REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 63, No. 2, 2022, Pages 475–488
Published online: December 13, 2022
https://doi.org/10.33044/revuma.2181

ORLICZ VERSION OF MIXED MOMENT TENSORS

CHANG-JIAN ZHAO

Abstract. Our main aim is to generalize the moment tensors $\mathbf{\Psi}_r(K)$ to the Orlicz space. Under the framework of the Orlicz–Brunn–Minkowski theory, we introduce a new affine geometric quantity $\mathbf{\Psi}_{\psi,r}(K,L)$, and call it *Orlicz mixed moment tensors* of convex bodies K and L. The fundamental notions and properties of the moment tensors as well as related Minkowski and Brunn–Minkowski inequalities are then extended to the Orlicz setting. Diverse inequalities for certain new L_p-mixed moment tensors $\mathbf{\Psi}_{p,r}(K,L)$ are also derived. The new Orlicz inequalities in special cases yield the Orlicz–Minkowski and the Orlicz–Brunn–Minkowski inequalities, respectively.

1. Introduction

One of the most important operations in geometry is vector addition. As an operation between sets K and L, defined by

$$K+L=\{x+y : x\in K,\, y\in L\},$$

it is usually called *Minkowski addition*, which, combined with volume, plays an important role in the Brunn–Minkowski theory. If K is a nonempty closed (not necessarily bounded) convex set in $\mathbb{R}^n$, then

$$h(K,x)=\max\{x\cdot y : y\in K\}$$

for $x\in\mathbb{R}^n$ defines the *support function* $h(K,x)$ of K. A nonempty closed convex set is uniquely determined by its support function. Since about 1970, the theory has been extended to the L_p-Brunn–Minkowski theory. A set operation called L_p *addition*, denoted by $+_p$, was defined by Firey [5]:

$$h(K+_p L,x)^p=h(K,x)^p+h(L,x)^p,\quad x\in\mathbb{R}^n, \tag{1.1}$$

where K and L are compact convex sets containing the origin and $1\le p\le\infty$. When $p=\infty$, (1.1) is interpreted as $h(K+_\infty L,x)=\max\{h(K,x),h(L,x)\}$, as is customary. The L_p addition and inequalities are the fundamental and core content in the L_p Brunn–Minkowski theory; we refer the reader to [6, 9, 10, 11, 12, 18, 20, 23, 24, 21, 22, 27, 28, 29, 32, 33, 34] and the references therein. In recent years, a new extension of the L_p-Brunn–Minkowski theory to the Orlicz–Brunn–Minkowski

2020 *Mathematics Subject Classification.* 46E30, 52A40.

Key words and phrases. L_p-addition, Orlicz addition, moment tensor of rank r, first variation, Orlicz–Minkowski inequality, Orlicz–Brunn–Minkowski inequality.

theory was initiated by Lutwak, Yang, and Zhang [25, 26]. Following this, Gardner, Hug, and Weil [7] introduced for the first time the Orlicz addition, constructed a general framework for the Orlicz–Brunn–Minkowski theory, and made clear the relation to Orlicz spaces and norms. The Orlicz addition of convex bodies was also introduced in different ways, and it extends the L_p-Brunn–Minkowski inequality to the Orlicz–Brunn–Minkowski inequality (see [35]). Advances in the theory can be found in [8, 14, 15, 16, 17, 30, 36, 37, 38, 39]. The Orlicz addition $K +_\psi L$ of compact convex sets K and L in $\mathbb{R}^n$ containing the origin is implicitly defined by (see [7])

$$\psi\left(\frac{h(K,x)}{h(K +_\psi L, x)}, \frac{h(L,x)}{h(K +_\psi L, x)}\right) = 1 \tag{1.2}$$

for $x \in \mathbb{R}^n$ if $h(K,x) + h(L,x) > 0$, and by $h(K +_\psi L, x) = 0$ if $h(K,x) = h(L,x) = 0$. Here $\psi \in \Phi_2$, the set of convex functions $\psi : [0,\infty)^2 \to [0,\infty)$ that are increasing in each variable and satisfy $\psi(0,0) = 0$, $\psi(1,0) = \psi(0,1) = 1$. The particular instance of interest corresponds to using (1.2) with $\psi(x_1,x_2) = \psi_1(x_1) + \varepsilon\psi_2(x_2)$ for $\varepsilon > 0$ and some $\psi_1, \psi_2 \in \Phi$, the set of convex functions $\psi_i : [0,\infty) \to (0,\infty)$ that are increasing and satisfy $\psi_i(1) = 1$ and $\psi_i(0) = 0$, where $i = 1,2$. The Orlicz addition reduces to the L_p addition, $1 \le p < \infty$, when $\psi(x_1,x_2) = x_1^p + x_2^p$, or to the L_∞ addition when $\psi(x_1,x_2) = \max\{x_1,x_2\}$. Moreover, Gardner, Hug, and Weil introduced the Orlicz mixed volume of K and L, obtaining the equation

$$V_\psi(K,L) = \frac{1}{n}\int_{S^{n-1}} \psi\left(\frac{h(L,u)}{h(K,u)}\right) h(K,u) S(K,du),$$

where $S(K,u)$ is the mixed surface area measure of K, and $\psi \in \Phi$. Here K is a convex body containing the origin in its interior, and L is a compact convex set containing the origin. Let $\mathcal{K}^n$ be the class of nonempty compact convex subsets of $\mathbb{R}^n$, let $\mathcal{K}_o^n$ be the class of members of $\mathcal{K}^n$ containing the origin, and let $\mathcal{K}_{oo}^n$ be those sets in $\mathcal{K}^n$ containing the origin in their interiors. A set $K \in \mathcal{K}^n$ is called a *convex body* if its interior is nonempty. If K, L are convex bodies containing the origin, $\varepsilon > 0$ and $\psi \in \Phi$, then the Orlicz linear combination of convex bodies K and L, denoted by $K +_{\psi,\varepsilon} L$, is defined by (see [7])

$$\psi\left(\frac{h(K,x)}{h(K +_{\psi,\varepsilon} L, x)}\right) + \varepsilon \cdot \psi\left(\frac{h(L,x)}{h(K +_{\psi,\varepsilon} L, x)}\right) = 1$$

for $x \in \mathbb{R}^n$ and $\varepsilon > 0$.

