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SHARP BOUNDS FOR FRACTIONAL TYPE OPERATORS WITH $L^{\alpha,s}$-HÖRMANDER CONDITIONS

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Abstract. We provide the sharp bound for a fractional type operator given by a kernel satisfying the $L^{\alpha,s}$-Hörmander condition and certain fractional size condition, $0<\alpha<n$ and $1<s\leq\infty$. In order to prove this result we use a new appropriate sparse domination. Examples of these operators include the fractional rough operators. For the case $s=\infty$ we recover the sharp bound of the fractional integral, I_α, proved by Lacey et al. [*J. Functional Anal.* **259** (2010), no. 5, 1073–1097].

1. Introduction and main results

Let $0<\alpha<n$. The *fractional integral operator* I_α on $\mathbb{R}^n$ is defined by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

This operator is bounded from $L^p(dx)$ into $L^q(dx)$ provided that $1<p<\frac{n}{\alpha}$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ (see [29] for this result).

In the study of weighted estimates for the fractional integral operator, the class of weights considered is the $A_{p,q}$ introduced by Muckenhoupt and Wheeden [23]. Recall that w is a weight if it is a non-negative locally integrable function. Given $1<p<q<\infty$, the weight w is in the class $A_{p,q}$ if

$$[w]_{A_{p,q}} := \sup_Q \left(\frac{1}{|Q|}\int_Q w^q\right)\left(\frac{1}{|Q|}\int_Q w^{-p'}\right)^{q/p'} < \infty.$$

If $w\in A_{p,q}$, then $w^q\in A_{1+q/p'}$ with $[w^q]_{1+q/p'}=[w]_{A_{p,q}}$, and $w^{-p'}\in A_{1+p'/q}$ with $[w^{-p'}]_{1+p'/q}=[w]_{A_{p,q}}^{p'/q}$, where A_s denotes the classical Muckenhoupt class of weights. Observe that $w\in A_{p,p}$ is equivalent to $w^p\in A_p$. The class $A_\infty=\cup_{p\geq 1}A_p$, and the statement $w\in A_{\infty,\infty}$ is equivalent to $w^{-1}\in A_1$.

There have been several works devoted to the study of quantitative weighted estimates; in other words, papers where the authors study how these estimates depend

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on the weight constant $[w]_{A_p}$ or $[w]_{A_{p,q}}$. The estimate for the Hardy–Littlewood maximal function was studied by Buckley [4] and for the maximal fractional operator by Pradolini and Salinas [27]. Buckley's result attracted renewed attention as a result of the work of Astala, Iwaniec and Saksman [2] on the theory of quasiregular mappings. They proved sharp regularity results for solutions to the Beltrami equation, assuming that the operator norm of the Beurling–Ahlfors transform grows linearly in terms of the A_p constant for $p \geq 2$. This linear growth was proved by Petermichl and Volberg [26]. This result opened up the possibility of considering some other operators. Petermichl [24, 25] proved the corresponding results for the Hilbert transform and the Riesz transforms. The A_2 theorem, namely the linear dependence on the A_2 constant for Calderón–Zygmund integral operators, proved by Hytönen [12], can be considered the most representative in this line. In the case of the fractional integral operator, the sharp dependence of the $A_{p,q}$ constants was obtained by Lacey, Moen, Pérez and Torres [16]. The precise statement is the following.

Theorem 1.1 ([16])**.** *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If $w \in A_{p,q}$, then*

$$\|I_\alpha f\|_{L^q(w^q)} \leq c_{n,\alpha}[w]_{A_{p,q}}^{(1-\frac{\alpha}{n})\max\left\{1,\frac{p'}{q}\right\}}\|f\|_{L^p(w^p)},$$

and the estimate is sharp in the sense that the inequality does not hold if we replace the exponent of the $A_{p,q}$ constant by a smaller one.

The Calderón–Zygmund integral operators can be generalized by taking other regularity conditions of the kernel, for example the $L^{r'}$-Hörmander condition. This integral operators are controlled in the L^p-norm sense by the maximal operator M_r, defined by L^r-average. For more details see, for example, [21, 22]. The operator I_α can be generalized in an analogous way by adding an assumption of boundedness, as in [15], or adding some fractional size condition, as in [3].

Now we give the definitions of the fractional size and Hörmander conditions. First we introduce some notation. We set

$$\|f\|_{s,B} = \left(\frac{1}{|B|}\int_B |f|^s\right)^{1/s},$$

where B is a ball. Observe that in these averages the balls B can be replaced by cubes Q. The notation $|x| \sim t$ means $t < |x| \leq 2t$, and we write

$$\|f\|_{s,|x|\sim t} = \|f\chi_{|x|\sim t}\|_{s,B(0,2t)}.$$

Let $0 < \alpha < n$ and $1 \leq s \leq \infty$. The function g is said to satisfy the *fractional size condition* $S_{\alpha,s}$ if there exists a constant $C > 0$ such that

$$\|g\|_{s,|x|\sim t} \leq Ct^{\alpha-n}.$$

We say that $g \in S_{\alpha,\infty}$ if g satisfies the previous condition with $\|\cdot\|_{L^\infty,|x|\sim t}$ in place of $\|\cdot\|_{s,|x|\sim t}$. For $s = 1$, we write $S_{\alpha,s} = S_\alpha$. Observe that if $g \in S_\alpha$, then there exists a constant $c > 0$ such that

$$\int_{|x|\sim t} |g(x)|\,dx \leq ct^\alpha.$$

The function h satisfies the $L^{\alpha,s}$*-Hörmander condition* ($h \in H_{\alpha,s}$) if there exist $c_s > 1$ and $C_s > 0$ such that, for all x and $R > c_s|x|$,

$$\sum_{m=1}^{\infty} (2^m R)^{n-\alpha} \|h(\cdot - x) - h(\cdot)\|_{s,|y|\sim 2^m R} \leq C_s.$$

We say that $h \in H_{\alpha,\infty}$ if h satisfies the previous condition with $\|\cdot\|_{L^\infty,|x|\sim 2^m R}$ in place of $\|\cdot\|_{s,|x|\sim 2^m R}$. For $\alpha = 0$, we write $H_{0,s} = H_s$, the classical L^s-Hörmander condition.

In this paper we consider the following fractional operator. Let $0 < \alpha < n$, $1 \leq r < \infty$, and let r' be the conjugated exponent of r. Let K_α be a measurable function defined away from 0, such that $K_\alpha \in S_{\alpha,r'} \cap H_{\alpha,r'}$. For any $f \in L^\infty_c(dx)$, we consider the operator

$$T_\alpha f(x) = \int K_\alpha(x-y) f(y)\, dy. \tag{1.1}$$

Observe that we do not assume that the operator is bounded.

If $K_\alpha(x) = |x|^{\alpha-n}$, then $T_\alpha = I_\alpha$ (the fractional integral operator) and $K_\alpha \in S_{\alpha,\infty} \cap H_{\alpha,\infty}$.

Remark 1.2. Let $1 < r < p < n/\alpha$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If $f \in L^\infty_c(dx)$ and $w^r \in A_{\frac{p}{r},\frac{q}{r}}$, then $T_\alpha f \in L^q(w^q)$. This remark is a particular case of Lemma 5.1 in [14].

Remark 1.3. This type of operators also appears in several works, for example [3, 8, 11].

