REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 63, No. 2, 2022, Pages 305–316
Published online: August 17, 2022
https://doi.org/10.33044/revuma.2247

BINOMIAL EDGE IDEALS OF COGRAPHS

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Abstract. We determine the Castelnuovo–Mumford regularity of binomial edge ideals of complement-reducible graphs (cographs). For cographs with n vertices the maximum regularity grows as $2n/3$. We also bound the regularity by graph-theoretic invariants and construct a family of counterexamples to a conjecture of Hibi and Matsuda.

1. Introduction

Let $G = ([n], E)$ be a simple undirected graph on the vertex set $[n] = \{1, \dots, n\}$. Let $X = \left(\begin{smallmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{smallmatrix}\right)$ be a generic $2 \times n$ matrix and let $S = \Bbbk\left[\begin{smallmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{smallmatrix}\right]$ be the ring of polynomials whose indeterminates are the entries of X and with coefficients in a field $\Bbbk$. The *binomial edge ideal of G* is $\mathrm{J}_G = \langle x_i y_j - y_i x_j : \{i, j\} \in E \rangle \subseteq S$, the ideal of 2×2 minors indexed by the edges of the graph. Since their inception in [5, 14], connecting combinatorial properties of G with algebraic properties of J_G or S/J_G has been a popular activity. Particular attention has been paid to the minimal free resolution of S/J_G as a standard $\mathbb{N}$-graded S-module [3, 11]. The data of a minimal free resolution is encoded in its graded Betti numbers $\beta_{i,j}(S/\mathrm{J}_G) = \dim_\Bbbk \mathrm{Tor}_i(S/\mathrm{J}_G, \Bbbk)_j$. An interesting invariant is the highest degree appearing in the resolution, the Castelnuovo–Mumford regularity $\mathrm{reg}(S/\mathrm{J}_G) = \max\{j - i : \beta_{ij}(S/\mathrm{J}_G) \neq 0\}$. It is a complexity measure, as low regularity implies favorable properties like vanishing of local cohomology. Binomial edge ideals have square-free initial ideals by [5, Theorem 2.1] and, using [1], this implies that the extremal Betti numbers and regularity can also be derived from those initial ideals. In this paper we rely on recursive constructions of graphs rather than Gröbner deformations.

At the time of writing it is unknown if the regularity of S/J_G depends on the characteristic $\mathrm{char}(\Bbbk)$ of the coefficient field. Indication that it is indeed independent comes, for example, from a purely combinatorial description of the linear strand of the determinantal facet ideals in [6]. Here we use only combinatorial constructions that are independent of $\mathrm{char}(\Bbbk)$, based on small graphs for which the

2020 *Mathematics Subject Classification.* Primary 05E40; Secondary 13D02, 05C99, 13P20.

Key words and phrases. Binomial edge ideal; undirected graph; Castelnuovo–Mumford regularity; Betti numbers.

Supported by the Deutsche Forschungsgemeinschaft (314838170, GRK 2297 MathCoRe).

minimal free resolutions are independent of $\Bbbk$. Our starting point are the following bounds due to Matsuda and Murai [13], which are valid independently of $\operatorname{char}(\Bbbk)$.

Theorem 1.1. *Let ℓ be the maximum length of an induced path in a graph G. Then*

$$\ell \leq \operatorname{reg}(S/\operatorname{J}_G) \leq n-1.$$

Our aim is to investigate families of graphs for which the lower bound is constant. We study the family of graphs with no induced path of length 3. These are the *complement-reducible graphs* (*cographs*). They have been characterized in [2, Theorem 2] as graphs such that for every connected induced subgraph with at least two vertices, the complement of that subgraph is disconnected. Cographs are hereditary in the sense that every induced subgraph of a cograph is a cograph [2, Lemma 1]. The graphs G for which $\operatorname{reg}(S/\operatorname{J}_G) \leq 2$ are all cographs by the ($\operatorname{char}(\Bbbk)$-independent) characterization in [16, Theorem 3.2].

