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THE ALGEBRAIC SETS OF VECTORS GENERATING PLANAR NORMAL SECTIONS OF ISOPARAMETRIC HYPERSURFACES OF FKM TYPE

CRISTIÁN U. SÁNCHEZ

ABSTRACT. We present a proof of the connectedness of the algebraic set of vectors generating planar normal sections for all isoparametric hypersurfaces, with *positive* multiplicities m_1 and m_2 of FKM type.

1. Introduction

The present paper is devoted to the study of *isoparametric hypersurfaces* $M \subset \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ with $g = 4$ principal curvatures of FKM type (also called of OT-FKM type).

Associated to each point p of any general isoparametric submanifold M, one has the *algebraic set* of unit tangent vectors *generating planar normal sections at* p, denoted by $\widehat{X}_p[M] \subset \mathbb{S}\left(T_p(M)\right)$ (see the definition below or [3]). The present paper studies these algebraic sets $\widehat{X}_p[M]$ (at any point $p \in M$) for an *isoparametric hypersurface* M on the unit sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$. We concentrate here on those with $g = 4$ principal curvatures, called of FKM type.

On the other hand, in [4] we studied $\widehat{X}_p[M]$ when $M \subset \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ is a *homogeneous isoparametric hypersurface* (i.e., it supports a transitive action of a compact Lie group). We proved that, for all of them, the algebraic sets $\widehat{X}_p[M] \subset \mathbb{S}\left(T_p(M)\right)$ (for $p \in M$) are connected by arcs[1]. Then what remains to be considered is the study of $\widehat{X}_p[M] \subset \mathbb{S}\left(T_p\left(M\right)\right)$ in the case of *non-homogeneous* isoparametric hypersurfaces in the sphere, and that is the theme of the present paper. In fact, we study here the *isoparametric hypersurfaces* with $g = 4$ of FKM type and obtain the following result.

Theorem 1.1. *Let $M^n \subset \mathbb{S}^{n+1}$ be an isoparametric hypersurface of FKM type with four principal curvatures and* positive *multiplicities m_1 and m_2. Then, for each point $p \in M$, the algebraic set $\widehat{X}_p[M] \subset \mathbb{S}\left(T_p\left(M\right)\right)$ is connected by arcs.*

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[1]We have also studied this, with a different proof, in our preprint "The algebraic sets of vectors generating planar normal sections of homogeneous isoparametric hypersurfaces".

The paper is organized as follows. In Section 2 we indicate some notation and recall the definitions of $\widehat{X}_p[M]$ and $Co(\widehat{X}_p[M])$ as well as basic facts from [3] needed here. In Section 3, to determine our manifold M and basic point E_0, we recall known information from [1] concerning (symmetric) Clifford systems. Also in that section we introduce some vectors in $T_{E_0}(M)$ required to determine the eigenspaces of the shape operator at the point E_0. In Section 4 we introduce certain subspaces of $T_{E_0}(M)$ and, using formula (7.2) established in the Appendix, show that they are in fact the eigenspaces of the shape operator at the point E_0, in agreement with [1, Cor. 3.75]. The knowledge of the eigenspaces is essential in Section 5, which is the central part of the paper and contains the required lemmata to prove Theorem 1.1. The proof makes essential use of the structure of the eigenspaces and of formula (7.4), which gives the condition to be satisfied by a unit tangent vector to generate a planar normal section at E_0 and whose proof is also part of the Appendix.

2. Planar normal sections

Let us consider an isoparametric hypersurface $M \subset \mathbb{S}^{2l-1} \subset \mathbb{R}^{2l}$ of FKM type with four principal curvatures and positive multiplicities m_1 and m_2. Let ∇^E be the Euclidean covariant derivative in $\mathbb{R}^{2l}$ and ∇ the Levi-Civita connection in M associated to the induced metric. We denote also by $\left(\overline{\nabla}_Y \alpha\right)$ the usual covariant derivative of the second fundamental form α of M in $\mathbb{R}^{2l}$. Let us consider, at $p \in M$, for each *unit vector* Y, the affine subspace of the ambient $\mathbb{R}^{2l}$ defined by $S(p, Y) = p + \operatorname{Span}\left\{Y, T_p^{\perp}(M)\right\}$. If U is a small enough neighborhood of p in M, then the intersection $U \cap S(p, Y)$ can be considered the image of a C^{∞} *regular* curve $\gamma(s)$ (parametrized by arc-length) such that $\gamma(0) = p$, $\gamma'(0) = Y$. This curve is called the *normal section of M at p in the direction of Y*. We say that this normal section γ is *planar* at p if its first three derivatives $\gamma'(0), \gamma''(0)$, and $\gamma'''(0)$ are *linearly dependent*. Every unit tangent vector $Y \in T_p(M)$ generates a normal section, but we consider only those whose normal section is *planar at $p \in M$*. Recall (see [3]) that the normal section γ of M at p in the direction of Y is *planar* at p if and only if Y satisfies the equation $\left(\overline{\nabla}_Y \alpha\right)(Y, Y) = 0$. For a point $p \in M$, we shall denote, as in [3],

$$\widehat{X}_p[M] = \left\{Y \in T_p(M) : \|Y\| = 1,\ \left(\overline{\nabla}_Y \alpha\right)(Y, Y) = 0\right\}.$$

We consider also the corresponding bundle $\Xi(M)$ union of all the sets $\widehat{X}_p[M]$ (for $p \in M$) which is contained in the *unit tangent bundle* $\mathbb{S}(T(M)) \subset T(M)$ of M. It is also convenient to have the notation

$$Co\left(\widehat{X}_p[M]\right) = \left\{Y \in T_p(M) : Y \neq 0,\ \left(\overline{\nabla}_Y \alpha\right)(Y, Y) = 0\right\}.$$

If V is a subspace of $T_p(M)$, we may write $Co(V) = (V - \{0\})$.

At this point we need to recall some basic facts from [3, Prop. 4.1] that will be needed below. Let M be a compact rank h, full, isoparametric submanifold of $\mathbb{R}^{n+h}$. Since the normal bundle of M is globally flat, all shape operators are simultaneously diagonalizable and we have common eigendistributions D_j $(j = 1, \ldots, g)$, that is, for any $\xi \in T_p^{\perp}(M)$, $A_\xi(X) = \lambda_j(\xi)X$ for all $X \in D_j(p)$. Each D_j is autoparallel, hence integrable with totally geodesic leaves. Let N_i be the leaf corresponding to the distribution D_i containing p. We have the following proposition.