Remark 1.4. It can be considered that T_α is not of convolution type. In this case, we need the corresponding Hörmander and size condition in both variables. In this paper, we only consider the convolution type operator, and the general case follows in an analogous way, with the obvious changes.

An interesting example of a kernel is the following. Let us consider $L = -\Delta + V$ the Schrödinger operator on $\mathbb{R}^n$, $n \geq 3$, where V satisfies a reverse Hölder condition RH_q, with $\frac{n}{2} < q < n$, and let K be the kernel associated to the Riesz transform $L^{-1/2}\nabla$. It can be proved that $K \in S_{0,r'} \cap H_{r'}$ for some $1 < r' < \infty$, see details in [5, 10, 20]. We define $K_\alpha(x,y) = |x-y|^\alpha K(x,y)$; then by [3, Proposition 4.1], $K_\alpha \in S_{\alpha,r'} \cap H_{\alpha,r'}$.

For $0 \leq \alpha < n$, $1 \leq r < \infty$ and $f \in L^1_{\text{loc}}(dx)$, the *maximal operator* $M_{\alpha,r}$ is defined by

$$M_{\alpha,r} f(x) = \sup_{B \ni x} |B|^{\alpha/n} \|f\|_{r,B},$$

where the supremum is taken over all the balls B containing x.

A more general case of this type of operator has been studied by Kurtz [15]. The author defines the following class of kernel, $K(r,\alpha)$. Let $0 < \alpha < n$ and $\frac{n}{n-\alpha} \leq r < \infty$. We say $h \in K(r,\alpha)$ if the following conditions are met:

(1) There is a non-decreasing function S on $(0,1)$ such that

$$\|(h(\cdot - x) - h(\cdot))\chi_{|\cdot|\sim R}\|_r \leq S\left(\frac{|x|}{|R|}\right) R^{\alpha - \frac{n}{r'}}, \qquad |x| < \frac{R}{2}.$$

(2) The convolution operator T, $Tf = h * f$, is bounded from $L^{r'}$ into L^q, where $\frac{1}{q} = \frac{1}{r'} - \frac{\alpha}{n}$.
(3) Finally,

$$\sum_{j=1}^{\infty} S(2^{-j}) < \infty.$$

Theorem 1.5 ([15])**.** *Let $0 < \alpha < n$ and $1 \leq r < n/\alpha$. Let $K_\alpha \in K(r', \alpha)$ and suppose T_α is bounded from $L^r(dx)$ into $L^q(dx)$ for $\frac{1}{q} = \frac{1}{r} - \frac{\alpha}{n}$. Then there exists a constant $C > 0$ such that, for $f \in L^r_{\text{loc}}(dx)$,*

$$M^\sharp(T_\alpha f)(x) \leq C M_{\alpha,r} f(x),$$

where $M^\sharp$ is the classical sharp maximal function.

Theorem 1.6 ([15])**.** *Let $0 < \alpha < n$ and $1 \leq r < n/\alpha$. Let $K_\alpha \in K(r', \alpha)$ and suppose T_α is bounded from $L^s(dx)$ into $L^q(dx)$ for all (s, q) with $\frac{1}{q} = \frac{1}{s} - \frac{\alpha}{n}$ and $n/(n-\alpha) < s < r'$. If $r < p < n/\alpha$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $w^r \in A_{\frac{p}{r}, \frac{q}{r}}$, then there exists a constant $C_w > 0$, independent of f but depending on w, such that*

$$\|T_\alpha f\|_{L^q(w^q)} \leq C_w \|f\|_{L^p(w^p)}. \tag{1.2}$$

Remark 1.7. It is easy to see that if $K_\alpha \in K(r', \alpha)$, then $K_\alpha \in H_{\alpha,r'}$ and the operator $T_\alpha f = K_\alpha * f$ is bounded from $L^r(dx)$ into $L^q(dx)$ for $\frac{1}{q} = \frac{1}{r} - \frac{\alpha}{n}$.

More recently, in [3] the authors proved a version of Theorem 1.5 without assuming that the operator is bounded.

Theorem 1.8 ([3])**.** *Let $0 < \alpha < n$ and $1 < r < \infty$. Let T_α be defined as in* (1.1) *and let $K_\alpha \in S_{\alpha,r'} \cap H_{\alpha,r'}$. Then there exists $C > 0$ such that, for $f \in L^\infty_c$,*

$$M^\sharp(T_\alpha f)(x) \leq C M_{\alpha,r} f(x),$$

where $M^\sharp$ is the classical sharp maximal function.

Remark 1.9. This result is stated in a different way in [3]. The authors consider the operator $M^\sharp_\delta(T_\alpha f) = M^\sharp(|T_\alpha f|^\delta)^{\frac{1}{\delta}}$, where $0 < \delta < 1$. For the case $0 < \alpha < n$, we observe that the proof holds with no changes for $\delta = 1$, so we can write $M^\sharp$ instead of $M^\sharp_\delta$.

From this result and the good-λ technique, we get the following proposition.

Proposition 1.10. *Let $0 < \alpha < n$ and $1 \leq r < n/\alpha$. Let T be defined as in* (1.1) *and let $K_\alpha \in S_{\alpha,r'} \cap H_{\alpha,r'}$. Then there exists a constant $C_w > 0$, depending on w, such that, for $f \in L^\infty_c(dx)$ and $w^r \in A_{1, \frac{n}{n-\alpha r}}$,*

$$\sup_{\lambda > 0} \lambda^r w^{\frac{rn}{n-\alpha r}} \{x \in \mathbb{R}^n : |(T_\alpha f)(x)| > \lambda\}^{\frac{n-\alpha r}{n}} \leq C_w \int |f|^r w^r.$$

The idea of the proof is the same as the one given in [28, Theorem 3.6], so we omit it.

From this result we know that if $w^r \in A_{\frac{p}{r}, \frac{q}{r}}$, then T_α is bounded from $L^p(w^p)$ into $L^q(w^q)$, and the dependence of the w constant was known only in the case

$T_\alpha = I_\alpha$, Theorem 1.1. The main result in this paper is the dependence of the constant $[w^r]_{A_{\frac{p}{r},\frac{q}{r}}}$ in the inequality (1.2) for a class of operators given by a kernel K_α less regular than the one on I_α. These kernels satisfy a $L^{\alpha,r'}$-Hörmander condition. The result is the following:

Theorem 1.11. *Let $0<\alpha<n$ and let T_α be defined as in* (1.1). *Let $1 \le r < p < n/\alpha$, $1/q = 1/p - \alpha/n$. Suppose $K_\alpha \in S_{\alpha,r'} \cap H_{\alpha,r'}$. If $w^r \in A_{\frac{p}{r},\frac{q}{r}}$, then*

$$\|T_\alpha f\|_{L^q(w^q)} \le c_n [w^r]_{A_{\frac{p}{r},\frac{q}{r}}}^{\max\left\{1-\frac{\alpha}{n}, \frac{(p/r)'}{q}\left(1-\frac{\alpha r}{n}\right)\right\}} \|f\|_{L^p(w^p)}.$$

This estimate is sharp in the following sense:

Proposition 1.12. *Let $0<\alpha<n$, $1 \le r < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Let $K_\alpha \in S_{\alpha,r'} \cap H_{\alpha,r'}$ and let T_α be defined as in* (1.1). *If there exists an increasing function $\Phi : [1,\infty) \to (0,\infty)$ such that*

$$\|T_\alpha\|_{L^p(w^p)\to L^q(w^q)} \lesssim \Phi\left([w^r]_{A_{\frac{p}{r},\frac{q}{r}}}\right)$$

for all $w^r \in A_{\frac{p}{r},\frac{q}{r}}$, then

$$\Phi(t) \gtrsim t^{\max\left\{1-\frac{\alpha}{n}, \frac{(p/r)'}{q}\left(1-\frac{\alpha r}{n}\right)\right\}}.$$

Remark 1.13. In the case of the fractional integral operator I_α, $r' = \infty$, we obtain the same sharp bound as in [16].

