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A GENERALIZATION OF THE ANNIHILATING IDEAL GRAPH FOR MODULES

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Abstract. We show that an R-module M is noetherian (resp., artinian) if and only if its annihilating submodule graph, $\mathbb{G}(M)$, is a non-empty graph and it has ascending chain condition (resp., descending chain condition) on vertices. Moreover, we show that if $\mathbb{G}(M)$ is a locally finite graph, then M is a module of finite length with finitely many maximal submodules. We also derive necessary and sufficient conditions for the annihilating submodule graph of a reduced module to be bipartite (resp., complete bipartite). Finally, we present an algorithm for deriving both $\Gamma(\mathbb{Z}_n)$ and $\mathbb{G}(\mathbb{Z}_n)$ by Maple, simultaneously.

1. Introduction

Throughout this article, all rings are commutative with identity and all modules are right unitary modules. Let M be an R-module. For each submodule N of M, define $(N : M) = \{r \in R \mid Mr \subseteq N\}$. The R-module M is called *reduced* provided that, for each $m \in M$ and $a \in R$, $ma^2 = 0$ implies that $ma = 0$. The set of all maximal submodules of M is denoted by $\mathrm{Max}(M)$. Let G be an undirected graph. We say that G is *connected* if there is a path between any two distinct vertices. A *cycle* of length n in G is a path of the form $x_1 - x_2 - x_3 \cdots - x_n - x_1$, where $x_i \neq x_j$ when $i \neq j$. A graph is *complete* if any two distinct vertices are adjacent. A complete graph with n vertices is denoted by K_n. A *bipartite* graph G is a graph whose vertices can be partitioned into two subsets V_1 and V_2 such that no edge has both endpoints in the same subset. A *complete bipartite* graph G is a bipartite graph with partitions V_1 and V_2 such that every possible edge that could connect vertices in different subsets is a part of the graph. That is, for every two vertices $v_1 \in V_1$ and $v_2 \in V_2$, v_1v_2 is an edge in G. A complete bipartite graph with partitions of size $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. A $K_{1,n}$ graph is often called a *star graph.* A graph is called *locally finite* whenever the degree of any vertex is finite. A *ray* is a simple path (a path with no repeated vertices) that begins at one vertex and continues from it through infinitely many vertices. Any

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unexplained terminology, and all the basic results on rings, modules and graphs that are used in what follows can be found in [10], [7], [19], [20] and [27].

Zero-divisor graphs of commutative rings and their related graphs (such as total graphs, annihilating ideal graphs, ...) have been extensively studied by many authors in recent decades (see [3], [5], [6], [7], [8], [9], [10], [11] and [22]). In [14], the classic zero-divisor graph has been generalized to modules over commutative rings. According to [14], two non-zero elements $m, n \in M$ are adjacent if and only if $(mR :_R M)(nR :_R M)M = 0$, which is a direct generalization of the classic zero-divisor graph. In [11] and [24], the authors have associated two different graphs to an R-module M and, accordingly, the "generalized" graphs of abelian groups have been studied in [12]. In [24], for a right R-module M, two elements x and y in M are considered as adjacent if $x*y = 0$, where by this the authors mean either $x(yR : M) = 0$ or $y(xR : M) = 0$. Also, $Z(M) = \{x \in M \mid \exists y \in M \text{ such that } x * y = 0\}$ and $Z(M)^* = Z(M) \setminus \{0\}$. The zero-divisor graph of an R-module M, denoted by $\Gamma(M_R)$, is an undirected graph with $Z(M)^*$ as vertices, and $x, y \in Z(M)^*$ are adjacent provided that $x * y = 0$. The graph $\mathbb{AG}(R)$, the annihilating ideal graph for a commutative ring R, has been introduced and extensively studied in [15], [16] and [1]. In [23], based on the aforementioned definition of the zero-divisor graph for modules, we have introduced the annihilating submodule graph for an R-module M, denoted by $\mathbb{G}(M)$, which is, in turn, a generalization of the annihilating-ideal graph. In this paper, for an R-module M, further aspects of $\mathbb{G}(M)$ are studied, and those results which have already been proved for the annihilating graph of the ring R are generalized to the module M. We will observe that the proofs become more transparent and new results are obtained.

This paper consists of four sections. In Section 2, we study relations between chain conditions on modules and locally finite graphs. Section 3 is essentially devoted to completeness. In Section 4, we examine conditions under which $\mathbb{G}(M)$ is bipartite. We begin immediately with the definition of our graph.

Definition 1.1. Let M be an R-module. The set of all submodules of M is denoted by $\mathbb{S}(M)$. For every two submodules N and K of M, we say that $N * K = 0$ provided that either $N(K : M) = 0$ or $K(N : M) = 0$. The submodule N of M is said to be *annihilating* if there exists a non-zero submodule K of M such that $N * K = 0$. The set of all annihilating submodules of M is denoted by $\mathbb{A}(M)$ and $\mathbb{A}^*(M) = \mathbb{A}(M) \setminus \{0\}$. The *annihilating submodule graph* of M, denoted by $\mathbb{G}(M)$, is an undirected graph with vertex set $\mathbb{A}^*(M)$ and such that $N, K \in \mathbb{A}^*(M)$ are adjacent if $N * K = 0$.

Remark 1.2. If I is an ideal of a ring R, it is obvious that $(I : R) = I$, and this implies that $\mathbb{G}(R)$ is precisely the annihilating-ideal graph of the commutative ring R, introduced in [15]. Inasmuch as [23, Lemma 2.5] has an essential role in this paper, we state it here for the sake of completeness.

Lemma 1.3 ([23, Lemma 2.5]). *Let M be an R-module and let $N, K \in \mathbb{A}^*(M)$. Then the following hold:*

(1) *If $N * K = 0$, then, for every non-zero submodule N' of N and every non-zero submodule K' of K, $N' * K' = 0$.*
(2) *If $N \cap K = 0$, then $N(K : M) = K(N : M) = \{0\}$.*

In Figures 1 and 2 we illustrate both the zero-divisor graph and the annihilating submodule graph ($\Gamma(M)$ and $\mathbb{G}(M)$, respectively) for some cyclic abelian groups, simultaneously. This will help readers to compare them with each other. In the following, for each positive integer n and $m \in \mathbb{Z}_n$, the cyclic subgroup of $\mathbb{Z}_n$ which is generated by m is denoted by $[m]$.

