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# A CLEMENS–SCHMID TYPE EXACT SEQUENCE OVER A LOCAL BASIS

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ABSTRACT. Let $k$ be a finite field of characteristic $p$ and let $X \to \operatorname{Spec} k[[t]]$ be a semistable family of varieties over $k$. We prove that there exists a Clemens–Schmid type exact sequence for this family. We do this by constructing a larger family defined over a smooth curve and using a Clemens–Schmid exact sequence in characteristic $p$ for this new family.

## 1. Introduction

Consider the following classical situation. Let $\Delta$ be the open disk around 0 in the complex plane and let $X$ be a smooth complex variety with Kähler total space. Suppose that $\pi\colon X \to \Delta$ is a semi-stable degeneration: a holomorphic, proper and flat map that is smooth outside the central fiber $X_0 = \pi^{-1}(0)$, which is a normal crossing divisor, i.e., it is a sum of irreducible components meeting transversally and such that each of them is smooth.

As stated in [11], in this situation we can associate a limit cohomology $H^m_{\lim}$ to the central fiber $X_0$, endowed with a monodromy operator $N$ and a weight filtration from a mixed Hodge structure. If we denote $X_t = \pi^{-1}(t)$, then we have $H^m_{\lim} \cong H^m(X_t)$ for $t \neq 0$, as vector spaces. We also have isomorphisms (of vector spaces) $H^m(X) \cong H^m(X_0) =: H^m$, and similarly for homology. Moreover, these can be inserted in a long exact sequence respecting mixed Hodge structures (see [11]):

$$\cdots \to H_{2n+2-m} \xrightarrow{\alpha} H^m \xrightarrow{i^*} H^m_{\lim} \xrightarrow{N} H^m_{\lim} \xrightarrow{\beta} H_{2n-m} \xrightarrow{\alpha} H^{m+2} \to \cdots \tag{1.1}$$

where $i^*$ is the natural map obtained by the inclusion $i\colon X_t \to X$, $\alpha$ is obtained via Poincaré duality, and $\beta$ by composing the Poincaré duality map with the dual map to $i^*$. The sequence (1.1) is called *Clemens–Schmid exact sequence*.

Then, we may consider an analogous situation in characteristic $p > 0$. Namely, let $k$ be a finite field of characteristic $p > 0$, $X$ a smooth variety of dimension $n+1$, and $C$ a smooth curve over $k$. Consider a proper and flat morphism $f\colon X \to C$, over $k$, and suppose moreover that for a $k$-rational point $s \in C$, the fiber of $f$ at $s$, denoted by $X_s$, is a normal crossing divisor (NCD) inside $X$, and that $f$ is smooth

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outside $X_s$. This allows us to define naturally a log-structure $M$ on $X$, and then by pull-back we can define a log-structure $M_s$ on $X_s$.

If we denote by $\mathcal{V}$ a complete and absolutely unramified discrete valuation ring of mixed characteristic with residue field $k$ and fraction field $K$, then in [5] it is obtained an arithmetic version of the Clemens–Schmid exact sequence:

$$\cdots \to H^m_{X_s,\mathrm{rig}}(X) \overset{\alpha}{\to} H^m_{\mathrm{rig}}(X_s) \overset{\gamma}{\to} H^m_{\text{log-crys}}((X_s, M_s)/\mathcal{V}^\times) \otimes K \xrightarrow{N_m}$$
$$H^m_{\text{log-crys}}((X_s, M_s)/\mathcal{V}^\times) \otimes K(-1) \overset{\delta}{\to} H^{m+2}_{X_s,\mathrm{rig}}(X) \overset{\alpha}{\to} H^{m+2}_{\mathrm{rig}}(X_s) \to \cdots ,$$

where $M$ is a log structure on $X$ associated to the NCD $X_s$, $M_s$ is the fiber of $M$ at $s$, and $(-1)$ denotes the $(-1)$-th Tate twist of the Frobenius structure.

In this article we consider another situation analogous to the classical one. Namely, let $k$ be a finite field of characteristic $p > 0$, and consider a proper and flat morphism

$$F \colon X \to \operatorname{Spec} k[[t]],$$

where $X$ is smooth over $k$, such that étale locally is étale over

$$\operatorname{Spec}(k[[t]][x_1, \dots, x_n]/(x_1 \cdots x_r - t)).$$

For this situation, we shall obtain an arithmetic version of the Clemens–Schmid exact sequence, similar to the one in [5]. In fact, we shall use that sequence, and for this purpose we shall see the special fiber of $F$ as a fiber of a family over a smooth curve.

