SEQUENTIAL OPTIMALITY CONDITIONS FOR OPTIMIZATION PROBLEMS WITH ADDITIONAL ABSTRACT SET CONSTRAINTS

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ABSTRACT. The positive approximate Karush–Kuhn–Tucker sequential condition and the strict constraint qualification associated with this condition for general scalar problems with equality and inequality constraints have recently been introduced. In this paper, we extend them to optimization problems with additional abstract set constraints. We also present an extension of the approximate Karush–Kuhn–Tucker sequential condition and its related strict constraint qualification. Furthermore, we explore the relations between the new constraint qualification and other constraint qualifications known in the literature as Abadie, quasi-normality and the approximate Karush–Kuhn–Tucker regularity constraint qualification. Finally, we introduce an augmented Lagrangian method for solving the optimization problem with abstract set constraints and we show that it is possible to obtain global convergence under the new condition.

1. INTRODUCTION

While equality and inequality constraints are more widely used in representing the constraint sets for many optimization problems, optimality conditions and numerical algorithms may become more convenient when some of these equality/inequality constraints as well as additional complicated side conditions are lumped into suitable abstract set constraints. For this reason we will consider the constrained nonlinear programming problem with additional abstract set constraints of the form

$$
\begin{aligned}
&\text{Minimize } f(x) \\
&\text{s. t. } h_i(x) = 0, \quad i = 1, \dots, m \\
&\qquad g_j(x) \leqslant 0, \quad j = 1, \dots, p \\
&\qquad x \in X,
\end{aligned}
\tag{1.1}
$$

where the functions $f : \mathbb{R}^n \to \mathbb{R}$, $h = (h_1, \ldots, h_m)^t : \mathbb{R}^n \to \mathbb{R}^m$, and $g = (g_1, \ldots, g_p)^t : \mathbb{R}^n \to \mathbb{R}^p$ are continuously differentiable in $\mathbb{R}^n$ and $X \subset \mathbb{R}^n$ is the additional abstract set. Throughout the paper, we assume that X is a nonempty, closed and regular set (see Definition 2.4) in $\mathbb{R}^n$. We write $\Omega = \{x \in \mathbb{R}^n : h(x) = 0, \, g(x) \leqslant 0 \text{ and } x \in X\}$. For each $x \in \Omega$, we define $I_g(x) = \{j \in \{1, \ldots, p\} : g_j(x) = 0\}$ as the index set of active inequality constraints at x.

The development of optimality conditions for optimization problems which include abstract set constraints started with [30, 34, 41] and has been followed by [17, 35, 18].

Sequential optimality conditions are asymptotic versions of the Karush–Kuhn–Tucker conditions and they play a central role in the design and analysis of numerical algorithms. Thus, several papers have been devoted to the study of such conditions in other contexts, for example: variational inequality problems [33], Nash equilibrium problems [28], mathematical programs with equilibrium constraints (MPECs) [40, 39], mathematical programs with complementarity constraints (MPCCs) [8], nonlinear vector optimization with conic constraints [43], general nonlinear conic programming [6], the multiobjective case [32], optimization problems in Banach spaces [26], variational problems in Banach spaces [36], quasi-equilibrium problems [27], the study of a KKT-proximity measure as a termination condition [29], among others.

The purpose of the present paper is to extend the positive approximate Karush–Kuhn–Tucker (PAKKT) sequential optimality condition and the companion strict constraint qualification associated with it —which is called PAKKT regular constraint qualification and is defined in [5]— to the scalar optimization problem with abstract set constraints (1.1). In order to fulfill this task, we first assume that X is a regular set (see Definition 2.4) and include the use of normal cone properties (see Definition 2.3). Secondly, we extend for (1.1) the well-known approximate Karush–Kuhn–Tucker (AKKT) sequential condition (see Definition 3.1) and its associated strict constraint qualification defined in [11] (see Definition 4.1). Thirdly, we establish the relationship between the new constraint qualification and other well-known conditions such as the quasi-normality [18], Abadie [1] and AKKT regularity [11] conditions.

Finally, we propose an augmented Lagrangian method (ALM) for solving (1.1) using the quadratic penalty function to penalize equality and inequality constraints, and we analyze the global convergence to stationary points using the new constraint qualification. In [4, 3, 21] the ALM with general lower-level constraints is considered. In [3] the lower-level constraint set is defined by a finite number of equalities and inequalities and the global convergence to stationary points is proved under the constant positive linear dependence constraint qualification (CPLD CQ) presented in [12, 38]. In [4, 21], the ALM for problems in which the lower-level constraint set is a box is considered and again the global convergence results are obtained using the CPLD CQ.

This paper is organized as follows. Section 2 is devoted to preliminaries and some basic definitions. In Section 3, we present the extension of the well-known

approximate sequential conditions for problems with abstract set constraints and the associated strict constraint qualification. In Section 4, the relationship between the new condition and other constraint qualifications is studied. Section 5 is devoted to the definition of the ALM and the analysis of the global convergence of the proposed method. Finally, in Section 6, conclusions and lines for future research are given.

Notation. Let us introduce the following notation.

$\mathbb{R}_+ = \{t \in \mathbb{R} : t \geqslant 0\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$; $\| \cdot \|$ denotes an arbitrary vector norm and $\| \cdot \|_\infty$ the supremum norm. $\mathbb{R}^p_+ := \{x \in \mathbb{R}^p : x_i \geqslant 0, \, i = 1, \ldots, p\}$ is the positive orthant.

The i-th component of the vector v is v_i.

If $K = \{k_0, k_1, k_2, \ldots\} \subset \mathbb{N}$ $(k_{j+1} > k_j \; \forall j)$, we write

$$\lim_{k \in K} x^k = \lim_{j \to \infty} x^{k_j}.$$

For all $y \in \mathbb{R}^n$, $y_+ = (\max\{0, y_1\}, \ldots, \max\{0, y_n\})$.

If $\{\gamma_k\} \subset \mathbb{R}$, $\gamma_k > 0$, and $\gamma_k \to 0$, we write $\gamma_k \downarrow 0$.

We define the *sign function* sgn by putting $\operatorname{sgn} a = 1$ if $a > 0$ and $\operatorname{sgn} a = -1$ if $a < 0$. We have $\operatorname{sgn}(a \cdot b) = \operatorname{sgn} a \cdot \operatorname{sgn} b$.

The Euclidean projection of $y \in \mathbb{R}^n$ onto a nonempty closed convex set X in $\mathbb{R}^n$ is denoted by $P_X(y)$.

2. Preliminaries and definitions

In this section we will provide some definitions and basic concepts which will be employed in the present paper.

We will use the definition of a usual local conical approximation to the constraint set, namely the tangent cone, which is particularly useful in characterizing local optimality of feasible solutions of (1.1), see for example [37].

Definition 2.1. A vector y is a *tangent* of a set $S \subset \mathbb{R}^n$ at a point $x \in S$ if either $y = 0$ or there exists a sequence $\{x^k\} \subset S$ such that $x^k \neq x$ for all k and

$$x^k \to x, \quad \frac{x^k - x}{\|x^k - x\|} \longrightarrow \frac{y}{\|y\|}.$$

The set of the tangent vectors of S at x is denoted by $T_S(x)$ and is called the *tangent cone* of S at x.

It can be proved that $T_S(x)$ is closed but not necessarily convex.

Definition 2.2. The *polar cone* of any nonempty set $T \subset \mathbb{R}^n$ is defined by

$$T^\circ = \{z \in \mathbb{R}^n : z^t y \leqslant 0, y \in T\}.$$

Note that T° is a closed convex cone.

Definition 2.3. The *limiting normal cone* $N_S(x)$ to a closed set $S \subset \mathbb{R}^n$ at $x \in S$ is defined as

$$N_S(x) = \big\{ z \in \mathbb{R}^n : \exists \{x^k\} \subset S,\ \{z^k\} \subset \mathbb{R}^n,\ x^k \to x,\ z^k \to z,$$
$$\text{and } z^k \in (T_S(x^k))^\circ\ \forall k \big\}. \quad (2.1)$$

In general, for $x \in S$, we have $(T_S(x))^\circ \subset N_S(x)$, but $N_S(x)$ may not always be equal to $(T_S(x))^\circ$. If we assume, for example, that S is convex and closed, we have that $N_S(x) = (T_S(x))^\circ$.

Definition 2.4 (see [42]). We say that $S \subset \mathbb{R}^n$ is *regular* at $x \in S$ if $N_S(x) = (T_S(x))^\circ$.

Regularity is, in fact, an important property because it distinguishes problems which have satisfactory Lagrange multiplier theory from those which do not, as it can be seen for example in [42].

As regards the tangent cone, we consider the linearized cone of the feasible set Ω at a point $x \in \Omega$ as follows:

$$L_\Omega(x) = \big\{ d \in T_X(x) : \nabla h_i(x)^t d = 0\ \forall i = 1, \dots, m \text{ and } \nabla g_j(x)^t d \leqslant 0\ \forall j \in I_g(x) \big\}.$$

When $X = \mathbb{R}^n$, $L_\Omega(x)$ can be considered to be the well-known first-order linear approximation of the tangent cone $T_\Omega(x)$ at $x \in \Omega$.

Let x^* be a local minimizer of (1.1). The geometrical first-order necessary optimality condition in [42] establishes that

$$\nabla f(x^*)^t w \geqslant 0 \qquad \forall w \in T_\Omega(x^*).$$

Although this is a necessary basic optimality condition for (1.1), it is generally difficult to apply because it is not easy to obtain a representation of the tangent cone $T_\Omega(x^*)$. Therefore, in this case we choose analytical conditions.

Assumption A. Throughout this paper, we assume that the abstract set X in (1.1) is a regular closed set (see Definition 2.4).

When X is regular, we apply the definition of the Karush–Kuhn–Tucker conditions for problems with abstract set constraints in the following alternative way.