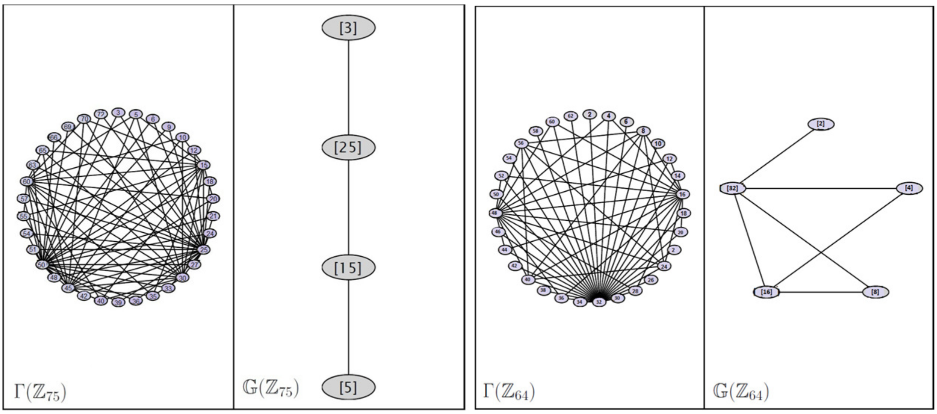

FIGURE 1

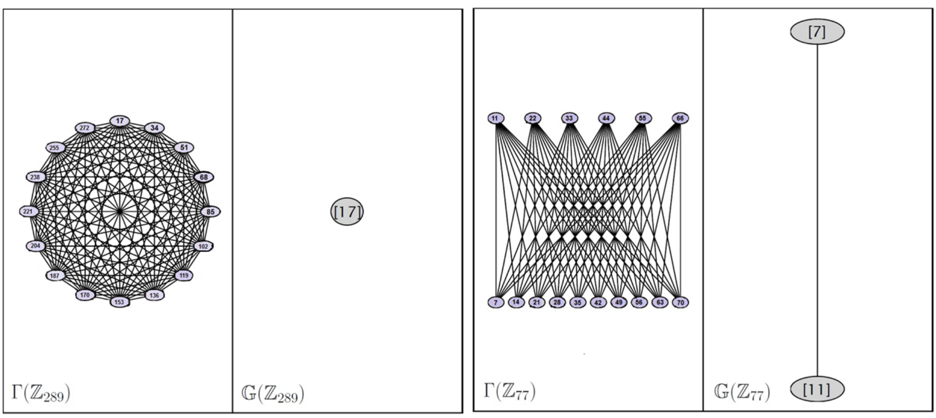

FIGURE 2

2. Finite conditions and locally finite graphs

In this section we proceed with the study of the relations between module theoretic properties of an R-module M and graph theoretic properties of $\mathbb{G}(M)$. Consequently, some main theorems in [1] and [15] are immediate outcomes of the results of this section with much simpler proofs. To prove the next proposition, we need the following lemma, which has been proved in [23].

Lemma 2.1 ([23, Proposition 2.6])**.** *Let M be an R-module. Then the following are equivalent:*

(1) $\mathbb{G}(M)$ *is an empty graph.*
(2) $\operatorname{ann}(M)$ *is a prime ideal of R and $\mathbb{A}^*(M) \neq \mathbb{S}(M) \setminus \{0\}$.*

The next proposition is a generalization of [15, Theorem 1.1].

Proposition 2.2. *Let R be a ring and M an R-module such that $\mathbb{G}(M)$ is a non-empty graph. Then $\mathbb{G}(M)$ has ascending (descending) chain condition over vertices if and only if M is a noetherian (an artinian) R-module.*

Proof. The "only if" part is obvious. Conversely, by Lemma 2.1, $\mathbb{G}(M)$ is a non-empty graph if and only if either $\operatorname{ann}(M)$ is not a prime ideal of R or $\mathbb{A}^*(M) = \mathbb{S}(M) \setminus \{0\}$. If $\mathbb{A}^*(M) = \mathbb{S}(M) \setminus \{0\}$, the proof is complete. Assume that $\operatorname{ann}(M)$ is not a prime ideal of R. There exist $a, b \in R$ such that $ab \in \operatorname{ann}(M)$ but neither $a \in \operatorname{ann}(M)$ nor $b \in \operatorname{ann}(M)$. Therefore there exist $m, n \in M$ such that both $ma \neq 0$ and $nb \neq 0$. It is clear that Ma and Mb are non-zero submodules of M such that $Ma(Mb : M) = 0$. Then $Ma \in A^*(M)$. By hypothesis, Ma is a noetherian (an artinian) submodule of M. On the other hand, the map $f : M \longrightarrow Ma$ with $f(m) = ma$ ($\forall m \in M$) is an R-epimorphism because R is a commutative ring. Then $\frac{M}{\ker f} \cong Ma$, and hence it is a noetherian (an artinian) R-module. If $\ker f = 0$, then M is a noetherian (an artinian) R-module. Assume that $\ker f \neq 0$. It is obvious that $\ker f = \{x \in M \mid xa = 0\}$. Moreover, $\ker f \in A^*(M)$ because

$$maR(\ker f : M) = m(\ker f : M)a \subseteq (\ker f)a = 0.$$

Therefore both $\ker f$ and $\frac{M}{\ker f}$ are noetherian (artinian). Hence M is a noetherian (an artinian) R-module. □

We now state a lemma which plays an important role in what follows. It is well known that in an artinian (commutative) ring, every maximal ideal is the annihilator of an element. The next lemma is also a generalization of this result.

Lemma 2.3. *Let M be an R-module of finite length. Then every maximal submodule of M is a vertex of $\mathbb{G}(M)$.*

Proof. Assume that N is a maximal submodule of M. Set $P = (N : M)$. Then $P = \operatorname{ann}\left(\frac{M}{N}\right)$ is a maximal ideal of R. Since M is artinian, there exists a positive integer n such that $MP^n = MP^{n+i}$ for each $i \geq 1$. Let k be the smallest integer number with this property. First, we show that $k > 0$. For, on the contrary, assume that $k = 0$. Therefore $M = MP$, and hence by the Nakayama lemma, since M is finitely generated, there exists $s \in R$ such that $1 - s \in P$ and $Ms = 0$. Therefore

$s \in \text{ann}(M) \subseteq (N : M) = P$, and hence $1 \in P$, a contradiction. Since M is a noetherian R-module, it follows that MP^k is finitely generated as an R-module. Again by the Nakayama lemma, $MP^k = MP^{k+1}$ implies that there exists $s \in R$ such that $1 - s \in P$ and $MP^k s = 0$. Set $T = MP^{k-1}s$. If $T = 0$, then, for each $y \in MP^{k-1}$, we have

$$y = y(1 - s) \in MP^{k-1}P = MP^k.$$

Therefore $MP^{k-1} \subseteq MP^k$, and hence $MP^{k-1} = MP^k$, which contradicts the minimality of k. Accordingly, $T \neq 0$, and hence

$$T(N : M) = TP = MP^{k-1}sP = MP^k s = 0.$$

Hence $N \in \mathbb{A}^*(M)$. □

In the following proposition, based on the properties of $\mathbb{G}(M)$, we give a necessary and sufficient condition under which the zero-divisor graph of a finite R-module M is a complete graph.

Proposition 2.4. *Let M be a finite R-module. The following statements are equivalent:*

1. *For some proper simple submodule N of M, $A^*(M) = \{N\}$.*
2. *The graph $\Gamma(M)$ is a complete graph such that, for each $x, y \in Z^*(M)$, $(xR : M) = (yR : M)$ and $x(yR : M) = y(xR : M) = (0)$.*

Proof. $(1 \Rightarrow 2)$. By Lemma 2.3, every maximal submodule of M, and hence every proper submodule of M, is a vertex of $\mathbb{G}(M)$. Therefore, M has the unique non-trivial submodule N. For each $x \in Z^*(M)$, there exists $z \in Z^*(M)$ such that $x * y = (0)$. Therefore $xR \in A^*(M)$, and hence $xR = N$. Inasmuch as $N * N = (0)$, then, for each $x, y \in Z^*(M)$, $(xR : M) = (yR : M)$ and $x(yR : M) = y(xR : M) = (0)$.

$(2 \Rightarrow 1)$. Put $N = Z(M)$. It is clear that N is a submodule. For each $K \in A^*(M)$, there exists the non-zero submodule L of M such that $K * L = (0)$. Therefore, for each non-zero element $Mk \in K$ and $z \in L$, $k * z = (0)$, and hence $K \subseteq N$. By Lemma 2.3, every proper submodule of M is a vertex of $\mathbb{G}(M)$. Hence N is both a maximal and a simple submodule of M. This implies that $A^*(M) = \{N\}$, as desired. □

Corollary 2.5. *Let M be a finite abelian group. The following assertions are equivalent:*

1. *$\mathbb{G}(M)$ is a graph with one vertex.*
2. *$\Gamma(M)$ is a finite complete graph.*
3. *There exists a prime number p such that $M \cong \mathbb{Z}_{p^2}$.*

In respect to the above Corollary, the following examples are considered.