Remark 1.14. For the singular integral operator with kernel $k \in H_{r'}$ ($\alpha = 0$), Li [20] gave the sparse domination. Following the same proof of Proposition 1.12 in Section 5, one can obtain the sharpness in this case.

The paper continues as follows: in the next section we present some particular operators as applications of these results. In Section 3 we give the sparse domination for T_α. In Section 4 we obtain the $L^p(w^p)$-$L^q(w^q)$ boundedness of the sparse operator with the dependence of the $[w^r]_{A_{\frac{p}{r},\frac{q}{r}}}$ constant. Finally, in Section 5 we give some examples to prove that the dependency of the constant given in Section 4 is optimal.

Throughout this paper, c and C will denote positive constants, not the same at each occurrence.

2. Applications

In this section, we give more examples of our results.

- Fractional rough operator:

  Let Ω be a function defined on S^{n-1}. We consider its extension to $\mathbb{R}^n \setminus \{0\}$, which is defined as $\Omega(x) = \Omega(x/|x|)$. Thus Ω is a homogeneous function of degree 0. For $1 \le s \le \infty$, the L^s-modulus of continuity of Ω is defined as

$$\bar{\omega}_s(t) = \sup_{|y|<t} \|\Omega(\cdot + y) - \Omega(\cdot)\|_{s,S^{n-1}}.$$

Let $0<\alpha<n$, $r'>\frac{n}{n-\alpha}$ and $\Omega\in L^{r'}(S^{n-1})$ such that $\int_0^1 \bar{\omega}_{r'}(t)\frac{dt}{t}<\infty$. Let

$$K_\alpha(x)=\frac{\Omega(x/|x|)}{|x|^{n-\alpha}}$$

and $T_\alpha f(x)=K_\alpha * f(x)$. It is proved in [3] that $K_\alpha\in H_{\alpha,r'}\cap S_{\alpha,r'}$. Since $r'>\frac{n}{n-\alpha}$, its conjugate exponent $r<n/\alpha$. Thus applying the main result, Theorem 1.11, we obtain that, for $1<r<p<n/\alpha$ and $1/q=1/p-\alpha/n$,

$$\|T_\alpha f\|_{L^q(w^q)}\leq c_n[w^r]_{A_{\frac{p}{r},\frac{q}{r}}}^{\max\left\{1-\frac{\alpha}{n},\frac{(p/r)'}{q}\left(1-\frac{\alpha r}{n}\right)\right\}}\|f\|_{L^p(w^p)}.$$

- Other kernels:

  Let $0<\alpha<1$, $\beta>0$, $1<r<p<1/\alpha$ and $\frac{1}{q}=\frac{1}{p}-\alpha$. For r' the conjugated exponent of r, let us consider

$$k(t)=\left(\frac{1}{t\log(e/t)^{1+\beta}}\right)^{1/r'}\chi_{(0,1)}(t).$$

  As shown in [14, 22], $k\in H_{r'}\cap S_{0,r'}$. Now, let

$$K_\alpha(t)=|t+4|^\alpha k(|t+4|);$$

  by [3, Proposition 4.1], $K_\alpha\in H_{\alpha,r'}\cap S_{\alpha,r'}$. Finally, let $T_\alpha f=K_\alpha * f$. Applying the main result, Theorem 1.11, we obtain, for $1<r<p<1/\alpha$ and $1/q=1/p-\alpha$,

$$\|T_\alpha f\|_{L^q(w^q)}\leq c_n[w^r]_{A_{\frac{p}{r},\frac{q}{r}}}^{\max\left\{1-\alpha,\frac{(p/r)'}{q}(1-\alpha r)\right\}}\|f\|_{L^p(w^p)}.$$

  For more details of the sharpness, see Subsection 5.2.

3. Sparse domination for T_α

In this section we present a sparse domination result for the operator T_α. Let us recall some well-known results.

A kernel K is said to belong to H_{Dini} if

$$|K(x)|\leq\frac{C_K}{|x|^n}$$

and

$$|K(x-y)-K(x'-y)|\leq\omega\left(\frac{|x-x'|}{|x-y|}\right)\frac{1}{|x-y|^n}$$

for $|x-y|>2|x-x'|$. The function $\omega:[0,1]\to[0,\infty)$ is continuous, increasing, submultiplicative with $\omega(0)=0$ and satisfies the Dini condition

$$\int_0^1\omega(t)\frac{dt}{t}<\infty.$$

Observe that

$$H_{\text{Dini}}\subset H_\infty\subset H_r\subset H_s\subset H_1,\quad 1<s<r<\infty.$$

In the case that T is a Calderón–Zygmund operator with $K \in H_{\mathrm{Dini}}$, the sparse domination was proved in [17]; for its commutators, in [19]; and for the vector-valued case, in [6]. The sparse domination for $K \in H_r$ was considered in [20] and, for K satisfying a Young type Hörmander condition, it was considered in [13]. Finally, for the case of I_α, the sparse domination was studied in [1]. It is possible to obtain a pointwise sparse domination that covers the general fractional operators that we are considering.

To state our result of sparse domination, we recall some definitions.

Given a cube $Q \in \mathbb{R}^n$, we denote by $\mathcal{D}(Q)$ the family of all dyadic cubes with respect to Q, that is, the cube obtained subdividing repeatedly Q and each of its descendants into 2^n subcubes of the same side lengths.

Given a dyadic family $\mathcal{D}$ we say that a family $\mathscr{S} \subset \mathcal{D}$ is an *η-sparse family*, with $0 < \eta < 1$, if, for every $Q \in \mathscr{S}$, there exists a measurable set $E_Q \subset Q$ such that $\eta|Q| \leq |E_Q|$ and the family $\{E_Q\}_{Q\in\mathscr{S}}$ are pairwise disjoint.