Definition 2.5. We say that a feasible point $x^* \in \Omega$ is a *Karush–Kuhn–Tucker (KKT) point* for problem (1.1) if there exist $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}_+^p$ such that

(a) $-\left(\nabla f(x^*) + \sum_{i=1}^m \lambda_i \nabla h_i(x^*) + \sum_{j=1}^p \mu_j \nabla g_j(x^*) \right) \in N_X(x^*);$

(b) $\mu_j g_j(x^*) = 0\ \forall j = 1, \dots, p.$

Observe that, by Assumption A, item (a) of Definition 2.5 is equivalent to the definition given in [37]:

$$\left(\nabla f(x^*) + \sum_{i=1}^m \lambda_i \nabla h_i(x^*) + \sum_{j=1}^p \mu_j \nabla g_j(x^*) \right)^t w \geqslant 0 \qquad \forall w \in T_X(x^*).$$

For the present paper, we prefer the definition which considers $N_X(x^*)$ instead of the previous one which considers $T_X(x^*)$, since it is more suitable to define an approximate KKT condition.

Given $x^* \in \Omega$, $x \in X$, $\tilde{x} \in X$, we consider the following convex cone:

$$K_\Omega(x, \tilde{x}) = \left\{ \sum_{i=1}^m \lambda_i \nabla h_i(x) + \sum_{j \in I_g(x^*)} \mu_j \nabla g_j(x) + \omega : \lambda \in \mathbb{R}^m,\, \mu \in \mathbb{R}_+^p,\, \omega \in N_X(\tilde{x}) \right\}.$$
(2.2)

Observe that $x^* \in \Omega$ is a KKT point of (1.1) if and only if $-\nabla f(x^*) \in K_\Omega(x^*, x^*)$.

We conclude this section by introducing the notion of outer semi-continuity, which will be used in the following sections.

Definition 2.6 (see [42]). Given a set-valued mapping (multifunction) $F : \mathbb{R}^s \rightrightarrows \mathbb{R}^d$, the outer limit of $F(z)$ as $z \to z^*$ is denoted by

$$\limsup_{z \to z^*} F(z) = \{ w^* \in \mathbb{R}^d : \exists\, (z^k, w^k) \to (z^*, w^*) \text{ with } w^k \in F(z^k) \}.$$

Then, the multifunction F is said to be *outer semi-continuous* at z^* if $\limsup_{z \to z^*} F(z) \subset F(z^*)$.

The outer limit is always a closed set. Furthermore, the normal cone enjoys the nice closedness property, as the proposition below shows.

Proposition 2.7 (see [42]). *The set-valued mapping $N_X : x \mapsto N_X(x)$ is outer semi-continuous at x^*: $N_X(x^*) = \limsup_{x \to x^*} N_X(x)$.*

3. Sequential optimality conditions for problems with additional abstract set

Sequential optimality conditions are properties of feasible points of nonlinear programming problems which are necessarily satisfied by any local minimizer x^* and are formulated in terms of sequences converging to x^*. In the case of $X = \mathbb{R}^n$ one of the most popular sequential optimality conditions is the approximate Karush–Kuhn–Tucker (AKKT), defined in [9]. For the abstract set $X \neq \mathbb{R}^n$ we propose as a natural extension of the AKKT condition for problems with abstract set constraints of the form (1.1) the following definition.

Definition 3.1. We say that a feasible point $x^* \in \Omega$ is an *approximate Karush–Kuhn–Tucker point* for (1.1) (AKKT) if there are sequences $\{x^k\} \subset X$, $\{\tilde{x}^k\} \subset X$, $\{\lambda^k\} \subset \mathbb{R}^m$, $\{\mu^k\} \subset \mathbb{R}_+^p$ and $\{\varepsilon^k\} \subset \mathbb{R}^n$ such that $\lim_{k \to \infty} x^k = \lim_{k \to \infty} \tilde{x}^k = x^*$, $\lim_{k \to \infty} \varepsilon^k = 0$ and

$$\varepsilon^k - \left(\nabla f(x^k) + \sum_{i=1}^m \lambda_i^k \nabla h_i(x^k) + \sum_{j \in I_g(x^*)} \mu_j^k \nabla g_j(x^k) \right) \in N_X(\tilde{x}^k), \qquad (3.1)$$

$$\lim_{k \to \infty} \| \min\{-g(x^k), \mu^k\} \| = 0. \qquad (3.2)$$

A new general sequential optimality condition called PAKKT has recently been introduced in [5] when $X = \mathbb{R}^n$. Thus, we propose the definition below as an extension of the PAKKT condition for (1.1).

Definition 3.2. We say that $x^* \in \Omega$ is a *positive approximate Karush–Kuhn–Tucker* point for problem (1.1) (PAKKT) if there are sequences $\{x^k\} \subset X$, $\{\tilde{x}^k\} \subset X$, $\{\lambda^k\} \subset \mathbb{R}^m$, $\{\mu^k\} \subset \mathbb{R}^p_+$ and $\{\varepsilon^k\} \subset \mathbb{R}^n$ such that $\lim_{k \to \infty} x^k = \lim_{k \to \infty} \tilde{x}^k = x^*$, $\lim_{k \to \infty} \varepsilon^k = 0$, (3.1) and (3.2) hold, and

$$\lambda_i^k h_i(x^k) > 0 \quad \text{if } \lim_{k \to \infty} \frac{|\lambda_i^k|}{\delta_k} > 0, \tag{3.3}$$

$$\mu_j^k g_j(x^k) > 0 \quad \text{if } \lim_{k \to \infty} \frac{\mu_j^k}{\delta_k} > 0, \tag{3.4}$$

where $\delta_k = \|(1, \lambda^k, \mu^k)\|_\infty$.

As mentioned in [5], the expressions (3.1) and (3.2) are related to the KKT conditions of the original problem and they are used in the AKKT optimality condition. The expressions (3.3) and (3.4) aim to control the sign of Lagrange multipliers, in the way mentioned in [5] when $X = \mathbb{R}^n$. Hence, when the abstract set $X = \mathbb{R}^n$, we have $N_X(x) = \{0\}$ for each $x \in X$, and Definition 3.2 is the PAKKT sequential condition introduced in [5].

Theorem 3.3. *PAKKT is a necessary optimality condition for problem* (1.1).

Proof. The proof follows that of [5, Theorem 2.2] by incorporating the abstract set X, as we show here for completeness.

Let x^* be a local minimizer of (1.1). Then we know that there exists $\alpha > 0$ such that x^* is a global minimizer of $f(x)$ on $\Omega \cap B(x^*, \alpha)$. Therefore x^* is the unique global minimizer of the problem

$$\text{Minimize } f(x) + \frac{1}{2}\|x - x^*\|_2^2$$
$$\text{s. t. } x \in \Omega \cap B(x^*, \alpha).$$

For $k \in \mathbb{N}$, consider the penalized problem

$$\text{Minimize } f(x) + \frac{1}{2}\|x - x^*\|_2^2 + \frac{\rho_k}{2}\left[\|h(x)\|_2^2 + \|g(x)_+\|_2^2\right]$$
$$\text{s. t. } x \in X, \|x - x^*\|_2 \leqslant \alpha, \tag{3.5}$$

$\rho_k > 0$. Since $B(x^*, \alpha) \cap X$ is a compact set, by the Bolzano–Weierstrass theorem this problem admits a solution $x^k \in X \cap B(x^*, \alpha)$. We suppose that $\rho_k \to \infty$. Since x^* is the unique global minimizer of $f(x) + \frac{1}{2}\|x - x^*\|_2^2$ subject to $\Omega \cap B(x^*, \alpha)$, we have that

$$\lim_{k \to \infty} x^k = x^*. \tag{3.6}$$

For k large enough (let us say for all $k \in K$), we can suppose that $\|x^k - x^*\|_2 < \alpha$. Then, by the stationarity condition for problem (3.5), we have

$$-\left(\nabla f(x^k) + \sum_{i=1}^{m} \rho_k h_i(x^k) \nabla h_i(x^k) + \sum_{j=1}^{p} \rho_k g_j(x^k)_+ \nabla g_j(x^k) \right) + x^* - x^k \in N_X(x^k).$$

(3.7)

For each $k \in K$, we define

$$\lambda^k = \rho_k h(x^k) \quad \text{and} \quad \mu^k = \rho_k g(x^k)_+ \geqslant 0.$$

(3.8)

Thus, by (3.6), (3.7) and (3.8) we obtain

$$\varepsilon^k - \left(\nabla f(x^k) + \sum_{i=1}^{m} \lambda_i^k \nabla h_i(x^k) + \sum_{j=1}^{p} \mu_j^k \nabla g_j(x^k) \right) \in N_X(x^k)$$

with $\varepsilon^k = x^* - x^k$. Therefore, (3.1) holds with $\tilde{x}^k = x^k$ for all k. By (3.8), we have that $\mu_i^k = 0$ if $g_i(x^k) < 0$. Hence, (3.2) is satisfied. If $\mu_i^k > 0$, $k \in K$, then $g_i(x^k) > 0$, and hence $\mu_i^k g_i(x^k) = \rho_k [g_i(x^k)]^2 > 0$. Analogously, if $\lambda_i^k \neq 0$, $k \in K$, then $h_i(x^k) \neq 0$, and hence $\lambda_i^k h_i(x^k) = \rho_k [h_i(x^k)]^2 > 0$. Thus, (3.3) and (3.4) are fulfilled independently of the limits of the dual sequences. $\square$

Remark 3.4. Note that the AKKT condition is exactly the same as the PAKKT one, but without (3.3) and (3.4). In consequence, PAKKT implies AKKT, as mentioned in [5] when $X = \mathbb{R}^n$.