One can quickly find that, among (connected) cographs, arbitrarily high regularity is possible (Proposition 2.2), but when the regularity is considered as a function of the number of vertices of the cograph, there is a stricter upper bound than that in Theorem 1.1. Our Theorem 2.7 shows that for cographs the regularity is essentially bounded by $2n/3$. The experimental results and also the proof methods leading to the $2n/3$ bound lead us to study regularity bounds in terms of other graph invariants. Theorem 2.12 bounds $\operatorname{reg}(S/\operatorname{J}_G)$ by the independence number $\alpha(G)$ and the number of maximal independent sets $\operatorname{s}(G)$. This stands in the context of a general upper bound of $\operatorname{reg}(S/\operatorname{J}_G)$ by the number of maximal cliques $\operatorname{c}(G)$, shown by Malayeri, Madani and Kiani [15]. Interestingly, for cographs the mentioned invariants are computable in linear time, while in general they are hard to compute.

The investigations for this paper started with a large experiment in which we tabulated properties of binomial edge ideals and the corresponding graphs for many small graphs. To this end we developed a database of algebraic properties of S/J_G and the necessary tools to extend the database. Our source code is available at

https://github.com/kruesemann/graph_ideals.

In the course of this work we found a counterexample to a conjecture of Hibi and Matsuda. We present this in Section 3, which also concerns graphs other than cographs.

Notation. All graphs in this paper are simple and undirected, meaning that they have only undirected edges with no loops and no multiple edges between two fixed vertices. We often consider binomial edge ideals J_G and J_H of graphs on different vertex sets $V(G) \neq V(H)$. In this case we can consider both ideals in a bigger ring with variables corresponding to $V(G) \cup V(H)$. Embedding the ideals in a ring with extra variables has no effect on the invariants under consideration and we still write J_G and J_H independently of the ambient ring. By the *regularity of G* we mean $\operatorname{reg}(S/\operatorname{J}_G)$. The *complement* $\overline{G}$ of a graph G is a graph on the same vertex set, but with the edge set $E(\overline{G}) = \{(i,j) : i \neq j, (i,j) \notin E(G)\}$. The *path of length $\ell - 1$* has ℓ vertices and is denoted by P_ℓ.

2. Regularity for cographs

Our first aim is to show that $\operatorname{reg}(S/\operatorname{J}_G)$ can take arbitrarily large values, even if G is restricted to connected cographs and the lower bound of Theorem 1.1 does not apply. If one allows disconnected graphs, this follows from the simple observation that regularity is additive under the disjoint union $G \sqcup H$ of two graphs:

$$\operatorname{reg}(S/\operatorname{J}_{G\sqcup H}) = \operatorname{reg}(S/\operatorname{J}_G) + \operatorname{reg}(S/\operatorname{J}_H).$$

As noted in [3, proof of Theorem 2.2], this follows from the fact that J_G and J_H use disjoint sets of variables. In particular, this is true independently of $\Bbbk$. To see that arbitrary regularity is possible for connected cographs, we employ the *join* of two simple undirected graphs G and H, which is

$$G * H = (V(G) \sqcup V(H), E(G) \cup E(H) \cup \{\{v,w\} : v \in V(G), w \in V(H)\}).$$

Here and in the following, $V(G)$ and $E(G)$ denote, respectively, the vertex set and the edge set of an undirected simple graph G. The join also behaves nicely with regularity as shown by Kiani and Madani [16, Theorem 2.1].

Theorem 2.1. *Let G and H be simple, undirected graphs and not both complete. Then, independently of* $\operatorname{char}(\Bbbk)$*,*

$$\operatorname{reg}(S/\operatorname{J}_{G*H}) = \max\{\operatorname{reg}(S/\operatorname{J}_G), \operatorname{reg}(S/\operatorname{J}_H), 2\}.$$

The join of two cographs is a cograph as the path P_4 of length 3 is not a join itself. It follows that the regularity can be made arbitrarily high by forming *cones*, that is, joins with single vertex graphs.

Proposition 2.2. *For any $r \geq 1$ there is a connected cograph with* $\operatorname{reg}(S/\operatorname{J}_G) = r$*. If r is even, then there is such a graph with $\frac{3}{2}r + 1$ vertices. If r is odd, then there is such a graph with $\frac{3}{2}(r+3)$ vertices.*

Proof. For $r = 1$ a single edge suffices. If $r = 2a$ is even, let G be a disjoint union of a copies of P_3. If $3 \leq r = 2a+1$ is odd, then let G be a disjoint union of a copies of P_3 and a single edge P_2. In both cases, $\operatorname{reg}(S/\operatorname{J}_G) = r$. Let $\operatorname{cone}(G)$ be the join of G with a single vertex graph. Theorem 2.1 implies that $\operatorname{reg}(S/\operatorname{J}_{\operatorname{cone}(G)}) = \operatorname{reg}(S/\operatorname{J}_G)$ and thus $\operatorname{cone}(G)$ is a connected cograph with regularity r. □

With the lower bound settled, we aim for a stricter upper bound on the regularity of cographs. For this we employ the original definition of complement-reducible graphs: they are constructed recursively by taking complements and disjoint unions of cographs, starting from the single-vertex graph. As a result, there is an unusual one-to-one relationship between connected and disconnected cographs of the same order.