Let us give an outline of the article. First we establish our notation and the situation that we want to study. In Section 3, we use Néron–Popescu desingularization (see [15]) to see our situation as a fiber inside a larger family of varieties $f \colon X_A \to Y$, in a similar way to what is done in [9, Section 4]. This allows us to use the relative cohomology theories defined by Shiho in [14], which give relative cohomology sheaves on a formal scheme, that is, a smooth lifting $\mathcal{Y}$ of $Y$. In Section 4 we state the results on relative cohomology that are useful for our purposes. In particular, we need the base change theorem and the comparison isomorphisms between the different cohomology theories, since these are needed to use the results in [5]. This means that the relative cohomology sheaves, defined on the large family, satisfy the desired properties. Then, in Section 5, we construct a smooth curve inside $Y$ in such a way that we can restrict the family of varieties, as well as the cohomology sheaves, over this curve. In particular, this shall allow us to use the main result in [5] (since the situation studied there is precisely this: a family over a smooth curve, with the same properties) and get the version of the Clemens–Schmid exact sequence for our setting.

## 2. Notation and setting

In this article we fix a finite field of characteristic $p > 0$, denoted by $k$. We denote by $W = W(k)$ its ring of Witt vectors and by $K$ the fraction field of $W$.

Recall that a divisor $Z \subset Y$ of a noetherian scheme is said to be a *strict normal crossing divisor* (SNCD) if $Z$ is a reduced scheme and, if $Z_i$, $i \in J$, are the irreducible components of $Z$, then, for any $I \subset J$, the intersection $Z_I = \cap_{i\in I} Z_i$ is a regular scheme of codimension equal to the number of elements of $I$. We shall say that $Y$ is a *normal crossing divisor* (NCD) if, étale locally on $Y$, it is a SNCD.

We consider a proper and flat morphism $F\colon X \to \operatorname{Spec} k[[t]]$ over $k$, where $X$ is a smooth scheme such that étale locally it is étale over

$$\operatorname{Spec}\left(k[[t]][x_1, \dots, x_n]/(x_1 \cdots x_r - t)\right).$$

We denote by $s$ the closed point of $\operatorname{Spec} k[[t]]$ and by $X_0$ its fiber, which is a NCD inside $X$. We denote by $(X, M)$ the scheme $X$ endowed with the log structure defined by $X_0$, by $\operatorname{Spec} k[[t]]^\times$ the scheme $\operatorname{Spec} k[[t]]$ endowed with the log structure defined by the point $s$ (i.e., by the NCD given by the ideal generated by $t$), and by $s^\times$ the log point given by the point $s$ and the log structure induced from $\operatorname{Spec} k[[t]]^\times$. Then, we have the following cartesian diagram of log schemes:

$$\begin{array}{ccc} (X_0, M_0) & \longrightarrow & (X, M) \\ \big\downarrow & & \big\downarrow F \\ s^\times & \longrightarrow & \operatorname{Spec} k[[t]]^\times \end{array}$$

where $(X_0, M_0)$ is obtained by taking the fiber product in the category of log schemes.

## 3. A construction using Néron–Popescu desingularization

In order to get the desired result, we need to study the cohomology of the special fiber $X_0$ of $X$ over $k[[t]]$, and for this, we use first the following theorem by Popescu (see [15]):

**Theorem 3.1.** *Let $f\colon R \to \Lambda$ be a morphism of rings. Then, $f$ is geometrically regular if and only if $\Lambda$ is a filtered colimit of smooth $R$-algebras.*

It can be checked that the natural morphism $k[t] \to k[[t]]$ is geometrically regular, according to the definition in [15]:

**Theorem 3.2.** *The natural morphism $k[t] \to k[[t]]$ is geometrically regular.*

*Proof.* The morphism is clearly flat, since it is a completion. Now there are only two prime ideals of $k[[t]]$, namely, 0 and $(t)$, and their respective counterpart in $k[t]$ are the only couples to consider in the definition of [15].

*Case 1 (the ideals generated by t):* In this case, we need to check that

$$k \otimes_{k[t]} k[[t]]_{(t)} \cong k$$

is geometrically regular over $k$, which is trivial.