In [5] the authors provide the weakest strict constraint qualification for the PAKKT condition, which they call *PAKKT regular*. Using the terminology in [11], any property of feasible points of a constrained optimization problem which guarantees that an AKKT point is already a KKT point is called a *strict constraint qualification*. We will extend the PAKKT regular condition to the case in which an abstract set constraint is considered. For this purpose, given $x^* \in \Omega$, for $x \in X$, $\tilde{x} \in X$ and $\alpha, \beta \geqslant 0$, we define the set

$K_\Omega^+(x, \tilde{x}, \alpha, \beta)$

$$= \left\{ \sum_{i=1}^{m} \lambda_i \nabla h_i(x) + \sum_{j \in I_g(x^*)} \mu_j \nabla g_j(x) + \omega \;\middle|\; \begin{array}{l} \lambda_i h_i(x) \geqslant \alpha \text{ if } |\lambda_i| > \beta \|(1, \lambda, \mu)\|_\infty, \\ \mu_j g_j(x) \geqslant \alpha \text{ if } \mu_j > \beta \|(1, \lambda, \mu)\|_\infty, \\ \lambda \in \mathbb{R}^m, \, \mu \in \mathbb{R}_+^p, \, \mu_j = 0 \, (j \notin I_g(x^*)), \\ \omega \in N_X(\tilde{x}) \end{array} \right\}.$$

Remark 3.5. Observe that:

(a) The KKT conditions for the problem (1.1) can be written as $-\nabla f(x^*) \in K_\Omega^+(x^*, x^*, 0, 0)$.

(b) When $\alpha = \beta = 0$, we have that $K_\Omega^+(x, \tilde{x}, 0, 0) = K_\Omega(x, \tilde{x})$.

The definition of the PAKKT regular constraint qualification for problems with abstract set constraints imposes an outer semicontinuity-like condition on the multifunction $(x, \tilde{x}, \alpha, \beta) \in X \times X \times \mathbb{R}_+ \times \mathbb{R}_+ \rightrightarrows K_\Omega^+(x, \tilde{x}, \alpha, \beta)$.

Definition 3.6. We say that $x^* \in \Omega$ satisfies the *PAKKT regular condition* for the problem with abstract set constraints (1.1) if

$$\limsup_{\substack{(x,\tilde{x}) \to (x^*,x^*) \\ \alpha \downarrow 0, \, \beta \downarrow 0}} K_\Omega^+(x, \tilde{x}, \alpha, \beta) \subset K_\Omega^+(x^*, x^*, 0, 0).$$

Next we prove the main result of this section, which guarantees that PAKKT regular is the weakest constraint qualification for the PAKKT sequential optimality condition of Definition 3.2.

Theorem 3.7. *If x^* is a PAKKT point which fulfills the PAKKT regular condition, then x^* is a KKT point for (1.1). Reciprocally, if for every continuously differentiable function f the PAKKT point x^* is also KKT, then x^* satisfies the PAKKT regular condition of Definition 3.6.*

Proof. If x^* is a PAKKT point, there are sequences $\{x^k\} \subset X$, $\{\tilde{x}^k\} \subset X$, $\{\lambda^k\} \subset \mathbb{R}^m$, $\{\mu^k\} \subset \mathbb{R}_+^p$ and $\{\varepsilon^k\}$ such that $x^k \to x^*$, $\tilde{x}^k \to x^*$, $\varepsilon^k \to 0$ and conditions (3.1), (3.2), (3.3) and (3.4) hold. Following (3.2), we can suppose without loss of generality that $\mu_j^k = 0$ whenever $j \notin I_g(x^*)$.

Consider the sequence y^k defined by

$$y^k = \sum_{i=1}^m \lambda_i^k \nabla h_i(x^k) + \sum_{j=1}^p \mu_j^k \nabla g_j(x^k).$$

Then, by (3.1),

$$\omega^k := \varepsilon^k - \nabla f(x^k) - y^k \in N_X(\tilde{x}^k).$$

We consider $\delta_k = \|(1, \lambda^k, \mu^k)\|_\infty$. Let us define the sets $I_+ = \{i \in \{1, \ldots, m\} : \lim_k |\lambda_i^k|/\delta_k > 0\}$ and $J_+ = \{j \in I_g(x^*) : \lim_k \mu_j^k/\delta_k > 0\}$, and, for each k, we take

$$\alpha_k = \min\left\{\frac{1}{k}, \min_{i \in I_+}\{\lambda_i^k h_i(x^k)\}, \min_{j \in J_+}\{\mu_j^k g_j(x^k)\}\right\}$$

and

$$\beta_k = \max\left\{\frac{1}{k}, \max_{i \notin I_+} \frac{|\lambda_i^k|}{\delta_k}, \max_{j \notin J_+} \frac{\mu_j^k}{\delta_k}\right\} + \frac{1}{k}.$$

We note that $\alpha_k \downarrow 0$, $\beta_k \downarrow 0$ and $y^k + \omega^k \in K_\Omega^+(x^k, \tilde{x}^k, \alpha_k, \beta_k)$ for all k large enough. As x^* fulfills the PAKKT regular condition, we have

$$-\nabla f(x^*) = \lim_{k \to \infty} -\nabla f(x^k) + \varepsilon^k = \lim_{k \to \infty} y^k + \omega^k \in \limsup_{k \to \infty} K_\Omega^+(x^k, \tilde{x}^k, \alpha_k, \beta_k)$$

$$\subset \limsup_{\substack{(x,\tilde{x}) \to (x^*,x^*) \\ \alpha \downarrow 0, \, \beta \downarrow 0}} K_\Omega^+(x, \tilde{x}, \alpha, \beta) \subset K_\Omega^+(x^*, x^*, 0, 0),$$

that is, x^* is a KKT point for (1.1). This proves the first statement.

Now let us show the reciprocal. Let $v^* \in \limsup\limits_{(x,\tilde{x}) \to (x^*,x^*),\, \alpha \downarrow 0,\, \beta \downarrow 0} K_\Omega^+(x,\tilde{x},\alpha,\beta)$.
Then there are sequences $\{x^k\} \subset X$, $\{\tilde{x}^k\} \subset X$, $\{v^k\} \subset \mathbb{R}^n$, $\{\alpha_k\} \subset \mathbb{R}_+$,
$\{\beta_k\} \subset \mathbb{R}_+$ such that $x^k \to x^*$, $\tilde{x}^k \to x^*$, $v^k \to v^*$, $\alpha_k \downarrow 0$, $\beta_k \downarrow 0$ and
$v^k \in K_\Omega^+(x^k,\tilde{x}^k,\alpha_k,\beta_k)$ for all k. Furthermore, for each k, there are vectors
$\lambda^k \in \mathbb{R}^m$, $\mu^k \in \mathbb{R}_+^p$ and $\omega^k \in N_X(\tilde{x}^k)$ such that $\mu_j^k = 0$ if $j \notin I_g(x^*)$ and

$$v^k = \sum_{i=1}^m \lambda_i^k \nabla h_i(x^k) + \sum_{j \in I_g(x^*)} \mu_j^k \nabla g_j(x^k) + \omega^k. \qquad (3.9)$$

We define $f(x) = -(v^*)^t x$. If $\lim\limits_k |\lambda_i^k|/\delta_k > 0$, then $|\lambda_i^k| > \beta_k \delta_k$ for all k sufficiently
large (the same happens with μ^k). In other words, the control over the sign of
the multipliers performed by (3.3) and (3.4) is encapsulated in the expression $v^k \in$
$K_\Omega^+(x^k,\tilde{x}^k,\alpha_k,\beta_k)$. Therefore, by taking $\varepsilon^k = v^k - v^* \to 0$, we conclude that x^* is
a PAKKT point of (1.1). By hypothesis, x^* is a KKT point for (1.1), and hence
$-\nabla f(x^*) = v^* = \lim\limits_{k \to \infty} v^k \in K_\Omega^+(x^*,x^*,0,0)$. This concludes the proof. $\qquad \square$

As a consequence of Theorems 3.3 and 3.7, we have that any local minimizer
of (1.1) which satisfies the PAKKT regular condition is a KKT point for (1.1).
Equivalently, we can conclude that PAKKT regular is a constraint qualification for
problems with abstract set constraints.

The following lemma, which has been inspired by the proof of [5, Lemma 2.6],
provides a powerful tool to show that every KKT point for (1.1) always admits
Lagrange multipliers with adequate signs for the PAKKT condition.

Lemma 3.8. *Let $x^* \in \Omega$. The inclusion*

$$K_\Omega^+(x^*,x^*,0,0) \subseteq \limsup\limits_{\substack{(x,\tilde{x}) \to (x^*,x^*),\\ \alpha \downarrow 0\, \beta \downarrow 0}} K_\Omega^+(x,\tilde{x},\alpha,\beta)$$

is always true.

Proof. Let $v \in K_\Omega^+(x^*,x^*,0,0)$. Then there exist $\omega \in N_X(x^*)$ and vectors $\lambda \in \mathbb{R}^m$,
$\mu \in \mathbb{R}_+^p$ such that $\mu_j g_j(x^*) = 0$ and

$$v = \sum_{i=1}^m \lambda_i \nabla h_i(x^*) + \sum_{j=1}^p \mu_j \nabla g_j(x^*) + \omega.$$

Let us consider the sets I and J of indexes of nonzero multipliers λ_i and μ_j,
respectively. If $I = J = \emptyset$, $v \in \limsup\limits_{(x,\tilde{x}) \to (x^*,x^*),\, \alpha \downarrow 0,\, \beta \downarrow 0} K_\Omega^+(x,\tilde{x},\alpha,\beta)$ by taking $v^k = v$,
$x^k = \tilde{x}^k = x^*$ for all k and any sequences $\alpha_k \downarrow 0$, $\beta_k \downarrow 0$.