Lemma 2.3. *Let G be a cograph with at least two vertices. Then G is connected if and only if $\overline{G}$ is disconnected.*

Proof. Let $\overline{G}$ be disconnected and $v, w \in V(G)$. If v and w are in different connected components of $\overline{G}$, then $\{v,w\} \in E(G)$, so there is a path from v to w in G. If v and w are in the same connected component of $\overline{G}$, then there exists another vertex $u \in V(G)$ in a different connected component of $\overline{G}$. In particular, $\{v,u\},\{w,u\} \notin E(\overline{G})$ and so $\{v,u\},\{w,u\} \in E(G)$, so (v,u,w) is a path from v to w in G. Thus G is connected.

The other implication follows from this stronger result [2, Theorem 2]: Every induced subgraph of a cograph with more than one vertex has disconnected complement. □

In Lemma 2.3, the implication "$\overline{G}$ disconnected $\Rightarrow$ G connected" holds for any graph, not just cographs. In the same generality, the join of graphs can be expressed using complement and disjoint union.

Lemma 2.4. *Let G_1 and G_2 be simple undirected graphs. Then $G_1 * G_2 = \overline{\overline{G_1} \sqcup \overline{G_2}}$.*

Proof. Let V denote the common vertex set of both graphs. Let $e = \{v,w\}$, where $v,w \in V$ are arbitrary vertices. If $e \not\subseteq V(G_i)$ for $i \in \{1,2\}$, then e is an edge in both $\overline{\overline{G_1} \sqcup \overline{G_2}}$ and $G_1 * G_2$. If $e \subseteq V(G_i)$ for one $i \in \{1,2\}$, then $e \in E(\overline{\overline{G_1} \sqcup \overline{G_2}})$ if and only if e is an edge in G_i, which is the case if and only if $e \in E(G_1 * G_2)$. □

Lemma 2.5. *A connected cograph G is the join of induced subgraphs $G_1,\dots,G_m$, which are exactly the complements of the connected components of $\overline{G}$.*

Proof. By Lemma 2.3, $\overline{G}$ is disconnected. Let $\overline{G_1},\dots,\overline{G_m}$ be the connected components of $\overline{G}$, and $G_1,\dots,G_m$ their complements. The G_i are induced subgraphs of G since their edge sets arise from that of G by complementing twice. Then, by Lemma 2.4, $G = \overline{\overline{G_1} \sqcup \cdots \sqcup \overline{G_m}} = G_1 * \cdots * G_m$, as both $\sqcup$ and $*$ are associative. □

Using Lemma 2.5, we can assume that any connected cograph G is written as a join of the complements of the connected components of its complement. Since any induced subgraph of a cograph is a cograph, the G_i in the lemma are cographs too and by Lemma 2.3 they are disconnected or have only one vertex.

Proposition 2.6. *Let G be a connected cograph that is not complete. Then*

$$\operatorname{reg}(S/\operatorname{J}_G) = \max(\{2\} \cup \{\operatorname{reg}(S/\operatorname{J}_{G_i}) : i \in [m]\}),$$

where $G = G_1 * \cdots * G_m$ as in Lemma 2.5 and the G_i are cographs each of which is either disconnected or a single vertex.

Proof. Since G is not complete, the G_i cannot all be complete, so the equation follows from Lemma 2.5 and Theorem 2.1. □

We now have all the ingredients for a recursive computation of $\operatorname{reg}(S/\operatorname{J}_G)$ for any cograph G. In the disconnected case, add the regularities of all connected components. In the connected case, compute the maximum regularity of the complements G_i of the connected components $\overline{G_i}$ of $\overline{G}$. We use this to bound the maximum regularity.

Theorem 2.7. *Let G be a cograph on $3k-a$ vertices, with $k \in \mathbb{N}$ and $a \in \{0,1,2\}$. Then*

$$\operatorname{reg}(S/\operatorname{J}_G) \le 2k - a.$$

If G is connected, $k > 1$, and $a \in \{0,1\}$, then $\operatorname{reg}(S/\operatorname{J}_G) \le 2k - a - 1$.