Theorem 3.1. *Let $0 < \alpha < n$, $1 \leq r < \infty$, and let T_α be defined as in* (1.1). *Suppose $K_\alpha \in S_{\alpha,r'} \cap H_{\alpha,r'}$. For any $f \in L^\infty_c(\mathbb{R}^n)$, there exist 3^n sparse families such that, for a.e. $x \in \mathbb{R}^n$,*

$$|T_\alpha f(x)| \leq c \sum_{j=1}^{3^n} \sum_{Q\in\mathscr{S}_j} |Q|^{\alpha/n} \|f\|_{r,Q} \chi_Q(x) =: c \sum_{j=1}^{3^n} \mathcal{A}^\alpha_{r,\mathscr{S}_j} f(x).$$

The *grand maximal truncated operator* M_{T_α} is defined by

$$M_{T_\alpha} f(x) = \sup_{Q\ni x} \operatorname*{ess\,sup}_{\xi\in Q} |T_\alpha(f\chi_{\mathbb{R}^n\setminus 3Q})(\xi)|,$$

where the supremum is taken over all the cubes $Q \subset \mathbb{R}^n$ containing x. For the proof of the preceding theorem we need to show that M_{T_α} maps $L^r(dx)$ into $L^{\frac{rn}{n-\alpha r},\infty}(dx)$. Also we need the following definitions:

- For a cube $Q_0 \subset \mathbb{R}^n$, a local version of M_{T_α} is defined as follows:
$$M_{T_\alpha,Q_0} f(x) = \sup_{x\in Q\subset Q_0} \operatorname*{ess\,sup}_{\xi\in Q} |T_\alpha(f\chi_{3Q_0\setminus 3Q})(\xi)|.$$
- Let $K_\alpha \in S_{\alpha,r'} \cap H_{\alpha,r'}$. We define
$$\tilde{T}_\alpha f(x) = \int |K_\alpha(x-y)| f(y)\, dy.$$
Observe that if $K_\alpha \in S_{\alpha,r'} \cap H_{\alpha,r'}$, then $|K_\alpha| \in S_{\alpha,r'} \cap H_{\alpha,r'}$ and Proposition 1.10 holds for $\tilde{T}_\alpha$.

Lemma 3.2. *Let $0 < \alpha < n$, $1 \leq r < \infty$, $K_\alpha \in S_{\alpha,r'} \cap H_{\alpha,r'}$ and let $Q_0 \subset \mathbb{R}^n$ be a cube. Let T_α be defined as in* (1.1) *and $f \in L^\infty_c(\mathbb{R}^n)$. Then,*

(1) *for a.e. $x \in Q_0$,*
$$|T_\alpha(f\chi_{3Q_0})(x)| \leq M_{T_\alpha,Q_0} f(x);$$
(2) *for all $x \in \mathbb{R}^n$,*
$$M_{T_\alpha}(f)(x) \lesssim M_{\alpha,r} f(x) + \tilde{T}_\alpha(|f|)(x).$$

From the last estimate and Proposition 1.10 it follows that M_{T_α} is bounded from $L^r(dx)$ into $L^{\frac{rn}{n-\alpha r},\infty}(dx)$.

Proof. (1) Let $Q(x,s)$ be a cube centered at x with side length s such that $Q(x,s) \subset Q_0$; then

$$|T_\alpha(f\chi_{3Q_0})(x)| \leq |T_\alpha(f\chi_{3Q(x,s)})(x)| + |T_\alpha(f\chi_{3Q_0\setminus 3Q(x,s)})(x)|.$$

For the first term, let us consider $B(x,R)$ with $R = 3\sqrt{n}s$; then $3Q(x,s) \subset B(x,R)$. As $K_\alpha \in S_{\alpha,r'}$, we have

$$\begin{aligned}
&|T_\alpha(f\chi_{3Q(x,s)})(x)| \\
&\quad \leq \int_{B(x,R)} |K_\alpha(x-y)||f(y)|\,dy \\
&\quad = \sum_{m=0}^{\infty} \frac{|B(x,2^{-m}R)|}{|B(x,2^{-m}R)|} \int_{B(x,2^{-m}R)} \chi_{B(x,2^{-m}R)\setminus B(x,2^{-m-1}R)} |K_\alpha(x-y)||f(y)|\,dy \\
&\quad \leq \sum_{m=0}^{\infty} |B(x,2^{-m}R)|\|K_\alpha\|_{r',|x|\sim 2^{-m-1}R}\|f\|_{r,B(x,2^{-m}R)} \\
&\quad \leq cM_r(f)(x)\sum_{m=0}^{\infty}(2^{-m}R)^n(2^{-m}R)^{\alpha-n} \\
&\quad = cM_r(f)(x)R^\alpha \sum_{m=0}^{\infty}(2^{-m})^\alpha = cM_r(f)(x)R^\alpha.
\end{aligned}$$

Then,

$$|T_\alpha(f\chi_{3Q_0})(x)| \leq c_n s^\alpha M_r f(x) + M_{T_\alpha,Q_0}f(x).$$

Observe that by hypothesis $M_r f < \infty$; then, letting $s \to 0$, we obtain the desired estimate.

(2) Let $x \in \mathbb{R}^n$ and let Q be a cube containing x. Let B_x be a ball with radius R such that $3Q \subset B_x$. For every $\xi \in Q$, we have

$$\begin{aligned}
|T_\alpha(f\chi_{\mathbb{R}^n\setminus 3Q})(\xi)| &\leq |T_\alpha(f\chi_{\mathbb{R}^n\setminus B_x})(\xi) - T_\alpha(f\chi_{\mathbb{R}^n\setminus B_x})(x)| \\
&\qquad + |T_\alpha(f\chi_{B_x\setminus 3Q})(\xi)| + |T_\alpha(f\chi_{\mathbb{R}^n\setminus B_x})(x)| \\
&\lesssim |T_\alpha(f\chi_{\mathbb{R}^n\setminus B_x})(\xi) - T_\alpha(f\chi_{\mathbb{R}^n\setminus B_x})(x)| \\
&\qquad + |T_\alpha(f\chi_{B_x\setminus 3Q})(\xi)| + \tilde{T}_\alpha(|f|)(x).
\end{aligned}$$

For the first term, as $K_\alpha \in H_{\alpha,r'}$, we get

$$\begin{aligned}
&|T_\alpha(f\chi_{\mathbb{R}^n\setminus B_x})(\xi) - T_\alpha(f\chi_{\mathbb{R}^n\setminus B_x})(x)| \\
&\quad \leq \int_{\mathbb{R}^n\setminus B_x} |K_\alpha(\xi-y) - K_\alpha(x-y)||f(y)|\,dy \\
&\quad = \sum_{m=1}^{\infty} \frac{|2^m B_x|}{|2^m B_x|} \int_{2^{m+1}B_x\setminus 2^m B_x} |K_\alpha(\xi-y) - K_\alpha(x-y)||f(y)|\,dy
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{m=1}^{\infty} (2^m R)^n \|K_\alpha(\xi - \cdot) - K_\alpha(x - \cdot)\|_{r',|y|\sim 2^m R} \|f\|_{r,2^{m+1}B_x} \\
&\leq \sum_{m=1}^{\infty} (2^m R)^{n-\alpha} \|K_\alpha(\xi - \cdot) - K_\alpha(x - \cdot)\|_{r',|y|\sim 2^m R} M_{\alpha,r} f(x) \\
&\leq c_r M_{\alpha,r} f(x).
\end{aligned}
$$

For the second term, observe that there exists $l \in \mathbb{N}$ such that $B(x, 2^{-l}R) \subset 3Q$; then, as $K_\alpha \in S_{\alpha,r'}$, we obtain

$$
\begin{aligned}
|T_\alpha(f\chi_{B_x\setminus 3Q})(\xi)| &\leq \int_{B_x\setminus 3Q} |K_\alpha(x-y)||f(y)|\,dy \\
&\leq \sum_{m=0}^{l-1} \int_{B(x,2^{-m}R)\setminus B(x,2^{-m-1}R)} |K_\alpha(x-y)||f(y)|\,dy \\
&\leq \sum_{m=0}^{l-1} |B(x,2^{-m}R)| \|K_\alpha\|_{r',|x|\sim 2^{-m-1}R} \|f\|_{r,B(x,2^{-m}R)} \\
&\leq c \sum_{m=0}^{l-1} (2^{-m}R)^n (2^{-m}R)^{\alpha-n} \|f\|_{r,B(x,2^{-m}R)} \\
&\leq c M_{\alpha,r} f(x).
\end{aligned}
$$

Finally, we get

$$
|T_\alpha(f\chi_{\mathbb{R}^n\setminus 3Q})(\xi)| \lesssim M_{\alpha,r} f(x) + \tilde{T}_\alpha(|f|)(x). \qquad \square
$$

The following lemma is the so-called 3^n dyadic lattices trick. This result was established in [18] and affirms the following:

Lemma 3.3 ([18])**.** *Given a dyadic family $\mathcal{D}$ there exist 3^n dyadic families $\mathcal{D}_j$ such that*

$$
\{3Q : Q \in \mathcal{D}\} = \bigcup_{j=1}^{3^n} \mathcal{D}_j,
$$

and, for every cube $Q \in \mathcal{D}$, we can find a cube R_Q in each $\mathcal{D}_j$ such that $Q \subset R_Q$ and $3l_Q = l_{R_Q}$.

