$

Proof. For $x, y \in C$, we have

$$\begin{aligned}
\|Tx - Ty\| &= \left\| \frac{1}{ab} \int_0^a \int_0^b (\alpha(s,t)x - \alpha(s,t)y) \, dt \, ds \right\| \\
&\le \frac{1}{ab} \int_0^a \int_0^b \|\alpha(s,t)x - \alpha(s,t)y\| \, dt \, ds \\
&\le \frac{1}{ab} \int_0^a \int_0^b \|x - y\| \, ds \, dt = \|x - y\|.
\end{aligned}$$

Hence T is nonexpansive. Clearly,

$$\bigcap_{\substack{0 \le s < a \\ 0 \le t < b}} F(\alpha(s,t)) \subset F(T).$$

We have to show that

$$F(T) \subset \bigcap_{\substack{0 \le s < a \\ 0 \le t < b}} F(\alpha(s,t)).$$

For this, let $z \in F(T)$ and $z \notin \bigcap_{\substack{0\le s<a\\0\le t<b}} F(\alpha(s,t))$; then there exist $(s_1,t_1) \in [0,a)\times[0,b)$ such that $\alpha(s_1,t_1)z \neq z$. By the Hahn–Banach theorem, there exists $f \in \mathcal{A}^*$ (the dual space of $\mathcal{A}$) with

$$\|f\| = 1, \qquad f(\alpha(s_1,t_1)z - z) = \|\alpha(s_1,t_1)z - z\|.$$

For some $n, m \in N$, we have $s_1 \in [a_n, a_{n+1})$ and $t_1 \in [b_m, b_{m+1})$. Let

$$l = \frac{1}{2}\min\{|s_1 - a_{n+1}|, |t_1 - b_{m+1}|\};$$

then, from the weak continuity assumption of the theorem, there exists $r \in (0,l]$ such that, for all $(s,t) \in [s_1, s_1+r)\times[t_1,t_1+r)$, we have

$$f(\alpha(s,t)z - z) > \frac{1}{2}\|\alpha(s_1,t_1)z - z\|.$$

Set $D_1 := [s_1, s_1+r)\times[t_1,t_1+r)$ and $D_2 := [0,a)\times[0,b)\setminus D_1$, and define mappings T_1 and T_2 on C by

$$T_1x = \frac{1}{r^2}\int\int_{D_1} \alpha(s,t)x\,dt\,ds,$$
$$T_2x = \frac{1}{ab - r^2}\int\int_{D_2} \alpha(s,t)x\,dt\,ds.$$

We have

$$Tx = \frac{r^2}{ab}T_1x + \frac{ab-r^2}{ab}T_2x$$

for all $x \in C$. Also

$$\begin{aligned}
f(T_1z - Tz) &= f\left(\frac{1}{r^2}\int\int_{D_1}\alpha(s,t)z\,dt\,ds - z\right)\\
&= f\left(\frac{1}{r^2}\int_{s_1}^{s_1+r}\int_{t_1}^{t_1+r}(\alpha(s,t)z - z)\,dt\,ds\right)\\
&= \frac{1}{r^2}\int_{s_1}^{s_1+r}\int_{t_1}^{t_1+r} f(\alpha(s,t)z - z)\,dt\,ds\\
&\ge \frac{1}{r^2}\int_{s_1}^{s_1+r}\int_{t_1}^{t_1+r}\frac{1}{2}\|\alpha(s_1,t_1)z - z\|\,dt\,ds\\
&= \frac{1}{2}\|\alpha(s_1,t_1)z - z\| > 0.
\end{aligned}$$

So

$$f(T_2z - Tz) = \frac{r^2}{ab - r^2}f(Tz - T_1z) < 0.$$

Hence $T_1 z \neq T_2 z$. Let $y \in \bigcap_{\substack{0\le s<a\\0\le t<b}} F(\alpha(s,t))$; then $T_1 y = T_2 y = y$, and we have

$$\begin{aligned}\|z-y\| = \|T_1 z - y\| &= \left\| \frac{r^2}{ab} T_1 z + \frac{ab-r^2}{ab} T_2 z - y \right\| \\ &\le \frac{r^2}{ab}\|T_1 z - y\| + \frac{ab-r^2}{ab}\|T_2 z - y\| \\ &= \frac{r^2}{ab}\|T_1 z - T_1 y\| + \frac{ab-r^2}{ab}\|T_2 z - T_2 y\| \\ &\le \frac{r^2}{ab}\|z-y\| + \frac{ab-r^2}{ab}\|z-y\| = \|z-y\|,\end{aligned}$$

and therefore

$$\|Tz-y\| = \|T_1 z - y\| = \|T_2 z - y\|.$$

This is a contradiction, because $\mathcal{A}$ is strictly convex. Hence the inclusion $F(T) \subset \bigcap_{\substack{0\le s<a\\0\le t<b}} F(\alpha(s,t))$ holds, and we conclude that

$$F(T) = \bigcap_{\substack{0\le s<a\\0\le t<b}} F(\alpha(s,t)).$$

□

The following two corollaries are a direct consequence of the previous theorem.