Proof. The proof is by induction over k. The cographs with at most three vertices are K_1, $K_2 = P_1$, $\overline{K_2}$, K_3, $\overline{K_3}$, P_2 and $\overline{P_2}$. Then $\operatorname{reg}(S/\operatorname{J}_G) \leq 2-a = \operatorname{reg}(S/\operatorname{J}_{P_{3-a}})$.

Now let $k > 1$. By Proposition 2.6, if G is connected, it either has regularity 2 or there is a smaller, disconnected cograph with the same regularity. So it can be assumed that G is disconnected. Let H be a connected component of G with $3k_H - a_H$ vertices and let $H' = G \setminus H$ have $3k_{H'} - a_{H'}$ vertices, where $k_H, k_{H'} \in \mathbb{N}$ and $a_H, a_{H'} \in \{0, 1, 2\}$. Then both H and H' have fewer vertices than G and

$$k_H + k_{H'} = \begin{cases} k & \text{if } a_H + a_{H'} = a, \\ k+1 & \text{if } a_H + a_{H'} = a+3. \end{cases}$$

By induction, $\operatorname{reg}(S/\operatorname{J}_H) \leq 2k_H - a_H$ and $\operatorname{reg}(S/\operatorname{J}_{H'}) \leq 2k_{H'} - a_{H'}$, and

$$\begin{aligned} \operatorname{reg}(S/\operatorname{J}_G) &\leq 2k_H - a_H + 2k_{H'} - a_{H'} \\ &= \begin{cases} 2k - a & \text{if } a_H + a_{H'} = a, \\ 2k - a - 1 & \text{if } a_H + a_{H'} = a+3 \end{cases} \\ &\leq 2k - a. \end{aligned}$$

If G is connected and $k > 1$, the regularity is either 2 or at most that of a disconnected cograph with fewer vertices. So if G has $3k - a$ vertices, with $a \in \{0, 1\}$, then $\operatorname{reg}(S/\operatorname{J}_G) \leq 2k - a - 1$. If G has $3k - 2$ vertices, then $\operatorname{reg}(S/\operatorname{J}_G) \leq 2(k-1) = 2k - 2$. ☐

The following example shows that the bounds for disconnected cographs in Theorem 2.7 can be realized.

Example 2.8. Let $n = 3k - a \in \mathbb{N}$ be a positive integer with $a \in \{0, 1, 2\}$. If $a = 0$, let G_n be a disjoint union of k copies of P_3. If $a = 1$, then let G_n be a disjoint union of $k - 1$ copies of P_3 and one of P_2. If $k = 2$, let G_n be a disjoint union of two copies of P_2 and $k - 2$ copies of P_3. Since P_n has regularity $n - 1$ and regularity is additive under disjoint union, G_n has regularity $2k - a$.

To make connected examples, use the cone construction from Proposition 2.2. Even more, in two of three cases the graphs from Example 2.8 are the only cographs of maximum regularity, as the following theorem shows.

Theorem 2.9. *Let G be a cograph with $3k-a$ vertices, where $k \in \mathbb{N}$ and $a \in \{0, 1\}$. Then $\operatorname{reg}(S/\operatorname{J}_G) = 2k - a$ if and only if G is a disjoint union of P_3 and at most one P_2.*

Proof. With Example 2.8 in place, just the only-if direction remains. Let G be a cograph. By Theorem 2.7, there is nothing to prove if G is connected, so we assume it is disconnected. We first show that any connected component has at most three vertices. To this end, let H be a connected component of G with $3k_H - a_H$ vertices, $k_H > 1$, and $a_H \in \{0, 1, 2\}$. Let $H' = G \setminus H$ have $3k_{H'} - a_{H'}$ vertices with $a_{H'} \in \{0, 1, 2\}$. Then

$$k_H + k_{H'} = \begin{cases} k & \text{if } a_H + a_{H'} = a, \\ k+1 & \text{if } a_H + a_{H'} = a+3. \end{cases}$$

If $a_H \neq 2$, then, by Theorem 2.7,

$$\begin{aligned}\operatorname{reg}(S/\operatorname{J}_G) &\leq 2k_H - a_H - 1 + 2k_{H'} - a_{H'} \\ &= \begin{cases} 2k - a - 1 & \text{if } a_H + a_{H'} = a, \\ 2k - a - 2 & \text{if } a_H + a_{H'} = a + 3 \end{cases} \\ &< 2k - a.\end{aligned}$$