Proof of Theorem 3.1. We claim that, for any cube $Q_0 \in \mathbb{R}^n$, there exists a $\frac{1}{2}$-sparse family $\mathcal{F} \subset \mathcal{D}(Q_0)$ such that, for a.e. $x \in Q_0$,

$$
|T_\alpha(f\chi_{3Q_0})(x)| \lesssim \sum_{Q\in\mathscr{F}} |3Q|^{\alpha/n} \|f\|_{r,3Q} \chi_Q(x). \tag{3.1}
$$

Suppose that we have already proved the claim (3.1). Let us take a partition of $\mathbb{R}^n$ by cubes Q_j such that $\mathrm{supp}(f) \subset 3Q_j$ for each j. We can do it as follows. We start with a cube Q_0 such that $\mathrm{supp}(f) \subset Q_0$ and cover $3Q_0 \setminus Q_0$ by $3^n - 1$ congruent cubes Q_j, with each of them satisfying $Q_0 \subset 3Q_j$. We do the same for $9Q_0 \setminus 3Q_0$ and so on. The union of all those cubes will satisfy the desired properties.

We apply the claim (3.1) to each cube Q_j. Then we have that since $\operatorname{supp}(f) \subset 3Q_j$ the following estimate holds for a.e. $x \in Q_j$:

$$|T_\alpha f(x)|\chi_{Q_j}(x) = |T_\alpha(f\chi_{3Q_j})(x)| \lesssim \sum_{Q\in\mathcal{F}_j} |3Q|^{\alpha/n}\|f\|_{r,3Q}\chi_Q(x),$$

where each $\mathcal{F}_j \subset \mathcal{D}(Q_j)$ is a $\frac{1}{2}$-sparse family. Taking $\mathcal{F} = \bigcup_j \mathcal{F}_j$, we have that $\mathcal{F}$ is a $\frac{1}{2}$-sparse family and, for a.e. $x \in \mathbb{R}^n$,

$$|T_\alpha f(x)| \lesssim \sum_{Q\in\mathcal{F}} |3Q|^{\alpha/n}\|f\|_{r,3Q}\chi_Q(x).$$

From Lemma 3.3 it follows that there exist 3^n dyadic families such that, for every cube Q of $\mathbb{R}^n$, there is a cube $R_Q \in \mathcal{D}_j$ for some j for which $3Q \subset R_Q$ and $|R_Q| \leq 3^n|3Q|$. Setting

$$\mathscr{S}_j = \{R_Q \in D_j : Q \in \mathcal{F}\},$$

and since $\mathcal{F}$ is $\frac{1}{2}$-sparse, we obtain that for each family $\mathscr{S}_j$ is $\frac{1}{2.9^n}$-sparse. Then we have that

$$|T_\alpha f(x)| \lesssim \sum_{j=1}^{3^n}\sum_{Q\in\mathscr{S}_j} |Q|^{\alpha/n}\|f\|_{r,Q}\chi_Q(x).$$

Proof of claim (3.1). To prove the claim it suffices to show the following recursive estimate: there exists a countable family $\{P_j\}_j$ of pairwise disjoint cubes in $\mathcal{D}(Q_0)$ such that $\sum_j P_j \leq \frac{1}{2}|Q_0|$ and

$$|T_\alpha(f\chi_{3Q_0})(x)|\chi_{Q_0}(x) \leq c|3Q_0|^{\alpha/n}\|f\|_{r,3Q_0}\chi_{Q_0}(x) + \sum_j |T_\alpha(f\chi_{3P_j})(x)|\chi_{P_j}(x) \tag{3.2}$$

for a.e. $x \in Q_0$. Iterating this estimate we obtain (3.1) with $\mathcal{F}$ being the union of all the families $\{P_j^k\}$ where $\{P_j^0\} = \{Q_0\}$, $\{P_j^1\} = \{P_j\}$ and the $\{P_j^k\}$ are the cubes obtained at the k-th stage of the iterative process. It is also clear that $\mathcal{F}$ is a $\frac{1}{2}$-sparse family. Indeed, for each P_j^k, it suffices to choose

$$E_{P_j^k} = P_j^k \setminus \bigcup_j P_j^{k+1}.$$

Let us prove the recursive estimate (3.2). Observe that, for any family $\{P_j\} \subset \mathcal{D}(Q_0)$ of disjoint cubes, we have

$$\begin{aligned}
&|T_\alpha(f\chi_{3Q_0})(x)|\chi_{Q_0}(x)\\
&\qquad\leq |T_\alpha(f\chi_{3Q_0})(x)|\chi_{Q_0\setminus\cup_j P_j}(x) + \sum_j |T_\alpha(f\chi_{3Q_0})(x)|\chi_{P_j}(x)\\
&\qquad\leq |T_\alpha(f\chi_{3Q_0})(x)|\chi_{Q_0\setminus\cup_j P_j}(x) + \sum_j |T_\alpha(f\chi_{3Q_0\setminus 3P_j})(x)|\chi_{P_j}(x)\\
&\qquad\qquad + \sum_j |T_\alpha(f\chi_{3P_j})(x)|\chi_{P_j}(x)
\end{aligned}$$

for a.e. $x \in \mathbb{R}^n$. So it suffices to show that we can choose a countable family $\{P_j\}_j$ of pairwise disjoint cubes in $\mathcal{D}(Q_0)$ such that $\sum_j P_j \leq \frac{1}{2}|Q_0|$ and, for a.e. $x \in Q_0$, we have

$$|T_\alpha(f\chi_{3Q_0})(x)|\chi_{Q_0\setminus\cup_j P_j}(x) + \sum_j |T_\alpha(f\chi_{3Q_0\setminus 3P_j})(x)|\chi_{P_j}(x) \lesssim |3Q_0|^{\alpha/n}\|f\|_{r,3Q_0}\chi_{Q_0}(x). \quad (3.3)$$

Now we define the following set:

$$E = \{x \in Q_0 : M_{T_\alpha,Q_0}f(x) > \beta_n c|3Q_0|^{\alpha/n}\|f\|_{r,3Q_0}\}.$$

By Lemma 3.2 we can choose β_n such that $|E| \leq \frac{1}{2^{n+2}}|Q_0|$.