Fact. Let R be an artinian ring. Then R_R is a module of finite length and $|\text{Max}(R)| < \infty$. Assume that $\mathbb{G}(R)$ is a locally finite graph. By Lemma 2.3, for each $M \in \text{Max}(R)$, there exists a non-zero ideal J of R such that $MJ = 0$. Since for every ideal $I \subseteq M$, we have $IJ = 0$ and $\deg J$ is finite, the number of sub-ideals

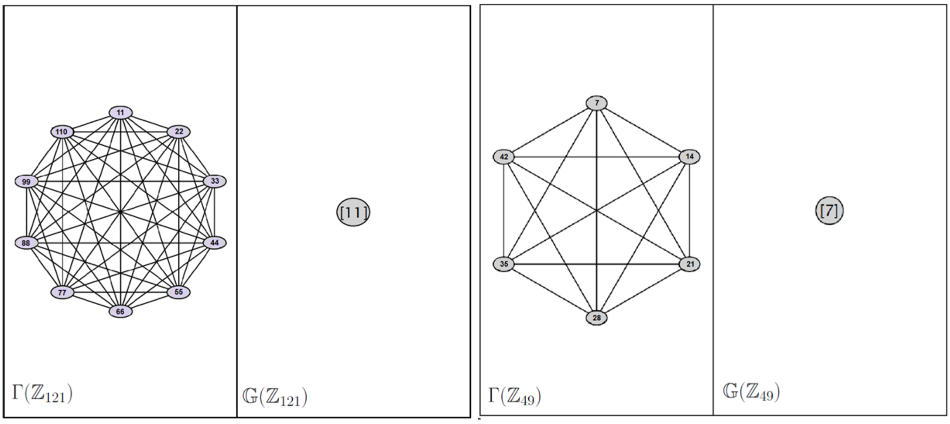

Figure 3

of M is finite. This implies that the number of ideals of R is finite. It provides a simpler proof for [15, Theorem 1.4 $(3 \to 1)$]. The next result concerns the situation in which the degree of any vertex $N \in \mathbb{A}^*(M)$ is finite.

Proposition 2.6. *Let M be an R-module and $\mathbb{G}(M)$ a locally finite graph. Then the following two statements hold:*

(1) *M is a module of finite length.*
(2) *The number of maximal submodules of M is finite.*

Proof. (1). This is proved through several steps.

(Step 1). We show that M is an artinian R-module. By Proposition 2.2, it is sufficient to show that M satisfies the descending chain condition on $\mathbb{A}^*(M)$. Assume that

$$M_1 \supseteq M_2 \supseteq \cdots$$

is a descending chain of elements of $\mathbb{A}^*(M)$. There exists a non-zero submodule N of M such that $N * M_1 = 0$. By Lemma 1.3 (1), for every $i \geq 1$, $M_i * N = 0$. Since $\deg N$ is finite, the number of N_i's is finite, as desired.

(Step 2). We show that the number of simple submodules of M is finite. Since M is artinian, it contains a simple submodule such as S. Assume that $\mathcal{S}$ is the set of all simple submodules of M. For each $T \in \mathcal{S}$, either $S = T$ or $S \cap T = 0$. Then by Lemma 1.3 (2), $|\mathcal{S}| - 1 \leq \deg S < \infty$. Then the number of simple submodules of M is finite. Suppose

$$\mathcal{S} = \{S_1, S_2, \ldots, S_n\}.$$

(Step 3). We show that M is a noetherian R-module. Assume that

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$$

is an ascending chain of elements of $\mathbb{A}^*(M)$. Set $\mathcal{A} = \{M_i \mid i = 1, 2, \ldots\}$. For each positive integer i, there exists a non-zero submodule N_i such that $N_i * M_i = 0$.

Since M is artinian, it follows that N_i contains a simple submodule. Then by Lemma 1.3 (1), for each $i \geq 1$, M_i is adjacent to a simple submodule of M. For each $1 \leq j \leq n$, put $\mathcal{M}_j = \{M_i \in \mathcal{A} \mid M_i * S_j = 0\}$. By hypothesis, $\deg S_j$ is finite for each j, and hence $|\mathcal{M}_j|$ is finite. Put $\mathcal{M} = \cup_{j=1}^{n} \mathcal{M}_j$. Since for every $i \geq 1$, $M_i \in \mathcal{M}$, we conclude that $\mathcal{A} \subseteq \mathcal{M}$, and hence $|\mathcal{A}|$ is finite, as desired.

(2). Since M is a module of finite length, by Lemma 2.3, $\mathrm{Max}(M) \subseteq \mathbb{A}^*(M)$. By a similar argument as that of item (1), for each $N \in \mathrm{Max}(M)$ there exists $1 \leq t \leq n$ such that $S_t * N = 0$. For each $1 \leq t \leq n$, set $\mathcal{B}_t = \{K \in \mathrm{Max}(M) \mid K * S_t = 0\}$. Since for each t, $\deg S_t$ is finite, we see that $|\mathcal{B}_t|$ is finite. On the other hand, $\mathrm{Max}(M) \subseteq \cup_{t=1}^{n} \mathcal{B}_t$, and hence $|\mathrm{Max}(M)|$ is finite. □

In [1, Theorem 19], the authors proved that if R is a noetherian reduced ring such that every proper ideal of R is an annihilating ideal, then R is a semisimple ring. Actually, this theorem is a direct consequence of the following result, which shows that the "noetherian" condition in [1, Theorem 19] is superfluous.

Proposition 2.7. *Let R be a ring and M a finitely generated reduced R-module such that $M \notin \mathbb{A}^*(M)$. If $\mathrm{Max}(M) \subseteq \mathbb{A}^*(M)$, then M is a semisimple R-module.*

Proof. Since M is finitely generated, M has a maximal submodule. Assume that N is a maximal submodule of M. By hypothesis, there exists a non-zero submodule K of M such that $N * K = 0$. Put $T = N \cap K$. If $T \neq 0$, then by Lemma 1.3 (1), $T(T : M) = 0$. Hence $M(T : M)^2 = 0$. Since M is reduced, we conclude that $M(T : M) = 0$, and hence $\mathbb{A}^*(M) = \mathbb{S}(M) \setminus \{0\}$, a contradiction. Therefore $T = 0$, and hence $N \oplus K = M$. Since every proper submodule of M is contained in a maximal submodule and maximal submodules of M are direct summands, it follows that M is a semisimple R-module. □

Corollary 2.8. *Let R be a ring and M an R-module. The following two statements hold:*

1. *If R is a reduced ring such that $\mathrm{Max}(R) \subseteq \mathbb{A}^*(R)$, then R is a semisimple ring.*
2. *If M is reduced and of finite length with $M \notin \mathbb{A}^*(M)$, then M is a semisimple R-module.*

Proof. (1). The verification is immediate.

(2). By Lemma 2.3, $\mathrm{Max}(M) \subseteq \mathbb{A}^*(M)$. Hence by Proposition 2.7, M is a semisimple R-module. □

Remark 2.9. Let R be a noetherian ring such that $\mathrm{Max}(R) \subseteq \mathbb{A}^*(R)$. We can show that R is a semi-local ring. This is a generalization of [15, Proposition 1.7] with a simpler proof. Since R is a noetherian ring, the ideal $\{0\}$ has a minimal primary decomposition, such as $\{0\} = \cap_{i=1}^{n} Q_i$ where the Q_i's are P_i-primary. By hypothesis, for each maximal ideal M of R there exists a non-zero ideal I of R such that $IM = 0 \subseteq \cap_{i=1}^{n} Q_i$. Since $I \neq 0$, there exists $1 \leq j \leq n$ such that $I \nsubseteq Q_j$. Then $M \subseteq P_j$, and hence $M = P_j$. This implies that

$$\mathrm{Max}(R) \subseteq \{P_1, P_2, \ldots, P_n\}.$$

The following result is a natural generalization of [15, Theorem 1.4].