Corollary 2.3. *Let C be a closed convex subset of a strictly convex Banach space $\mathcal{A}$. Let $a, b > 0$ and let $\{\alpha(s,t) : (s,t) \in [0,a) \times [0,b)\}$ be a family of nonexpansive mappings on C such that the mappings $(s,t) \mapsto \alpha(s,t)x$ are weakly continuous on $[0,a) \times [0,b)$ for all $x \in C$ and*

$$\bigcap_{\substack{0\le s<a\\0\le t<b}} F(\alpha(s,t)) \neq \emptyset.$$

Define a nonexpansive mapping T on C by

$$Tx = \frac{1}{ab}\int_0^a \int_0^b \alpha(s,t)x \, dt \, ds$$

for all $x \in C$. Then

$$F(T) = \bigcap_{\substack{0\le s<a\\0\le t<b}} F(\alpha(s,t)).$$

Corollary 2.4. *Let C be a closed convex subset of a strictly convex Banach space $\mathcal{A}$. Let $\{T_{mn} : m,n \in \mathbb{N}\}$ be a double sequence of nonexpansive mappings on C. Suppose*

$$\bigcap_{\substack{m=1\\n=1}}^{\infty} F(T_{mn}) \neq \emptyset.$$

Let $\{c_{mn}\}$ *be a double sequence such that* $c_{mn} = a_m b_n$*, where* $\{a_m\}$ *and* $\{b_n\}$ *are sequences of positive numbers with* $\sum_{m=1}^{\infty} a_m = \sum_{n=1}^{\infty} b_n = 1$*. Define a nonexpansive mapping* T *on* C *by*

$$Tx = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} T_{mn} x$$

for all $x \in C$*. Then*

$$F(T) = \bigcap_{\substack{m=1\\n=1}}^{\infty} F(T_{mn}).$$

Proof. We define, inductively, strictly increasing sequences $\{\alpha_m\}$ and $\{\beta_n\}$ in $[0, 1)$ as follows: $\alpha_1 = \beta_1 = 0$ and, for $m \geq 2$ and $n \geq 2$ in $\mathbb{N}$,

$$\alpha_m = \sum_{k=1}^{m-1} a_k, \qquad \beta_n = \sum_{k=1}^{n-1} b_k.$$

Then we have $\lim_{m\to\infty} \alpha_m = \lim_{n\to\infty} \beta_n = 1$.

Let $\{T(s, t) : (s, t) \in [0, 1) \times [0, 1)\}$ be a family of two-parameter nonexpansive maps defined by

$$T(s, t)x = T_{mn}x \qquad \text{if } (s, t) \in [\alpha_m, \alpha_{m+1}) \times [\beta_n, \beta_{n+1})$$

for all $x \in C$. We have

$$Tx = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} T_{mn} x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{\beta_n}^{\beta_{n+1}} \int_{\alpha_m}^{\alpha_{m+1}} T(s, t)x \, ds \, dt$$
$$= \int_0^1 \int_0^1 T(s, t)x \, ds \, dt.$$

Therefore, by

$$\bigcap_{\substack{m=1\\n=1}} F(T_{mn}) = \bigcap_{\substack{0\leq s<1\\0\leq t<1}} F(T(s, t))$$

and Theorem 2.2, we have

$$F(T) = \bigcap_{\substack{m=1\\n=1}} F(T_{mn}).$$

□

Up to here, semigroup properties weren't necessary. From now on, we assume $\{\alpha(s, t)\}$ is a nonexpansive semigroup.

Theorem 2.5. *Let* C *be a closed convex subset of a strictly convex Banach space* $\mathcal{A}$ *and let* $\{\alpha(s, t) : (s, t) \in \mathbb{R}_+ \times \mathbb{R}_+\}$ *be a strongly continuous semigroup of nonexpansive mappings on* C *such that*

$$\bigcap_{\substack{0\leq s<\infty\\0\leq t<\infty}} F(\alpha(s, t)) \neq \emptyset.$$

Suppose $s_1, s_2, t_1, t_2 \in R_+$ with $s_1 < s_2$ and $t_1 < t_2$. Define a nonexpansive mapping T on C by

$$Tx = \frac{1}{(s_2 - s_1)(t_2 - t_1)} \int_{s_1}^{s_2} \int_{t_1}^{t_2} \alpha(s,t)x \, dt \, ds$$

for all $x \in C$. Then

$$F(T) = \bigcap_{\substack{0 \le s < \infty \\ 0 \le t < \infty}} F(\alpha(s,t)).$$

Proof. Clearly,

$$\bigcap_{\substack{0 \le s < \infty \\ 0 \le t < \infty}} F(\alpha(s,t)) \subseteq F(T).$$