Proposition 2.1. *At any $p \in N_i$, take X, Y unitary in $D_i(p) = T_p(N_i)$, and any $Z \in T_p(M)$. Then $\left(\overline{\nabla}_X \alpha\right)(Y, Z) = 0$.*

Proposition 2.1 for the case $X = Y = Z$ yields the following corollaries.

Corollary 2.2. *If $X \in T_p(N_i)$, then $\left(\overline{\nabla}_X \alpha\right)(X, X) = 0$. Hence $\|X\| = 1$ yields $X \in \widehat{X}_p[M]$.*

Corollary 2.3. *If $X \in T_p(N_i) \oplus T_p(N_j)$ $(i \neq j)$, then $\left(\overline{\nabla}_X \alpha\right)(X, X) = 0$. Hence $\|X\| = 1$ implies $X \in \widehat{X}_p[M]$.*

3. Required facts and notation

To study the isoparametric hypersurfaces of FKM type we are going to use some well-known facts from the book [1]. There, in Section 3.9, the authors include everything that is needed here for our study of these hypersurfaces.

Let $H(2l, \mathbb{R})$ be the space of symmetric real $2l \times 2l$ matrices with the standard inner product $(A, B) = \left(\frac{1}{2l}\right) \operatorname{trace}(AB)$. For positive integers l and m, the $(m+1)$-tuple $\{P_0, \ldots, P_m\}$ with $P_j \in H(2l, \mathbb{R})$ is called a *(symmetric) Clifford system* on $\mathbb{R}^{2l}$ if the P_j satisfy

$$P_j^2 = I, \quad P_k P_j = -P_j P_k, \qquad k \neq j, \ 0 \leq k, j \leq m; \tag{3.1}$$

furthermore, they are orthogonal since

$$\langle P_j x, P_j y \rangle = \left\langle x, P_j^2 y \right\rangle = \langle x, Iy \rangle = \langle x, y \rangle.$$

We assume that $\{P_0, P_1, \ldots, P_m\}$ is an *orthonormal basis* for the given Clifford system $\{P_0, P_1, \ldots, P_m\}$ such that $m_1 = m$ and $m_2 = l - m - 1$ *are both positive*; then one can construct $F : \mathbb{R}^{2l} \longrightarrow R$ defined by

$$F(X) = \|X\|^4 - 2 \sum_{j=0}^{m} \langle P_j(X), X \rangle^2.$$

We take

$$M = \{X : F(X) = 0\}. \tag{3.2}$$

Associated to the Clifford system $\{P_0, P_1, \ldots, P_m\}$ one has its *Clifford sphere* $\Sigma(P_0, P_1, \ldots, P_m)$, which is a fundamental associated object. It is defined as the *unit sphere* in the subspace $\operatorname{Span}_{\mathbb{R}} \{P_0, P_1, \ldots, P_m\} \subset H(2l, \mathbb{R})$, which has many important properties that the reader can find in [1, Theorem 3.71]. We shall recall only what is needed here.

It is important to notice that the gradient of F on the points X in M is of the form

$$\nabla F(X) = 4\|X\|^2 X - 8 \sum_{i=0}^{m} \langle P_i(X), X \rangle P_i(X); \tag{3.3}$$

and at the points $X \in M$ in (3.2) we have

$$\|\nabla F(X)\| = 4 \quad \text{and} \quad \langle \nabla F(X), X\rangle = 0 \qquad \forall X \in M.$$

We shall assume, as in [1, p. 174], that $m_2 > 0$ and consider the focal submanifold $M_+ = F^{-1}(1)$ of codimension $(m+1)$, where $m = m_1$. From the definition of F we have that

$$M_+ = \left\{x \in \mathbb{S}^{2l-1} : \langle P_j(x), x\rangle = 0,\ 0 \le j \le m\right\}.$$

Then, on any $x \in M_+$, the operators $\{P_0, \dots, P_m\}$ satisfy

$$\langle P_j(x), x\rangle = 0, \qquad 0 \le j \le m, \tag{3.4}$$

and it follows (see [1, (3.228) and (3.229)]) that the *normal bundle* of M_+ is *trivial* with a global orthonormal frame $\{P_0(x), \dots, P_m(x)\}$ as x is moving along M_+; that is, the isoparametric hypersurfaces of the family containing M *are trivial sphere bundles* over M_+. Furthermore, for each $x \in M_+$,

$$T_x^{\perp}(M_+) = \{Q(x) : Q \in \operatorname{Span}_{\mathbb{R}}(P_0, P_1, \dots, P_m)\}.$$

We have a fixed member $M = F^{-1}(0)$ of the isoparametric family associated to the focal manifold M_+ and we want to fix a basic point in M. To that end we apply one of the properties of the Clifford sphere indicated in [2, 4.2 (iii)][2], which is:

> For $x \in M_+$ and $P \in \Sigma(P_0, P_1, \dots, P_m)$, on the normal great circle $c(t) = \cos(\theta)\, x + \sin(\theta)\, P(x)$ we have $F(c(t)) = \cos(4t)$.

We fix a point $x_0 \in M_+$ (which we shall keep fixed below) and also fix

$$E_0 = \cos(\theta)\, x_0 + \sin(\theta)\, P_0(x_0). \tag{3.5}$$

Since the solutions of the equation $\cos(4\theta) = 0$ are $\left\{\frac{1}{8}\pi + \frac{1}{4}\pi k : k \in \mathbb{Z}\right\}$, we may take the smallest positive one, that is, $\theta = \frac{\pi}{8}$, *and maintain this θ also fixed below.* Since $\langle x_0, x_0\rangle = 1$ and P_0 is orthogonal, (3.4) yields $\langle E_0, E_0\rangle = 1$. We also fix the notation

$$H(E_0) = \frac{1}{4}\nabla F(E_0).$$

A straightforward computation shows that

$$\langle P_k(E_0), E_0\rangle = 0 \qquad \text{for } 1 \le k \le m, \tag{3.6}$$

and, on the other hand, for $k = 0$, we have

$$\langle P_0(E_0), E_0\rangle = 2\sin(\theta)\cos(\theta) = \frac{1}{2}\sqrt{2}. \tag{3.7}$$

Hence, $F(E_0) = 1 + (-8)\sin^2(\theta)\cos^2(\theta) = \cos(4\theta) = 0$, so we have $E_0 \in M = \{X : F(X) = 0\}$.

Now, recalling (3.3), we have that

$$H(E_0) = E_0 - 2\sum_{j=0}^{m} \langle P_j(E_0), E_0\rangle\, P_j(E_0).$$

[2] P. 485 in the original German version and p. 9 in the English translation.