Theorem 2.10. *Let M be an R-module. The following assertions are equivalent:*

(1) *The number of submodules of M is finite.*
(2) *The graph $\mathbb{G}(M)$ is a finite graph (i.e., $|\mathbb{A}^*(M)| < \infty$).*
(3) *$\mathbb{G}(M)$ is a locally finite graph.*

Proof. $(1 \Rightarrow 2)$ and $(1 \Rightarrow 3)$ are obvious.

$(2 \Rightarrow 1)$. Assume that $\mathbb{G}(M)$ is finite. By Proposition 2.6, M is a module of finite length. We proceed by induction on the length of M. If $\text{length}(M) = 1$, then every maximal submodule of M is a simple submodule too. Let N be a maximal simple submodule of M. For each proper submodule K of M, either $N \cap K = 0$ or $N \cap K = N$. Therefore, by Lemma 1.3 (2), either $N * K = 0$ or $N = K$. Then by Lemma 2.3, $\mathbb{A}^*(M) = \mathbb{S}(M) \setminus \{0\}$. Assume that, for every R-module N with $\text{length}(N) < n$ and $|\mathbb{G}(N)| < \infty$, we have $|\mathbb{S}(N)| < \infty$. Suppose that M is an R-module with $\text{length}(M) = n$ and $|\mathbb{G}(M)| < \infty$. By Lemma 2.3, the number of maximal submodules of M is finite. Assume that $\text{Max}(M) = \{M_1, M_2, \dots, M_t\}$ is the set of all maximal submodules of M. By Lemma 1.3 (1), for each $1 \leq i \leq t$, every submodule of M_i belongs to $\mathbb{A}^*(M)$, and hence $|\mathbb{G}(M_i)|$ is finite. On the other hand, since $\text{length}(M_i) = n - 1$, by hypothesis, the number of submodules of M_i is finite for each $1 \leq i \leq t$. Since M is finitely generated, every proper submodule of M is contained in a maximal submodule. Therefore

$$|\mathbb{S}(M)| \leq \sum_{i=1}^{t} |\mathbb{S}(M_i)| + 1,$$

which is finite.

$(3 \Rightarrow 1)$. By Proposition 2.6, M is a module of finite length and $|\text{Max}(M)| < \infty$. Moreover, by Lemma 2.3, $\text{Max}(M) \subseteq \mathbb{A}^*(M)$. Suppose that $\text{Max}(M) = \{M_1, M_2, \dots, M_n\}$. For each $1 \leq i \leq n$, there exists a non-zero submodule N_i of M such that $M_i * N_i = 0$. By Lemma 1.3 (1), for each $1 \leq i \leq n$, every submodule of M_i is adjacent to N_i, in $\mathbb{G}(M)$. Since $\deg N_i$ is finite, $|\mathbb{S}(M_i)|$ is finite for each i. On the other hand, since M is noetherian, any proper submodule of M is contained in a maximal submodule. Hence

$$|\mathbb{S}(M)| \leq \sum_{i=1}^{n} |\mathbb{S}(M_i)| + 1,$$

as desired. □

Remark 2.11. In the above theorem, for $(3 \Rightarrow 2)$, we may give a graph-theoretical proof. We need some well-known results in graph theory. It is well known that a locally finite graph has infinite diameter if and only if it contains a ray. On the other hand, Konig's lemma states that an infinite graph which is connected and locally finite has a ray. Now inasmuch as $\mathbb{G}(M)$ is connected and always has finite diameter ([23, Proposition 2.7]), if $\mathbb{G}(M)$ is locally finite, by Konig's lemma, it has a ray whenever it is infinite. Now by the above fact it must have an infinite diameter, which is a contradiction.

Let n be a positive integer. The graph $G = \langle V, E \rangle$ is called *n-regular* provided that, for each $x \in V$, $\deg x = n$. If M is an R-module such that, for some positive integer number n, the graph $\mathbb{G}(M)$ is an n-regular graph, then by Lemma 2.3, Proposition 2.6 and Theorem 2.10, M is a module of finite length, the number of submodules of M is finite and $\mathbb{S}(M) \setminus \{\{0\}, M\} \subseteq \mathbb{A}^*(M)$. Now we want to investigate, when is $\mathbb{G}(M)$ an n-regular graph? The next theorem is a generalization of [1, Theorem 8].

Theorem 2.12. *Let n be a positive integer, R a ring and M an R-module such that $\mathbb{G}(M)$ is an n-regular graph. Then $\mathbb{G}(M)$ is a complete graph with $|\mathbb{A}^*(M)| = n+1$.*

Proof. If $M \in \mathbb{A}^*(M)$, there exists a non-zero submodule K of M such that $K * M = 0$. Since $K(M : M) \neq 0$, it follows that $M(K : M) = 0$ or equivalently $(K : M) \subseteq \operatorname{ann}(M)$. Hence for every non-zero submodule N of M, we have $N(K : M) = 0$. Thus K is adjacent to any non-zero submodule of M. Since $\mathbb{G}(M)$ is an n-regular graph, for each $N \in \mathbb{S}^*(M)$, $\deg N = \deg K$. Therefore any two non-equal non-zero submodules of M are adjacent in $\mathbb{G}(M)$. Accordingly, $\mathbb{G}(M)$ is a complete graph. Now assume that $\mathbb{A}^*(M) = \mathbb{S}(M) \setminus \{\{0\}, M\}$. Let S be a simple submodule of M and let $\mathcal{A} = \{S_1, S_2, \dots, S_n\}$ be the subset of submodules of M which are adjacent to S in $\mathbb{G}(M)$. If S is the only simple submodule of M, then by Proposition 2.6, Lemma 2.3 and Lemma 1.3 (1), any member of $\mathbb{A}^*(M) \setminus \{S\}$ is adjacent to S. Therefore $\mathbb{G}(M)$ is a complete graph. Now suppose that the number of simple submodules of M is greater than or equal to 1. If T is a simple submodule of M such that $S \neq T$, then $S \cap T = 0$, and hence by Lemma 1.3 (2), S and T are adjacent in $\mathbb{G}(M)$. This implies that $T \in \mathcal{A}$. Without loss of generality assume that, for some $1 \leq t \leq n$, the set $\{S, S_1, S_2, \dots, S_t\}$ is the set of all simple submodules of M. First, we show that, for each $N \in \mathbb{A}^*(M) \setminus \{S, S_1, S_2, \dots, S_n\}$, N is adjacent to $S_1, S_2, \dots, S_n$. Two cases may occur.

(Case 1). Assume that $\mathbb{A}^*(M) \setminus \{S, S_1, S_2, \dots, S_n\} = \{N\}$. Since $\deg S = \deg N = n$ and S is not adjacent to N, N must be adjacent to S_i for all $1 \leq i \leq n$.

(Case 2). Assume that $|\mathbb{A}^*(M) \setminus \{S, S_1, S_2, \dots, S_n\}| \geq 2$. On the contrary, suppose that $\{N_1, N_2\} \subseteq \mathbb{A}^*(M) \setminus \{S, S_1, S_2, \dots, S_n\}$ such that $N_1 * N_2 = 0$. If $S \cap N_1 = 0$, then by Lemma 1.3 (2), S and N_1 are adjacent, a contradiction. Let $S \cap N_1 \neq 0$. Since S is a simple submodule of M, S is a submodule of N_1. By Lemma 1.3 (1), N_2 is adjacent to S in $\mathbb{G}(M)$, again a contradiction.

Now we show that $\mathbb{A}^*(M) = \{S, S_1, S_2, \dots, S_n\}$. On the contrary, assume that $N \in \mathbb{A}^*(M) \setminus \{S, S_1, S_2, \dots, S_n\}$. We know that $\operatorname{Soc}(M) = (\oplus_{i=1}^{t} S_i) \oplus S$ is a submodule of M. Again two cases may occur.