If at least one of the sets I and J is nonempty, according to [10, Lemma 1], there
are sets $\mathcal{I} \subset I$ and $\mathcal{J} \subset J$ as well as vectors $\hat{\lambda}_\mathcal{I}$, $\hat{\mu}_\mathcal{J}$ such that

$$v = \sum_{i \in \mathcal{I}} \hat{\lambda}_i \nabla h_i(x^*) + \sum_{j \in \mathcal{J}} \hat{\mu}_j \nabla g_j(x^*) + \omega,$$

with $\omega \in N_X(x^*)$, $\hat{\lambda}_i \neq 0$ for all $i \in \mathcal{I}$, $\hat{\mu}_j > 0$ for all $j \in \mathcal{J}$, and the set of gradients

$$\{\nabla h_i(x^*)\}_{i \in \mathcal{I}} \cup \{\nabla g_j(x^*)\}_{j \in \mathcal{J}}$$

is linearly independent.

Thus, using the same arguments as in the proof presented in [5, Lemma 2.6] we can show that there exists a sequence $\{x^k\}$ converging to x^* such that (3.3) and (3.4) are satisfied for the sequences $\{\tilde{x}^k\}$, $\{\lambda^k\}$, $\{\mu^k\}$ and $\{\varepsilon^k\}$ given by $\tilde{x}^k = x^*$, $\lambda^k = \hat{\lambda}$, $\mu^k = \hat{\mu}$ and $\varepsilon^k = 0$ for all k large enough, where we consider $\hat{\lambda}_i = 0$, $i \notin \mathcal{I}$ and $\hat{\mu}_j = 0$, $j \notin \mathcal{J}$. Since (3.3) and (3.4) hold, by the proof of Theorem 3.7, we observe that there are sequences $\alpha_k \downarrow 0$, $\beta_k \downarrow 0$ and a sequence $\{v^k\}$ such that $v^k \in K_\Omega^+(x^k, \tilde{x}^k, \alpha_k, \beta_k)$ and $v^k \to v$. This concludes the proof. $\qquad\square$

Corollary 3.9. *Every KKT point is a PAKKT point for* (1.1).

4. Relation between the PAKKT-regular condition and other constraint qualifications

In this section, as we have already mentioned in the Introduction, we analyze the relation between the PAKKT regular condition and the following well-known constraint qualifications, namely AKKT regularity, Abadie and quasi-normality.

In the following exposition, we generalize the definition of the AKKT regular constraint qualification introduced in [11] for problems with abstract set constraints.

Definition 4.1 (see [11]). We say that $x^* \in \Omega$ satisfies the *AKKT regular constraint qualification* for problem (1.1) if, given $x \in X$, $\tilde{x} \in X$, the multifunction $(x, \tilde{x}) \rightrightarrows K_\Omega(x, \tilde{x})$ is outer semicontinuous at (x^*, x^*), where $K_\Omega(x, \tilde{x})$ is given in (2.2). That is,

$$\limsup_{(x,\tilde{x}) \to (x^*, x^*)} K_\Omega(x, \tilde{x}) \subset K_\Omega(x^*, x^*).$$

Therefore, it can be proved that, as in the case in which $X = \mathbb{R}^n$, AKKT regular (see Definition 4.1) is the least stringent constraint qualification associated with the AKKT sequential condition (see Definition 3.1). Definition 4.1 corresponds to the natural extension of AKKT regular for problems which include an additional abstract set X.

Definition 4.2 (see [31, 37]). We say that a feasible point $x^* \in \Omega$ of problem (1.1) satisfies the *Abadie constraint qualification* if

$$L_\Omega(x^*) = T_\Omega(x^*)$$

holds and $K_\Omega(x^*, x^*)$ is closed.

The condition that the set $K_\Omega(x^*, x^*)$ must be closed is necessary in order to apply the Farkas lemma and characterize a solution to problem (1.1) using KKT conditions.

Definition 4.3 (see [37])**.** We say that $x^* \in \Omega$ is a *quasi-normal point* if there are no vectors $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}^p_+$ and no sequence $\{x^k\} \subset X$ such that

(1) $-\left(\sum_{i=1}^m \lambda_i \nabla h_i(x^*) + \sum_{j=1}^p \mu_j \nabla g_j(x^*) \right) \in N_X(x^*).$

(2) $\mu_j = 0$ if $j \notin I_g(x^*)$ for all $j = 1, \dots, p$.

(3) $\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_p$ are not all equal to 0.

(4) $\{x^k\}$ converges to x^* and, for each k, $\lambda_i h_i(x^k) > 0$ for all i with $\lambda_i \neq 0$ and $\mu_j g_j(x^k) > 0$ for all j with $\mu_j \neq 0$.

Quasi-normality is a general constraint qualification, introduced in [35] for classical nonlinear optimization problems with $X = \mathbb{R}^n$, and extended for problems with $X \neq \mathbb{R}^n$ in [37].

The following theorem is an extension of [5, Theorem 3.4] to (1.1).

Theorem 4.4. *If x^* is a quasi-normal point of* (1.1)*, then x^* is a PAKKT regular point of* (1.1)*.*

Proof. We suppose that x^* is not PAKKT regular. Then there exists

$$v^* \in \left(\limsup_{(x,\tilde{x}) \to (x^*,x^*),\, \alpha \downarrow 0,\, \beta \downarrow 0} K_\Omega^+(x, \tilde{x}, \alpha, \beta) \right) \setminus K_\Omega^+(x^*, x^*, 0, 0).$$

Let us take sequences $\{x^k\} \subset X$, $\{\tilde{x}^k\} \subset X$, $\{v^k\} \subset \mathbb{R}^n$, $\{\alpha_k\} \subset \mathbb{R}_+$ and $\{\beta_k\} \subset \mathbb{R}_+$ such that $x^k \to x^*$, $\tilde{x}^k \to x^*$, $v^k \to v^*$, $\alpha_k \downarrow 0$, $\beta_k \downarrow 0$ and $v^k \in K_\Omega^+(x^k, \tilde{x}^k, \alpha_k, \beta_k)$ for all $k \geqslant 1$. Then, for each k, there exist $\lambda^k \in \mathbb{R}^m$, $\mu^k \in \mathbb{R}^p_+$ and $\omega^k \in N_X(\tilde{x}^k)$ such that $\mu_j^k = 0$ if $j \notin I_g(x^*)$ and v^k is as in (3.9). Then, we have that

$$v^k - \sum_{i=1}^m \lambda_i^k \nabla h_i(x^k) - \sum_{j \in I_g(x^*)} \mu_j^k \nabla g_j(x^k) = \omega^k \in N_X(\tilde{x}^k). \qquad (4.1)$$

We define $\tilde{\delta}_k = \|(\lambda^k, \mu^k)\|_\infty$. The sequence $\{\tilde{\delta}_k\}$ must be unbounded because, otherwise, after taking an adequate sequence, we would have $v^* \in K_\Omega^+(x^*, x^*, 0, 0)$ since $v^k \in K_\Omega^+(x^k, \tilde{x}^k, \alpha_k, \beta_k)$ for all k. Thus, dividing (4.1) by $\tilde{\delta}_k$ and taking the limit on an appropriate subsequence, by Proposition 2.7, we obtain

$$-\left(\sum_{i=1}^m \lambda_i^* \nabla h_i(x^*) + \sum_{j \in I_g(x^*)} \mu_j^* \nabla g_j(x^*) \right) \in N_X(x^*),$$

where $(\lambda^*, \mu^*) \neq 0$ is a limit point of $\dfrac{(\lambda^k, \mu^k)}{\tilde{\delta}_k}$.

Given a neighborhood $\mathcal{B}(x^*)$ of x^*, we have, for some k large enough, $x^k \in \mathcal{B}(x^*)$ and $\mathrm{sgn}(\lambda_i^* h_i(x^k)) = \mathrm{sgn}(\lambda_i^k h_i(x^k)) = 1$ whenever $\lambda_i^* \neq 0$ (note that $\lim_k \lambda_i^k / \tilde{\delta}_k = \lambda_i^* \neq 0$ implies $|\lambda_i^k| > \beta_k \tilde{\delta}_k = \beta_k \|(\lambda^k, \mu^k)\|_\infty = \beta_k \|(1, \lambda^k, \mu^k)\|_\infty$ for all k sufficiently large, where the last equality holds since the sequence $\{\tilde{\delta}_k\}$ is unbounded). The same happens with μ^*. Hence, x^* does not satisfy the quasi-normality CQ, which completes the proof. $\qquad \square$

The relationship between the AKKT regular condition, for the case $X = \mathbb{R}^n$, and other constraint qualifications as Abadie's (presented in [1]) and quasi-normality (described in [16, 35]) can be found in [11].

To prove the relation between AKKT regular and Abadie for problems with abstract set constraints, we need the following lemmas.

Lemma 4.5. $(K_\Omega(x^*, x^*))^{\circ\circ} \subset (L_\Omega(x^*))^\circ$.

Proof. To justify this inclusion, we will prove that $L_\Omega(x^*) \subset (K_\Omega(x^*, x^*))^\circ$. Let $d \in L_\Omega(x^*)$. We want to prove that $d^t s \leqslant 0$ for any $s \in K_\Omega(x^*, x^*)$.