Clearly (3.6) yields $H(E_0) = E_0 - 2\langle P_0(E_0), E_0\rangle P_0(E_0)$, and by (3.7) we get that

$$H(E_0) = E_0 - \sqrt{2}P_0(E_0). \tag{3.8}$$

Now we observe that, for $1 \le j \le m$, we have

$$\begin{aligned}\langle P_j(E_0), P_0(E_0)\rangle &= (-1)\langle E_0, P_0P_j(E_0)\rangle \\ &= (-1)\langle P_0(E_0), P_j(E_0)\rangle;\end{aligned}$$

hence

$$\langle P_j(E_0), P_0(E_0)\rangle = 0 \qquad \text{for } 1 \le j \le m. \tag{3.9}$$

Then by (3.6), (3.9) and (3.8) we see that

$$\begin{aligned}\langle P_k(E_0), E_0\rangle = 0 = \langle P_k(E_0), H(E_0)\rangle \qquad &\text{for } 1 \le k \le m,\\ \langle E_0, P_0P_j(x_0)\rangle = 0 = \langle H(E_0), P_0P_j(x_0)\rangle \qquad &\text{for } 1 \le j \le m,\end{aligned}$$

and therefore, for $1 \le j \le m$, $P_j(E_0)$ and $P_0P_j(x_0)$ are in $T_{E_0}(M)$. By definition, the vectors $P_j(E_0)$ $(1 \le j \le m)$ are of the form

$$\begin{aligned}P_j(E_0) &= \cos(\theta) P_j(x_0) + \sin(\theta) P_jP_0(x_0)\\ &= \cos(\theta) P_j(x_0) - \sin(\theta) P_0P_j(x_0).\end{aligned} \tag{3.10}$$

4. Eigenspaces

We consider these two subspaces contained in $T_{E_0}(M)$:

$$W_1 = \mathrm{Span}_{\mathbb{R}}\{P_j(x_0) : 1 \le j \le m\}, \tag{4.1}$$

$$W_2 = \mathrm{Span}_{\mathbb{R}}\{P_0P_j(x_0) : 1 \le j \le m\}; \tag{4.2}$$

their dimensions are clearly

$$\dim W_1 = \dim W_2 = m.$$

4.1. **W_1 and W_2 are orthogonal.** We have to understand $\langle P_jP_0(x_0), P_k(x_0)\rangle$. By (3.4) this product is zero if $j = k$. Furthermore, if $j = 0$ and $k \ne 0$, or $j \ne 0$ and $k = 0$, the product is also zero. So we study the case $j \ne k$, $1 \le j, k \le m$. We have that

$$\langle P_jP_0(x_0), P_k(x_0)\rangle = \langle P_jP_k(x_0), P_0(x_0)\rangle$$

but also

$$\begin{aligned}\langle P_jP_0(x_0), P_k(x_0)\rangle &= (-1)\langle P_j(x_0), P_0P_k(x_0)\rangle\\ &= (-1)(-1)\langle P_j(x_0), P_kP_0(x_0)\rangle\\ &= \langle P_kP_j(x_0), P_0(x_0)\rangle.\end{aligned}$$

Using these two equalities we may write

$$\begin{aligned}2\langle P_jP_0(x_0), P_k(x_0)\rangle &= \langle P_jP_k(x_0), P_0(x_0)\rangle + \langle P_kP_j(x_0), P_0(x_0)\rangle\\ &= \langle [P_jP_k(x_0) + P_kP_j(x_0)], P_0(x_0)\rangle\\ &= \langle 0, P_0(x_0)\rangle = 0.\end{aligned}$$

So we get

$$\langle P_jP_0(x_0), P_k(x_0)\rangle = 0 \qquad \text{for } 0 \le j, k \le m. \tag{4.3}$$

Furthermore, we also have

$$\begin{aligned} \langle P_j P_0 (x_0), P_j P_0 (x_0)\rangle &= \langle (x_0), (x_0)\rangle = 1, \\ \langle P_k P_0 (x_0), P_0 P_j (x_0)\rangle &= 0 \qquad \text{for } j \neq k,\ 0 \leq j, k \leq m. \end{aligned} \tag{4.4}$$

So, by (4.3), we have that (4.1) and (4.2) *are two orthogonal subspaces in the tangent space* $T_{E_0}(M)$ and, furthermore, the spanning sets are orthonormal bases for each of them. We shall show that W_1 and W_2 are the two eigenspaces of $A_{H(E_0)}$ of dimension m in $T_{E_0}(M)$.

4.2. **Study of W_1.** Let us consider the formula (7.2) indicated in the Appendix. Take the vector $U = P_j (x_0)$ $(1 \leq j \leq m)$; then (7.2) becomes

$$\begin{aligned} A_{H(E_0)} (P_j (x_0)) = (-P_j (x_0)) &+ 2 \sum_{k=0}^{m} 2 \langle P_k (E_0), P_j (x_0)\rangle P_k (E_0) \\ &+ 2 \sum_{k=0}^{m} \langle P_k (E_0), E_0\rangle P_k (P_j (x_0)). \end{aligned}$$

By (3.6) and (3.7), we have

$$\langle P_k (E_0), E_0\rangle = \begin{cases} 0 & \text{if } 0 < k \leq m, \\ \frac{1}{2}\sqrt{2} & \text{if } k = 0. \end{cases} \tag{4.5}$$

Then the second sum above is

$$+2 \sum_{k=0}^{m} \langle P_k (E_0), E_0\rangle P_k (P_j (x_0)) = 2 \left(\frac{1}{2}\sqrt{2}\right) P_0 (P_j (x_0)) = \sqrt{2} P_0 (P_j (x_0)).$$

Now we study the first sum. So we need $\langle P_k (E_0), P_j (x_0)\rangle$ and we have

$$\langle P_k (E_0), P_j (x_0)\rangle = \begin{cases} 0 & \text{if } k \neq j \text{ including } k = 0, \\ \cos(\theta) & \text{if } k = j. \end{cases}$$

Then, by (3.10), we have

$$\begin{aligned} +2 \sum_{k=0}^{m} 2 \langle P_k (E_0), P_j (x_0)\rangle P_k (E_0) &= 4 \cos (\theta) P_j (E_0) \\ &= 4 \cos^2 (\theta) P_j (x_0) + 4 \cos (\theta) \sin (\theta) P_j P_0 (x_0). \end{aligned}$$