(Case 1). If $\operatorname{Soc}(M) = M$, then M is a semisimple R-module, and hence N is a proper semisimple submodule of M. Since S and N are not adjacent, $S \cap N \neq 0$, and hence S is a proper submodule of N. Therefore there exists $1 \leq j \leq t$ such that $S \oplus S_j$ is a submodule of N. For each $t+1 \leq r \leq n$, we know S_r is adjacent to N, and then by Lemma 1.3 (1), S_r is adjacent to any non-zero submodule of N; in particular, S_r is adjacent to S_j for each $t+1 \leq r \leq n$. On the other hand, every

$B \in \{N, S, S_1, S_2, \dots, S_{j-1}, S_{j+1}, \dots, S_t\}$ is adjacent to S_j. Thus

$$\begin{aligned}\deg S_j &\geq |\{N, S, S_1, S_2, \dots, S_{j-1}, S_{j+1}, \dots, S_t\}| + |\{S_{t+1}, \dots, S_n\}| \\ &= (t+1) + (n-t) = n+1,\end{aligned}$$

a contradiction.

(Case 2). Assume that $\mathrm{Soc}(M)$ is a proper submodule of M. It is obvious that $\mathrm{Soc}(M) \neq S_i$ for each $1 \leq i \leq t$ and $\mathrm{Soc}(M) \neq S$. If, for some $t+1 \leq r \leq n$, $\mathrm{Soc}(M) = S_r$, then N is adjacent to $\mathrm{Soc}(M)$, and hence by Lemma 1.3 (1), N is adjacent to any non-zero submodule of $\mathrm{Soc}(M)$; in particular, N is adjacent to S, a contradiction. Hence

$$\mathrm{Soc}(M) \in \mathbb{A}^*(M) \setminus \{S, S_1, S_2, \dots, S_n\}.$$

Therefore, for every $t+1 \leq l \leq n$, S_l is adjacent to $\mathrm{Soc}(M)$, and again by Lemma 1.3 (1), S_l is adjacent to S_i for each $1 \leq i \leq t$. Then, for each $1 \leq i \leq t$,

$$\begin{aligned}\deg S_i &\geq |\{\mathrm{Soc}(M), S, S_1, S_2, \dots, S_{i-1}, S_{i+1}, \dots, S_t\}| + |\{S_{t+1}, \dots, S_n\}| \\ &= (t+1) + (n-t) = n+1,\end{aligned}$$

which is a contradiction. Therefore $\mathbb{A}^*(M) = \{S, S_1, S_2, \dots, S_n\}$, and hence $\mathbb{G}(M)$ is a complete graph. □

Corollary 2.13. *Let M be an R-module. Then $\mathbb{G}(M)$ cannot be a cycle.*

Proof. On the contrary, suppose that $\mathbb{G}(M)$ is a cycle. By Theorem 2.12, $|\mathbb{A}^*(M)| = 3$. Let $\mathbb{A}^*(M) = \{S_1, S_2, S_3\}$. If $M \in \mathbb{A}^*(M)$, then $M = S_i$ for some $1 \leq i \leq 3$. Without loss of generality, suppose $M = S_1$. By Proposition 2.6 and Lemma 2.3 every proper submodule of M is a vertex of $\mathbb{G}(M)$. Without loss of generality, assume that S_2 is a maximal submodule of M. Because $S_2 \neq 0$, $M(S_2 : M) = 0$. Since $(S_2 : M)$ is a maximal ideal of R, for every non-zero element $m \in M$, $(S_2 : M) = \mathrm{ann}(m)$. Therefore M is a semisimple R-module. Since S_2 and S_3 are the only proper submodules of M, they are simple and $M = S_2 \oplus S_3$. By hypothesis, $M(S_2 : M) = M(S_3 : M) = 0$. This implies that $S_2 \cong S_3$. Let $f : S_2 \longrightarrow S_3$ be an isomorphism. Put

$$S = \{s + f(s) \mid s \in S_2\}.$$

It is obvious that S is a non-zero submodule of M which contains neither S_2 nor S_3, a contradiction. Hence $M \notin \mathbb{A}^*(M)$. Three cases may occur.

(Case 1). If, for each distinct $i, j \in \{1, 2, 3\}$, $S_i \not\subseteq S_j$, then the S_i's are simple submodules of M. Therefore $S_i(S_j : M) = 0$, and hence $M(S_j : M) = 0$. This is a contradiction.

(Case 2). If, for some $i \neq j$, S_i and S_j are maximal submodules of M, by hypothesis, either $S_i(S_j : M) = 0$ or $S_j(S_i : M) = 0$. Then either S_i or S_j are semisimple but not simple because $S_t = S_i \cap S_j$ for $t \in \{1, 2, 3\} \setminus \{i, j\}$. This is a contradiction.

(Case 3). Assume that, only for one $i \in \{1, 2, 3\}$, S_i is a maximal submodule of M. Without loss of generality, assume that $i = 3$. Then S_1 and S_2 are simple submodules of M which are contained in S_3. Since $M(S_1 : M) \neq 0$, there exists

$a \in (S_1 : M)$ such that $Ma \neq 0$. Define the map $g : M \longrightarrow S_1$ with $g(m) = ma$. It is clear that g is an R-epimorphism. Since $\frac{M}{\ker g} \cong S_1$ and S_1 is a simple module, $\ker g$ is a maximal submodule of M, and hence $\ker g = S_3$. This implies that $\frac{M}{S_3} \cong S_1$. With a similar argument we can show that $\frac{M}{S_3} \cong S_2$. Therefore $S_1 \cong S_2$. Let $f \in \mathrm{Hom}_R(S_1, S_2)$ be an isomorphism. Put $S = \{s + f(s) \mid s \in S_1\}$. It is clear that S is a non-zero submodule of M which contains neither S_1 nor S_2. This is a contradiction. □

3. Completeness and related topics

The main goal of the present section is the study of those modules whose annihilating submodule graphs contain a vertex which is adjacent to all other vertices. This study will naturally lead to investigating complete and star graphs. To prove our results, we will need the following lemma, which is a generalization of an important fact in ring theory. A well-known lemma due to Richard Brauer states that every minimal ideal of a ring is either nilpotent or a direct summand. The next lemma is a generalization of Brauer's lemma to modules over commutative rings.

Lemma 3.1. *Let M be an R-module and S a simple submodule of M. Then either $S * S = 0$ or S is a direct summand of M.*

Proof. Suppose that $S * S \neq 0$. Then $S(S : M) \neq 0$, and hence there exists a non-zero element $a \in S$ such that $a(S : M) \neq 0$. Since S is simple, we conclude that $a(S : M) = S$. Therefore there exists $b \in (S : M)$ such that $ab = a$. Put

$$N = \{x \in S \mid xb = 0\}.$$

Since R is a commutative ring, N is a submodule of M which is contained in S. Since $a \in S \setminus N$, it follows that $N = 0$. Define $\phi : M \longrightarrow S$, with $\phi(m) = mb$, for each $m \in M$. Since $\phi(a) = a \neq 0$, ϕ is an epimorphism, and hence $\frac{M}{\ker \phi} \cong S$, where

$$\ker \phi = \{m \in M \mid mb = 0\} = \mathrm{ann}_M(b).$$

Since S is a simple R-module, we conclude that $\mathrm{ann}_M(b)$ is a maximal submodule of M. On the other hand, $S \cap \mathrm{ann}_M(b) = N = 0$. Therefore $M = S \oplus \mathrm{ann}_M(b)$. □

It is obvious that finite commutative domains are fields. A simple generalization of this fact is that artinian domains are fields. This can also be generalized as: every domain with a minimal ideal is a field. The next result is a generalization of these observations to modules.