For $K \in \mathcal{K}^n$ and $r \in \mathbb{N}_0$, the moment tensor of rank r, denoted by $\mathbf{\Psi}_r(K)$, is defined by (see [31, p. 316])

$$\mathbf{\Psi}_r(K) := \frac{1}{r!}\int_K x^r \, dx. \tag{1.3}$$

Complying with the basic spirit of Aleksandrov [1], Fenchel and Jensen's [4] introduction of mixed quermassintegrals, and the introduction of Lutwak's [19] p-mixed quermassintegrals, we base our study on the first order Orlicz variation of the moment tensor. In Section 3, we prove that the first order Orlicz variation of the moment tensor can be expressed as

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon\to 0^+} \mathbf{\Psi}_r(K +_{\psi,\varepsilon} L) = \frac{1}{(\psi_1)'_l(1)} \cdot \mathbf{\Psi}_{\psi_2,r}(K, L),$$

where $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$, $\psi_1, \psi_2 \in \Phi$ and $r \in \mathbb{N}_0$, and $(\psi_1)'_l(1)$ denotes the value of the left derivative of the convex function ψ_1 at point 1. In this first order variational equation, we find a new geometric quantity. Based on this, we extract the required geometric quantity, denoted by $\mathbf{\Psi}_{\psi_2,r}(K, L)$, and call it *Orlicz mixed moment tensor* of K and L, defined by

$$\mathbf{\Psi}_{\psi_2,r}(K, L) := (\psi_1)'_l(1) \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon\to 0^+} \mathbf{\Psi}_r(K +_{\psi,\varepsilon} L).$$

We also prove the new affine geometric quantity $\mathbf{\Psi}_{\psi,r}(K, L)$ has an integral representation:

$$\mathbf{\Psi}_{\psi,r}(K, L) = \frac{1}{n + r} \int_{S^{n-1}} \psi\left(\frac{h(L, u)}{h(K, u)}\right) h(K, u) S^r_{n-1}(K, du), \tag{1.4}$$

where $\psi \in \Phi$ and $S^r_{n-1}(K, \cdot)$ denotes the $\mathbb{T}^r$-valued Borel measure on S^{n-1} (see [31, p. 319]). In (1.4), if $K = L$, the Orlicz mixed moment tensor $\mathbf{\Psi}_{\psi,r}(K, L)$ becomes the moment tensor $\mathbf{\Psi}_r(K)$. When $r = 0$, the Orlicz mixed moment tensor $\mathbf{\Psi}_{\psi,r}(K, L)$ becomes the Orlicz mixed volume $V_\psi(K, L)$. Moreover, taking $\psi(t) = t^p$ and $p \geq 1$ in (1.4), $\mathbf{\Psi}_{\psi,r}(K, L)$ becomes a new mixed moment tensor in L_p-space, denoted by $\mathbf{\Psi}_{p,r}(K, L)$ and called L_p*-mixed moment tensor* of K and L:

$$\mathbf{\Psi}_{p,r}(K, L) = \frac{1}{n + r} \int_{S^{n-1}} h(L, u)^p h(K, u)^{1-p} S^r_{n-1}(K, du).$$

When $r = 0$, $\mathbf{\Psi}_{p,r}(K, L)$ becomes the well-known L_p-mixed volume $V_p(K, L)$, defined by (see [21])

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p h(K, u)^{1-p} S_{n-1}(K, du).$$

When $p = 1$, we write $\mathbf{\Psi}_{p,r}(K, L)$ as $\mathbf{\Psi}_r(K, L)$ and call it *mixed moment tensor* of K and L:

$$\mathbf{\Psi}_r(K, L) = \frac{1}{n + r} \int_{S^{n-1}} h(L, u) S^r_{n-1}(K, du). \tag{1.5}$$

In Section 4, we establish the following inequality:

Orlicz–Minkowski inequality for mixed moment tensors. *If* $K \in \mathcal{K}^n_{oo}$*,* $L \in \mathcal{K}^n_o$*,* $\psi \in \Phi$*, and* $r \in \mathbb{N}_0$*, then*

$$\mathbf{\Psi}_{\psi,r}(K, L) \geq \mathbf{\Psi}_r(K) \cdot \psi\left(\frac{\mathbf{\Psi}_r(K, L)}{\mathbf{\Psi}_r(K)}\right). \tag{1.6}$$

If ψ *is strictly convex, equality holds if and only if* K *and* L *are dilates or* $L = \{o\}$*.*

Obviously, a special case of (1.6) is the following well-known L_p-Minkowski inequality. If $K, L \in \mathcal{K}^n_{oo}$, $p \geq 1$, and $0 \leq i < n - 1$, then

$$V_p(K, L)^n \geq V(K)^{n-p} V(L)^p,$$

with equality if and only if K and L are dilates or $L=\{o\}$. On the other hand, if $r=0$, then (1.6) changes to the Orlicz–Minkowski inequality, which was obtained by Gardner, Hug, and Weil [7]. Let $\psi\in\Phi$. If $K\in\mathcal{K}^n_{oo}$ and $L\in\mathcal{K}^n_o$, then

$$V_\psi(K,L)\geq V(K)\cdot\psi\left(\left(\frac{V(L)}{V(K)}\right)^{1/n}\right).$$

If ψ is strictly convex, equality holds if and only if K and L are dilates or $L=\{o\}$.

In Section 5, we establish the following inequality:

Orlicz–Brunn–Minkowski inequality for mixed moment tensors. *Let $K_\varepsilon = K+_{\psi,\varepsilon}L$. If $K\in\mathcal{K}^n_{oo}$, $L\in\mathcal{K}^n_o$, $\psi\in\Phi_2$, and $r\in\mathbb{N}_0$, then, for $\varepsilon>0$,*

$$1\geq\psi\left(\frac{\mathbf{\Psi}_r(K_\varepsilon,K)}{\mathbf{\Psi}_r(K+_{\psi,\varepsilon}L)},\frac{\mathbf{\Psi}_r(K_\varepsilon,L)}{\mathbf{\Psi}_r(K+_{\psi,\varepsilon}L)}\right).\tag{1.7}$$

If ψ is strictly convex, equality holds if and only if K and L are dilates or $L=\{o\}$.

Obviously, a special case of (1.7) is the following well-known L_p-Brunn–Minkowski inequality. If $K\in\mathcal{K}^n_{oo}$, $L\in\mathcal{K}^n_o$, and $p\geq1$, then

$$V(K+_pL)^{p/n}\geq V(K)^{p/n}+V(L)^{p/n},$$

with equality if and only if K and L are dilates or $L=\{o\}$. If $r=0$, then (1.7) changes to the Orlicz–Brunn–Minkowski inequality, which was recently established by Gardner, Hug, and Weil [7]:

$$1\geq\psi\left(\left(\frac{V(K)}{V(K+_{\psi,\varepsilon}L)}\right)^{1/n},\left(\frac{V(L)}{V(K+_{\psi,\varepsilon}L)}\right)^{1/n}\right).$$

If ψ is strictly convex, equality holds if and only if K and L are dilates or $L=\{o\}$.