Putting the three terms together we get

$$\begin{aligned} A_{H(E_0)} (P_j (x_0)) = (-P_j (x_0)) &+ 4 \cos^2 (\theta) P_j (x_0) \\ &+ 4 \cos (\theta) \sin (\theta) P_j P_0 (x_0) + \sqrt{2} P_0 (P_j (x_0)). \end{aligned}$$

Now, since $\theta = \frac{\pi}{8}$, we have $4 \cos (\theta) \sin (\theta) = \sqrt{2}$ and this yields

$$4 \cos (\theta) \sin (\theta) P_j P_0 (x_0) + \sqrt{2} P_0 (P_j (x_0)) = \sqrt{2} P_j P_0 (x_0) + \sqrt{2} P_0 (P_j (x_0)) = 0.$$

On the other hand, $4 \cos^2 (\theta) - 1 = \sqrt{2} + 1$, and then we finally have

$$A_{H(E_0)} (P_j (x_0)) = \left(\sqrt{2} + 1\right) P_j (x_0) = \cot \left(\frac{\pi}{8}\right) P_j (x_0). \tag{4.6}$$

4.3. **Study of W_2.** Using again (7.2) from the Appendix, for $j = 1, \dots, m$, we have

$$A_{H(E_0)}\left(P_0P_j\left(x_0\right)\right) = \left(-P_0P_j\left(x_0\right)\right) + 2\sum_{k=0}^{m} 2\left\langle P_k\left(E_0\right), P_0P_j\left(x_0\right)\right\rangle P_k\left(E_0\right) + 2\sum_{k=0}^{m}\left\langle P_k\left(E_0\right), E_0\right\rangle P_k\left(P_0P_j\left(x_0\right)\right).$$

Again, we have (4.5), so the second sum is

$$+2\sum_{k=0}^{m}\left\langle P_k\left(E_0\right), E_0\right\rangle P_k\left(P_0P_j\left(x_0\right)\right) = 2\left(\frac{1}{2}\sqrt{2}\right) P_0P_0\left(P_j\left(x_0\right)\right) = \sqrt{2}P_j\left(x_0\right).$$

Now we must study the first sum, where we need $\left\langle P_k\left(E_0\right), P_0P_j\left(x_0\right)\right\rangle$ for our fixed j and $k = 0, 1, \dots, m$.

Recalling (3.5), we have

$$\left\langle P_k\left(E_0\right), P_0P_j\left(x_0\right)\right\rangle = \cos\left(\theta\right)\left\langle P_k\left(x_0\right), P_0P_j\left(x_0\right)\right\rangle + \sin\left(\theta\right)\left\langle P_kP_0\left(x_0\right), P_0P_j\left(x_0\right)\right\rangle.$$

By (4.3), $\left\langle P_k\left(x_0\right), P_0P_j\left(x_0\right)\right\rangle = 0$ for $0 \leq j, k \leq m$, and by (4.4), for $j \neq k$, also $\left\langle P_kP_0\left(x_0\right), P_0P_j\left(x_0\right)\right\rangle = 0$. Then only the term corresponding to $k = j$ remains, and therefore

$$+2\sum_{k=0}^{m} 2\left\langle P_k\left(E_0\right), P_0P_j\left(x_0\right)\right\rangle P_k\left(E_0\right) = 4\left\langle P_j\left(E_0\right), P_0P_j\left(x_0\right)\right\rangle P_j\left(E_0\right).$$

Now, since

$$P_j\left(E_0\right) = \cos\left(\theta\right) P_j\left(x_0\right) + \sin\left(\theta\right) P_jP_0\left(x_0\right) \tag{4.7}$$

we have

$$\begin{aligned}\left\langle P_j\left(E_0\right), P_0P_j\left(x_0\right)\right\rangle &= \left(-1\right)\left\langle P_j\left(E_0\right), P_jP_0\left(x_0\right)\right\rangle \\ &= \left(-1\right)\left\langle E_0, P_0\left(x_0\right)\right\rangle \\ &= \left(-1\right)\sin\left(\theta\right)\left\langle P_0\left(x_0\right), P_0\left(x_0\right)\right\rangle \\ &= \left(-1\right)\sin\left(\theta\right),\end{aligned}$$

and so the value of the first sum is

$$+2\sum_{k=0}^{m} 2\left\langle P_k\left(E_0\right), P_0P_j\left(x_0\right)\right\rangle P_k\left(E_0\right) = 4\left(-1\right)\sin\left(\theta\right) P_j\left(E_0\right).$$

Now, replacing $P_j\left(E_0\right)$ by its expression (4.7) and interchanging P_j and P_0, the right-hand side above is

$$\begin{aligned}&4\left(-1\right)\sin\left(\theta\right) P_j\left(E_0\right) \\ &\quad = 4\left(-1\right)\sin\left(\theta\right)\cos\left(\theta\right) P_j\left(x_0\right) + 4\left(-1\right)\sin^2\left(\theta\right) P_jP_0\left(x_0\right) \\ &\quad = 4\left(-1\right)\sin\left(\theta\right)\cos\left(\theta\right) P_j\left(x_0\right) + 4\sin^2\left(\theta\right) P_0P_j\left(x_0\right).\end{aligned}$$

Now, since $\theta = \frac{\pi}{8}$, we may evaluate the coefficients of this last expression, which are

$$4\,(-1)\sin(\theta)\cos(\theta) = \left(-\sqrt{2}\right), \qquad 4\sin^2(\theta) = \left(2-\sqrt{2}\right),$$

and replacing everything in the formula, we finally have

$$A_{H(E_0)}\left(P_0P_j\left(x_0\right)\right) = -P_0P_j\left(x_0\right) + \left(2-\sqrt{2}\right)P_0P_j\left(x_0\right) + \left(-\sqrt{2}\right)P_j\left(x_0\right) + \sqrt{2}P_j\left(x_0\right),$$

that is,

$$A_{H(E_0)}\left(P_0P_j\left(x_0\right)\right) = \left(1-\sqrt{2}\right)P_0P_j\left(x_0\right). \tag{4.8}$$

Remark 4.1. Comparing (4.6) and (4.8) with [1, Cor. 3.75], we see that, since we have taken here $(-t) = \theta = \frac{\pi}{8}$, we have

$$\cot(-t) = \cot\left(\frac{\pi}{8}\right) = 1+\sqrt{2},$$
$$\cot\left(\frac{\pi}{2}-t\right) = \cot\left(\frac{\pi}{2}+\frac{\pi}{8}\right) = 1-\sqrt{2},$$

which correspond to the two eigenspaces indicated there, with multiplicities m.

We have to study now the other two eigenspaces of $A_{H(E_0)}$.