Proposition 3.2. *Let R be a ring and M an R-module such that $\mathbb{G}(M) = \emptyset$.*

1. *If R is an artinian ring, then M is a simple R-module.*
2. *If M contains a simple submodule, then M is a simple R-module.*

Proof. (1). By [23, Proposition 2.5], $\mathrm{ann}(M)$ is a prime ideal of R. Since R is an artinian ring, we conclude that $\mathrm{ann}(M)$ is a maximal ideal. Thus M is a semisimple R-module. There exists a family $\{S_i\}_{i \in I}$ of simple submodules M such that $M = \oplus_{i \in I} S_i$. If i and j are two distinct elements of I, then S_i and S_j are two distinct elements of $\mathbb{A}^*(M)$ which are adjacent in $\mathbb{G}(M)$. This is a contradiction. Therefore $|I| = 1$, and hence M is a simple R-module.

(2). Assume that S is a simple submodule of M. We show that $S = M$. On the contrary, assume that $S \neq M$. By Lemma 3.1, either $S * S = 0$ or S is a direct summand of M. This implies that $S \in \mathbb{A}^*(M)$, which is a contradiction. □

Notation. Let M be an R-module. For each $m \in M$, put

$$\mathrm{N}(m) = \{n \in M \mid m * n = 0\}.$$

Theorem 3.3. *Let M be an R-module. There exists a submodule $N \in \mathbb{A}^*(M)$ which is adjacent to any other vertices of $\mathbb{G}(M)$ if and only if one of the following conditions holds:*

1. *There exist a simple submodule S and a submodule K of M such that $M = S \oplus K$, $S \cap A = \{0\}$ for each $A \in \mathbb{A}^*(M) \setminus \{S\}$, and $\Gamma(K) = \emptyset$.*
2. *There exists a non-zero element $m \in M$ such that $Z(M) = \mathrm{N}(m)$.*

Proof. Suppose $N \in \mathbb{A}^*(M)$ is adjacent to any other vertices of $\mathbb{G}(M)$ and, for each $m \in M$, $Z(M) \neq \mathrm{N}(m)$. Assume that $0 \neq n \in N$. Since, for each $m \in Z^*(M)\setminus\{n\}$, we have $mR \in \mathbb{A}^*(M)$, it follows that $N * mR = 0$, and hence by Lemma 1.3 (1), $nR * mR = 0$. This implies that $m \in \mathrm{N}(n)$, and hence $Z(M) \setminus \{n\} \subseteq \mathrm{N}(n)$. Then by hypothesis, $n * n \neq 0$. If xR is a proper submodule of nR, then xR is adjacent to any other vertices of $\mathbb{G}(M)$. On the other hand, Lemma 1.3 (1) and $xR*nR = 0$ imply that $xR*xR = 0$, and hence $x*x = 0$. Therefore $Z(M) = \mathrm{N}(x)$, a contradiction. This shows that $x = 0$. Then $S = nR$ is a simple submodule of M such that $S * S \neq 0$. Moreover, if $A \in \mathbb{A}^*(M) \setminus \{S\}$ and $S \cap A \neq \{0\}$, then $S \subseteq A$, and hence $S * A = 0$ implies that $S * S = 0$, which is a contradiction. By Lemma 3.1, there exists a submodule K of M such that $M = S \oplus K$. Now we show that $\Gamma(K) = \emptyset$. On the contrary, assume that there exists an element $k \in Z^*(K)$. Then there exists a non-zero element $y \in K$ such that either $k(yR : K) = 0$ or $y(kR : K) = 0$. Without loss of generality, assume that $k(yR : K) = 0$. Then

$$(S \oplus kR)(yR : M) = S(yR : M) \oplus (kR)(yR : M) \subseteq (S \cap yR) + (kR)(yR : K) = \{0\},$$

and therefore $S \oplus kR \in \mathbb{A}^*(M)$. By hypothesis, $S * (S \oplus kR) = 0$. Two cases may hold:

(Case 1). Let $S(S \oplus kR : M)$. Since $(S : M) \subseteq (S \oplus kR : M)$, we conclude that $S(S : M) \subseteq S(S \oplus kR : M) = 0$. Hence $S * S = 0$, a contradiction.

(Case 2). Let $(S \oplus kR)(S : M) = 0$. Then

$$0 = (S \oplus kR)(S : M) = S(S : M) \oplus kR(S : M) = S(S : M).$$

Then $S * S = 0$, a contradiction. This implies that $S \oplus kR$ is a vertex of $\mathbb{G}(M)$ which is not adjacent to S. This is a contradiction. If $y(kR : M) = 0$, then by a similar argument we can show that $S \oplus yR$ is a vertex of $\mathbb{G}(M)$ which is not adjacent to S. Therefore $\Gamma(K) = \emptyset$. □

Corollary 3.4. *Let M be an R-module. There exists a submodule $N \in \mathbb{A}^*(M)$ which is adjacent to any other vertices of $\mathbb{G}(M)$ if and only if either $M = S \oplus K$, where S is a simple submodule of M, $S \cap A = \{0\}$ for each $A \in \mathbb{A}^*(M) \setminus \{S\}$ and K is a submodule of M such that $\mathbb{G}(K) = \emptyset$ or $\mathbb{A}(M) = \mathrm{N}(A)$ for some non-zero submodule A of M.*

Lemma 3.5. *Let R be an artinian local ring and M a finitely generated R-module. Then $\mathbb{S}(M) \setminus \{0, M\} \subseteq \mathbb{A}^*(M)$ and there exists a submodule of M which is adjacent to any other vertices of $\mathbb{G}(M)$.*

Proof. Let P be the unique maximal ideal of R. Since R is a noetherian ring, by [20, Corollary 3.58], $\mathrm{Ass}(M) \neq \emptyset$. Therefore $\mathrm{Ass}(M) = \{P\}$ because prime ideals of R are maximal ideals. By [20, Lemma 3.56], $P = \mathrm{ann}(m)$ for some non-zero element $m \in M$. On the other hand, for each maximal submodule N of M, $(N : M)$ is a maximal ideal of R, and hence $P = (N : M)$. Therefore, for each maximal submodule N of M, we have $mR(N : M) = mRP = 0$. Since M is a finitely generated R-module, every proper submodule of M is contained in a maximal submodule. Therefore by Lemma 1.3 (1), mR is adjacent to any proper submodule of M. □

Lemma 3.6. *Let M be an R-module such that $M = S_1 \oplus S_2$, where S_1 and S_2 are two non-isomorphic simple submodules of M. Then $\mathbb{S}(M) = \{\{0\}, S_1, S_2, M\}$. Moreover, $\mathbb{G}(M)$ is an edge between S_1 and S_2.*

Proof. Assume that S is a non-trivial submodule of M. By [7, Lemma 9.2], either $M = S \oplus S_1$ or $M = S \oplus S_2$. Hence S is a simple submodule of M such that either $S \cong S_1$ or $S \cong S_2$. Assume that $S \cong S_1$. We show that $S = S_1$. If $S \neq S_1$, then $S \oplus S_1 = M$. Hence $S \cong M/S_1 \cong S_2$, which implies that $S_1 \cong S_2$. This is a contradiction. □

Theorem 3.7. *Let R be an artinian ring and M a finitely generated faithful R-module such that $M \notin \mathbb{A}^*(M)$. The following are equivalent:*

(1) *There exists an annihilating submodule which is adjacent to any other vertices of $\mathbb{G}(M)$.*
(2) *One of the following conditions hold: a- $M = S_1 \oplus S_2$, where S_1 and S_2 are two simple submodules of M. b- R is a local ring.*

Proof. ($2 \Rightarrow 1$). Assume that $M = S_1 \oplus S_2$, where S_1 and S_2 are two simple submodules of M. If $S_1 \cong S_2$, then by [23, Corollary 2.11)], $\mathbb{G}(M)$ is a complete graph. If $S_1 \ncong S_2$, then by Lemma 3.6, $\mathbb{G}(M)$ is an edge.