Let

$$s = \sum_{i=1}^m \lambda_i \nabla h_i(x^*) + \sum_{j=1}^p \mu_j \nabla g_j(x^*) + \omega,$$

with $\mu_j \geqslant 0$, $\mu_j g_j(x^*) = 0$ and $\omega \in N_X(x^*)$. Then,

$$s^t d = \sum_{i=1}^m \lambda_i \nabla h_i(x^*)^t d + \sum_{j=1}^p \mu_j \nabla g_j(x^*)^t d + \omega^t d.$$

Since $d \in L_\Omega(x^*)$, we know that $\nabla h_i(x^*)^t d = 0$ for all $i = 1, \ldots, m$, and $\nabla g_j(x^*)^t d \leqslant 0$ for all $j = 1, \ldots, p$. Then,

$$s^t d \leqslant \omega^t d.$$

By Assumption A, we have that $N_X(x^*) = (T_X(x^*))^\circ$ and $\omega^t d \leqslant 0$ since $d \in T_X(x^*)$ and $\omega \in N_X(x^*)$. In consequence, $s^t d \leqslant 0$ and $d \in (K_\Omega(x^*, x^*))^\circ$. Thus, the lemma is obtained by the polarity relation: if $A \subset B$ implies $B^\circ \subset A^\circ$, for A, B cones. $\qquad\square$

Lemma 4.6. *For all $x^* \in \Omega$ and $v \in (T_\Omega(x^*))^\circ$, there exist sequences $\{x^k\} \subset X$, $\{\lambda^k\} \subset \mathbb{R}^m$, $\{\mu^k\} \subset \mathbb{R}_+^p$ and $\tilde{\omega}^k \in N_X(x^k)$ for each $k \geqslant 1$ such that*

(a) $\displaystyle\sum_{i=1}^m \lambda_i^k \nabla h_i(x^k) + \sum_{j=1}^p \mu_j^k \nabla g_j(x^k) + \tilde{\omega}^k \to v$ *as $k \to \infty$,*

(b) $\lambda^k = kh(x^k)$ *and* $\mu^k = kg(x^k)_+$.

Proof. Let $v \in (T_\Omega(x^*))^\circ$. Following [42, Theorem 6.11], there exists a smooth function F such that $-\nabla F(x^*) = v$, where x^* is the unique global minimum of F relative to Ω.

From now on, the proof of (a) and (b) follows directly that of [37, Proposition 2.1]. $\qquad\square$

In the following theorem, we will prove that PAKKT regular is stronger than Abadie's constraint qualification.

Theorem 4.7. *If $x^* \in \Omega$ satisfies the PAKKT regular condition, then it satisfies Abadie's constraint qualification.*

Proof. Firstly, observe that, according to Lemma 3.8, equality holds in Definition 3.6: PAKKT regularity implies that

$$\limsup_{\substack{(x,\tilde{x}) \to (x^*,x^*) \\ \alpha \downarrow 0,\, \beta \downarrow 0}} K_\Omega^+(x,\tilde{x},\alpha,\beta) = K_\Omega^+(x^*,x^*,0,0).$$

Thus, since the outer limit is always a closed set, this yields that $K_\Omega^+(x^*,x^*,0,0)$ is closed. Then, by Remark 3.5 (b), $K_\Omega(x^*,x^*)$ is a closed set.

Now our aim is to prove that $T_\Omega(x^*) = L_\Omega(x^*)$. The inclusion $T_\Omega(x^*) \subset L_\Omega(x^*)$ is always true. However, to prove the opposite inclusion, $L_\Omega(x^*) \subset T_\Omega(x^*)$, we will first prove that $N_\Omega(x^*) \subset K_\Omega^+(x^*,x^*,0,0)$.

Let $z \in N_\Omega(x^*)$. By (2.1), there exist sequences $\{x^l\} \subset \Omega$, $\{z^l\} \subset \mathbb{R}^n$ such that $x^l \to x^*$, $z^l \to z$ and $z^l \in \left(T_\Omega(x^l)\right)^\circ$. Then, following Lemma 4.6, for each $l \in \mathbb{N}$, there exist sequences $\{x_l^k\}_k \subset X$, $\{\lambda_l^k\}_k \subset \mathbb{R}^m$, $\{\mu_l^k\}_k \subset \mathbb{R}_+^p$ and $\{\tilde{\omega}_l^k\}_k$, $\tilde{\omega}_l^k \in N_X(x_l^k)$ for each $k \geqslant 1$ such that items (a) and (b) hold. This means that

$$v_l^k := \sum_{i=1}^m k h_i(x_l^k) \nabla h_i(x_l^k) + \sum_{j=1}^p k g_j(x_l^k)_+ \nabla g_j(x_l^k) + \tilde{\omega}_l^k \to z^l \quad \text{as } k \to \infty.$$

The multipliers in item (b) of Lemma 4.6 have the same sign of their corresponding constraints for all k: $\mathrm{sgn}(k h_i(x_l^k)) = \mathrm{sgn}(h_i(x_l^k))$ and $\mathrm{sgn}(k g_j(x_l^k)_+) = \mathrm{sgn}(g_j(x_l^k)_+)$. Therefore, there are appropriate sequences $\{x_{l(k)}^k\}_k$, $\{\lambda_{l(k)}^k\}_k$, $\{\mu_{l(k)}^k\}_k$, $\{v_{l(k)}^k\}_k$ and sequences of scalars $\{\alpha_k\}$, $\{\beta_k\}$ such that $v_{l(k)}^k \in K_\Omega^+(x_{l(k)}^k, x_{l(k)}^k, \alpha_k, \beta_k)$, $\lim_{k\to\infty} x_{l(k)}^k = x^*$ and $\lim_{k\to\infty} v_{l(k)}^k = z$. Thus, using the PAKKT regularity condition, we obtain $z \in \limsup_{k\to\infty} K_\Omega^+(x_{l(k)}^k, x_{l(k)}^k, \alpha_k, \beta_k) \subset K_\Omega^+(x^*,x^*,0,0)$. Therefore, $N_\Omega(x^*) \subset K_\Omega^+(x^*,x^*,0,0) = K_\Omega(x^*,x^*)$, as we wanted to prove.

Then, on the one hand, following Remark 3.5 (b) and Lemma 4.5, we conclude that

$$N_\Omega(x^*) \subset K_\Omega(x^*,x^*) \subset (K_\Omega(x^*,x^*))^{\circ\circ} \subset (L_\Omega(x^*))^\circ. \tag{4.2}$$

On the other hand, the inclusion $T_\Omega(x^*) \subset L_\Omega(x^*)$ always holds, so, by polarity, $(L_\Omega(x^*))^\circ \subset (T_\Omega(x^*))^\circ$. Thus, using (4.2), we have that

$$N_\Omega(x^*) \subset (T_\Omega(x^*))^\circ, \tag{4.3}$$

which means that Ω is regular at x^*, since $(T_\Omega(x^*))^\circ \subset N_\Omega(x^*)$ always holds by [42, Theorem 6.28 (a)].

Finally, according to the polarity in (4.2), (4.3) and [42, Corollary 6.30], we obtain

$$L_\Omega(x^*) \subset (L_\Omega(x^*))^{\circ\circ} \subset (N_\Omega(x^*))^\circ \subset T_\Omega(x^*),$$

as we wanted to prove. $\square$

Note that $K_\Omega^+(x,\tilde{x},\alpha,\beta) \subset K_\Omega^+(x,\tilde{x},0,0) = K_\Omega(x,\tilde{x})$ for all $x \in \mathbb{R}^n$, $\tilde{x} \in X$ and $\alpha, \beta > 0$. Then, we obtain the next result.

Theorem 4.8. *If $x^* \in \Omega$ satisfies the AKKT regular condition, then it satisfies the PAKKT regular constraint qualification.*

As it happens when $X = \mathbb{R}^n$, AKKT regularity and quasi-normality are independent constraint qualifications, as described in [11].