4.4. **The other eigenspaces.** Recalling the two eigenspaces (4.1) and (4.2), we obviously have $P_0\left(W_1\right) = W_2$ and vice versa, and therefore

$$P_0\left(W_1 \oplus W_2\right) = W_1 \oplus W_2. \tag{4.9}$$

Now, by definition, $T_{E_0}(M)$ is orthogonal to E_0 and $H\left(E_0\right)$, and since we have (3.8), it is clear that $P_0\left(E_0\right) = \left(1/\sqrt{2}\right)\left(E_0 - H\left(E_0\right)\right)$ and also $P_0\left(H\left(E_0\right)\right) = P_0\left(E_0\right) - \sqrt{2}E_0$, so we see that the normal space $T_{E_0}^{\perp}(M)$ at E_0 *is invariant* by P_0 and, since P_0 is orthogonal, we have

$$P_0\left(T_{E_0}(M)\right) = T_{E_0}(M).$$

Let Q be the orthogonal complement of $\left(W_1 \oplus W_2\right)$ in $T_{E_0}(M)$. Then, $T_{E_0}(M) = W_1 \oplus W_2 \oplus Q$.

Clearly, (4.9) and the fact that P_0 is orthogonal yield

$$P_0(Q) = Q,$$

and furthermore, it is clear that

$$Q = \left\{X \in T_{E_0}(M) : \left\langle X, P_j\left(x_0\right)\right\rangle = 0,\ \left\langle X, P_0P_j\left(x_0\right)\right\rangle = 0,\ 1 \le j \le m\right\}.$$

Let now Q_- and Q_+ be the two eigenspaces of P_0 in Q. Each of these orthogonal subspaces is invariant by P_0. We call them

$$W_3 = Q_-, \qquad W_4 = Q_+,$$

and, in fact, we may write

$$W_3 = \left\{X \in T_{E_0}(M) : P_0(X) = (-X),\ \left\langle X, \left(W_1 \oplus W_2\right)\right\rangle = 0\right\}, \tag{4.10}$$
$$W_4 = \left\{X \in T_{E_0}(M) : P_0(X) = X,\ \left\langle X, \left(W_1 \oplus W_2\right)\right\rangle = 0\right\}. \tag{4.11}$$

Since the tangent and normal spaces are invariant by P_0, we may also write

$$T_{E_0}(M) = T_{E_0}^{(+)}(M) \oplus T_{E_0}^{(-)}(M) \qquad \text{tangent,}$$
$$T_{E_0}^{\perp}(M) = T_{E_0}^{\perp(+)}(M) \oplus T_{E_0}^{\perp(-)}(M) \qquad \text{normal,}$$

decomposing $T_{E_0}(M)$ and $T_{E_0}^{\perp}(M)$ respectively in the two eigenspaces of P_0.

Let us now take $Z \in W_3$ and consider $P_q(Z)$ for each $1 \leq q \leq m$. Since $Z \in W_3 \Longrightarrow P_0(Z) = -Z$, we have

$$P_0(P_q(Z)) = (-1) P_q(P_0(Z)) = (-1)(-1) P_q(Z) = P_q(Z).$$

Then $P_q(Z) \in T_{E_0}^{\perp(+)}(M) \oplus T_{E_0}^{(+)}(M)$, and since $W_3 \subset T_{E_0}^{(-)}(M)$, we see that

$$Z \in W_3 \Longrightarrow \langle P_q(Z), Z\rangle = 0 \text{ for every } 1 \leq q \leq m, \tag{4.12}$$

and similarly

$$Z \in W_4 \Longrightarrow \langle P_q(Z), Z\rangle = 0 \text{ for every } 1 \leq q \leq m. \tag{4.13}$$

4.4.1. *The subspace* W_3. Let us study now the shape operator on the subspace W_3, which is of the form (4.10). Take $X \in W_3$ and consider again (7.2) for $U = X$; we have

$$\begin{aligned} A_{H(E_0)}(X) = (-X) &+ 2\sum_{i=0}^{m} 2\langle P_i(E_0), X\rangle P_i(E_0) \\ &+ 2\sum_{i=0}^{m} \langle P_i(E_0), E_0\rangle P_i(X). \end{aligned}$$

For $X \in W_3$, by (4.10) we have $+2\sum_{i=0}^{m} 2\langle P_i(E_0), X\rangle P_i(E_0) = 0$ and hence

$$A_{H(E_0)}(X) = (-X) + 2\sum_{i=0}^{m} \langle P_i(E_0), E_0\rangle P_i(X).$$

On the other hand, by (4.5) and since $X \in W_3$, we have

$$\begin{aligned} A_{H(E_0)}(X) &= (-X) + 2\frac{1}{2}\sqrt{2}P_0(X) = (-X) + \sqrt{2}P_0(X) \\ &= \left(-1 - \sqrt{2}\right) X. \end{aligned}$$

Since we have $(-t) = \frac{\pi}{8}$, we have

$$\cot\left(\frac{3\pi}{4} - t\right) = \cot\left(\frac{3\pi}{4} + \frac{\pi}{8}\right) = -\sqrt{2} - 1.$$

4.4.2. *The subspace* W_4. Similarly, for $X \in W_4$, the same computation above by (4.11) shows that

$$\begin{aligned} A_{H(E_0)}(X) &= (-X) + 2\frac{1}{2}\sqrt{2}P_0(X) = (-X) + \sqrt{2}P_0(X) \\ &= \left(\sqrt{2} - 1\right) X. \end{aligned}$$

Again, since $(-t) = \frac{\pi}{8}$, we have

$$\cot\left(\frac{\pi}{4} - t\right) = \cot\left(\frac{\pi}{4} + \frac{\pi}{8}\right) = \sqrt{2} - 1.$$

Then, as in Remark 4.1, these two facts agree with [1, Cor. 3.75]. *In this way we have identified the four eigenspaces of $A_{H(E_0)}$ in $T_{E_0}(M)$.*

5. Required lemmata

Lemma 5.1. *If M is a submanifold as above and we have in $T_{E_0}(M)$ three independent non-zero vectors X_1, X_2, X_3 such that*

$$X_1, X_2, X_3, (X_1 + X_2), (X_1 + X_3), (X_2 + X_3) \in Co\big(\widehat{X}_p[M]\big),$$

then $\big(\overline{\nabla}_{(X_1+X_2+X_3)}\alpha\big)\left((X_1 + X_2 + X_3), (X_1 + X_2 + X_3)\right) = 6\overline{\nabla}_{X_1}\alpha\,(X_2, X_3)$.