If R is a local ring, then by Lemma 3.5, the proof is complete.

($1 \Rightarrow 2$). By Theorem 3.3, either there exist a simple submodule S and a submodule K of M such that $M = S \oplus K$, $S \cap A = \{0\}$ for each $A \in \mathbb{A}^*(M) \setminus \{S\}$, and $\Gamma(K) = \emptyset$ or there exists a non-zero element $m \in M$ such that $Z(M) = \mathrm{N}(m)$. If $M = S \oplus K$, where $\Gamma(K) = \emptyset$, then by Lemma 3.2, K is a simple R-module, as desired. Now, assume that $Z(M) = \mathrm{N}(m)$ for some $m \in M$. Since R is an artinian ring, M is a module of finite length, and hence any non-zero submodule of M contains a simple submodule. Therefore without loss of generality, assume that $S = mR$ is a simple submodule of M. We show that R is a local ring through several steps.

(Step 1). We show that, for each maximal submodule N of M, $S(N : M) = 0$. Since M is a module of finite length, by Lemma 2.3, $N \in \mathbb{A}^*(M)$, and hence $S * N = 0$. Therefore $M(S : M)(N : M) = 0$. Since M is faithful, we conclude that $(S : M)(N : M) = 0$. Since R is a noetherian ring and $(N : M)$ is a maximal ideal, $(S : M)$ is a non-zero finitely generated semisimple ideal of R such that $(N : M) = \operatorname{ann}((S : M))$. Therefore $(S : M) = T_1 \oplus T_2 \oplus \cdots \oplus T_k$, where T_i's are minimal ideals of R with $\operatorname{ann}(T_i) = (N : M)$. Therefore MT_1 is a non-zero submodule of S, and hence $MT_1 = S$. Hence $S(N : M) = MT_1(N : M) = 0$. This implies that, for each maximal submodule N of M, $(N : M) = \operatorname{ann}(S)$.

(Step 2). We show that $\operatorname{Max}(R) = \{(N : M) \mid N \in \operatorname{Max}(M)\} = \{\operatorname{ann}(S)\}$, and hence R is a local ring. Let I be a maximal ideal of R. If $MI = M$, then by the Nakayama lemma, there exists an element $x \in R$ such that $1-x \in M$ and $Mx = 0$. Since M is faithful, it follows that $x = 0$, and hence $1 \in I$. This is a contradiction. Therefore MI is a proper submodule of M and $I = \operatorname{ann}(M/MI)$. Hence M/MI is a finitely generated semisimple R-module. Thus

$$\frac{M}{MI} = \frac{T_1}{MI} \oplus \frac{T_2}{MI} \oplus \cdots \oplus \frac{T_k}{MI},$$

where T_i/MI's are simple submodules of M/MI with $\operatorname{ann}(T_i/MI) = I$. Put $A = T_1 + T_2 + \cdots + T_{k-1}$. It is clear that $M/A \cong T_n/MI$. Therefore A is a maximal submodule of M and

$$(A : M) = \operatorname{ann}\left(\frac{M}{A}\right) = \operatorname{ann}\left(\frac{T_n}{MI}\right) = I,$$

as desired. □

4. Bipartite graphs

When one looks for conditions under which $\mathbb{G}(M)$ is a bipartite graph, one finds that reduced modules are, as our main result shows, one of the best kind of modules which give us profound results in this regard. An R-module M is called *atomic* provided that any two non-zero cyclic submodules are isomorphic. Two submodules N and K of an R-module M are *orthogonal*, written as $N \perp K$, in case that they don't have isomorphic submodules. A module M has *finite type dimension* n, denoted by $\operatorname{t.dim} M = n$, if M contains an essential direct sum of n pairwise orthogonal atomic submodules of M. In the main result of this section, Theorem 4.2, the reader can observe that when the type dimension of M is 2, $\mathbb{G}(M)$ is bipartite and vice versa. An example is given to show that this result is no longer true when we replace 2 by other natural numbers. The reader is referred to [17] and [25] for undefined terms and concepts on type theory of modules. Regarding reduced modules, Lemma 1.3 finds a more thorough form. In what follows, we will frequently use this lemma:

Lemma 4.1. *Let M be a reduced R-module such that $M \notin \mathbb{A}^*(M)$, and let N and K be two submodules. The following are equivalent:*

(1) $N \perp K$.
(2) $N \cap K = \{0\}$.
(3) $N(K : M) = K(N : M) = \{0\}$.

Proof. ($1 \Rightarrow 2 \Rightarrow 3$). It is clear that, if $N \perp K$, then $N \cap K = \{0\}$, and hence by Lemma 1.3 (2), $N(K : M) = K(N : M) = \{0\}$.

($3 \Rightarrow 1$). On the contrary, assume that $N * K = 0$ and N' and K' are non-zero isomorphic submodules of N and K, respectively. There exists an isomorphism $f : N' \longrightarrow K'$. Without loss of generality, suppose that $K(N : M) = 0$. By Lemma 1.3 (1), $K'(N' : M) = 0$. Thus

$$0 = K'(N' : M) = f(N')(N' : M) = f(N'(N' : M)).$$

Since f is a monomorphism, $N'(N' : M) = 0$ and hence $M(N' : M)^2 = 0$. Therefore $M(N' : M) = 0$ because M is a reduced R-module. This implies that $M \in \mathbb{A}^*(M)$, which is a contradiction. □

In the following we characterize reduced modules for which their annihilating graphs are bipartite.

Theorem 4.2. *Let M be a reduced R-module such that $M \notin \mathbb{A}^*(M)$. The following statements are equivalent:*

(1) $\mathbb{G}(M)$ *is a bipartite graph.*
(2) $\mathbb{G}(M)$ *is a complete bipartite graph.*
(3) *There exist prime submodules N and K of M such that $N \cap K = 0$ and $(N : M) \cap (K : M) = \operatorname{ann}(M)$.*

Furthermore, if M is a semi-artinian module, the above conditions are equivalent to

(4) $\operatorname{t.dim} M = 2$.

Proof. ($1 \Rightarrow 2$). There exist the non-empty subsets V_1 and V_2 of $\mathbb{A}^*(M)$ such that $V_1 \cap V_2 = \emptyset$, $\mathbb{A}^*(M) = V_1 \cup V_2$ and no element of V_i is adjacent to another element of V_i for $i = 1, 2$. On the contrary, assume that $A \in V_1$ and $B \in V_2$ such that $A(B : M) \neq 0$. There exist $C \in V_2$ and $D \in V_1$ such that $A(C : M) = 0$ and $B(D : M) = 0$. Therefore $A(B : M)(C : M) = 0$. Since $A(B : M)$ is a non-zero submodule of B, by Lemma 1.3 (1), $A(B : M)(D : M) = 0$. Since $D \in V_1$, we conclude that $A(B : M) \in V_2$. If $A(B : M) \neq C$, then two elements of V_2 are adjacent, which is a contradiction. Therefore $A(B : M) = C$, and hence $C(C : M) = 0$. This implies that $M(C : M)^2 = 0$, and hence $M(C : M) = 0$ (M is a reduced module). Thus $M \in \mathbb{A}^*(M)$, a contradiction.