If otherwise $a_H = 2$, then since $a \in \{0, 1\}$, G must have another connected component H' with $3k_{H'} - a_{H'}$ vertices and $a_{H'} \in \{1, 2\}$. Let $H'' = G \setminus (H \sqcup H')$ with $3k_{H''} - a_{H''}$ vertices, where $k_{H''} \in \mathbb{N}_0$ and $a_{H''} \in \{0, 1, 2\}$. Then

$$k_H + k_{H'} + k_{H''} = \begin{cases} k + 1 & \text{if } a_H + a_{H'} + a_{H''} = a + 3, \\ k + 2 & \text{if } a_H + a_{H'} + a_{H''} = a + 6. \end{cases}$$

Therefore, Theorem 2.7 implies

$$\begin{aligned}\operatorname{reg}(S/\operatorname{J}_G) &\leq 2k_H - a_H + 2k_{H'} - a_{H'} + 2k_{H''} - a_{H''} \\ &= \begin{cases} 2k - a - 1 & \text{if } a_H + a_{H'} + a_{H''} = a + 3, \\ 2k - a - 2 & \text{if } a_H + a_{H'} + a_{H''} = a + 6 \end{cases} \\ &< 2k - a.\end{aligned}$$

We have thus shown that if G has a connected component with more than three vertices, it cannot have maximum regularity. Since the only cographs of maximum regularity with three and two vertices are, respectively, P_3 and P_2, it follows that if G has maximum regularity, then it is a disjoint union of 2-paths P_3, single edges P_2 and isolated vertices. We now analyze these cases separately for two possible values of a.

Suppose first that $a = 0$. If G has a connected component H with one or two vertices, that is $3 - a_H$ vertices and $a_H \in \{1, 2\}$, then G must have another connected component H' with $3 - a_{H'}$ vertices, where $a_{H'} \in \{1, 2\}$. With a similar computation as above, we find that $\operatorname{reg}(S/\operatorname{J}_G) < 2k$. Therefore, if G has $3k$ vertices and maximal regularity, each connected component must have exactly three vertices and be equal to P_3.

Finally, consider the case $a = 1$. If G has an isolated vertex, then G must have another connected component with fewer than three vertices. Then G cannot have maximal regularity as isolated vertices contribute no regularity, and a disjoint union of an edge and a vertex has regularity 1. If there are two isolated edges, then the subgraph on these four vertices contributes regularity only 2 and thus G cannot have maximal regularity. □

Remark 2.10. Graphs with $3k - 2$ vertices and maximal regularity, which would constitute the $a = 2$ case in Theorem 2.9, do not have a simple characterization. For example, the class contains cones over disjoint unions of 2-paths as well as other types of joins and disjoint unions of joins, paths, and isolated vertices.

The following corollary partially characterizes connected maximizers of regularity.

Corollary 2.11. *Let G be a connected cograph with maximum regularity among connected cographs on $3k-a$ vertices with $k>1$ and $a \in \{0,2\}$, that is,* $\operatorname{reg}(S/\operatorname{J}_G) = 2k-1$ *for $a=0$ and* $\operatorname{reg}(S/\operatorname{J}_G) = 2k-2$ *for $a=2$. Then G is a cone.*

Proof. Let G be a connected cograph with $3k-a$ vertices and maximum regularity, where $k>1$ and $a \in \{0,2\}$. By Lemma 2.5, $G = G_1 * \cdots * G_m$, where the G_i are disconnected or single vertices. Proposition 2.6 shows that if G is not a cone, that is, neither of the G_i is a single vertex, then $\operatorname{reg}(S/\operatorname{J}_G) = \operatorname{reg}(S/\operatorname{J}_{G'})$ for some G' which has at least two vertices fewer than G. So assume G' has $3k-a-2$ vertices and apply Theorem 2.7. If $a=0$, then G' has $3k-2$ vertices and thus regularity at most $2k-2$, but G had maximum regularity $2k-1$, a contradiction. If $a=2$, then G' has $3(k-1)-1$ vertices and thus regularity at most $2(k-1)-1 = 2k-3$, but G had maximum regularity $2k-2$, another contradiction. By these contradictions, either G is a cone after all, or its regularity was not maximal. ☐

Applying the reasoning in the proof to the $a=1$ case does not yield the contradiction. In this case, G has $3k-1$ vertices and thus a regularity bound of $2k-1-1 = 2k-2$ while G', with its $3(k-1)$ vertices, has regularity at most $2(k-1)$.