*Case 2 (the ideals 0):* In this case, we need to check that $k(t) \otimes_{k[t]} k((t))$ is geometrically regular over $k(t)$. Take a finite extension $k'$ of $k(t)$ such that

$$(k')^p \subset k(t).$$

Note that this is necessarily $k(t^{1/p})$. Indeed, it is a finite extension of degree $p$ (hence it does not have any subextension) and $(k')^p = k(t)$. Then, we only need to check that $k(t^{1/p}) \otimes_{k(t)} (k(t) \otimes_{k[t]} k((t)))$ is a regular local ring, but

$$k(t^{1/p}) \otimes_{k(t)} (k(t) \otimes_{k[t]} k((t))) \cong k((t^{1/p})),$$

which is clearly regular. □

We get

$$k[[t]] = \varinjlim_{\alpha} A_\alpha,$$

where the $A_\alpha$'s are smooth $k[t]$-algebras (and in particular smooth $k$-algebras). Since $X$ is proper over $k[[t]]$, there exist a smooth $k[t]$-algebra $A$, a scheme $X_A$, proper over $\operatorname{Spec} A$, Zariski locally étale over $\operatorname{Spec} A[x_1, \ldots, x_n]/(x_1 \cdots x_r - t)$, and such that the following diagram is cartesian:

$$\begin{array}{ccc} X & \xrightarrow{u} & X_A \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} k[[t]] & \xrightarrow[v]{} & \operatorname{Spec} A \end{array}$$

Note that the composition $v \circ F$ is flat, hence $f$ is flat in an open subset of $X_A$ containing the image of $X$ under $u$. Thus, we may assume that $f\colon X_A \to \operatorname{Spec} A$ is flat.

Consider the divisor of $Y = \operatorname{Spec} A$, defined by $Y_0 = (t = 0)$, and the fiber product $X_{A,t=0} = Y_0 \times_Y X_A$, which is a NCD divisor in $X_A$. These define fine log structures $M_A$ and $N$ on $X_A$ and $Y$, respectively. Then,

$$f\colon (X_A, M_A) \to (Y, N)$$

is a morphism of log schemes. Moreover, we have the following:

**Lemma 3.3.** *The morphism $f\colon (X_A, M_A) \to (Y, N)$ is log-smooth.*

*Proof.* We use Theorem 3.5 in [10]. First note that $f$ has (étale locally on $X_A$) a chart $(P_{X_A} \to M_A, Q_Y \to N, Q \to P)$ given by $Q = \mathbb{N}$, $P = \mathbb{N}^r$, and the diagonal map $Q \to P$.

We can easily see also that the kernel and the torsion part of the cokernel of $Q^{gp} \to P^{gp}$ (which is just the diagonal map $\mathbb{Z} \to \mathbb{Z}^r$) are both trivial. Here $Q^{gp}$ and $P^{gp}$ are the Grothendieck groups of $Q$ and $P$, respectively.

It remains to prove that the induced morphism $X_A \to Y \times_{\operatorname{Spec}\mathbb{Z}[Q]} \operatorname{Spec}\mathbb{Z}[P]$ is smooth. Recall that $X_A$ is locally étale over

$$V = \operatorname{Spec}(A[x_1, \dots, x_n]/(x_1 \cdots x_r - t)),$$

and note that

$$\begin{aligned} W = \operatorname{Spec} A \times_{\operatorname{Spec}\mathbb{Z}[Q]} \operatorname{Spec}\mathbb{Z}[P] &\cong \operatorname{Spec} A \times_{\operatorname{Spec}\mathbb{Z}[u]} \operatorname{Spec}\mathbb{Z}[u_1, \dots, u_r] \\ &\cong \operatorname{Spec}(A[u_1, \dots, u_r]/(u_1 \cdots u_r - t)). \end{aligned}$$

The last isomorphism can be verified by checking directly that the ring

$$A[u_1, \dots, u_r]/(u_1 \cdots u_r - t)$$

satisfies the universal property of the tensor product $A \otimes_{\mathbb{Z}[u]} \mathbb{Z}[u_1, \dots, u_r]$.

Now note that there are natural closed immersions

$$j_V : V \hookrightarrow \mathbb{A}^n_A, \quad j_W : W \hookrightarrow \mathbb{A}^r_A.$$

Moreover, the following diagram is cartesian:

$$\begin{array}{ccc} V & \xrightarrow{j_V} & \mathbb{A}^n_A \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle p} \\ W & \xrightarrow[j_W]{} & \mathbb{A}^r_A \end{array}$$

where $h$ is defined by sending each $u_i$ to $x_i$ for $i = 1, \dots, r$, and $p$ is the natural projection from the first $r$ components. Since $p$ is smooth, we get that $h$ is smooth. Since $X_A \to W$ is the composition of an étale and a smooth morphism, we conclude that it is smooth (in the classical sense). □

Then, we have the following diagram of log schemes:

$$\begin{array}{ccccc} (X_0, M_0) & \longrightarrow & (X, M) & \longrightarrow & (X_A, M_A) \\ {\scriptstyle f_s}\downarrow & & {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f} \\ s^\times & \longrightarrow & \operatorname{Spec} k[[t]]^\times & \longrightarrow & (Y, N) \end{array}$$

where the horizontal arrows are exact closed immersions. In particular, note that $s$ is a closed point inside $Y$, hence $(X_0, M_0)$ is a fiber of the log smooth family $(X_A, M_A) \to (Y, N)$. This means that we can study the cohomology of $X_0$ using relative log-cohomology sheaves for this family. These are studied in the next section.