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┌──────────────┐                    ┌──────────────────┐
│ AKKT regular │                    │ Quasi-normality  │
└──────────────┘                    └──────────────────┘
         ╲                              ╱
          ╲              ┌──────────────────┐
           ──────────────│  PAKKT regular   │
                         └──────────────────┘
                                  │
                          ┌──────────────┐
                          │    Abadie    │
                          └──────────────┘
```
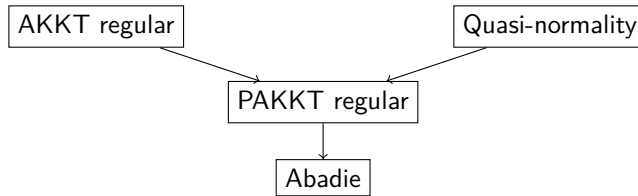
FIGURE 1. Relations between the PAKKT regularity condition and other constraint qualifications. An arrow between two conditions means that one is strictly stronger than the other.

The present paper contributes mainly to the development of first-order optimality conditions in the sequential approximate form (AKKT) for (1.1) when the additional abstract set X satisfies Assumption A. However, in recent years many optimality conditions and constraint qualifications have been successfully generalized to minimize objective functions on smooth manifolds; see for example [15, 44] and references therein.

We assume, as in [44], that X is a d-dimensional Riemannian manifold in $\mathbb{R}^n$ given by
$$X = \{x \in \mathbb{R}^n : \Phi(x) = 0\},$$
where $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth mapping for which $J\Phi(x)$ (Jacobian matrix of Φ at x) is of full row rank m (where $m = n - d$) for all $x \in X$. Thus, by [42, Example 6.8], we have that X is regular and
$$T_X(x) = \{w \in \mathbb{R}^n : J\Phi(x)w = 0\}$$
$$N_X(x) = \{J\Phi(x)^t y : y \in \mathbb{R}^m\},$$
which shows that the tangent and the normal cones are linear subspaces orthogonally complementary to each other.

Therefore, in this particular case, we have that condition (a) of Definition 2.5 is equivalent to
$$-\left(\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla h_i(x^*) + \sum_{j=1}^{p} \mu_j \nabla g_j(x^*)\right) = \sum_{i=1}^{d} \gamma_i \nabla \Phi_i(x^*), \qquad (4.4)$$
where $\gamma \in \mathbb{R}^d$.

Then, taking into account that the results of the present work are developed for the differentiable case, we can state that the first-order optimality condition (5.20) in [44] directly follows from equality (4.4) and condition (b) of Definition 2.5.

From our research, we realize that many real-world problems are formulated as (1.1) when X is a manifold constraint, not necessarily given in a functional equality and/or inequality constraint form. For example, the group of special orthogonal matrices $SO(n)$, the Grassmannian manifold and the Stiefel manifold problems developed in [2]. Consequently, the study of sequential optimality conditions for optimization subject to smooth manifolds is a challenge for future analysis, as the previously mentioned examples show.

5. An augmented Lagrangian method for problems with abstract constraints

The authors in [5] have recently proved that PAKKT regular constraint qualification and quasi-normality can be used to validate the convergence of the ALM for general nonlinear optimization problems with $X = \mathbb{R}^n$. As mentioned in [5], it can be proved that the ALM generates PAKKT limit points but not necessarily PAKKT sequences.

In this section, we are interested in the analysis of the convergence of the ALM applied to (1.1).

To penalize the set of equality and inequality constraints we consider the quadratic penalty augmented Lagrangian function (see [21]). Thus, we have the following form:

$$L(x, \lambda, \mu, \rho) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{i=1}^{m} \frac{\rho}{2}(h_i(x))^2 + \sum_{j=1}^{p} \frac{1}{2\rho}(\max\{0, \mu_j + \rho g_j(x)\}^2 - \mu_j^2).$$

We focus on the case in which X is a closed and convex set, perhaps not described by a finite number of equalities or inequalities as in [3]. However, we consider that computing the Euclidean projection onto X is affordable.

Assumption B. Throughout this section, we assume that the abstract set X in (1.1) is closed and convex.

Algorithm 1. Let $x^0 \in X$ be an arbitrary initial point. The given parameters for the execution of the algorithm are $\tau \in [0, 1)$, $\gamma > 1$, $-\infty < \bar{\lambda}_i^{\min} < \bar{\lambda}_i^{\max} < \infty$ $\forall i = 1, \ldots, m$, $\bar{\lambda}_i^0 \in [\bar{\lambda}_i^{\min}, \bar{\lambda}_i^{\max}]$ $\forall i = 1, \ldots, m$, $0 \leqslant \bar{\mu}_i^{\max} < \infty$ $\forall i = 1, \ldots, p$, $\rho_0 \in \mathbb{R}_{++}$, $\bar{\mu}_i^0, \bar{\mu}_i^1 \in [0, \bar{\mu}_i^{\max}]$, $\sigma_i^0 = \dfrac{\bar{\mu}_i^1 - \bar{\mu}_i^0}{\rho_0}$ $\forall i = 1, \ldots, p$, $\{\epsilon_k\} \subset \mathbb{R}$, $\lim\limits_{k \to \infty} \epsilon_k = 0$.

Initialize $k \leftarrow 1$.

Step 1. Find an approximate solution $x^k \in X$ of the subproblem

$$\begin{aligned} &\text{Minimize } L(x, \bar{\lambda}^k, \bar{\mu}^k, \rho_k) \\ &\text{s. t. } x \in X, \end{aligned} \tag{5.1}$$

that is, compute a point $x^k \in X$ satisfying

$$\left\| P_X\left(x^k - \nabla_x L\left(x^k, \bar{\lambda}^k, \bar{\mu}^k, \rho_k\right)\right) - x^k \right\|_\infty \leqslant \epsilon_k. \tag{5.2}$$

If this is not possible, stop the execution of the algorithm.

Step 2. Estimate new multipliers and define a new infeasibility and complementarity measure.

For $i = 1, \ldots, m$ compute

$$\lambda_i^{k+1} = \bar{\lambda}_i^k + \rho_k h_i(x^k) \quad \text{and} \quad \bar{\lambda}_i^{k+1} := P_{[\bar{\lambda}_i^{\min}, \bar{\lambda}_i^{\max}]}(\lambda_i^{k+1}). \tag{5.3}$$

For $j = 1, \ldots, p$ compute

$$\mu_j^{k+1} = \max\left\{0, \bar{\mu}_j^k + \rho_k g_j(x^k)\right\} \quad \text{and} \quad \bar{\mu}_j^{k+1} := P_{[0, \bar{\mu}_j^{\max}]}(\mu_j^{k+1}). \tag{5.4}$$

Define

$$\sigma_j^k := \max\left\{g_j(x^k), -\frac{\bar{\mu}_j^k}{\rho_k}\right\}.$$

Step 3. Update the penalty parameter.

If $k = 0$ or

$$\max\{\|h(x^k)\|_\infty, \|\sigma^k\|_\infty\} \leqslant \tau \max\{\|h(x^{k-1})\|_\infty, \|\sigma^{k-1}\|_\infty\},$$

define $\rho_{k+1} = \rho_k$. Else, define $\rho_{k+1} = \gamma\rho_k$.

Step 4. Set $k \leftarrow k + 1$ and go to Step 1.

Let us denote by $l(x, \lambda, \mu)$ the usual Lagrangian function for the optimization problem (1.1):

$$l(x, \lambda, \mu) = f(x) + \sum_{i=1}^m \lambda_i h_i(x) + \sum_{j=1}^p \mu_j g_j(x).$$

By Assumption B, the projection is well defined and condition (a) of Definition 2.5 is equivalent to

$$P_X(x^* - \nabla_x l(x^*, \lambda, \mu)) - x^* = 0. \tag{5.5}$$

In this case, the projection satisfies the following property.

Proposition 5.1 (see [42])**.** *Let S be a nonempty convex closed set and $x \in S$. Then, $\omega \in N_S(x)$ if and only if $P_S(x + \omega) = x$.*

Observe that, by combining (5.5) and Proposition 5.1, (5.2) is a natural approximate condition for the subproblem (5.1).

When X is a box ($X = \{x \in \mathbb{R}^n : l \leqslant x \leqslant u\}$), the active set method GENCAN can be used to obtain (5.2); see for example [21, Chapter 9] and [19, 20]. If n is large, the spectral projected gradient (SPG) method mentioned in [24, 25, 22, 23] is an alternative to solve the subproblem (5.1).

We aim for the feasibility of the limit points, but we know that it could be impossible since we realize that feasible points might not exist at all. In the following theorem, we prove a "feasibility result" which determines that those limit points of the sequence generated by Algorithm 1 are stationary points for some infeasibility measure of the constraints of the feasible set Ω.

Theorem 5.2. *Assume that the sequence $\{x^k\}$ is an infinite sequence generated by Algorithm 1. Then, every limit point x^* is a KKT point of*

$$Minimize \ \tfrac{1}{2}\left(\|h(x)\|_2^2 + \|g(x)_+\|_2^2\right)$$
$$s. \ t. \ x \in X. \tag{5.6}$$

Proof. Let $\{x^k\}$, $\{\lambda^k\}$, $\{\mu^k\}$ and $\{\rho_k\}$ be sequences generated by Algorithm 1. Let x^* be a limit point of $\{x^k\}$. Then, there exists a subset $K \subset \mathbb{N}$ such that $\lim_{k \in K} x^k = x^*$.

Since $x^k \in X$ for all k and X is closed, we have that $x^* \in X$.

Now, we consider two possibilities: (a) the sequence $\{\rho_k\}$ is bounded, and (b) the sequence $\{\rho_k\}$ is unbounded.

In case (a), from some iteration on, the penalty parameter is not updated. Therefore, $\lim_{k \to \infty} \|h(x^k)\|_\infty = 0$ and $\lim_{k \to \infty} \|\sigma^k\|_\infty = 0$. Now, if $g_j(x^*) > 0$, then $g_j(x^k) > c > 0$ for $k \in K$ large enough and this would contradict the fact that $\lim_{k \to \infty} \|\sigma^k\| = 0$. Therefore, $g_j(x^*) \leqslant 0$ for $j = 1, \ldots, p$.

Consider now case (b). Let $y^k = P_X\left(x^k - \nabla_x L\left(x^k, \bar{\lambda}^k, \bar{\mu}^k, \rho_k\right)\right) - x^k$. Then, $\|y^k\| \leqslant \epsilon_k$, and by Proposition 5.1, we have that