We apply the Calderón–Zygmund decomposition to the function χ_E on Q_0 at height $\lambda = \frac{1}{2^{n+1}}$. Then, there exists a family $\{P_j\} \subset \mathcal{D}(Q_0)$ of pairwise disjoint cubes such that

$$\left\{x \in Q_0 : \chi_E(x) > \frac{1}{2^{n+1}}\right\} = \bigcup_j P_j.$$

From this it follows that $|E \setminus \cup_j P_j| = 0$,

$$\sum_j |P_j| \leq 2^{n+1}|E| \leq \frac{1}{2}|Q_0|,$$

and

$$\frac{1}{2^{n+1}} \leq \frac{|P_j \cap E|}{|P_j|} \leq \frac{1}{2},$$

from which we obtain $|P_j \cap E^c| > 0$.

Since $P_j \cap E^c \neq \emptyset$, we have $M_{T_\alpha,Q_0}(f)(x) \leq \beta_n c|3Q_0|^{\alpha/n}\|f\|_{r,3Q_0}$ for some $x \in P_j$ and this implies that

$$\operatorname*{ess\,sup}_{\xi \in P_j} |T_\alpha(f\chi_{3Q_0\setminus 3P_j})(\xi)| \leq \beta_n c|3Q_0|^{\alpha/n}\|f\|_{r,3Q_0},$$

which allows us to control the second term in (3.3).

By (1) in Lemma 3.2, for a.e. $x \in Q_0$, we have

$$|T_\alpha(f\chi_{3Q_0})(x)|\chi_{Q_0\setminus\cup_j P_j}(x) \leq M_{T_\alpha,Q_0}f(x)\chi_{Q_0\setminus\cup_j P_j}(x).$$

Since $|E \setminus \cup_j P_j| = 0$ and by the definition of E, we obtain, for a.e. $x \in Q_0 \setminus \cup_j P_j$,

$$M_{T_\alpha,Q_0}(f)(x) \leq \beta_n c|3Q_0|^{\alpha/n}\|f\|_{r,3Q_0}.$$

Then, for a.e. $x \in Q_0 \setminus \cup_j P_j$, we get

$$|T_\alpha(f\chi_{3Q_0})(x)| \leq \beta_n c|3Q_0|^{\alpha/n}\|f\|_{r,3Q_0}.$$

Thus we obtain the estimate in (3.3). $\square$

4. Sharp bounds for norm inequality

Since the sparse domination is a pointwise estimate, it suffices to prove Theorem 1.11 and Proposition 1.12 for the sparse operator $A^{\alpha}_{r,\mathscr{S}}$ for any sparse family $\mathscr{S}$.

Theorem 4.1. *Let $0 \le \alpha < n$, $1 \le r < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. If $w^r \in A_{\frac{p}{r},\frac{q}{r}}$, then*

$$\|A^{\alpha}_{r,\mathscr{S}} f\|_{L^q(w^q)} \le c_n [w^r]_{A_{p/r,q/r}}^{\max\left\{1-\frac{\alpha}{n},\frac{(p/r)'}{q}\left(1-\frac{\alpha r}{n}\right)\right\}} \|f\|_{L^p(w^p)}.$$

This estimate is sharp in the following sense:

If there exists an increasing function $\Phi : [1,\infty) \to (0,\infty)$ such that

$$\|A^{\alpha}_{r,\mathscr{S}}\|_{L^p(w^p)\to L^q(w^q)} \lesssim \Phi\left([w^r]_{A_{\frac{p}{r},\frac{q}{r}}}\right)$$

for all $w^r \in A_{\frac{p}{r},\frac{q}{r}}$, then

$$\Phi(t) \gtrsim t^{\max\left\{1-\frac{\alpha}{n},\frac{(p/r)'}{q}\left(1-\frac{\alpha r}{n}\right)\right\}}.$$

Remark 4.2. The first approximation of this type for the fractional integral operator, using the sparse technique, appears in [7]. In this paper the author does not prove the sharpness of the constant. In the case $r = 1$, the appropriate sparse operator for the fractional integral operator I_α, we obtain the same sharp bound as in [1]. If $\alpha = 0$, we get the same sharp bound as in [17].

We consider the following sparse operator defined in [9], for $\mathscr{S}$ a sparse family, $0 < s < \infty$ and $0 < \beta \le 1$:

$$\tilde{A}^{\beta}_{s,\mathscr{S}} g(x) = \left(\sum_{Q\in\mathscr{S}} \left(|Q|^{-\beta}\int_Q g\right)^s \chi_Q(x)\right)^{1/s}.$$

Theorem 4.3 ([9])**.** *Let $1 \le r < p \le q < \infty$, and $0 < \beta \le 1$. Let us consider the weights $u, \sigma \in A_\infty$. The sparse operator $\tilde{A}^{\beta}_{s,\mathscr{S}}(\cdot\sigma)$ maps $L^p(\sigma) \to L^q(u)$ if and only if the two-weight $A^{\beta}_{p,q}$-characteristic*

$$[u,\sigma]_{A^{\beta}_{p,q}(\mathscr{S})} := \sup_{Q\in\mathscr{S}} |Q|^{-\beta} u(Q)^{1/q} \sigma(Q)^{1/p'}$$

is finite, and in this case,

$$1 \le \frac{\|\tilde{A}^{\beta}_{s,\mathscr{S}}(\cdot\sigma)\|_{L^p(\sigma)\to L^q(u)}}{[u,\sigma]_{A^{\beta}_{p,q}(\mathscr{S})}} \lesssim [\sigma]_{A_\infty}^{1/q} + [u]_{A_\infty}^{\frac{1}{s}-\frac{1}{p}}.$$

Proof of Theorem 4.1. Let $\sigma = w^{-(p/r)'r}$. Observe that

$$A^{\alpha}_{r,\mathscr{S}}(f) = \left(\tilde{A}^{1-\alpha/n}_{1/r,\mathscr{S}}(f^r)\right)^{1/r}.$$

Then,

$$\begin{aligned}\left\|A^{\alpha}_{r,\mathscr{S}}(f)\right\|^{r}_{L^q(w^q)} &= \left\|\tilde{A}^{1-\alpha/n}_{1/r,\mathscr{S}}(f^r)\right\|_{L^{q/r}(w^q)} = \left\|\tilde{A}^{1-\alpha/n}_{1/r,\mathscr{S}}(f^r\sigma^{-1}\sigma)\right\|_{L^{q/r}(w^q)} \\ &\lesssim [w^q,\sigma]_{A^{1-\alpha/n}_{\frac{p}{r},\frac{q}{r}}(\mathscr{S})}\left([\sigma]^{r/q}_{A_\infty} + [w^q]^{r-\frac{r}{p}}_{A_\infty}\right)\|f^r\sigma^{-1}\|_{L^{p/r}(\sigma)}.\end{aligned}$$