## 4. Relative cohomology

By Theorem 7 in [6, Section 4], there exists a $W[t]$-algebra $A_0$ such that $A_0/pA_0 = A$, which is smooth over $W$.[1] Let $\hat{A}$ be the $p$-adic completion of $A_0$, and let $\mathcal{Y} = \operatorname{Spf} \hat{A}$. We can define a log structure $\mathcal{N}$ on $\mathcal{Y}$ by $1 \mapsto t$, and then we have the following diagram:

$$\begin{array}{ccccc} (X_0, M_0) & \longrightarrow & (X_A, M_A) & & \\ \downarrow{\scriptstyle f_s} & & \downarrow{\scriptstyle f} & & \\ s^\times & \longrightarrow & (Y, N) & \longrightarrow & (\mathcal{Y}, \mathcal{N}) \end{array} \tag{4.1}$$

where the lower row consists of two exact closed immersions. Now we are in the situation studied in [14], and we can use all the results there. We shall state the results on relative log crystalline, log convergent and log analytic cohomology that are useful in applying the main result in [5].

**Relative log crystalline cohomology.** In the situation of diagram (4.1), Shiho defined in [14], for any sheaf $\mathcal{F}$ on the log crystalline site $(X/\mathcal{Y})^{\log}_{\mathrm{crys}}$, the sheaves of relative log crystalline cohomology of $(X_A, M_A)/(\mathcal{Y}, \mathcal{N})$ with coefficient $\mathcal{F}$, denoted by $R^m f_{X_A/\mathcal{Y},\mathrm{crys}*}\mathcal{F}$, and for an isocrystal $\mathcal{E} = \mathbb{Q} \otimes \mathcal{F}$, denoted by $R^m f_{X_A/\mathcal{Y},\mathrm{crys}*}\mathcal{E}$. Here we will work only with the trivial log isocrystal $\mathcal{E} = \mathcal{O}_{X_A/\mathcal{Y},\mathrm{crys}}$.

In order to study the sheaves $R^m f_{X_A/\mathcal{Y},\mathrm{crys}*}\mathcal{O}_{X/\mathcal{Y},\mathrm{crys}}$, we fix a Hyodo–Kato embedding system $(\mathcal{P}_\bullet, \mathcal{M}_\bullet)$ of an étale hypercovering $(X_\bullet, M_\bullet)$ of $(X_A, M_A)$. Such a system always exists, as stated in [8, 2.18] (the definition of simplicial schemes and étale hypercoverings can be found in [4]). Then, we have the following diagram:

$$\begin{array}{ccccc} (X_{0,\bullet}, M_{0,\bullet}) & \longrightarrow & (X_\bullet, M_\bullet) & \xrightarrow{i_\bullet} & (\mathcal{P}_\bullet, \mathcal{M}_\bullet) \\ \downarrow{\scriptstyle \theta_s} & & \downarrow{\scriptstyle \theta} & & \Big\downarrow \\ (X_0, M_0) & \longrightarrow & (X_A, M_A) & & \Big\downarrow{\scriptstyle g} \\ \downarrow{\scriptstyle f_s} & & \downarrow{\scriptstyle f} & & \Big\downarrow \\ s^\times & \longrightarrow & (Y, N) & \longrightarrow & (\mathcal{Y}, \mathcal{N}) \end{array} \tag{4.2}$$

where $(X_{0,\bullet}, M_{0,\bullet})$ is the fiber product in the upper left square.

We want to see that the sheaves $R^m f_{X_A/\mathcal{Y},\mathrm{crys}*}(\mathcal{O}_{X/\mathcal{Y},\mathrm{crys}})$ satisfy some finiteness properties. For each $n \in \mathbb{N}$, denote by $\mathcal{Y}_n$ the reduction of $\mathcal{Y}$ modulo $p^n$, and by

[1] Note that $A_0$ might be not smooth over $W[t]$.