The results so far give a fairly clear picture of the asymptotic behaviour of regularity in the class of cographs and individually for graphs with known complement-reducible decomposition. If the exact structure of a cograph is unknown, it can be useful to bound regularity using graph-theoretic invariants. We consider here $\alpha(G)$, the size of the largest independent set; $\operatorname{s}(G)$, the number of maximal independent sets in G; and $\operatorname{c}(G)$, the number of maximal cliques of G. The recursive construction of cographs yields bounds because these invariants satisfy simple formulas under disjoint union and join:

$$\operatorname{s}(G \sqcup H) = \operatorname{s}(G)\operatorname{s}(H), \tag{2.1}$$

$$\operatorname{s}(G * H) = \operatorname{s}(G) + \operatorname{s}(H), \tag{2.2}$$

$$\alpha(G \sqcup H) = \alpha(G) + \alpha(H), \tag{2.3}$$

$$\alpha(G * H) = \max\{\alpha(G), \alpha(H)\}. \tag{2.4}$$

We find the following bounds which are independent of the number of vertices.

Theorem 2.12. *Let G be a cograph. Then* $\operatorname{reg}(S/\operatorname{J}_G) \leq \min\{\operatorname{s}(G), \alpha(G)\}$.

Proof. The proof is by induction on the number of vertices of G. An isolated vertex has exactly one maximal independent set with one vertex and $\operatorname{reg}(S/\operatorname{J}_{K_1}) = 0$, so the statement holds. Now let G be any cograph. If G is connected, it is the join of smaller cographs $G_1, \ldots, G_m$ and by induction

$$\operatorname{reg}(S/\operatorname{J}_{G_i}) \leq \min\{\operatorname{s}(G_i), \alpha(G_i)\} \quad \text{for all } i = 1, \ldots, m.$$

If G is complete, both inequalities are trivial. In the other case, it follows by Theorem 2.1 and (2.2) that

$$\operatorname{reg}(S/\operatorname{J}_G) = \max\{\{2\} \cup \{\operatorname{reg}(S/\operatorname{J}_{G_i}) : i = 1, \ldots, m\}\} \leq \sum_{i=1}^{m} \operatorname{s}(G_i) = \operatorname{s}(G)$$

and, since $\alpha(G)=1$ if and only if G is complete, (2.4) leads to

$$\begin{aligned}\operatorname{reg}(S/\operatorname{J}_G) &= \max\{\{2\}\cup\{\operatorname{reg}(S/\operatorname{J}_{G_i}) : i=1,\dots,m\}\}\\ &\le \max\{\alpha(G_i) : i=1,\dots,m\} = \alpha(G).\end{aligned}$$

If G is disconnected, it has connected components $G_1,\dots,G_m$ which by induction satisfy

$$\operatorname{reg}(S/\operatorname{J}_{G_i}) \le \min\{\operatorname{s}(G_i),\alpha(G_i)\} \quad \text{for all } i=1,\dots,m.$$

In this case, since G is the disjoint union of its connected components, and since regularity is additive, with (2.1) we have

$$\operatorname{reg}(S/\operatorname{J}_G) = \sum_i \operatorname{reg}(S/\operatorname{J}_{G_i}) \le \prod_i \operatorname{s}(G_i) = \operatorname{s}(G).$$

Finally, by (2.3) we have

$$\operatorname{reg}(S/\operatorname{J}_G) = \sum_i \operatorname{reg}(S/\operatorname{J}_{G_i}) \le \sum_i \alpha(G_i) = \alpha(G). \qquad \square$$

The method of bounding by $\operatorname{s}(G)$ seems coarse as the maximum and the sum over a set of integers are respectively replaced by the sum and the product over those integers. Nevertheless $s(G)$ can be a good bound as discussed in Remark 2.15.

Since independent vertices cannot be in the same clique, we have $\alpha(G)\le \operatorname{c}(G)$. Thus $\operatorname{reg}(S/\operatorname{J}_G)\le \operatorname{c}(G)$ holds for cographs, but in fact it holds for all graphs by [15].

Our last bound uses the maximum vertex degree $\delta(G)$ of a connected cograph G.