Proof. By Codazzi's equation we have

$$\begin{aligned}
&\big(\overline{\nabla}_{(X_1+X_2+X_3)}\alpha\big)\left((X_1 + X_2 + X_3), (X_1 + X_2 + X_3)\right)\\
&\quad = \big(\overline{\nabla}_{X_1}\alpha\big)(X_1, X_1) + \big(\overline{\nabla}_{X_2}\alpha\big)(X_2, X_2) + \big(\overline{\nabla}_{X_3}\alpha\big)(X_3, X_3)\\
&\qquad + 3\big(\overline{\nabla}_{X_2}\alpha\big)(X_1, X_1) + 3\big(\overline{\nabla}_{X_1}\alpha\big)(X_2, X_2) \qquad \text{(a)}\\
&\qquad + 3\big(\overline{\nabla}_{X_1}\alpha\big)(X_3, X_3) + 3\big(\overline{\nabla}_{X_3}\alpha\big)(X_1, X_1) \qquad \text{(b)}\\
&\qquad + 3\big(\overline{\nabla}_{X_2}\alpha\big)(X_3, X_3) + 3\big(\overline{\nabla}_{X_3}\alpha\big)(X_2, X_2) \qquad \text{(c)}\\
&\qquad + 6\big(\overline{\nabla}_{X_1}\alpha\big)(X_2, X_3).
\end{aligned} \tag{5.1}$$

Now, since X_1, X_2, and X_3 are in $Co\big(\widehat{X}_p[M]\big)$, we have that

$$\big(\overline{\nabla}_{X_k}\alpha\big)(X_k, X_k) = 0 \qquad \text{for } 1 \leq k \leq 3. \tag{5.2}$$

Since $(X_1 + X_2)$, $(X_1 + X_3)$, and $(X_2 + X_3)$ are also in $Co\big(\widehat{X}_p[M]\big)$, (5.2) yields that each one of the lines marked with (a), (b), and (c) in (5.1) vanishes. Hence we have the indicated equality. ☐

Lemma 5.2. *Let us assume that M is an isoparametric hypersurface of $\mathbb{S}^{2l-1}$ with four principal curvatures, and that E_0 is a point in M. Let W_k ($1 \leq k \leq 4$) be the four eigenspaces of the shape operators in $T_{E_0}(M)$, and take three non-zero vectors X, Y, Z in $T_{E_0}(M)$, each contained in one of three different eigenspaces. Let us furthermore assume that $(X + Y + Z) \in Co\big(\widehat{X}_{E_0}[M]\big)$. Then, for each $t \in [0, 1]$, we have $(tX + Y + Z) \in Co\big(\widehat{X}_{E_0}[M]\big)$.*

Proof. By assumption we have

$$\big(\overline{\nabla}_{(X+Y+Z)}\alpha\big)\left((X + Y + Z), (X + Y + Z)\right) = 0,$$

and by Corollary 2.2 we have $X, Y, Z \in Co\big(\widehat{X}_{E_0}[M]\big)$. In turn, Corollary 2.3 yields that $(X + Y)$, $(X + Z)$, and $(Y + Z)$ are also in $Co\big(\widehat{X}_{E_0}[M]\big)$. Hence by Lemma 5.1 we have

$$\overline{\nabla}_X\alpha\,(Y, Z) = 0. \tag{5.3}$$

Now (again by Corollaries 2.2 and 2.3), for each $t \in (0,1]$, the triple $((tX), Y, Z)$ satisfies the hypothesis of Lemma 5.1, that is,

$$(tX),\ Y,\ Z,\ ((tX)+Y),\ ((tX)+Z),\ (Y+Z) \in Co\big(\widehat{X}_p[M]\big), \tag{5.4}$$

and then Lemma 5.1 yields the equality

$$\big(\overline{\nabla}_{(tX+Y+Z)}\alpha\big)\left((tX+Y+Z),(tX+Y+Z)\right) = 6\overline{\nabla}_{(tX)}\alpha\,(Y,Z) \qquad \text{for } t \in (0,1].$$

Now, since

$$\overline{\nabla}_{tX}\alpha\,(Y,Z) = t\overline{\nabla}_X\alpha\,(Y,Z) \qquad \text{for } t \in (0,1]\,,$$

as a consequence of (5.3) we have

$$\big(\overline{\nabla}_{(tX+Y+Z)}\alpha\big)\left((tX+Y+Z),(tX+Y+Z)\right) = 0 \qquad \text{for } t \in [0,1],$$

and then $(tX+Y+Z) \in Co\big(\widehat{X}_{E_0}[M]\big)$ for $t \in (0,1]$. We already know, by (5.4), that, for $t=0$, $(Y+Z) \in Co\big(\widehat{X}_{E_0}[M]\big)$, so the lemma is proved. □

Corollary 5.3. *Lemma 5.2 means that if a vector of the form* $(X+Y+Z)$ *is in* $Co\big(\widehat{X}_{E_0}[M]\big)$ *(with each term contained in one of three different eigenspaces), then* $(X+Y+Z)$ *can be joined to* $(Y+Z) \in Co\big(\widehat{X}_{E_0}[M]\big)$ *by a segment* totally contained in $Co\big(\widehat{X}_{E_0}[M]\big)$.

Remark 5.4. So far we have established that all points of the form $(X+Y+Z) \in Co\big(\widehat{X}_{E_0}[M]\big)$ (with each term contained in one of three different eigenspaces) form an arc-wise connected set in $Co\big(\widehat{X}_{E_0}[M]\big)$. We do not know if vectors of this type exist in $Co\big(\widehat{X}_{E_0}[M]\big)$ at all, but we have just proved that if any one does then it can be joined to those of the form $(Y+Z)$ that are in $Co\big(\widehat{X}_{E_0}[M]\big)$ by Corollary 2.3.

Remark 5.5. It remains to consider the case of points of the form $X+Y+Z+T$ with $X \in W_1$, $Y \in W_2$, $Z \in W_3$, and $T \in W_4$ (none of them zero) such that $(X+Y+Z+T) \in Co\big(\widehat{X}_{E_0}[M]\big)$. There may not exist such a vector in $Co\big(\widehat{X}_{E_0}[M]\big)$, but we must consider the possibility that there is one. We study this in the next section.