Now observe that

$$[w^q,\sigma]_{A^{1-\alpha/n}_{\frac{p}{r},\frac{q}{r}}(\mathscr{S})} \le [w^r]^{r/q}_{A_{p/r,q/r}}$$

and

$$\|f^r\sigma^{-1}\|_{L^{p/r}(\sigma)} = \|f\|^r_{L^p(w^p)}.$$

Since

$$[\sigma]_{A_{1+(p/r)'r/q}} = [w^r]^{(p/r)'r/q}_{A_{p/r,q/r}} \qquad \text{and} \qquad [w^q]_{A_{1+\frac{q}{r(p/r)'}}} = [w^r]_{A_{p/r,q/r}},$$

we have

$$\begin{aligned}\|A^{\alpha}_{r,\mathscr{S}}(f)\|_{L^q(w^q)} &\lesssim [w^r]^{1/q}_{A_{p/r,q/r}}\left([\sigma]^{r/q}_{A_\infty} + [w^q]^{r-\frac{r}{p}}_{A_\infty}\right)^{1/r}\|f\|_{L^p(w^p)} \\ &\le [w^r]^{1/q}_{A_{p/r,q/r}}\left([w^r]^{(p/r)'(r/q)^2}_{A_{p/r,q/r}} + [w^r]^{\frac{r}{p'}}_{A_{p/r,q/r}}\right)^{1/r}\|f\|_{L^p(w^p)} \\ &\le [w^r]^{1/q+\max\{(p/r)'r/q^2,\frac{1}{p'}\}}_{A_{p/r,q/r}}\|f\|_{L^p(w^p)} \\ &\le [w^r]^{\max\{(1-\frac{\alpha r}{n})(p/r)'/q,1-\alpha/n\}}_{A_{p/r,q/r}}\|f\|_{L^p(w^p)},\end{aligned}$$

where the last inequality holds since $(1+(p/r)'r/q) = \left(1-\frac{\alpha r}{n}\right)(p/r)'$ and $1/q + 1/p' = 1-\alpha/n$. □

5. Examples

5.1. **Sparse operator $A^{\alpha}_{r,\mathscr{S}}$.** In this subsection, we prove the sharpness of Theorem 4.1.

Proof. Let $A = A^{\alpha}_{r,\mathscr{S}}$ be the sparse operator. Let $0<\varepsilon<1$. If

$$w_\varepsilon(x) = |x|^{\frac{n-\varepsilon}{r(p/r)'}} \qquad \text{and} \qquad f(x) = |x|^{\frac{\varepsilon-n}{r}}\chi_{B(0,1)},$$

then

$$[w^r_\varepsilon]_{A_{\frac{p}{r},\frac{q}{r}}} \simeq \varepsilon^{-\frac{q}{r(p/r)'}} \qquad \text{and} \qquad \|fw_\varepsilon\|_{L^p} \simeq \varepsilon^{-1/p}.$$

Let $\{Q_k\}$ be the cube of center 0 and length 2^{-k} and observe that $B(0,1)\subset Q_0$. This family $\{Q_k\}$ is a $\frac{1}{2}$-sparse family with $E_{Q_k} = Q_k\setminus Q_{k+1}$.

Now, if $x\in E_{Q_k}$, $k\in\mathbb{N}$, then we have that

$$\begin{aligned}Af(x) &\ge |Q_k|^{\alpha/n-1/r}\left(\int_{Q_k}|y|^{\varepsilon-n}\right)^{1/r} \gtrsim (2^{-kn})^{\alpha/n-1/r}\left(\frac{2^{-k\varepsilon}}{\varepsilon}\right)^{1/r} \\ &\gtrsim \varepsilon^{-1/r}|x|^{\alpha-n/r+\varepsilon/r}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int Af^q w_\varepsilon^q &\geq \sum_{k=1}^{\infty} \int_{E_{Q_k}} Af^q w_\varepsilon^q \\ &\gtrsim \varepsilon^{-q/r} \int_{B(0,\frac{1}{2})} |x|^{q(\alpha-n/r+\varepsilon/r)+q\frac{n-\varepsilon}{r(p/r)'}}\,dx \\ &\simeq \varepsilon^{-q/r-1}\end{aligned}$$

since $q(\alpha/n-1/r+\varepsilon/r)+q\frac{n-\varepsilon}{r(p/r)'} \leq \varepsilon q/p-n$. Then

$$\varepsilon^{-\frac{1}{r(p/r)'}-1/q} \lesssim \varepsilon^{-1/r-1/q+1/p} \lesssim \|A\|_{L^p(w_\varepsilon^p)\to L^q(w_\varepsilon^q)} \lesssim \Phi\Big(\varepsilon^{-\frac{q}{r(p/r)'}}\Big).$$

Now, taking $t=\varepsilon^{-\frac{q}{r(p/r)'}}$, we obtain

$$\Phi(t) \gtrsim t^{(p/r)'1/q(1-\alpha r/n)}.$$

Let $0<\varepsilon<1$. If

$$w_\varepsilon(x)=|x|^{\frac{\varepsilon-n}{q}} \qquad \text{and} \qquad f(x)=|x|^{\frac{\varepsilon-n}{r}}\chi_{B(0,1)},$$

then

$$[w_\varepsilon^r]_{A_{\frac{p}{r},\frac{q}{r}}} \simeq \varepsilon^{-1} \qquad \text{and} \qquad \|fw_\varepsilon\|_{L^p} \lesssim \varepsilon^{-1/p}.$$

Since $1/r+1/q=1/r-\alpha/n+1/p\geq 1/p$,

$$\begin{aligned}\int f^p w_\varepsilon^p = \int_{B(0,1)} |x|^{\frac{\varepsilon-n}{r}p+\frac{\varepsilon-n}{q}p} &= \int_{B(0,1)} |x|^{p(\varepsilon(1/r+1/q)-n(1/r+1/q))} \\ &\leq \int_{B(0,1)} |x|^{p(\varepsilon(1/r+1/q)-n/p)} \simeq \varepsilon^{-1}.\end{aligned}$$

Now, if $x\in Q_0$,

$$Af(x) \geq |Q_0|^{\alpha/n-1/r}\left(\int_{Q_0}|y|^{\varepsilon-n}\right)^{1/r} \gtrsim \left(\frac{1}{\varepsilon}\right)^{1/r} \simeq \varepsilon^{-1/r} \gtrsim \varepsilon^{-1}.$$

Since $B(0,1)\subset Q_0$, we get

$$\int Af^q w_\varepsilon^q \gtrsim \varepsilon^{-q}\int_{B(0,1)}|x|^{\varepsilon-n}\,dx \gtrsim \varepsilon^{-q-1};$$

then,

$$\varepsilon^{-1-1/q} \lesssim \|Af\|_{L^q(w_\varepsilon^q)} \lesssim \Phi(\varepsilon^{-1})\|f\|_{L^p(w_\varepsilon^p)} \lesssim \Phi(\varepsilon^{-1})\varepsilon^{-1/p}.$$

Now, if we take $t=\varepsilon^{-1}$, then

$$t^{1-\alpha/n} \lesssim \Phi(t). \qquad \square$$

5.2. **An operator T_α.** In this subsection, we give an example of an operator to prove the sharpness of the Proposition 1.12.

Proof of Proposition 1.9. Let $0<\alpha<1$, $\beta>\max\{0,q/r'-1\}$, $1<r<p<1/\alpha$, $\frac{1}{q}=\frac{1}{p}-\alpha$ and $\frac{1}{r}+\frac{1}{r'}=1$. Let us consider

$$k(t)=\left(\frac{1}{t\log(e/t)^{1+\beta}}\right)^{1/r'}\chi_{(0,1)}(t).$$

As shown in [14, 22], we know that $k\in H_{r'}\cap S_{r'}$. Now, let

$$K_\alpha(t)=|t+4|^\alpha k(|t+4|);$$

by [3, Proposition 4.1], $K_\alpha\in H_{\alpha,r'}\cap S_{\alpha,r'}$. Let us consider $T_\alpha f=K_\alpha * f$.