$C_{X_\bullet/\mathcal{Y}_n}$ the logarithmic de Rham complex of the log PD-envelope of the closed immersion $i_\bullet$ over $(\mathcal{Y}_n, \mathcal{N}_n)$. Then, we have the following:

**Lemma 4.1.** (a) *For each $n$, there is a canonical quasi-isomorphism*

$$R(f\theta)_* C_{X_\bullet/\mathcal{Y}_n} \otimes^L_{\mathcal{O}_{\mathcal{Y}_n}} \mathcal{O}_{\mathcal{Y}_{n-1}} \xrightarrow{\sim} R(f\theta)_* C_{X_\bullet/\mathcal{Y}_{n-1}}.$$

(b) *For each $n$, $R(f\theta)_* C_{X_\bullet/\mathcal{Y}_n}$ is bounded and has finitely generated cohomologies.*

*Proof.* In [14, Section 1], it is proved that

$$R(f\theta)_* C_{X_\bullet/\mathcal{Y}_n} \cong Rf_{X_\bullet/\mathcal{Y}_n,\mathrm{crys},*}(\mathcal{O}_{X_\bullet/\mathcal{Y}_n,\mathrm{crys}}),$$

and so part (a) follows from the claim in the proof of [14, Theorem 1.15].

For part (b), we proceed inductively. Note that for $n = 1$, $\mathcal{Y}_1 = Y$, and so the result follows by properness of $f$. The inductive step is direct by the second part of the claim used in the proof of part (a). □

The preceding lemma says that $\{R(f\theta)_* C_{X_\bullet/\mathcal{Y}_n}\}_n$ is a consistent system, as defined in [2, B.4]. Then, by [2, Corollary B.9], it follows that

$$Rf_{X_A/\mathcal{Y},\mathrm{crys}*}(\mathcal{O}_{X_A/\mathcal{Y},\mathrm{crys}}) = R\varprojlim Rf_{X_A/\mathcal{Y}_n,\mathrm{crys},*}(\mathcal{O}_{X_A/\mathcal{Y}_n,\mathrm{crys}})$$

is bounded above and has finitely generated cohomologies. Thus, we have the following:

**Theorem 4.2.** *$Rf_{X_A/\mathcal{Y},\mathrm{crys}*}(\mathcal{O}_{X_A/\mathcal{Y},\mathrm{crys}})$ is a perfect complex of isocoherent sheaves on $\mathcal{Y}$. Moreover, the isocoherent cohomology sheaf $R^m f_{X_A/\mathcal{Y},\mathrm{crys}*}(\mathcal{O}_{X_A/\mathcal{Y},\mathrm{crys}})$ admits a Frobenius structure for each $m$.*

*Proof.* The first assertion follows from the above paragraph and [14, Theorem 1.16]. The Frobenius structure is given by [8, 2.24], since $f$ is of Cartier type. Indeed, recall that $f$ has a local chart $(P_{X_A} \to M_A, Q_Y \to N, Q \to P)$ given by $Q = \mathbb{N}$, $P = \mathbb{N}^r$, and $Q \to P$ the diagonal map. □

Now let us consider the following commutative diagram, where all squares are cartesian:

$$\begin{array}{ccccc}
(X_0, M_0) & \xrightarrow{f_s} & s^\times & \longrightarrow & \mathrm{Spf}\, W^\times \\
\downarrow & & \downarrow & & \downarrow \varphi \\
(X_A, M_A) & \xrightarrow[f]{} & (Y, N) & \xrightarrow[\iota]{} & (\mathcal{Y}, \mathcal{N})
\end{array} \tag{4.3}$$

By [14, Theorem 1.19], we have the following base change property.

**Theorem 4.3.** *In diagram* (4.3)*, there is a quasi-isomorphism*

$$L\varphi^* Rf_{X_A/\mathcal{Y},\mathrm{crys}*}(\mathcal{O}_{X_A/\mathcal{Y},\mathrm{crys}}) \xrightarrow{\sim} Rf_{s,X_0/W,\mathrm{crys},*}(\mathcal{O}_{X_0/W,\mathrm{crys}}).$$

Note that $Rf_{s,X_0/W,\mathrm{crys},*}(\mathcal{O}_{X_0/W,\mathrm{crys}})$ is a perfect $K$-complex that gives the cohomology

$$H^i_{\text{log-crys}}((X_0, M_0)/W^\times) \otimes K.$$

**Relative log convergent cohomology.** Following [14], we study the relative log convergent cohomology sheaves there defined. Again, we work only with the trivial isocrystal $\mathcal{O}_{X_A/\mathcal{Y},\mathrm{conv}}$, on the log convergent site, and denote the sheaves of relative cohomology by $Rf_{X_A/\mathcal{Y},\mathrm{conv}*}(\mathcal{O}_{X_A/\mathcal{Y},\mathrm{conv}})$.