6. The case of four terms

Let us consider, again, the two eigenspaces

$$W_1 = \mathrm{Span}_{\mathbb{R}}\left\{\cos(\theta)\,P_j(x_0) : 1 \le j \le m\right\},$$
$$W_2 = \mathrm{Span}_{\mathbb{R}}\left\{(-\sin(\theta))\,P_0P_j(x_0) : 1 \le j \le m\right\},$$

which are those of dimension $m = m_1$, and consider also the subspace

$$\begin{aligned}\Delta &= \mathrm{Span}_{\mathbb{R}}\left\{(\cos(\theta)\,P_j(x_0) + (-\sin(\theta))\,P_0P_j(x_0)) : 1 \le j \le m\right\}\\ &= \mathrm{Span}_{\mathbb{R}}\left\{P_j(E_0) : 1 \le j \le m\right\},\end{aligned} \tag{6.1}$$

whose dimension is clearly $\dim(\Delta) = \dim(W_1) = \dim(W_2) = m$.

Let $\Delta^{\perp}$ be the *orthogonal complement* of Δ in $(W_1 \oplus W_2)$, whose dimension is, obviously, also m. Let us consider also the two eigenspaces W_3 and W_4 and take now any $X \in T_{E_0}(M)$ such that $X \in \Delta^{\perp} \oplus W_3 \oplus W_4$; then we have that X is orthogonal to Δ and so, for $1 \leq j \leq m$, satisfies $\langle P_j(E_0), X\rangle = 0$. Furthermore, since (by (3.8)) $P_0(E_0) \in T_{E_0}^{\perp}(M)$, we also have $\langle P_0(E_0), X\rangle = 0$. Then considering formula (7.4) in the Appendix, that is,

$$\Gamma(X) = \sum_{i=0}^{m} \langle P_i(E_0), X\rangle \langle P_i(X), X\rangle,$$

we see that $\Gamma(X) = 0$, and therefore $X \in Co\big(\widehat{X}_{E_0}[M]\big)$. Then we have proved the following lemma.

Lemma 6.1. *$Co\left(\Delta^{\perp} \oplus W_3 \oplus W_4\right) \subset Co\big(\widehat{X}_{E_0}[M]\big)$.*

Corollary 6.2. *Also, $Co\left(\Delta^{\perp} \oplus W_h\right) \subset Co\big(\widehat{X}_{E_0}[M]\big)$ for $h = 3, 4$ and $Co\left(\Delta^{\perp}\right) \subset Co\big(\widehat{X}_{E_0}[M]\big)$.*

Now we take a vector in $T_{E_0}(M)$ of the form $X + Y + Z + T$ such that $X \in W_1$, $Y \in W_2$, $Z \in W_3$, and $T \in W_4$, as indicated in Remark 5.5. That is,

$$X + Y + Z + T \in Co\big(\widehat{X}_{E_0}[M]\big). \tag{6.2}$$

If there is such a vector, then $(X + Y) \in (W_1 \oplus W_2)$, and we may write it as $(X + Y) = U + V$ with $U \in \Delta$ and $V \in \Delta^{\perp}$. Then we have

$$X + Y + Z + T = U + V + Z + T.$$

Now, by Lemma 6.1, we have that any point of the form $(V + Z + T)$ with $V \in \Delta^{\perp}$, $Z \in W_3$, and $T \in W_4$ satisfies

$$V + Z + T \in Co\big(\widehat{X}_{E_0}[M]\big), \tag{6.3}$$

and, in particular, we may have $Z = T = 0$ or just $Z = 0$ or $T = 0$. Then, for our given V, Z, T, we have

$$V,\ Z,\ T,\ (V + Z),\ (V + T),\ (Z + T) \in Co\big(\widehat{X}_{E_0}[M]\big). \tag{6.4}$$

Now, *by assumption* (6.2), we have

$$\left(\overline{\nabla}_{(U+V+Z+T)}\alpha\right)\left((U + V + Z + T), (U + V + Z + T)\right) = 0$$

and we notice that, by Corollary 2.3, $Co\,(W_1 \oplus W_2) \subset Co\big(\widehat{X}_{E_0}[M]\big)$, so we may add the pair U, $(U + V)$ to the set indicated in (6.4) and write

$$\begin{aligned} &V,\ Z,\ T,\ (V + Z),\ (V + T),\ (Z + T) \in Co\big(\widehat{X}_{E_0}[M]\big), \\ &U,\ (U + V) \in Co\big(\widehat{X}_{E_0}[M]\big). \end{aligned} \tag{6.5}$$

Notice that we do not know whether $(U + Z)$ and $(U + T)$ belong to $Co\big(\widehat{X}_{E_0}[M]\big)$ or not. Our next lemma takes care of this question.

Lemma 6.3. *For every $U \in \Delta$, $Z \in W_3$, and $T \in W_4$, we have that $(U + Z)$ and $(U + T)$ belong to $Co\big(\widehat{X}_{E_0}[M]\big)$.*

Proof. By the definition of Δ in (6.1), an arbitrary $U \in \Delta$ can be written as

$$U = \sum_{q=1}^{m} a_q P_q(E_0). \tag{6.6}$$

Let Z be an arbitrary vector in W_3. In order to prove that $X = U + Z = \sum_{q=1}^{m} a_q P_q(E_0) + Z \in Co\big(\widehat{X}_{E_0}[M]\big)$, we shall use again formula (7.4). Let us replace, in (7.4), X by $\left(\sum_{q=1}^{m} a_q P_q(E_0) + Z\right)$ and start by computing $\langle P_k(E_0), X\rangle$ *for each k such that* $1 \le k \le m$. That is,

$$\begin{aligned}\langle P_k(E_0), X\rangle &= \left\langle P_k(E_0), \sum_{q=1}^{m} a_q P_q(E_0) + Z\right\rangle\\ &= \sum_{q=1}^{m} a_q \langle P_k(E_0), P_q(E_0)\rangle + \langle P_k(E_0), Z\rangle.\end{aligned}$$

Here, since $P_k(E_0) \in W_1 \oplus W_2$ for $1 \le k \le m$ and $Z \in W_3$, we have

$$\langle P_k(E_0), Z\rangle = 0 \qquad \text{for } 1 \le k \le m, \tag{6.7}$$

and then we have

$$\langle P_k(E_0), X\rangle = \sum_{q=1}^{m} a_q \langle P_k(E_0), P_q(E_0)\rangle.$$