$$-\nabla_x L\left(x^k, \bar{\lambda}^k, \bar{\mu}^k, \rho_k\right) - y^k \in N_X(x^k + y^k).$$

Since $\rho_k \to \infty$ and $\{\bar{\mu}^k\}$ is a bounded sequence, we have that, for $k \in K$ large enough, $\max\left\{0, \bar{\mu}_j^k + \rho_k g_j(x^k)\right\} = 0$ whenever $g_j(x^*) < 0$. Therefore,

$$-\nabla f(x^k) - \sum_{i=1}^m (\bar{\lambda}_i^k + \rho_k h_i(x^k))\nabla h_i(x^k)$$
$$- \sum_{\{j: g_j(x^*) \geqslant 0\}} \max\left\{0, \bar{\mu}_j^k + \rho_k g_j(x^k)\right\} \nabla g_j(x^k) - y^k \in N_X(x^k + y^k). \tag{5.7}$$

Dividing by $\rho_k \geqslant 1$ in (5.7) and using the continuity of ∇f, ∇h_i, ∇g_j, the boundedness of the sequences $\{\bar{\lambda}^k\}$, $\{\bar{\mu}^k\}$, the definition of the normal cone and the fact that $\lim_{k \to \infty} \epsilon_k = 0$, taking the limit on an appropriate subsequence, we obtain

$$-\sum_{i=1}^m h_i(x^*)\nabla h_i(x^*) - \sum_{j: g_j(x^*) \geqslant 0} \max\left\{0, g_j(x^*)\right\} \nabla g_j(x^*) \in N_X(x^*),$$

and x^* is a KKT point of (5.6). $\qquad\square$

These kinds of feasibility results were previously proved in [3, 21] considering that the lower-level set, given the structure

$$\{x \in \mathbb{R}^n : \tilde{h}(x) = 0, \ \tilde{g}(x) \leqslant 0\} \tag{5.8}$$

for $\tilde{h} = (\tilde{h}_1, \ldots, \tilde{h}_{\tilde{m}})^t : \mathbb{R}^n \to \mathbb{R}^{\tilde{m}}$, $\tilde{g} = (\tilde{g}_1, \ldots, \tilde{g}_{\tilde{p}})^t : \mathbb{R}^n \to \mathbb{R}^{\tilde{p}}$ continuously differentiable in $\mathbb{R}^n$, satisfies the CPLD CQ. Then, if an abstract set X has structure (5.8), the results of Theorem 5.2 and [3, Theorem 4.2] are independent, since the

following conditions are not mutually implied: (i) X is convex and closed; (ii) the CPLD CQ with respect to X given by (5.8) holds.

In the following theorem, we prove that under the PAKKT regular constraint qualification, feasible limit points are stationary (KKT) points of the original problem (1.1). This is the strongest result about global convergence of the ALM for problems with abstract set constraints which can be proved.

Theorem 5.3. *Assume that the sequence $\{x^k\}$ is an infinite sequence generated by Algorithm 1. Then, every limit point x^* which satisfies the PAKKT regular constraint qualification is a KKT point for* (1.1).

Proof. Let $\{x^k\}$, $\{\lambda^k\}$, $\{\mu^k\}$ and $\{\rho_k\}$ be sequences generated by Algorithm 1. Let x^* be a limit point of $\{x^k\}$. Then, there exists a subset $K \subset \mathbb{N}$ such that $\lim_{k \in K} x^k = x^*$.

Following (5.2) and Proposition 5.1, we have

$$\omega^k := -\nabla_x L\left(x^k, \bar{\lambda}^k, \bar{\mu}^k, \rho_k\right) - y^k \in N_X(x^k + y^k),$$

where $y^k = P_X\left(x^k - \nabla_x L\left(x^k, \bar{\lambda}^k, \bar{\mu}^k, \rho_k\right)\right) - x^k$ and $\left\|y^k\right\| \leqslant \epsilon_k$.

Then, by (5.3) and (5.4), we have

$$\nabla f(x^k) + \sum_{i=1}^m \lambda_i^{k+1}\nabla h_i(x^k) + \sum_{j=1}^p \mu_j^{k+1}\nabla g_j(x^k) + \omega^k + y^k = 0 \qquad (5.9)$$

with $\omega^k \in N_X(x^k + y^k)$.

If $\rho_k \to \infty$, then $\mu_j^{k+1} = 0$ whenever $g_j(x^*) < 0$ and k is sufficiently large. If $\{\rho_k\}$ is bounded, then $\lim_{k \to \infty} \sigma^k = 0$, and thus $\lim_{k \to \infty} \mu_j^{k+1} = 0$ whenever $g_j(x^*) < 0$.

Thus, we can suppose without loss of generality that $\mu_j^{k+1} = 0$ whenever $g_j(x^*) < 0$.

Let us define $\delta_{k+1} = \|(1, \lambda^{k+1}, \mu^{k+1})\|_\infty$. If we assume that $\{\delta_{k+1}\}$ is bounded, then, by (5.9), we have

$$\omega^k = -\nabla f(x^k) - \sum_{i=1}^m \lambda_i^{k+1}\nabla h_i(x^k) - \sum_{j=1}^p \mu_j^{k+1}\nabla g_j(x^k) - y^k$$

with $\omega^k \in N_X(x^k + y^k)$. Since $\{\delta_{k+1}\}$ is bounded, we can extract a convergent subsequence, and there exist $K_1 \subset K$ and $\lambda \in \mathbb{R}^m, \mu \in \mathbb{R}_+^p$ such that

$$\lim_{k \in K_1} \omega^k = -\nabla f(x^*) - \sum_{i=1}^m \lambda_i\nabla h_i(x^*) - \sum_{j=1}^p \mu_j\nabla g_j(x^*) \qquad (5.10)$$

due to the continuity of the gradients and the fact that $\left\|y^k\right\| \leqslant \epsilon_k$, $\lim_{k \to \infty} \epsilon_k = 0$. Thus, according to Proposition 2.7, (5.10) implies that x^* is a KKT point of problem (1.1).

If we assume that $\{\delta_{k+1}\}$ is unbounded, we define the sets

$$I_+ = \left\{ i \in \{1, \ldots, m\} : \lim_k \frac{|\lambda_i^{k+1}|}{\delta_{k+1}} > 0 \right\},$$

$$J_+ = \left\{ j \in I_g(x^*) : \lim_k \frac{\mu_j^{k+1}}{\delta_{k+1}} > 0 \right\},$$

and for each k we take

$$\alpha_k = \min \left\{ \frac{1}{k}, \min_{i \in I_+} \{\lambda_i^{k+1} h_i(x^k)\}, \min_{j \in J_+} \{\mu_j^{k+1} g_j(x^k)\} \right\}$$

and

$$\beta_k = \max \left\{ \frac{1}{k}, \max_{i \notin I_+} \frac{|\lambda_i^{k+1}|}{\delta_{k+1}}, \max_{j \notin J_+} \frac{\mu_j^{k+1}}{\delta_{k+1}} \right\} + \frac{1}{k}.$$

We note that $\alpha_k \downarrow 0$, $\beta_k \downarrow 0$, and we have that

$$\begin{aligned}
\lambda_i^{k+1} h_i(x^k) &\geqslant \alpha_k \quad \text{when } |\lambda_i^{k+1}| > \beta_k \delta_{k+1}, \\
\mu_j^{k+1} g_j(x^k) &\geqslant \alpha_k \quad \text{when } \mu_j^{k+1} > \beta_k \delta_{k+1}.
\end{aligned} \tag{5.11}$$

Thus, by (5.9) and (5.11), if we define $\tilde{x}^k = x^k + y^k$, we can affirm that, for all k large enough, $-\nabla f(x^k) - y^k \in K_\Omega^+(x^k, \tilde{x}^k, \alpha_k, \beta_k)$. As x^* fulfills the PAKKT regular condition and $\lim_{k \to \infty} \epsilon_k = 0$, we have

$$\begin{aligned}
-\nabla f(x^*) = \lim_{k \to \infty} -\nabla f(x^k) - y^k &\in \limsup_{k \to \infty} K_\Omega^+(x^k, \tilde{x}^k, \alpha_k, \beta_k) \\
&\subset \limsup_{\substack{(x, \tilde{x}) \to (x^*, x^*) \\ \alpha \downarrow 0, \beta \downarrow 0}} K_\Omega^+(x, \tilde{x}, \alpha, \beta) \subset K_\Omega^+(x^*, x^*, 0, 0),
\end{aligned}$$

that is, x^* is a KKT point of problem (1.1). $\qquad\square$

It is worth mentioning that the global convergence result for the proposed ALM has been obtained using the least restrictive constraint qualification known in the literature in this context. In [3, 21], the authors established global convergence using the CPLD CQ. Since PAKKT regularity is weaker than CPLD CQ, the theoretical result given in the previous theorem is stronger than the theoretical results proved with CPLD CQ.

In [7], a scaled version of the PAKKT condition is defined when the abstract set X is a box, and a variation of the ALGENCAN (the well-established safeguarded ALM has been defined in [3, 21]) is studied by carrying out a vast amount of numerical experiments.

We know that the global convergence of the ALM has been substantially improved over the last years due to the introduction of sequential optimality conditions. For $X = \mathbb{R}^n$, there are stronger sequential optimality conditions than AKKT such as the CAKKT [13] and PCAKKT [14] conditions:

- A feasible point $x^* \in \Omega$, $X = \mathbb{R}^n$ is a complementary approximate KKT (CAKKT) point [13] if there are sequences $\{x^k\} \subset X$, $\{\lambda^k\} \subset \mathbb{R}^m$, and $\{\mu^k\} \subset \mathbb{R}_+^p$ such that $\lim_{k \to \infty} x^k = x^*$ and

$$\lim_{k \to \infty} \nabla f(x^k) + \sum_{i=1}^m \lambda_i^k \nabla h_i(x^k) + \sum_{j=1}^p \mu_j^k \nabla g_j(x^k) = 0,$$

$$\lim_{k \to \infty} \lambda_i^k h_i(x^k) = 0 \quad \text{and} \quad \lim_{k \to \infty} \mu_j^k g_j(x^k) = 0.$$

- A feasible point $x^* \in \Omega$, $X = \mathbb{R}^n$ is a positive complementary AKKT (PCAKKT) point if x^* is a CAKKT point and conditions (3.3) and (3.4) of Definition 3.2 hold.

In [14], it is proved that PCAKKT is stronger than CAKKT and PAKKT, and in [5], the authors prove that PAKKT is independent of the CAKKT condition.

In [13, Theorem 5.1] (respectively, in [14, Theorem 3.1]), the following is demonstrated: If x^* is a feasible limit point of the sequence $\{x^k\}$ generated by the safeguarded ALM (Algorithm 1 for $X = \mathbb{R}^n$), then x^* is a CAKKT (respectively, PCAKKT) point, provided that a generalized Lojasiewicz (GL) inequality is satisfied at the limit point (see [13, Section 3]).

We aim to study suitable extensions and applications of CAKKT and PCAKKT for problem (1.1) in the future.

6. Conclusions

In this work, we have shown that the positive approximate KKT condition can be generalized for problems with abstract set constraints. We have also presented the appropriate strict constraint qualification associated with the PAKKT sequential optimality condition.

As necessary sequential optimality conditions provide a natural stopping criterion for nonlinear iterative methods, we have studied the convergence of an ALM for solving problems with equality, inequality and abstract set constraints under the new constraint qualification. Furthermore, we have shown that every limit point generated by the algorithm is a stationary point of a problem of minimizing the infeasibility of the equality and inequality constraints subject to the abstract set.