2. Notations and preliminaries

The setting for this paper is the n-dimensional Euclidean space $\mathbb{R}^n$. We denote by B the unit ball centered at the origin, whose surface is S^{n-1}, and by $u\in S^{n-1}$ we denote unit vectors. For a compact set K, we write $V(K)$ for the (n-dimensional) Lebesgue measure of K and call this the *volume* of K. If K is a nonempty closed (not necessarily bounded) convex set, then

$$h(K,x)=\sup\{x\cdot y:y\in K\},$$

for $x\in\mathbb{R}^n$, defines the *support function* of K, where $x\cdot y$ denotes the usual inner product of x and y in $\mathbb{R}^n$. A nonempty closed convex set is uniquely determined by its support function. The support function is homogeneous of degree 1, that is,

$$h(K,rx)=rh(K,x)$$

for all $x\in\mathbb{R}^n$ and $r\geq0$. Let d denote the Hausdorff metric on $\mathcal{K}^n$, i.e., for $K,L\in\mathcal{K}^n$,

$$d(K,L)=|h(K,u)-h(L,u)|_\infty,$$

where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$.

2.1. **Mixed volumes.** If $K_i \in \mathcal{K}^n$ $(i = 1, 2, \dots, r)$ and λ_i $(i = 1, 2, \dots, r)$ are nonnegative real numbers, then of fundamental importance is the fact that the volume of $\sum_{i=1}^{r} \lambda_i K_i$ is a homogeneous polynomial in λ_i given by (see e.g. [2])

$$V(\lambda_1 K_1 + \dots + \lambda_n K_n) = \sum_{i_1,\dots,i_n} \lambda_{i_1} \dots \lambda_{i_n} V_{i_1 \dots i_n}, \tag{2.1}$$

where the sum is taken over all n-tuples $(i_1, \dots, i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1 \dots i_n}$ depends only on the bodies $K_{i_1}, \dots, K_{i_n}$ and is uniquely determined by (2.1); it is called the *mixed volume* of $K_i, \dots, K_{i_n}$ and is written as $V(K_{i_1}, \dots, K_{i_n})$. If $K_1 = \dots = K_{n-i} = K$ and $K_{n-i+1} = \dots = K_n = B$, then the mixed volumes $V(K_1, \dots, K_n)$ are written as $W_i(K)$ and called *quermassintegrals* (or *i-th quermassintegrals*) of K. If $K_1 = \dots = K_{n-i-1} = K$, $K_{n-i} = \dots = K_{n-1} = B$ and $K_n = L$, then the mixed volumes $V(K_1, \dots, K_n)$ are written as $W_i(K, L)$ and called *mixed quermassintegrals* of K and L.

When $i = 0$, the mixed quermassintegrals $W_i(K, L)$ become the well-known mixed volume $V_1(K, L)$.

For $K \in \mathcal{K}^n_{oo}$, and $i = 0, 1, \dots, n-1$, there exists a regular Borel measure $S_{n-1}(K, \cdot)$ on S^{n-1}, such that the mixed volume $V_1(K, L)$ has the following representation:

$$V_1(K, L) = \frac{1}{n} \lim_{\varepsilon \to 0^+} \frac{V(K + \varepsilon \cdot L) - V(K)}{\varepsilon} = \frac{1}{n} \int_{S^{n-1}} h(L, u)\, dS_{n-1}(K, u).$$

The Minkowski inequality for the mixed volumes states that, if $K, L \in \mathcal{K}^n$, then

$$V_1(K, L)^n \geq V(K)^{n-1} V(L), \tag{2.2}$$

with equality if and only if K and L are dilates (see [3]).

Associated with $K_1, \dots, K_n \in \mathcal{K}^n$ is a Borel measure $S(K_1, \dots, K_{n-1}, \cdot)$ on S^{n-1}, called the *mixed surface area measure* of $K_1, \dots, K_{n-1}$, which has the property that, for each $K \in \mathcal{K}^n$,

$$V(K_1, \dots, K_{n-1}, K) = \frac{1}{n} \int_{S^{n-1}} h(K, u)\, dS(K_1, \dots, K_{n-1}, u). \tag{2.3}$$

(see e.g. [8, p. 353]). In fact, the measure $S(K_1, \dots, K_{n-1}, \cdot)$ can be defined by the property that (2.3) holds for all $K \in \mathcal{K}^n$. Let $K_1 = \dots = K_{n-1} = K$; then the mixed surface area measure $S(K_1, \dots, K_{n-1}, \cdot)$ becomes the surface area measure $S_{n-1}(K, \cdot)$.

2.2. L_p**-mixed volumes.** The L_p-mixed volumes $V_p(K, L)$, as the first variation of the ordinary volumes with respect to Firey's addition, are defined by (see e.g. [21])

$$V_p(K, L) = \frac{p}{n} \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}$$

for $K \in \mathcal{K}^n_{oo}, L \in \mathcal{K}^n_o$, and real $p \geq 1$. The L_p-mixed volume $V_p(K, L)$ has the following integral representation:

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p\, dS_p(K, u),$$

where $S_p(K,\cdot)$ denotes the Borel measure on S^{n-1}. The measure $S_p(K,\cdot)$ is absolutely continuous with respect to $S_{n-1}(K,\cdot)$, and has Radon–Nikodym derivative

$$\frac{dS_p(K,\cdot)}{dS_{n-1}(K,\cdot)}=h(K,\cdot)^{1-p}.$$

A fundamental inequality for the L_p-mixed volumes states that, for $K,L\in\mathcal{K}^n_{oo}$ and $p\geq 1$,

$$V_p(K,L)^n\geq V(K)^{n-p}V(L)^p, \tag{2.4}$$

with equality if and only if K and L are dilates or $L=\{o\}$. The L_p-Brunn–Minkowski inequality states that, if $K\in\mathcal{K}^n_{oo}$, $L\in\mathcal{K}^n_o$ and $p\geq 1$, then

$$V(K+_pL)^{p/n}\geq V(K)^{p/n}+V(L)^{p/n}, \tag{2.5}$$

with equality if and only if K and L are dilates or $L=\{o\}$.

2.3. **Orlicz linear combination.** Throughout the paper, the standard orthonormal basis for $\mathbb{R}^n$ will be $\{e_1,\ldots,e_n\}$. Let Φ_n, $n\in\mathbb{N}$, denote the set of convex functions $\psi:[0,\infty)^n\to[0,\infty)$ that are strictly increasing in each variable and satisfy $\psi(0)=0$ and $\psi(e_j)=1>0$, $j=1,\ldots,n$. When $n=1$, we shall write Φ instead of Φ_1. The left derivative and right derivative of a real-valued function f are denoted by $(f)'_l$ and $(f)'_r$, respectively.