Proposition 2.13. *Let G be a connected cograph. Then*

$$\operatorname{reg}(S/\operatorname{J}_G) \le \delta(G) = \max\{\delta(v) : v\in V(G)\}.$$

Proof. If $G=K_n$, then $\operatorname{reg}(S/\operatorname{J}_G)\le 1$, $\delta(G)=n-1$, and the inequality holds. If G is not complete but connected, it is the join of induced subgraphs $G_1,\dots,G_m$ as in Lemma 2.5. Then $\operatorname{reg}(S/\operatorname{J}_G) = \max\{2,\operatorname{reg}(S/\operatorname{J}_{G_1}),\dots,\operatorname{reg}(S/\operatorname{J}_{G_m})\}$ by Proposition 2.6. Writing n_i for the number of vertices of G_i, Theorem 1.1 gives

$$\operatorname{reg}(S/\operatorname{J}_G) \le \max\{2, n_1-1,\dots,n_m-1\}.$$

Let $n_{\max}=\max\{n_1,\dots,n_m\}$. Then $\operatorname{reg}(S/\operatorname{J}_G)\le n_{\max}$, since G is not complete and thus $n_{\max}\ge 2$. Since G is a join of the G_i, the maximum vertex degree satisfies $n_{\max}\le \max\{\delta(v) : v\in V(G)\}$, which gives the desired inequality. $\square$

Remark 2.14. The bound in Proposition 2.13 does not give anything new for cone graphs since in this case it agrees with Theorem 1.1.

Remark 2.15. One can ask if one of the bounds in this section is generally preferable over the others. Table 1 shows that any bound can beat any other bound, with the exception of $\alpha(G)\le \operatorname{c}(G)$. On the other hand, if one asks for the best bound, it can be confirmed that among the 2,341 cographs in our database, for 505 of them $\operatorname{s}(G)$ is strictly the best bound, and for 724 $\alpha(G)$ is strictly the best bound. No other bound is ever strictly the best and for all remaining graphs there is a tie for the best bound.

	order bound	c(G)	s(G)	$\alpha(G)$	max deg
order bound	0	968	968	146	1,090
c(G)	918	0	1,049	0	724
s(G)	920	1,050	0	514	837
$\alpha(G)$	1,830	1,139	1,522	0	1,150
max deg	5	362	201	1	0

Table 1. Comparisons of five regularity bounds for all 2,341 cographs in our database. Each figure is the number of cographs for which the bound in the corresponding row header is strictly better than the bound in the corresponding column header. 'Order bound' stands for the bound in Theorem 2.7 and 'max deg' denotes the maximum vertex degree in Proposition 2.13. Comparisons with 'max deg' are made only for the 1,171 connected cographs.

Remark 2.16. The questions about regularity in this paper can also be asked about the regularity of $S/\mathcal{I}_G$, where $\mathcal{I}_G$ is the *parity binomial edge ideal* of [10]. Using our database we observed the following inequality, slightly weaker than the one in Theorem 1.1:

$$\ell \leq \operatorname{reg}(S/\mathcal{I}_G) \leq n.$$

Based on our computations we conjecture that the maximum regularity is achieved exactly for disjoint unions of odd cycles. Minimal free resolutions of parity binomial edge ideals contain many interesting patterns that remain to be investigated. At the time of the first posting of this paper, explaining even the minimal free resolution of $S/\mathcal{I}_{K_n}$ was open and we conjectured that $\operatorname{reg}(S/\mathcal{I}_{K_n}) = 3$. In the meantime this has been confirmed in [8].

3. Regularity versus h-polynomials

As a standard graded $\Bbbk$-algebra, the Hilbert series of S/J_G takes the form $\frac{h_G(t)}{(1-t)^d}$, where d is the Krull dimension and $h_G(t) \in \mathbb{Z}[t]$. The numerator h_G is known as the *h-polynomial.* In the first arXiv version of [7], Hibi and Matsuda conjectured that for binomial edge ideals its degree bounds the regularity from above. The conjecture was removed from a subsequent version of their paper after we informed them of the following minimal counterexample on eight vertices:

Example 3.1. Let G be the graph in Figure 1, that is, the graph on the vertex set $\{1,\ldots,8\}$ with edges $\{1,8\}$, $\{2,6\}$, $\{3,7\}$, $\{3,8\}$, $\{4,5\}$, $\{4,8\}$, $\{5,6\}$, $\{5,7\}$, $\{6,7\}$, $\{6,8\}$, $\{7,8\}$. Then $\operatorname{reg}(S/\operatorname{J}_G) = 4$ and $\deg(h_G) = 3$.