For the converse, let $y \in F(T)$; then from Corollary 2.3, we have

$$\bigcap_{\substack{s_1 \le s < s_2 \\ t_1 \le t < t_2}} F(\alpha(s,t)) = F(T),$$

and so, for $(s,t) \in [s_1, s_2) \times [t_1, t_2)$, we have $\alpha(s,t)y = y$. Therefore, for every $(s,t) \in \left[0, \frac{s_2 - s_1}{2}\right] \times \left[0, \frac{t_2 - t_1}{2}\right]$,

$$\alpha(s,t)y = \alpha(s,t)\alpha(s_1, t_1)y = \alpha(s + s_1, t + t_1)y = y.$$

Fix $(s,t) \in R_+ \times R_+$; then there exist $m, n \in N \cup \{0\}$ and $(u,v) \in \left[0, \frac{s_2 - s_1}{2}\right) \times \left[0, \frac{t_2 - t_1}{2}\right)$ such that

$$s = u + m(s_2 - s_1)/2 \qquad \text{and} \qquad t = v + n(t_2 - t_1)/2.$$

Without loss of generality, we can assume that $n \le m$. Hence we have $n = \frac{m}{\beta}$ for some $\beta \ge 1$. Therefore

$$\begin{aligned}
\alpha(s,t)y &= \alpha\left(u + m\frac{(s_2 - s_1)}{2}, v + n\frac{(t_2 - t_1)}{2}\right) y \\
&= \alpha(u,v)\alpha\left(m\frac{(s_2 - s_1)}{2}, n\frac{(t_2 - t_1)}{2}\right) y \\
&= \alpha(u,v)\alpha\left(m\frac{(s_2 - s_1)}{2}, m\frac{(t_2 - t_1)}{2\beta}\right) y \\
&= \alpha(u,v)\left(\alpha\left(\frac{(s_2 - s_1)}{2}, \frac{(t_2 - t_1)}{2\beta}\right)\right)^m y \\
&= \alpha(u,v)y = y,
\end{aligned}$$

where $\left(\alpha\left(\frac{(s_2 - s_1)}{2}, \frac{(t_2 - t_1)}{2\beta}\right)\right)^0$ is the identity mapping on C. Hence y is a common fixed point of $\{\alpha(s,t) : (s,t) \in R_+ \times R_+\}$. Therefore

$$F(T) = \bigcap_{\substack{0 \le s < \infty \\ 0 \le t < \infty}} F(\alpha(s,t)).$$

□

From Corollary 2.4 and Theorem 2.5, we have the following theorem.

Theorem 2.6. *Let C be a closed convex subset of a strictly convex Banach space $\mathcal{A}$. Let $\{T_{mn}(s,t) : (s,t) \in \mathbb{R}_+ \times \mathbb{R}_+,\ m,n \in \mathbb{N}\}$ be a double sequence of strongly continuous two-parameter semigroups of nonexpansive mappings on C, and let $\{U_{mn} : m,n \in \mathbb{N}\}$ be a double sequence of nonexpansive mappings on C such that*

$$\bigcap_{\substack{m=1\\n=1}}^{\infty} \bigcap_{\substack{0\le s<\infty\\0\le t<\infty}} F\left(T_{mn}(s,t)\right) \cap \bigcap_{\substack{m=1\\n=1}}^{\infty} F(U_{mn}) \neq \emptyset.$$

Suppose $\{s_n\}$, $\{t_n\}$, $\{u_n\}$, $\{v_n\}$, $\{\alpha_{mn}\}$ and $\{\beta_{mn}\}$ are real sequences such that $0 \le s_n < u_n$, $0 \le t_n < v_n$, $\alpha_{mn} > 0$ and $\beta_{mn} > 0$ for all $m,n \in \mathbb{N}$, and $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_{mn} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\beta_{mn} = 1$. Define a nonexpansive mapping T on C by

$$Tx = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\alpha_{mn}}{(u_m - s_m)(v_n - t_n)} \int_{s_n}^{u_n}\int_{t_n}^{v_n} T_{mn}(s,t)x\,dt\,ds + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \beta_{mn}U_{mn}x$$

for all $x \in C$. Then

$$\bigcap_{\substack{m=1\\n=1}}^{\infty} \bigcap_{\substack{0\le s<\infty\\0\le t<\infty}} F\left(T_{mn}(s,t)\right) \cap \bigcap_{\substack{m=1\\n=1}}^{\infty} F(U_{mn}) = F(T).$$

Proof. For all $m,n \in \mathbb{N}$, let

$$S_{mn}x = \frac{1}{(u_m - s_m)(v_n - t_n)} \int_{t_n}^{v_n}\int_{s_m}^{u_m} T_{mn}(s,t)x\,ds\,dt.$$

Hence

$$Tx = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \alpha_{mn}S_{mn}x + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \beta_{mn}U_{mn}x.$$

From Theorem 2.5 we obtain the result. □

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Received: August 18, 2020
Accepted: June 30, 2021