Let $m\geq 2$, $\psi\in\Phi_m$, $K_j\in\mathcal{K}^n_0$ and $j=1,\ldots,m$; we define the *Orlicz addition* of $K_1,\ldots,K_m$, denoted by $+_\psi(K_1,\ldots,K_m)$, as follows (see [7]):

$$h(+_\psi(K_1,\ldots,K_m),x)=\inf\left\{\lambda>0:\psi\left(\frac{h(K_1,x)}{\lambda},\ldots,\frac{h(K_m,x)}{\lambda}\right)\leq 1\right\}$$

for all $x\in\mathbb{R}^n$. Equivalently, the Orlicz addition $+_\psi(K_1,\ldots,K_m)$ can be defined implicitly (and uniquely) by

$$\psi\left(\frac{h(K_1,x)}{h(+_\psi(K_1,\ldots,K_m),x)},\ldots,\frac{h(K_m,x)}{h(+_\psi(K_1,\ldots,K_m),x)}\right)=1 \tag{2.6}$$

for all $x\in\mathbb{R}^n$. An important special case is obtained when

$$\psi(x_1,\ldots,x_m)=\sum_{j=1}^m\psi_j(x_j)$$

for some fixed $\psi_j\in\Phi$ such that $\psi_1(1)=\cdots=\psi_m(1)=1$. We then write $+_\psi(K_1,\ldots,K_m)=K_1+_\psi\cdots+_\psi K_m$. This means that $K_1+_\psi\cdots+_\psi K_m$ is defined either by

$$h(K_1+_\psi\cdots+_\psi K_m,u)=\inf\left\{\lambda>0:\sum_{j=1}^m\psi_j\left(\frac{h(K_j,x)}{\lambda}\right)\leq 1\right\}$$

for all $x\in\mathbb{R}^n$, or by the corresponding special case of (2.6).

For real $p\geq 1$, $K,L\in\mathcal{K}^n_{oo}$ and $\alpha,\beta\geq 0$ (not both zero), the Firey linear combination $\alpha\cdot K+_p\beta\cdot L\in\mathcal{K}^n_o$ can be defined by (see [5])

$$h(\alpha\cdot K+_p\beta\cdot L,\cdot)^p=\alpha h(K,\cdot)^p+\beta h(L,\cdot)^p.$$

Obviously, the Firey and Minkowski scalar multiplications are related by $\alpha \cdot K = \alpha^{1/p}K$. The Orlicz linear combination, denoted by $+_\psi(K,L,\alpha,\beta)$ for $K, L \in \mathcal{K}_o^n$, $\psi_1, \psi_2 \in \Phi$ and $\alpha, \beta \geq 0$, is defined by (see [7])

$$\alpha\psi_1\left(\frac{h(K,x)}{h(+_\psi(K,L,\alpha,\beta),x)}\right) + \beta\psi_2\left(\frac{h(L,x)}{h(+_\psi(K,L,\alpha,\beta),x)}\right) = 1 \tag{2.7}$$

if $\alpha h(K,x)+\beta h(L,x) > 0$, and by $h(+_\psi(K,L,\alpha,\beta),x) = 0$ if $\alpha h(K,x)+\beta h(L,x) = 0$, for all $x \in \mathbb{R}^n$. It is easy to verify that when $\psi_1(t) = \psi_2(t) = t^p$, $p \geq 1$, the Orlicz linear combination $+_\psi(K,L,\alpha,\beta)$ equals the Firey combination $\alpha \cdot K +_p \beta \cdot L$. Henceforth we shall write $K +_{\psi,\varepsilon} L$ instead of $+_\psi(K,L,1,\varepsilon)$ for $\varepsilon \geq 0$, $\alpha = 1$, and $\beta = \varepsilon$, and assume throughout that this is defined by (2.7).

2.4. **Mixed moment tensors.** For $K \in \mathcal{K}^n$ and $r \in \mathbb{N}_0$, the *moment tensor of rank* r, denoted by $\mathbf{\Psi}_r(K)$, is defined by (see [31, p. 316])

$$\mathbf{\Psi}_r(K) := \frac{1}{r!}\int_K x^r\,dx. \tag{2.8}$$

The normalizing factor $\frac{1}{r!}$ has the effect that the formula for the polynomial behaviour of $\mathbf{\Psi}_r$ under translations takes a simple form, namely

$$\mathbf{\Psi}_r(K+t) = \sum_{j=0}^{r}\frac{1}{j!}\mathbf{\Psi}_{r-j}(K)t^j \quad \text{for } t \in \mathbb{R}^n.$$

We can write (2.8) in the form (see [30, p. 319])

$$\mathbf{\Psi}_r(K) = \frac{1}{n+r}\int_{S^{n-1}} h(K,u)S_{n-1}^r(K,du),$$

where $K \in \mathcal{K}^n$, $r \in \mathbb{N}_0$, and $S_{n-1}^r(K,\cdot)$ denotes the $\mathbb{T}^r$-valued Borel measure on S^{n-1}.

3. Orlicz mixed moment tensors

In the following, for brevity, let

$$K_\varepsilon := K +_{\psi,\varepsilon} L,$$

where $\varepsilon > 0$, $K \in \mathcal{K}_{oo}^n$, and $L \in \mathcal{K}_o^n$.

Lemma 3.1. *If $K, L \in \mathcal{K}_o^n$, then*

$$K_\varepsilon \to K \tag{3.1}$$

in the Hausdorff metric as $\varepsilon \to 0^+$.

Lemma 3.1 was first proved in [7].

Lemma 3.2. *Let $\psi_1, \psi_2 \in \Phi$. If $K \in \mathcal{K}_{oo}^n$, $L \in \mathcal{K}_o^n$, and $r \in \mathbb{N}_0$, then*

$$\lim_{\varepsilon\to 0^+}\frac{\mathbf{\Psi}_r(K,K_\varepsilon) - \mathbf{\Psi}_r(K)}{\varepsilon} = \frac{1}{(n+r)(\psi_1)_l'(1)}\cdot\int_{S^{n-1}}\psi_2\left(\frac{h(L,u)}{h(K,u)}\right)h(K,u)S_{n-1}^r(K,du). \tag{3.2}$$