Finally, it is important to mention that the study of sequential optimality conditions for optimization problems subject to particular abstract constraints, such as positive definiteness or Grassmannian/Stiefel manifolds, smooth or nonsmooth, remains a challenge for future analysis.

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References

[1] J. Abadie, On the Kuhn-Tucker theorem, in *Nonlinear Programming (NATO Summer School, Menton, 1964)*, North-Holland, Amsterdam, 1967, pp. 19–36. MR Zbl

[2] P.-A. Absil, R. Mahony, and R. Sepulchre, *Optimization algorithms on matrix manifolds*, Princeton University Press, Princeton, NJ, 2008. DOI MR Zbl

[3] R. Andreani, E. G. Birgin, J. M. Martínez, and M. L. Schuverdt, On augmented Lagrangian methods with general lower-level constraints, *SIAM J. Optim.* **18** no. 4 (2007), 1286–1309. DOI MR Zbl

[4] R. Andreani, E. G. Birgin, J. M. Martínez, and M. L. Schuverdt, Augmented Lagrangian methods under the constant positive linear dependence constraint qualification, *Math. Program.* **111** no. 1-2, Ser. B (2008), 5–32. DOI MR Zbl

[5] R. Andreani, N. S. Fazzio, M. L. Schuverdt, and L. D. Secchin, A sequential optimality condition related to the quasi-normality constraint qualification and its algorithmic consequences, *SIAM J. Optim.* **29** no. 1 (2019), 743–766. DOI MR Zbl

[6] R. Andreani, W. Gómez, G. Haeser, L. M. Mito, and A. Ramos, On optimality conditions for nonlinear conic programming, *Math. Oper. Res.* **47** no. 3 (2022), 2160–2185. DOI MR Zbl

[7] R. Andreani, G. Haeser, M. L. Schuverdt, L. D. Secchin, and P. J. S. Silva, On scaled stopping criteria for a safeguarded augmented Lagrangian method with theoretical guarantees, *Math. Program. Comput.* **14** no. 1 (2022), 121–146. DOI MR Zbl

[8] R. Andreani, G. Haeser, L. D. Secchin, and P. J. S. Silva, New sequential optimality conditions for mathematical programs with complementarity constraints and algorithmic consequences, *SIAM J. Optim.* **29** no. 4 (2019), 3201–3230. DOI MR Zbl

[9] R. Andreani, G. Haeser, and J. M. Martínez, On sequential optimality conditions for smooth constrained optimization, *Optimization* **60** no. 5 (2011), 627–641. DOI MR Zbl

[10] R. Andreani, G. Haeser, M. L. Schuverdt, and P. J. S. Silva, A relaxed constant positive linear dependence constraint qualification and applications, *Math. Program.* **135** no. 1-2, Ser. A (2012), 255–273. DOI MR Zbl

[11] R. Andreani, J. M. Martínez, A. Ramos, and P. J. S. Silva, A cone-continuity constraint qualification and algorithmic consequences, *SIAM J. Optim.* **26** no. 1 (2016), 96–110. DOI MR Zbl

[12] R. Andreani, J. M. Martinez, and M. L. Schuverdt, On the relation between constant positive linear dependence condition and quasinormality constraint qualification, *J. Optim. Theory Appl.* **125** no. 2 (2005), 473–485. DOI MR Zbl

[13] R. Andreani, J. M. Martínez, and B. F. Svaiter, A new sequential optimality condition for constrained optimization and algorithmic consequences, *SIAM J. Optim.* **20** no. 6 (2010), 3533–3554. DOI MR Zbl

[14] R. Andreani, A. Ramos, A. A. Ribeiro, L. D. Secchin, and A. R. Velazco, On the convergence of augmented Lagrangian strategies for nonlinear programming, *IMA J. Numer. Anal.* **42** no. 2 (2022), 1735–1765. DOI MR Zbl

[15] R. Bergmann and R. Herzog, Intrinsic formulation of KKT conditions and constraint qualifications on smooth manifolds, *SIAM J. Optim.* **29** no. 4 (2019), 2423–2444. DOI MR Zbl

[16] D. P. Bertsekas, *Nonlinear programming*, second ed., Athena Scientific, Belmont, MA, 1999. MR Zbl

[17] D. P. Bertsekas, A. Nedić, and A. E. Ozdaglar, *Convex analysis and optimization*, Athena Scientific, Belmont, MA, 2003. MR Zbl

[18] D. P. Bertsekas and A. E. Ozdaglar, Pseudonormality and a Lagrange multiplier theory for constrained optimization, *J. Optim. Theory Appl.* **114** no. 2 (2002), 287–343. DOI MR Zbl

[19] E. G. Birgin and J. M. Martínez, A box-constrained optimization algorithm with negative curvature directions and spectral projected gradients, in *Topics in numerical analysis*, Comput. Suppl. 15, Springer, Vienna, 2001, pp. 49–60. DOI MR Zbl

[20] E. G. Birgin and J. M. Martínez, Large-scale active-set box-constrained optimization method with spectral projected gradients, *Comput. Optim. Appl.* **23** no. 1 (2002), 101–125. DOI MR Zbl

[21] E. G. Birgin and J. M. Martínez, *Practical augmented Lagrangian methods for constrained optimization*, Fundamentals of Algorithms 10, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2014. DOI MR Zbl

[22] E. G. Birgin, J. M. Martínez, and M. Raydan, Nonmonotone spectral projected gradient methods on convex sets, *SIAM J. Optim.* **10** no. 4 (2000), 1196–1211. DOI MR Zbl

[23] E. G. Birgin, J. M. Martínez, and M. Raydan, Algorithm 813: SPG – software for convex-constrained optimization, *ACM Trans. Math. Softw.* **27** no. 3 (2001), 340–349. DOI Zbl

[24] E. G. Birgin, J. M. Martínez, and M. Raydan, Inexact spectral projected gradient methods on convex sets, *IMA J. Numer. Anal.* **23** no. 4 (2003), 539–559. DOI MR Zbl

[25] E. G. Birgin, J. M. Martínez, and M. Raydan, Spectral projected gradient methods, in *Encyclopedia of optimization*, Springer, Boston, 2009, pp. 3652–3659. DOI

[26] E. Börgens, C. Kanzow, P. Mehlitz, and G. Wachsmuth, New constraint qualifications for optimization problems in Banach spaces based on asymptotic KKT conditions, *SIAM J. Optim.* **30** no. 4 (2020), 2956–2982. DOI MR Zbl

[27] L. F. Bueno, G. Haeser, F. Lara, and F. N. Rojas, An augmented Lagrangian method for quasi-equilibrium problems, *Comput. Optim. Appl.* **76** no. 3 (2020), 737–766. DOI MR Zbl

[28] L. F. Bueno, G. Haeser, and F. N. Rojas, Optimality conditions and constraint qualifications for generalized Nash equilibrium problems and their practical implications, *SIAM J. Optim.* **29** no. 1 (2019), 31–54. DOI MR Zbl

[29] J. Dutta, K. Deb, R. Tulshyan, and R. Arora, Approximate KKT points and a proximity measure for termination, *J. Global Optim.* **56** no. 4 (2013), 1463–1499. DOI MR Zbl

[30] W. Fenchel, Convex cones, sets and functions, Logistic Project Report, Department of Mathematics, Princeton University, 1953. Zbl

[31] M. L. Flegel, C. Kanzow, and J. V. Outrata, Optimality conditions for disjunctive programs with application to mathematical programs with equilibrium constraints, *Set-Valued Anal.* **15** no. 2 (2007), 139–162. DOI MR Zbl

[32] G. Giorgi, B. Jiménez, and V. Novo, Approximate Karush–Kuhn–Tucker condition in multiobjective optimization, *J. Optim. Theory Appl.* **171** no. 1 (2016), 70–89. DOI MR Zbl

[33] G. Haeser and M. L. Schuverdt, On approximate KKT condition and its extension to continuous variational inequalities, *J. Optim. Theory Appl.* **149** no. 3 (2011), 528–539. DOI MR Zbl

[34] M. R. Hestenes, *Calculus of variations and optimal control theory*, John Wiley & Sons, New York-London-Sydney, 1966. MR Zbl

[35] M. R. Hestenes, *Optimization theory: The finite dimensional case*, Pure and Applied Mathematics, John Wiley & Sons, New York, 1975. MR Zbl

[36] C. Kanzow, D. Steck, and D. Wachsmuth, An augmented Lagrangian method for optimization problems in Banach spaces, *SIAM J. Control Optim.* **56** no. 1 (2018), 272–291. DOI MR Zbl

[37] A. E. OZDAGLAR and D. P. BERTSEKAS, The relation between pseudonormality and quasiregularity in constrained optimization, *Optim. Methods Softw.* **19** no. 5 (2004), 493–506. DOI MR Zbl

[38] L. QI and Z. WEI, On the constant positive linear dependence condition and its application to SQP methods, *SIAM J. Optim.* **10** no. 4 (2000), 963–981. DOI MR Zbl

[39] A. RAMOS, Two new weak constraint qualifications for mathematical programs with equilibrium constraints and applications, *J. Optim. Theory Appl.* **183** no. 2 (2019), 566–591. DOI MR Zbl

[40] A. RAMOS, Mathematical programs with equilibrium constraints: a sequential optimality condition, new constraint qualifications and algorithmic consequences, *Optim. Methods Softw.* **36** no. 1 (2021), 45–81. DOI MR Zbl

[41] R. T. ROCKAFELLAR, *Convex analysis*, Princeton Mathematical Series, No. 28, Princeton University Press, Princeton, NJ, 1970. MR Zbl

[42] R. T. ROCKAFELLAR and R. J.-B. WETS, *Variational analysis*, Grundlehren der mathematischen Wissenschaften 317, Springer-Verlag, Berlin, 1998. DOI MR Zbl

[43] N. V. TUYEN, Y.-B. XIAO, and T. Q. SON, On approximate KKT optimality conditions for cone-constrained vector optimization problems, *J. Nonlinear Convex Anal.* **21** no. 1 (2020), 105–117. MR Zbl

[44] W. H. YANG, L.-H. ZHANG, and R. SONG, Optimality conditions for the nonlinear programming problems on Riemannian manifolds, *Pac. J. Optim.* **10** no. 2 (2014), 415–434. MR Zbl

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