Recall that there is a canonical functor (see [14, Proposition 2.35]) from the category of isocrystals on the relative log convergent site to that on the log crystalline site,

$$\Phi\colon I_{\mathrm{conv}}((X_A/\mathcal{Y})^{\log}_{\mathrm{conv}}) \to I_{\mathrm{crys}}((X_A/\mathcal{Y})^{\log}_{\mathrm{crys}}),$$

sending locally free isocrystals to locally free isocrystals. In particular,

$$\Phi(\mathcal{O}_{X_A/\mathcal{Y},\mathrm{conv}}) = \mathcal{O}_{X_A/\mathcal{Y},\mathrm{crys}}.$$

Now let us go back to the situation in diagram (4.2). Let $]X_\bullet[^{\log}_{\mathcal{P}_\bullet}$ be the log tube of the closed immersion $i_\bullet$, and let $\widehat{\mathcal{P}_\bullet}$ be the completion of $\mathcal{P}_\bullet$ along $X_\bullet$. Then, as in [5], we have a specialization map

$$\mathrm{sp}\colon\, ]X_\bullet[^{\log}_{\mathcal{P}_\bullet} \to \widehat{\mathcal{P}_\bullet}.$$

Moreover, if we denote by $\Omega^\bullet_{]X_\bullet[^{\log}_{\mathcal{P}_\bullet}/\mathcal{Y}_K}\langle \mathcal{M}_\bullet/\mathcal{N}\rangle$ the logarithmic de Rham complex of the simplicial rigid analytic space $]X_\bullet[^{\log}_{\mathcal{P}_\bullet}$ over the generic fiber $\mathcal{Y}_K$ of $\mathcal{Y}$, then by [14, Corollary 2.34] we have

$$Rf_{X_A/\mathcal{Y},\mathrm{conv}*}(\mathcal{O}_{X_A/\mathcal{Y},\mathrm{conv}}) \cong R(f\theta)_*\mathrm{sp}_*\Omega^\bullet_{]X_\bullet[^{\log}_{\mathcal{P}_\bullet}/\mathcal{Y}_K}\langle \mathcal{M}_\bullet/\mathcal{N}\rangle.$$

Now, by using the techniques introduced in [14] before Theorem 2.36, and passing to the projective limit, we have a canonical morphism of complexes

$$\mathrm{sp}_*\Omega^\bullet_{]X_\bullet[^{\log}_{\mathcal{P}_\bullet}/\mathcal{Y}_K}\langle \mathcal{M}_\bullet/\mathcal{N}\rangle \longrightarrow \varprojlim_n C_{X_\bullet/\mathcal{Y}_n}, \tag{4.4}$$

which by [14, Theorem 2.36] gives the following:

**Theorem 4.4.** *The canonical morphism* (4.4) *induces an isomorphism*

$$R^m f_{X_A/\mathcal{Y},\mathrm{conv}*}(\mathcal{O}_{X_A/\mathcal{Y},\mathrm{conv}}) \cong R^m f_{X_A/\mathcal{Y},\mathrm{crys}*}(\mathcal{O}_{X_A/\mathcal{Y},\mathrm{crys}})$$

*of isocoherent sheaves on* $\mathcal{Y}$.

By Theorem 4.2, this result allows us to prove that $Rf_{X_A/\mathcal{Y},\mathrm{conv}*}(\mathcal{O}_{X_A/\mathcal{Y},\mathrm{conv}})$ is a perfect complex of isocoherent sheaves, and a base change theorem:

**Theorem 4.5.** *With the same notation as in diagram* (4.3), *there is a natural isomorphism*

$$L\varphi^* Rf_{X_A/\mathcal{Y},\mathrm{conv}*}(\mathcal{O}_{X_A/\mathcal{Y},\mathrm{conv}}) \cong Rf_{sX_0/W,\mathrm{conv}*}(\mathcal{O}_{X/\mathcal{Y},\mathrm{conv}}).$$

This complex gives the cohomology $H^i_{\text{log-conv}}((X_0, M_0)/W^\times)$.