Proof. From (1.3), (1.5), (2.7), (3.1) and in view of the continuity of ψ_1^{-1} and ψ_2, we obtain

$$\lim_{\varepsilon\to 0^+}\frac{\mathbf{\Psi}_r(K,K_\varepsilon)-\mathbf{\Psi}_r(K)}{\varepsilon}$$
$$=\frac{1}{n+r}\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_{S^{n-1}}\left(\frac{h(K,u)}{\psi_1^{-1}\left(1-\varepsilon\psi_2\left(\frac{h(L,u)}{h(K_\varepsilon,u)}\right)\right)}-h(K,u)\right)S_{n-1}^r(K,du)$$
$$=\frac{1}{n+r}$$
$$\times\lim_{\varepsilon\to 0^+}\int_{S^{n-1}}\frac{h(K,u)\cdot\frac{d\psi_1^{-1}(y)}{dy}\cdot\left(\psi_2\left(\frac{h(L,u)}{h(K_\varepsilon,u)}\right)-\varepsilon\cdot\frac{d\psi_2(z)}{dz}\cdot\frac{h(L,u)}{h(K_\varepsilon,u)^2}\frac{dh(K_\varepsilon,u)}{d\varepsilon}\right)}{\left(\psi_1^{-1}\left(1-\varepsilon\psi_2\left(\frac{h(L,u)}{h(K_\varepsilon,u)}\right)\right)\right)^2}$$
$$\times S_{n-1}^r(K,du),$$

where

$$y=1-\varepsilon\psi_2\left(\frac{h(L,u)}{h(K_\varepsilon,u)}\right)$$

and

$$z=\frac{h(L,u)}{h(K_\varepsilon,u)}.$$

Noting that $y\to 1^-$ as $\varepsilon\to 0^+$, and

$$\frac{d\psi_1^{-1}(y)}{dy}=\lim_{y\to 1^-}\frac{\psi_1^{-1}(y)-\psi_1^{-1}(1)}{y-1}=\frac{1}{(\psi_1)_l'(1)},$$

we obtain

$$\lim_{\varepsilon\to 0^+}\frac{\mathbf{\Psi}_r(K,K_\varepsilon)-\mathbf{\Psi}_r(K)}{\varepsilon}$$
$$=\frac{1}{(n+r)(\psi_1)_l'(1)}\cdot\int_{S^{n-1}}\psi_2\left(\frac{h(L,u)}{h(K,u)}\right)h(K,u)S_{n-1}^r(K,du).$$

□

Lemma 3.3. *Let $\psi_1,\psi_2\in\Phi$. If $K\in\mathcal{K}_{oo}^n$, $L\in\mathcal{K}_o^n$, and $r\in\mathbb{N}_0$, then*

$$\lim_{\varepsilon\to 0^+}\frac{\mathbf{\Psi}_r(K_\varepsilon)-\mathbf{\Psi}_r(K_\varepsilon,K)}{\varepsilon}$$
$$=\frac{1}{(n+r)(\psi_1)_l'(1)}\int_{S^{n-1}}h(K,u)\psi_2\left(\frac{h(L,u)}{h(K,u)}\right)S_{n-1}^r(K,du).\quad(3.3)$$

Proof. This follows immediately from Lemma 3.2. □

Setting $\mathbf{\Psi}_{\psi,r}(K,L)$ for any $\psi\in\Phi$ and $r\in\mathbb{N}_0$, the integral on the right-hand side of (3.3) with ψ_2 replaced by ψ, we see that either side of the equation (3.3) is equal to $\mathbf{\Psi}_{\psi_2,r}(K,L)$, and therefore this new geometric quantity $\mathbf{\Psi}_{\psi,r}(K,L)$ has been born; we call it *Orlicz mixed moment tensor*.

Definition 3.4 (Orlicz mixed moment tensor)**.** For $\psi \in \Phi$ and $r \in \mathbb{N}_0$, the *Orlicz mixed moment tensor*, denoted by $\mathbf{\Psi}_{\psi,r}(K,L)$, is defined by

$$\mathbf{\Psi}_{\psi,r}(K,L) := \frac{1}{n+r} \int_{S^{n-1}} \psi\left(\frac{h(L,u)}{h(K,u)}\right) h(K,u) S_{n-1}^{r}(K,du) \tag{3.4}$$

for all $K \in \mathcal{K}_{oo}^n$ and $L \in \mathcal{K}_o^n$.

It is worth noting that we write ψ_2 as ψ in equation (3.4), and hence $\psi_2(K,L)$ and $\psi(K,L)$ have the same meaning here.

Lemma 3.5. *If $\psi_1, \psi_2 \in \Phi$, $K \in \mathcal{K}_{oo}^n, L \in \mathcal{K}_o^n$, and $r \in \mathbb{N}_0$, then*

$$\mathbf{\Psi}_{\psi_2,r}(K,L) = (\psi_1)_l'(1) \cdot \lim_{\varepsilon \to 0^+} \frac{\mathbf{\Psi}_r(K_\varepsilon) - \mathbf{\Psi}_r(K_\varepsilon, K)}{\varepsilon}. \tag{3.5}$$

Proof. This follows immediately from (3.3) and (3.4). □

4. Orlicz–Minkowski inequality for Orlicz mixed moment tensors

Definition 4.1 (Normalized tensor measure)**.** If $K \in \mathcal{K}_o^n$ and $r \in \mathbb{N}_0$, the *normalized tensor measure*, denoted by $\bar{\mathbf{\Psi}}_r(K,\upsilon)$, is defined by

$$\bar{\mathbf{\Psi}}_r(K,d\upsilon) = \frac{h(K,\upsilon)}{(n+r)\mathbf{\Psi}_r(K)} S_{n-1}^r(K,d\upsilon). \tag{4.1}$$

For $r = 0$, the normalized tensor measure $\bar{\mathbf{\Psi}}_r(K,\upsilon)$ becomes the normalized cone measure $\bar{V}_n(K,\upsilon)$, by (see [7])

$$\bar{V}_n(K,d\upsilon) = \frac{h(K,\upsilon)}{nV(K)} S_{n-1}(K,d\upsilon).$$

In the following, we start with two auxiliary results, which will be the base for the remainder of our study.

Lemma 4.2 (Jensen's inequality [13])**.** *Suppose that μ is a probability measure on a space X and $g : X \to I \subset \mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. If $\phi : I \to \mathbb{R}$ is a convex function, then*

$$\int_X \phi(g(x))\, d\mu(x) \geq \phi\left(\int_X g(x)\, d\mu(x)\right).$$

If ϕ is strictly convex, equality holds if and only if $g(x)$ is constant for μ-almost all $x \in X$.

Lemma 4.3. *Let $0 < a \leq \infty$ be an extended real number, and let $I = [0,a)$ be a possibly infinite interval. Suppose that $\psi : I \to [0,\infty)$ is convex with $\psi(0) = 0$. If $K \in \mathcal{K}_{oo}^n$ and $L \in \mathcal{K}_o^n$ are such that $L \subset \operatorname{int}(aK)$, and $r \in \mathbb{N}_0$, then*

$$\frac{1}{(n+r)\mathbf{\Psi}_r(K)} \int_{S^{n-1}} \psi\left(\frac{h(L,u)}{h(K,u)}\right) h(K,u) S_{n-1}^r(K,du) \geq \psi\left(\frac{\mathbf{\Psi}_r(K,L)}{\mathbf{\Psi}_r(K)}\right). \tag{4.2}$$

If ψ is strictly convex, equality holds if and only if K and L are homothetic or $L = \{o\}$.