Observe that there exists $0<t_0<1$ such that k is decreasing in $(0,t_0)$.

Let $0<\varepsilon<1$. If $w_\varepsilon(x)=|x|^{\frac{1-\varepsilon}{r(p/r)'}}$ and $f(x)=|x+4|^{\frac{\varepsilon}{r}-1}\chi_{(-5,-3)}(x)$, then

$$[w_\varepsilon^r]_{A_{\frac{p}{r},\frac{q}{r}}}\simeq\varepsilon^{-\frac{q}{r(p/r)'}}\qquad\text{and}\qquad\|fw_\varepsilon\|_{L^p}\lesssim\varepsilon^{-1/p}.$$

Observe that if $|x-y|\le 1$ and $|y|\le 1$, then $|x|\le 2$ and $\operatorname{supp}(Tf)\subset[-2,2]$. Let $x\in\operatorname{supp}(Tf)$, $|x-y|\le 1$, and $0\le|y|\le|x|/2$; then $\frac{1}{2}|x|\le|x-y|\le\frac{3}{2}|x|$ and $|x-y|^\alpha\gtrsim|x|^\alpha$.

For $|x|\le\frac{2}{3}t_0\le 2$, since k is decreasing in $(0,t_0)$, we have

$$\begin{aligned}T_\alpha f(x)&\ge\int_{|y|\le|x|/2}|x-y|^{\alpha-1/r'}\left(\frac{1}{\log(e/|x-y|)}\right)^{\frac{1+\beta}{r'}}\chi_{(0,1)}(|x-y|)|y|^{\frac{\varepsilon}{r}-1}\,dy\\&\gtrsim|x|^\alpha k\left(\frac{3}{2}|x|\right)\int_{|y|\le|x|/2}|y|^{\frac{\varepsilon}{r}-1}\,dy\\&\gtrsim\varepsilon^{-1}|x|^{\alpha+\frac{\varepsilon}{r}}k\left(\frac{3}{2}|x|\right).\end{aligned}$$

Then, using that $\log(t)<\frac{t^\varepsilon}{\varepsilon}$ for $\varepsilon>0$ and $t>1$, we get

$$\begin{aligned}\int_{\mathbb{R}}|Tf(x)|^q w_\varepsilon^q(x)\,dx&\gtrsim\varepsilon^{-q}\int_{|x|\le\frac{2}{3}t_0}|x|^{q\left(\alpha+\frac{\varepsilon}{r}\right)}k\left(\frac{3}{2}|x|\right)^q|x|^{q\left(\frac{1-\varepsilon}{r(p/r)'}\right)}\,dx\\&=\varepsilon^{-q}\int_{|x|\le\frac{2}{3}t_0}k\left(\frac{3}{2}|x|\right)^q|x|^{\frac{q}{r}-1}|x|^{\varepsilon\frac{q}{p}}\,dx\\&\gtrsim\varepsilon^{-q}\int_{|x|\le\frac{2}{3}t_0}\left(\frac{|x|^\varepsilon}{\varepsilon}\right)^{\frac{q}{r'}(1+\beta)}|x|^{\frac{q}{r}+\varepsilon\frac{q}{p}-1-\frac{q}{r'}}\,dx\\&\gtrsim\varepsilon^{-q-1}\int_{|x|\le\frac{2}{3}t_0}|x|^{\frac{q}{r}-\frac{q}{r'}+\varepsilon\left(\frac{q}{r'}(1+\beta)+\frac{q}{p}\right)-1}\,dx.\end{aligned}$$

The last inequality holds since $\beta>\max\{0,q/r'-1\}$.

If $r\ge 2$,

$$\frac{q}{r}-\frac{q}{r'}=q\left(\frac{1}{r}-\frac{1}{r'}\right)\le 0.$$

Then

$$\begin{aligned}\int_{\mathbb{R}} |Tf(x)|^q w_\varepsilon^q(x)\,dx &\gtrsim \varepsilon^{-q-1} \int_{|x|\le \frac{2}{3}t_0} |x|^{\frac{q}{r}-\frac{q}{r'}+\varepsilon\left(\frac{q}{r'}(1+\beta)+\frac{q}{p}\right)-1}\,dx \\ &\gtrsim \varepsilon^{-q-1} \int_{|x|\le \frac{2}{3}t_0} |x|^{\varepsilon\left(\frac{q}{r'}(1+\beta)+\frac{q}{p}\right)-1}\,dx \\ &\gtrsim \varepsilon^{-q-2} \ge \varepsilon^{-q-1}.\end{aligned}$$

If $r < 2$,

$$\frac{q}{r} - \frac{q}{r'} = q\left(\frac{1}{r} - \frac{1}{r'}\right) > 0.$$

Then

$$\begin{aligned}\int_{\mathbb{R}} |Tf(x)|^q w_\varepsilon^q(x)\,dx &\gtrsim \varepsilon^{-q-1} \int_{|x|\le \frac{2}{3}t_0} |x|^{\frac{q}{r}-\frac{q}{r'}+\varepsilon\left(\frac{q}{r'}(1+\beta)+\frac{q}{p}\right)-1}\,dx \\ &\gtrsim \varepsilon^{-q-1} \frac{t_0^{\frac{q}{r}-\frac{q}{r'}+\varepsilon\left(\frac{q}{r'}(1+\beta)+\frac{q}{p}\right)}}{\frac{q}{r}-\frac{q}{r'}+\varepsilon\left(\frac{q}{r'}(1+\beta)+\frac{q}{p}\right)} \\ &\gtrsim \varepsilon^{-q-1}.\end{aligned}$$

Then, for $1 < r < \infty$, we obtain

$$\int_{\mathbb{R}} |Tf(x)|^q w_\varepsilon^q(x)\,dx \gtrsim \varepsilon^{-q-1}.$$

Therefore,

$$\|Tf\|_{L^q(w_\varepsilon^q)} \gtrsim \varepsilon^{-1-1/q}.$$

Then,

$$\varepsilon^{-\frac{1}{r(p/r)'}-1/q} \lesssim \varepsilon^{-1-1/q+1/p} \lesssim \|T\|_{L^p(w_\varepsilon^p)\to L^q(w_\varepsilon^q)} \lesssim \Phi(\varepsilon^{-\frac{q}{r(p/r)'}})$$

and

$$\Phi(t) \gtrsim t^{(p/r)'1/q(1-\alpha r)}.$$

Let $0 < \varepsilon < 1$. If $w_\varepsilon(x) = |x|^{\frac{\varepsilon-1}{q}}$ and $f(x) = |x+4|^{\varepsilon/r-1}\chi_{(-5,-3)}$, then

$$[w_\varepsilon^r]_{A_{\frac{p}{r},\frac{q}{r}}} \simeq \varepsilon^{-1} \quad \text{and} \quad \|fw_\varepsilon\|_{L^p} \lesssim \varepsilon^{-1/p}.$$

In an analogous way we obtain, for $0 < |x| < \frac{2}{3}t_0$, $Tf(x) \gtrsim \varepsilon^{-1}|x|^{\alpha+\frac{\varepsilon}{r}}k\left(\frac{3}{2}|x|\right)$ and

$$\|Tf\|_{L^q(w_\varepsilon^q)} \gtrsim \varepsilon^{-1-1/q}.$$

Hence

$$t^{1-\alpha/n} \lesssim \Phi(t). \qquad \square$$

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