**Relative log analytic cohomology.** Now we study the sheaves of relative log analytic cohomology. Note that $g$ in diagram (4.2) induces a morphism

$$g_K^{\mathrm{ex}}\colon \,]X_\bullet[^{\log}_{\mathcal{P}_\bullet} \to \mathcal{Y}_K.$$

Then, the log analytic cohomology sheaves of $(X_A, M_A)/(Y,N)$ with respect to $(\mathcal{Y},\mathcal{N})$ can be computed by (see [14, Definition 4.1])

$$R^m f_{X_A/\mathcal{Y},an*}(\mathcal{O}_{X_A/\mathcal{Y},an}) = R^m g^{\mathrm{ex}}_{K*}\Omega^\bullet_{]X_\bullet[^{\log}_{\mathcal{P}_\bullet}/\mathcal{Y}_K}\langle \mathcal{M}_\bullet/\mathcal{N}\rangle.$$

Then, by applying [14, Theorem 4.6], we have the following comparison theorem:

**Theorem 4.6.** *Let $sp$ be the specialization map $\mathcal{Y}_K \to \mathcal{Y}$. Then for each $m$, $R^m f_{X_A/\mathcal{Y},an*}(\mathcal{O}_{X_A/\mathcal{Y},an})$ is a coherent sheaf on $\mathcal{Y}_K$, and there is an isomorphism*

$$sp_* R^m f_{X_A/\mathcal{Y},an*}(\mathcal{O}_{X_A/\mathcal{Y},an}) \cong R^m f_{X_A/\mathcal{Y},\mathrm{conv}*}(\mathcal{O}_{X_A/\mathcal{Y},\mathrm{conv}}).$$

## 5. Reduction to the case of a family over a curve

Now that we have relative cohomology sheaves defined for the family over $Y$, we want to restrict those sheaves to a smaller family. Namely, a family over a curve, in order to be in the same situation as in [5].

Let us first construct the curve that we shall use. As stated at the beginning of the preceding section, $A_0$ is a smooth $W$-algebra. Let $\widetilde{Y} = \operatorname{Spec} A_0$ and $S = \operatorname{Spec} W$. Since $Y \to \widetilde{Y}$ is a closed immersion, the image $\hat{s}$ of $s$ inside $\widetilde{Y}$ is a closed point. Since the natural morphism $\widetilde{Y} \to S$ is smooth, there exists an affine open neighborhood $\widetilde{U}$ of $\hat{s}$ and an étale morphism $\sigma\colon \widetilde{U} \to \mathbb{A}^d_W$ such that $\widetilde{W} \to S$ factorizes in the following way:

$$\begin{array}{ccc} \widetilde{U} & \xrightarrow{\ \sigma\ } & \mathbb{A}^d_W \\ \downarrow & \swarrow & \\ S & & \end{array}$$

Let us recall this construction. There exists an open affine subset $\widetilde{U} = \operatorname{Spec}(A_0)_g$ of $\widetilde{Y}$ such that the restriction of $\widetilde{Y} \to S$ is standard smooth. Moreover, we may assume that we can write

$$(A_0)_g = W[x_1, \dots, x_r, t]/(f_1, \dots, f_c),$$

in such a way that the morphism $W[x_{c+1}, \dots, x_r, t] \to (A_0)_g$ is étale, and we get the desired factorization with $d = r + 1 - c$.

Using this description, it is clear how to construct a smooth curve $C_W$ inside $\widetilde{U}$, transversal to $(t = 0)$ and passing through the point $\hat{s}$: by pulling back a curve with these properties inside $\mathbb{A}^d_W$. In particular, its reduction $C$ modulo $p$ is a smooth curve inside $Y$, transversal to $(t = 0)$ and passing throught the point $s$.

Let $N_C$ be the log structure on $C$ defined to make the closed immersion

$$(C, N_C) \to (Y, N)$$

exact. Then, we have a sequence of exact closed immersions

$$s^{\times} \to (C, N_C) \to (Y, N).$$

Let $(X_C, M_C) = (X_A, M_A) \times_{(Y,N)} (C, N_C)$. Then, we have the following diagram, where all the squares are cartesian:

$$\begin{array}{ccccc} (X_0, M_0) & \longrightarrow & (X_C, M_C) & \longrightarrow & (X_A, M_A) \\ \downarrow & & \downarrow & & \downarrow \\ s^{\times} & \longrightarrow & (C, N_C) & \longrightarrow & (Y, N) \end{array}$$