Proof. Since $L \subset \mathrm{int}(aK)$, we have $0 \le \frac{h(L,u)}{h(K,u)} < a$ for all $u \in S^{n-1}$, and

$$\int_{S^{n-1}} \bar{\mathbf{\Psi}}_r(K, dv) = 1, \tag{4.3}$$

so (4.3) defines a Borel probability measure $\bar{\mathbf{\Psi}}_r(K, v)$ on S^{n-1}. Hence, from (1.5), (4.1), and Jensen's inequality, we obtain

$$\begin{aligned}
&\frac{1}{(n+r)\mathbf{\Psi}_r(K)} \int_{S^{n-1}} \psi\left(\frac{h(L,u)}{h(K,u)}\right) h(K,u) S^r_{n-1}(K, du) \\
&\quad = \int_{S^{n-1}} \psi\left(\frac{h(L,u)}{h(K,u)}\right) \bar{\mathbf{\Psi}}_r(K, dv) \\
&\quad \ge \psi\left(\frac{1}{(n+r)\mathbf{\Psi}_r(K)} \int_{S^{n-1}} h(L,u) S^r_{n-1}(K, dv)\right) \\
&\quad = \psi\left(\frac{\mathbf{\Psi}_r(K,L)}{\mathbf{\Psi}_r(K)}\right).
\end{aligned}$$

Next, we discuss the equality condition of (4.2). If ψ is strictly convex, suppose L and K are homothetic or $L = \{o\}$; then there exists $r \ge 0$ such that

$$h(L,u)/h(K,u) = r.$$

This shows that

$$r = \frac{\mathbf{\Psi}_r(K,L)}{\mathbf{\Psi}_r(K)}.$$

From (1.3) and (3.4), we obtain

$$\begin{aligned}
&\frac{1}{(n+r)\mathbf{\Psi}_r(K)} \int_{S^{n-1}} \psi\left(\frac{h(L,u)}{h(K,u)}\right) h(K,u) S^r_{n-1}(K, du) \\
&\quad = \frac{1}{(n+r)\mathbf{\Psi}_r(K)} \int_{S^{n-1}} \psi\left(\frac{\mathbf{\Psi}_r(K,L)}{\mathbf{\Psi}_r(K)}\right) h(K,u) S^r_{n-1}(K, du) \\
&\quad = \psi\left(\frac{\mathbf{\Psi}_r(K,L)}{\mathbf{\Psi}_r(K)}\right).
\end{aligned}$$

Conversely, if ψ is strictly convex, suppose the equality holds in (4.2); from the equality condition of Jensen's inequality, it follows that this equality holds if $h(L,u)/h(K,u)$ are constant. This yields that if this equality holds then K and L must be homothetic or $L = \{o\}$. □

Theorem 4.4 (Orlicz–Minkowski inequality for Orlicz mixed moment tensors)**.** *Let $\psi \in \Phi$. If $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$, and $r \in \mathbb{N}_0$, then*

$$\mathbf{\Psi}_{\psi,r}(K,L) \ge \mathbf{\Psi}_r(K) \cdot \psi\left(\frac{\mathbf{\Psi}_r(K,L)}{\mathbf{\Psi}_r(K)}\right). \tag{4.4}$$

If ψ is strictly convex, equality holds if and only if K and L are homothetic or $L = \{o\}$.

Proof. This follows immediately from (3.4) and Lemma 4.3 with $a = \infty$. □

Corollary 4.5 (Minkowski inequality for L_p-mixed moment tensors)**.** *If $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$, $p \geq 1$, and $r \in \mathbb{N}_0$, then*

$$\mathbf{\Psi}_{p,r}(K, L) \geq \mathbf{\Psi}_r(K)^{1-p} \cdot \mathbf{\Psi}_r(K, L)^p, \tag{4.5}$$

with equality if and only if K and L are homothetic or $L = \{o\}$.

Proof. This follows immediately from (2.3) and (4.4) with $\psi(t) = t^p$ and $p \geq 1$. □

For $r = 0$, and from (2.2), (4.5) becomes Lutwak's L_p-Minkowski inequality (2.4) stated in Section 2.

5. Orlicz–Brunn–Minkowski inequality for mixed moment tensors

In this section, our main goal is to establish an Orlicz–Brunn–Minkowski inequality for mixed moment tensors. As an application, we use this new inequality to prove the previous Minkowski inequality for Orlicz mixed moment tensors.

Lemma 5.1 ([7])**.** *Let $K, L \in \mathcal{K}^n_o$, $\varepsilon > 0$, and $\psi \in \Phi$.*

(1) *If K and L are dilates, then K and $K +_{\psi,\varepsilon} L$ are dilates.*
(2) *If K and $K +_{\psi,\varepsilon} L$ are dilates, then K and L are dilates.*

Theorem 5.2. *Let $\psi \in \Phi_2$. If $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$, and $r \in \mathbb{N}_0$, then, for $\varepsilon > 0$,*

$$1 \geq \psi\left(\frac{\mathbf{\Psi}_r(K +_{\psi,\varepsilon} L, K)}{\mathbf{\Psi}_r(K +_{\psi,\varepsilon} L)}, \frac{\mathbf{\Psi}_r(K +_{\psi,\varepsilon} L, L)}{\mathbf{\Psi}_r(K +_{\psi,\varepsilon} L)}\right). \tag{5.1}$$

If ψ is strictly convex, equality holds if and only if K and L are dilates or $L = \{o\}$.