($2 \Rightarrow 3$). There exist the non-empty subsets V_1 and V_2 of $\mathbb{A}^*(M)$ such that $V_1 \cap V_2 = \emptyset$, $\mathbb{A}^*(M) = V_1 \cup V_2$ and no element of V_i is adjacent to another element of V_i for $i = 1, 2$. Set

$$N = \sum_{A \in V_1} A \qquad \text{and} \qquad K = \sum_{B \in V_2} B.$$

We show that N and K are two prime submodules of M. For, assume that $rx \in N$ for some $r \in R$ and $x \in M$. We show that either $x \in N$ or $r \in (N : M)$. There exist submodules $A_1, A_2, \ldots, A_n$ contained in V_1 such that $rx \in A_1 + A_2 + \cdots + A_n$. Set $A = A_1 + A_2 + \cdots + A_n$. For each $B \in V_2$ and $1 \leq i \leq n$, $A_i(B : M) = 0$, and hence $A(B : M) = 0$. This implies that $A \in \mathbb{V}_1$. So for each $B \in V_2$, $rx(B : M) = 0$. Fix $B \in V_2$. Therefore $rx(B : M) = 0$ implies that

$$Br(B : M)(xR : M) = B(xR : M)r(B : M) \subseteq xRr(B : M) = xr(B : M) = 0.$$

Two cases may occur.

(Case 1). Assume that $Br(B : M) = 0$. We show that $r \in (N : M)$. If $Br(B : M) = 0$, then

$$Mr(B : M)^2 = M(B : M)r(B : M) \subseteq Br(B : M) = 0.$$

Since M is a reduced R-module and $Mr(B : M)^2 = 0$, we have $Mr(B : M) = 0$. If $Mr = 0$, then $r \in (N : M)$. If $Mr \neq 0$, then Mr is a vertex of $\mathbb{G}(M)$ which is adjacent to $B \in V_2$. Therefore $Mr \in V_1$, and hence $Mr \subseteq N$. This implies that $r \in (N : m)$.

(Case 2). Assume that $Br(B : M) \neq 0$. We show that $x \in N$. Inasmuch as $Br(B : M)(xR : M) = 0$ and $Br(B : M) \neq 0$, then xR is a vertex of $\mathbb{G}(M)$. Since A is adjacent to B, by Lemma 1.3 (1), A is adjacent to $Br(B : M)$. Then $Br(B : M) \in V_2$, and hence $xR \in V_1$. This implies that $x \in N$. With the same method we can show that K is a prime submodule.

Now we show that $N \cap K = 0$. On the contrary, assume that $0 \neq y \in N \cap K$. There exist submodules $C \in V_1$ and $D \in V_2$ such that $y \in C \cap D$. By Lemma 1.3 (1), $yR(yR : M) = 0$. Thus $M(yR : M)^2 = 0$, and hence $M(yR : M) = 0$ (M is a reduced module). This implies that $M \in \mathbb{A}^*(M)$, a contradiction. Hence $N \cap K = 0$. It is clear that $\operatorname{ann}(M) \subseteq (N : M) \cap (K : M)$. Assume that $s \in (N : M) \cap (K : M)$. Then $Ms \subseteq N \cap K = 0$. So the equality holds.

($3 \Rightarrow 1$). Assume that N and K are two submodules of M which satisfy the second condition. Let V_1 be the set of all non-zero submodules of M which are contained in N and V_2 the set of all non-zero submodules of M which are contained in K. For each $A \in V_1$ and $B \in V_2$, we have $A \cap B \subseteq N \cap K = 0$. By Lemma 1.3 (2), A and B are vertices of $\mathbb{G}(M)$ which are adjacent too. Hence $V_1 \cup V_2 \subseteq \mathbb{A}^*(M)$. Now assume that $A \in \mathbb{A}^*(M)$. There exists a non-zero submodule B of M such that $A(B : M) = 0$. Since $A(B : M) \subseteq N$ and N is a prime submodule, either $A \subseteq N$ or $(B : M) \subseteq (N : M)$. With the same argument we can show that either $A \subseteq K$ or $(B : M) \subseteq (K : M)$. If $A \subseteq N$ or $A \subseteq K$, then $A \in V_1 \cup V_2$, as desired. Otherwise, $(B : M) \subseteq (N : M) \cap (K : M) = \operatorname{ann}(M)$. Thus $M(B : M) = 0$, and hence $M \in \mathbb{A}^*(M)$, a contradiction. Hence $\mathbb{A}^*(M) = V_1 \cup V_2$. If $C \in V_1 \cap V_2$, then $C \subseteq N \cap K = 0$, a contradiction. So $V_1 \cap V_2 = \emptyset$.

If A_1 and A_2 are two elements of V_1 that are adjacent in $\mathbb{G}(M)$, then $A_1(A_2 : M) = 0 \subseteq K$. Since K is a prime submodule, either $A_1 \subseteq K$ or $(A_2 : M) \subseteq (K : M)$. Inasmuch as $A_1 \nsubseteq K$, then $(A_2 : M) \subseteq (K : M)$. Since $A_2 \subseteq N$, $(A_2 : M) \subseteq (N : M)$. Hence $(A_2 : M) \subseteq (N : M) \cap (K : M) = \operatorname{ann}(M)$. Therefore $M(A_2 : M) = 0$. This is a contradiction because $M \notin \mathbb{A}^*(M)$. With a similar method we can show that no two elements of V_2 are adjacent. Hence $\mathbb{G}(M)$ is a complete bipartite graph. Now, suppose that M is a semi-artinian module.

$(1 \Leftrightarrow 4)$. By Lemma 4.1 and [25, Theorem 4.6], the verification is immediate. □

Remark 4.3. Let $M = \mathbb{Z}_{16}$ as a $\mathbb{Z}$-module. It is clear that M is not reduced. One may also observe that the type dimension of M is 1 (it is atomic) but $\mathbb{G}(M)$ is a star graph with 3 vertices.

5. An algorithm for generating the annihilating graph and the zero-divisor graph of cyclic finite abelian groups, simultaneously

In [18], J. Krone gave an algorithm which illustrates the zero-divisor graphs of finite commutative rings. This algorithm is recursive in nature and constructs the graph for a given ring from subgraphs which themselves are zero-divisor graphs of rings of smaller orders. She put forward algorithms to derive zero-divisor graphs of rings of integers modulo n, i.e., $\mathbb{Z}_n$, and (finite) products of $\mathbb{Z}_n$'s, and furthermore, E_n (all even integers with the usual addition and multiplication mod n). Here we present a new algorithm for deriving both zero-divisor graphs and annihilating graphs of finite abelian groups $\mathbb{Z}_n$ by Maple, simultaneously. Here too, as the aforementioned case, our algorithm is recursive.

```
> with(GraphTheory); with(numtheory);
> CreateGraph := proc (n :: integer)
local A, B, L, H, G, k, i, j, VerNum;
VerNum := 0;
G := Graph();
for i from 1 to n − 1 do
    for j from 1 to n − 1 do
        if (i.j mod n = 0) and not (evalb(i in Vertices(G))) then
            VerNum := VerNum + 1;
            G := AddVertex(G, i);
        end if;
    end do;
end do;
for i from 1 to n − 1 do
    for j from i to n − 1 do
        if (i.j mod n = 0) and evalb(i in Vertices(G)) and
        evalb(j in Vertices(G)) and not (i = j) then
            G := AddEdge(G, {i, j});
        end if;
    end do;
end do;
A := DrawGraph(G);
H := Graph();
L := divisors(n);
```

for i **from** 2 **to** nops(L) $-$ 1 **do**
    $H := AddVertex(H, \text{cat}(\text{“[”}, L[i], \text{“]”}))$;
end do;
for i **from** 2 **to** nops(L) $-$ 1 **do**
    for j **from** 2 **to** nops(L) $-$ 1 **do**
        if $(L[i].L[j] \bmod n) = 0$ **and not** $(i = j)$ **then**
            $H := AddEdge(H, \{\text{cat}(\text{“[”}, L[i], \text{“]”}), \text{cat}(\text{“[”}, L[j], \text{“]”})\})$;
        end if;
    end do;
end do;
$B := DrawGraph(H)$;
$plots[display](Array([A, B]))$;
end proc;

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