Note that the family $(X_C, M_C) \to (C, N_C)$ is in the situation studied in [5]. We denote by $\mathcal{C}$ the $p$-adic completion of $C_W$ along the special fiber $C$. Then $1 \mapsto t$ defines a log structure $\mathcal{N}_{\mathcal{C}}$ on $\mathcal{C}$ and we have the diagram

$$\begin{array}{ccccc} (X_C, M_C) & \xrightarrow[f_C]{} & (C, N) & \longrightarrow & (\mathcal{C}, \mathcal{N}_{\mathcal{C}}) \\ \downarrow & & \downarrow & & \downarrow \iota \\ (X_A, M_A) & \xrightarrow{f} & (Y, N) & \longrightarrow & (\mathcal{Y}, \mathcal{N}) \end{array}$$

Then, by [14, Theorem 1.19 and Corollary 2.38], we have an isomorphism

$$L\iota^* Rf_{X_A/\mathcal{Y},\mathrm{crys}*}(\mathcal{O}_{X_A/\mathcal{Y},\mathrm{crys}}) \xrightarrow{\sim} Rf_{C,X_C/\mathcal{C},\mathrm{crys}*}(\mathcal{O}_{X_C/\mathcal{C},\mathrm{crys}}). \tag{5.1}$$

Now consider the diagram

$$\begin{array}{ccccc} (X_0, M_0) & \longrightarrow & s^{\times} & \longrightarrow & \operatorname{Spf} W^{\times} \\ \downarrow & & \downarrow & & \downarrow \psi \\ (X_C, M_C) & \longrightarrow & (C, N_C) & \longrightarrow & (\mathcal{C}, \mathcal{N}_{\mathcal{C}}) \end{array}$$

where $\iota \circ \psi = \varphi$. Then we have an isomorphism

$$L\psi^* Rf_{C,X_C/\mathcal{C},\mathrm{crys}*}(\mathcal{O}_{X_C/\mathcal{C},\mathrm{crys}}) \xrightarrow{\sim} Rf_{s,X_0/W,\mathrm{crys}*}(\mathcal{O}_{X_0/W,\mathrm{crys}}). \tag{5.2}$$

By combining the isomorphisms (5.1) and (5.2) and the fact that

$$L\psi^* L\iota^* \cong L(\psi^* \iota^*) \cong L((\iota \circ \psi)^*) = L\psi^*,$$

we get that $Rf_{s,X_0/W,\mathrm{crys}*}(\mathcal{O}_{X_0/W,\mathrm{crys}})$ can be obtained from the family over $Y$ or over $C$. In particular, by the main result in [5], we get the following Clemens–Schmid type exact sequence:

$$\begin{aligned}\cdots \to H^m_{X_0,\mathrm{rig}}(X_C) \to H^m_{\mathrm{rig}}(X_0) \to H^m_{\text{log-crys}}((X_0,M_0)/W^\times)\otimes K \xrightarrow{N} \\ H^m_{\text{log-crys}}((X_0,M_0)/W^\times)\otimes K(-1) \to H^{m+2}_{X_0,\mathrm{rig}}(X_C) \to H^{m+2}_{\mathrm{rig}}(X_0) \to \cdots\end{aligned}$$

The terms of the form $H^{m+2}_{X_0,\mathrm{rig}}(X_C)$ depend a priori on the choice of the curve $C$, but if we choose a different smooth curve $C'$, by Poincaré duality ([1, Theorem 2.4]), we have isomorphisms

$$\begin{aligned}H^{m+2}_{X_0,\mathrm{rig}}(X_C) \cong H^{m+2}_{X_0,\mathrm{rig}}(X_{C'}) \cong H^{2\dim X-m-2}_{c,\mathrm{rig}}(X_0)^\vee(-\dim X) \\ \cong H^{\mathrm{rig}}_{2\dim X_0-m}(X_0)(-\dim X),\end{aligned}$$

and we get a Clemens–Schmid type exact sequence that depends only on $X$ and the special fiber $X_0$ for our starting situation.

**Remark 5.1.** The Clemens–Schmid type exact sequence that we have obtained can be used to prove a good reduction criterion for $K3$ surfaces. Namely, if $K$ is a finite extension of $\mathbb{Q}_p$ and $X_K$ is a smooth, projective $K3$ surface, with a minimal semistable model $X$ over the ring of integers $O_K$ of $K$, then $X_K$ has good reduction if and only if the monodromy on $H^2_{DR}(X_K)$ is trivial. This criterion is obtained by first realizing the special fiber of $X$ inside a family over $k[[t]]$ (this can be done by [12]). Then we apply our Clemens–Schmid type exact sequence to this family and proceed in a similar fashion as in [7]. The obtained criterion is directly equivalent to the one obtained in [13], which is obtained using a trascendental argument and $p$-adic Hodge theory. Moreover, these criteria can be extended to more cases, as is done in [3].

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