Proof. From (1.3), (1.5), (2.6), (2.7), (3.4), and (4.4), we obtain, for $\psi_1, \psi_2 \in \Phi$,

$$\begin{aligned}
\mathbf{\Psi}_r(K_\varepsilon) &= \frac{1}{n}\int_{S^{n-1}} \psi\left(\frac{h(K,u)}{h(K_\varepsilon,u)}, \frac{h(L,u)}{h(K_\varepsilon,u)}\right) h(K_\varepsilon,u) S^r_{n-1}(K_\varepsilon, du) \\
&= \frac{1}{n}\int_{S^{n-1}} \left(\psi_1\left(\frac{h(K,u)}{h(K_\varepsilon,u)}\right) + \varepsilon \cdot \psi_2\left(\frac{h(L,u)}{h(K_\varepsilon,u)}\right)\right) h(K_\varepsilon,u) S^r_{n-1}(K_\varepsilon, du) \\
&= \mathbf{\Psi}_{\psi_1,r}(K_\varepsilon, K) + \varepsilon \cdot \mathbf{\Psi}_{\psi_2,r}(K_\varepsilon, L) \\
&\geq \mathbf{\Psi}_r(K_\varepsilon)\left(\psi_1\left(\frac{\mathbf{\Psi}_r(K_\varepsilon, K)}{\mathbf{\Psi}_r(K_\varepsilon)}\right) + \varepsilon \cdot \psi_2\left(\frac{\mathbf{\Psi}_r(K_\varepsilon, L)}{\mathbf{\Psi}_r(K_\varepsilon)}\right)\right) \\
&= \mathbf{\Psi}_r(K_\varepsilon) \cdot \psi\left(\frac{\mathbf{\Psi}_r(K_\varepsilon, K)}{\mathbf{\Psi}_r(K_\varepsilon)}, \frac{\mathbf{\Psi}_r(K_\varepsilon, L)}{\mathbf{\Psi}_r(K_\varepsilon)}\right).
\end{aligned} \tag{5.2}$$

Obviously, (5.2) yields (5.1).

If ψ is strictly convex, from the equality of (4.4) it follows that equality holds in (5.2) if and only if K and K_ε are homothetic, and L and K_ε are homothetic. Combining this with Lemma 5.1, it follows that if ψ is strictly convex then equality holds if and only if K and L are dilates or $L = \{o\}$. □

Corollary 5.3. *If $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$, $p \geq 1$, and $r \in \mathbb{N}_0$, then*

$$\mathbf{\Psi}_r(K +_{\psi,\varepsilon} L)^p \geq \mathbf{\Psi}_r(K +_{\psi,\varepsilon} L, K)^p + \mathbf{\Psi}_r(K +_{\psi,\varepsilon} L, L)^p, \tag{5.3}$$

with equality if and only if K and L are dilates or $L = \{o\}$.

Proof. This follows immediately from Theorem 5.2 with $\psi(x_1, x_2) = x_1^p + x_2^p$ and $p \geq 1$. □

Using $r = 0$ and (2.2), (5.3) becomes the L_p-Brunn–Minkowski inequality (2.5) stated in Section 2.

Corollary 5.4. *Let $\psi \in \Phi$. If $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$, and $r \in \mathbb{N}_0$, then*

$$\mathbf{\Psi}_{\psi,r}(K, L) \geq \mathbf{\Psi}_r(K) \cdot \psi\left(\frac{\mathbf{\Psi}_r(K, L)}{\mathbf{\Psi}_r(K)}\right).$$

If ψ is strictly convex, equality holds if and only if K and L are dilates or $L = \{o\}$.

Proof. From (3.2) and (3.5), and by using the Orlicz–Brunn–Minkowski inequality (5.1), we obtain

$$\begin{aligned}
&\frac{1}{(\psi_1)'_l(1)} \cdot \mathbf{\Psi}_{\psi_2,r}(K, L) \\
&\quad= \lim_{\varepsilon \to 0^+} \frac{\mathbf{\Psi}_r(K_\varepsilon) - \mathbf{\Psi}_r(K_\varepsilon, K)}{\varepsilon} \\
&\quad= \lim_{\varepsilon \to 0^+} \frac{1 - \frac{\mathbf{\Psi}_r(K_\varepsilon, K)}{\mathbf{\Psi}_r(K_\varepsilon)}}{1 - \psi_1\left(\frac{\mathbf{\Psi}_r(K_\varepsilon, K)}{\mathbf{\Psi}_r(K_\varepsilon)}\right)} \cdot \frac{1 - \psi_1\left(\frac{\mathbf{\Psi}_r(K_\varepsilon, K)}{\mathbf{\Psi}_r(K_\varepsilon)}\right)}{\varepsilon} \cdot \mathbf{\Psi}_r(K_\varepsilon) \\
&\quad= \lim_{t \to 1^-} \frac{1 - t}{1 - \psi_1(t)} \cdot \lim_{\varepsilon \to 0^+} \left(\frac{1 - \psi_1\left(\frac{\mathbf{\Psi}_r(K_\varepsilon, K)}{\mathbf{\Psi}_r(K_\varepsilon)}\right)}{\varepsilon} \cdot \mathbf{\Psi}_r(K_\varepsilon)\right) \\
&\quad\geq \frac{n}{(\psi_1)'_l(1)} \cdot \lim_{\varepsilon \to 0^+} \psi_2\left(\frac{\mathbf{\Psi}_r(K_\varepsilon, L)}{\mathbf{\Psi}_r(K_\varepsilon)}\right) \cdot \mathbf{\Psi}_r(K) \\
&\quad= \frac{1}{(\psi_1)'_l(1)} \cdot \psi_2\left(\frac{\mathbf{\Psi}_r(K, L)}{\mathbf{\Psi}_r(K)}\right) \cdot \mathbf{\Psi}_r(K).
\end{aligned} \tag{5.4}$$

Obviously, (5.4) is just the Orlicz–Minkowski inequality for mixed moment tensors. From the equality conditions of the Orlicz–Brunn–Minkowski inequality (5.1), it follows that if ϕ is strictly convex, the equality in (5.4) holds if and only if K and L are dilates or $L = \{o\}$.

This proof is complete. □

6. Conclusion

We introduced a new affine geometric quantity, the Orlicz mixed moment tensors $\mathbf{\Psi}_{\psi,r}(K, L)$, which is an important generalization of the classical moment tensors $\mathbf{\Psi}_r(K)$. The Minkowski and Brunn–Minkowski inequalities for the Orlicz mixed moment tensors were established. The new Orlicz inequalities in special cases yield the Orlicz–Minkowski and the Orlicz–Brunn–Minkowski inequalities, respectively.

Moreover, diverse inequalities for certain new L_p-mixed moment tensors $\mathbf{\Psi}_{p,r}(K, L)$ are also derived. Hence, our results here are meaningful.

Acknowledgment. The author expresses his grateful thanks to the production editor for his many excellent suggestions and comments.

References

[1] A. D. Aleksandrov, On the theory of mixed volumes. I. Extension of certain concepts in the theory of convex bodies, *Mat. Sb. (N.S.)* **2** (1937), 947–972 (in Russian).

[2] Yu. D. Burago and V. A. Zalgaller, *Geometric Inequalities*, Grundlehren der mathematischen Wissenschaften, 285, Springer-Verlag, Berlin, 1988. MR 0936419.

[3] H. Busemann, *Convex Surfaces*, Interscience Tracts in Pure and Applied Mathematics, no. 6, Interscience, New York, 1958. MR 0105155.