At the time of writing, our database contains 39 counterexamples and none shows a difference greater than 1 between $\operatorname{reg}(S/\operatorname{J}_G)$ and $\deg(h_G)$. However, gluing two copies of the counterexample in Figure 1 at vertex 1 yields a graph G (visible in

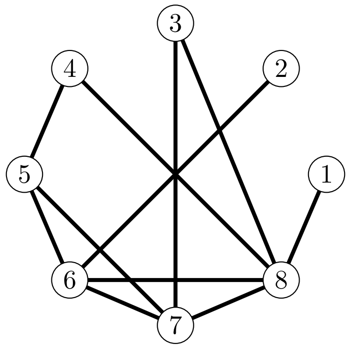

FIGURE 1. A graph with $\operatorname{reg}(S/\operatorname{J}_G) > \deg(h_G)$.

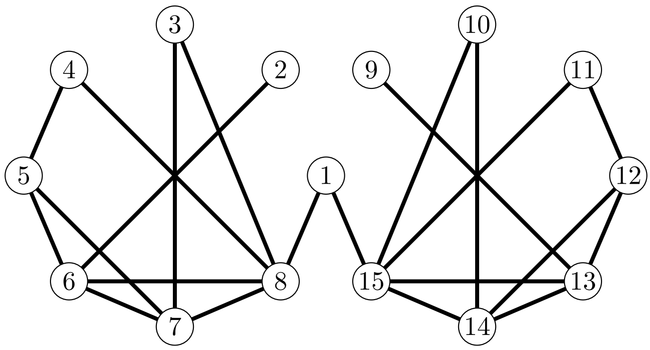

FIGURE 2. A graph with $\operatorname{reg}(S/\operatorname{J}_G) > \deg(h_G) + 1$.

Figure 2) which satisfies $\operatorname{reg}(S/\operatorname{J}_G) = 8$ and $\deg(h_G) = 6$. We now show that the difference can be made arbitrarily large. To this end we employ the following two theorems that explain the behaviour of the regularity and the Hilbert series upon gluing two graphs G_1 and G_2 over a vertex which is a free vertex in both graphs. If G is a gluing like this, then G_1 and G_2 are a *split* of G.

Theorem 3.2 ([9, Theorem 3.1])**.** *Let G_1 and G_2 be a split of a graph G at a vertex v. If v is a free vertex in both G_1 and G_2, then* $\operatorname{reg}(S/\operatorname{J}_G) = \operatorname{reg}(S/\operatorname{J}_{G_1}) + \operatorname{reg}(S/\operatorname{J}_{G_2})$.

Theorem 3.3 ([12, Theorem 3.2])**.** *Let G_1 and G_2 be the decomposition of a graph G at a vertex v. If v is a free vertex in both G_1 and G_2, then*

$$\operatorname{Hilb}_{S/\operatorname{J}_G}(t) = (1-t)^2 \operatorname{Hilb}_{S/\operatorname{J}_{G_1}}(t) \operatorname{Hilb}_{S/\operatorname{J}_{G_2}}(t).$$

Theorem 3.4. *Let $k \in \mathbb{N}$. Then there exists a graph G such that*

$$\operatorname{reg}(S/\operatorname{J}_G) = \deg(h_G) + k.$$

Proof. Let G_1 be the graph in Figure 1. The reduced Hilbert series of $S/\,\mathrm{J}_{G_1}$ can be computed with Macaulay2 [4] as

$$\mathrm{Hilb}_{S/\,\mathrm{J}_{G_1}}(t) = \frac{1+7t+17t^2+13t^3}{(1-t)^9}$$

and its regularity as $\mathrm{reg}(S/\,\mathrm{J}_{G_1}) = 4$. Since G_1 has two free vertices $1, 2$, we can glue a chain of k copies of G_1 along free vertices (see Figure 2 for the case $k = 2$ in which the vertices 2 and 9 are available for further gluing). By Theorem 3.3, the Hilbert series of the resulting graph is

$$\mathrm{Hilb}_{S/\,\mathrm{J}_{G_2}}(t) = \frac{(1+7t+17t^2+13t^3)^k}{(1-t)^{7k+2}}$$

and, by Theorem 3.2, $\mathrm{reg}(S/\,\mathrm{J}_{G_2}) = 4k$. Thus $\mathrm{reg}(S/\,\mathrm{J}_G) - \deg(h_G) = k$. □

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Received: October 26, 2020
Accepted: March 9, 2021