[4] W. Fenchel and B. Jessen, Mengenfunktionen und konvexe Körper, *Danske Vid. Selskab. Mat.-fys. Medd.* **16** (1938), no. 3, 1–31.

[5] Wm. J. Firey, p-means of convex bodies, *Math. Scand.* **10** (1962), 17–24. MR 0141003.

[6] R. J. Gardner, *Geometric Tomography*, second edition, Encyclopedia of Mathematics and its Applications, 58, Cambridge University Press, New York, 2006. MR 2251886.

[7] R. J. Gardner, D. Hug, and W. Weil, The Orlicz-Brunn-Minkowski theory: a general framework, additions, and inequalities, *J. Differential Geom.* **97** (2014), no. 3, 427–476. MR 3263511.

[8] C. Haberl, E. Lutwak, D. Yang, and G. Zhang., The even Orlicz Minkowski problem, *Adv. Math.* **224** (2010), no. 6, 2485–2510. MR 2652213.

[9] C. Haberl and L. Parapatits, The centro-affine Hadwiger theorem, *J. Amer. Math. Soc.* **27** (2014), no. 3, 685–705. MR 3194492.

[10] C. Haberl and F. E. Schuster, Asymmetric affine L_p Sobolev inequalities, *J. Funct. Anal.* **257** (2009), no. 3, 641–658. MR 2530600.

[11] C. Haberl and F. E. Schuster, General L_p affine isoperimetric inequalities, *J. Differential Geom.* **83** (2009), no. 1, 1–26. MR 2545028.

[12] C. Haberl, F. E. Schuster, and J. Xiao, An asymmetric affine Pólya-Szegö principle, *Math. Ann.* **352** (2012), no. 3, 517–542. MR 2885586.

[13] J. Hoffmann-Jørgensen, *Probability with a View Toward Statistics. Vol. I*, Chapman & Hall Probability Series, Chapman & Hall, New York, 1994. MR 1278485.

[14] Q. Huang and B. He, On the Orlicz Minkowski problem for polytopes, *Discrete Comput. Geom.* **48** (2012), no. 2, 281–297. MR 2946448.

[15] M. A. Krasnosel′skiĭ and Ja. B. Rutickiĭ, *Convex Functions and Orlicz Spaces*, P. Noordhoff, Groningen, 1961. MR 0126722.

[16] J. Li and D. Ma, Laplace transforms and valuations, *J. Funct. Anal.* **272** (2017), no. 2, 738–758. MR 3571907.

[17] Y. Lin, Affine Orlicz Pólya-Szegö principle for log-concave functions, *J. Funct. Anal.* **273** (2017), no. 10, 3295–3326. MR 3695895.

[18] M. Ludwig and M. Reitzner, A classification of SL(n) invariant valuations, *Ann. of Math. (2)* **172** (2010), no. 2, 1219–1267. MR 2680490.

[19] E. Lutwak, The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem, *J. Differential Geom.* **38** (1993), no. 1, 131–150. MR 1231704.

[20] E. Lutwak, The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas, *Adv. Math.* **118** (1996), no. 2, 244–294. MR 1378681.

[21] E. Lutwak, D. Yang, and G. Zhang, L_p affine isoperimetric inequalities, *J. Differential Geom.* **56** (2000), no. 1, 111–132. MR 1863023.

[22] E. Lutwak, D. Yang, and G. Zhang, Sharp affine L_p Sobolev inequalities, *J. Differential Geom.* **62** (2002), no. 1, 17–38. MR 1987375.

[23] E. Lutwak, D. Yang, and G. Zhang, On the L_p-Minkowski problem, *Trans. Amer. Math. Soc.* **356** (2004), no. 11, 4359–4370. MR 2067123.

[24] E. Lutwak, D. Yang, and G. Zhang, L_p John ellipsoids, *Proc. London Math. Soc. (3)* **90** (2005), no. 2, 497–520. MR 2142136.

[25] E. Lutwak, D. Yang, and G. Zhang, Orlicz projection bodies, *Adv. Math.* **223** (2010), no. 1, 220–242. MR 2563216.

[26] E. Lutwak, D. Yang, and G. Zhang, Orlicz centroid bodies, *J. Differential Geom.* **84** (2010), no. 2, 365–387. MR 2652465.

[27] E. Lutwak, D. Yang, and G. Zhang, The Brunn-Minkowski-Firey inequality for nonconvex sets, *Adv. in Appl. Math.* **48** (2012), no. 2, 407–413. MR 2873885.

[28] L. Parapatits, SL(n)-covariant L_p-Minkowski valuations, *J. Lond. Math. Soc. (2)* **89** (2014), no. 2, 397–414. MR 3188625.

[29] L. Parapatits, SL(n)-contravariant L_p-Minkowski valuations, *Trans. Amer. Math. Soc.* **366** (2014), no. 3, 1195–1211. MR 3145728.

[30] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Monographs and Textbooks in Pure and Applied Mathematics, 146, Marcel Dekker, New York, 1991. MR 1113700.

[31] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, second expanded edition, Encyclopedia of Mathematics and its Applications, 151, Cambridge University Press, Cambridge, 2014. MR 3155183.

[32] C. Schütt and E. Werner, Surface bodies and p-affine surface area, *Adv. Math.* **187** (2004), no. 1, 98–145. MR 2074173.

[33] E. M. Werner, Rényi divergence and L_p-affine surface area for convex bodies, *Adv. Math.* **230** (2012), no. 3, 1040–1059. MR 2921171.

[34] E. Werner and D. Ye, New L_p affine isoperimetric inequalities, Adv. Math. **218** (2008), no. 3, 762–780. MR 2414321.

[35] D. Xi, H. Jin, and G. Leng, The Orlicz Brunn-Minkowski inequality, *Adv. Math.* **260** (2014), 350–374. MR 3209355.

[36] C.-J. Zhao, Orlicz dual mixed volumes, *Results Math.* **68** (2015), no. 1-2, 93–104. MR 3391494.

[37] C.-J. Zhao, Orlicz dual affine quermassintegrals, *Forum Math.* **30** (2018), no. 4, 929–945. MR 3824799.

[38] C.-J. Zhao, Orlicz-Brunn-Minkowski inequality for radial Blaschke-Minkowski homomorphisms, *Quaest. Math.* **41** (2018), no. 7, 937–950. MR 3882049.

[39] C.-J. Zhao, Orlicz-Aleksandrov-Fenchel inequality for Orlicz multiple mixed volumes, *J. Funct. Spaces* **2018**, Art. ID 9752178, 16 pp. MR 3859786.

Chang-Jian Zhao
Department of Mathematics, China Jiliang University, Hangzhou 310018, P. R. China
chjzhao@163.com; chjzhao@cjlu.edu.cn

Received: October 1, 2020
Accepted: March 2, 2021