But since

$$\begin{aligned}\langle P_k(E_0), P_q(E_0)\rangle &= 0 \qquad \text{for } k \neq q,\\ \langle P_q(E_0), P_q(E_0)\rangle = \langle E_0, E_0\rangle &= 1 \qquad \text{for } k = q,\end{aligned}$$

we see that

$$\langle P_k(E_0), X\rangle = a_k \qquad \text{for } 1 \le k \le m. \tag{6.8}$$

On the other hand, since X is tangent to M at E_0, by (3.8) we see that

$$\langle P_0(E_0), X\rangle = 0.$$

Then, by (6.7) and (6.8), it turns out that $\Gamma(X)$ in (7.4) takes the form

$$\Gamma(X) = \sum_{i=0}^{m} \langle P_i(E), X\rangle \langle P_i(X), X\rangle = \sum_{k=1}^{m} a_k \langle P_k(X), X\rangle,$$

that is,

$$\Gamma(X) = \left\langle \left(\sum_{k=1}^{m} a_k P_k(X)\right), X\right\rangle. \tag{6.9}$$

Let us study the first factor in (6.9). Since $X = \sum_{q=1}^{m} a_q P_q(E_0) + Z$, we have

$$\begin{aligned} \sum_{k=1}^{m} a_k P_k(X) &= \sum_{k=1}^{m} a_k P_k \left(\sum_{q=1}^{m} a_q P_q(E_0) + Z \right) \\ &= \sum_{k=1}^{m} \sum_{q=1}^{m} a_k a_q P_k P_q(E_0) + \sum_{k=1}^{m} a_k P_k(Z), \end{aligned} \tag{6.10}$$

and considering the first term of the last line in (6.10) (since $1 \leq k, q \leq m$) we clearly have

$$\sum_{k=1}^{m} \sum_{q=1}^{m} a_k a_q P_k P_q(E_0) = \sum_{j=1}^{m} a_j^2 E_0 + \sum_{k<q} a_k a_q \left[P_k P_q(E_0) + P_q P_k(E_0) \right];$$

but, by (3.1), we have

$$\left[P_k P_q(E_0) + P_q P_k(E_0) \right] = 0 \qquad \text{for } k \neq q,$$

and then the first term of the last line in (6.10) is

$$\sum_{k=1}^{m} \sum_{q=1}^{m} a_k a_q P_k P_q(E_0) = \sum_{j=1}^{m} a_j^2 E_0 = G E_0, \tag{6.11}$$

where we are setting $G := \sum_{j=1}^{m} a_j^2 \neq 0$, which is fixed by the definition of U, (6.6). Hence, going back to (6.10), we have

$$\sum_{k=1}^{m} a_k P_k(X) = G E_0 + \sum_{k=1}^{m} a_k P_k(Z),$$

and therefore $\Gamma(X)$ becomes

$$\Gamma(X) = \left\langle \sum_{k=1}^{m} a_k P_k(X), X \right\rangle = \left\langle G E_0 + \sum_{k=1}^{m} a_k P_k(Z), \sum_{h=1}^{m} a_h P_h(E_0) + Z \right\rangle.$$

Now, $\sum_{h=1}^{m} a_h P_h(E_0) + Z \in T_{E_0}(M)$, therefore its product with $G E_0$ vanishes. On the other hand, since $Z \in W_3$, by (4.12) we clearly have $\left\langle \sum_{k=1}^{m} a_k P_k(Z), Z \right\rangle = 0$, and therefore our $\Gamma(X)$ takes the form

$$\Gamma(X) = \left\langle \sum_{k=1}^{m} a_k P_k(X), X \right\rangle = \left\langle \sum_{k=1}^{m} a_k P_k(Z), \sum_{h=1}^{m} a_h P_h(E_0) \right\rangle.$$

Now we may write this as

$$\Gamma(X) = \sum_{k=1}^{m} \sum_{h=1}^{m} a_k a_h \left\langle P_k(Z), P_h(E_0) \right\rangle = \left\langle Z, \sum_{k=1}^{m} \sum_{h=1}^{m} a_k a_h P_k P_h(E_0) \right\rangle$$

and, recalling (6.11), we see that

$$\Gamma(X) = \langle Z, G E_0 \rangle = 0,$$

because $Z \in W_3 \subset T_{E_0}(M)$. The proof for $T \in W_4$ is the same (changing Z by T) because we only used here (4.12), and for T, we have (4.13). Then we have proved Lemma 6.3. □

As a consequence of Lemma 6.3, we see that, for our point $(U+V+Z+T)$, we may modify condition (6.5) to the condition

$$\begin{aligned} &V,\, Z,\, T,\, (V+Z),\, (V+T),\, (Z+T) \in Co\big(\widehat{X}_{E_0}[M]\big),\\ &U,\, (U+V),\, (U+Z),\, (U+T) \in Co\big(\widehat{X}_{E_0}[M]\big). \end{aligned} \tag{6.12}$$

Let us go back now to our vector

$$X+Y+Z+T = U+V+Z+T,$$

which, *as we are assuming, is contained in* $Co\big(\widehat{X}_{E_0}[M]\big)$. Then it satisfies the condition

$$\big(\overline{\nabla}_{(U+V+Z+T)}\alpha\big)\left((U+V+Z+T),(U+V+Z+T)\right)=0.$$

By expanding this expression, using Codazzi's equation, we see that it is

$$\begin{aligned} 0 = {} & \big(\overline{\nabla}_U\alpha\big)(U,U) + \big(\overline{\nabla}_V\alpha\big)(V,V) + \big(\overline{\nabla}_Z\alpha\big)(Z,Z) + \big(\overline{\nabla}_T\alpha\big)(T,T)\\ & + 3\big(\overline{\nabla}_U\alpha\big)(V,V) + 3\big(\overline{\nabla}_V\alpha\big)(U,U)\\ & + 3\big(\overline{\nabla}_U\alpha\big)(Z,Z) + 3\big(\overline{\nabla}_Z\alpha\big)(U,U)\\ & + 3\big(\overline{\nabla}_U\alpha\big)(T,T) + 3\big(\overline{\nabla}_T\alpha\big)(U,U)\\ & + 3\big(\overline{\nabla}_V\alpha\big)(Z,Z) + 3\big(\overline{\nabla}_Z\alpha\big)(V,V)\\ & + 3\big(\overline{\nabla}_V\alpha\big)(T,T) + 3\big(\overline{\nabla}_T\alpha\big)(V,V)\\ & + 3\big(\overline{\nabla}_Z\alpha\big)(T,T) + 3\big(\overline{\nabla}_T\alpha\big)(Z,Z)\\ & + 6\left(\big(\overline{\nabla}_U\alpha\big)(V,Z) + \big(\overline{\nabla}_U\alpha\big)(V,T)\right)\\ & + 6\left(\big(\overline{\nabla}_U\alpha\big)(Z,T) + \big(\overline{\nabla}_V\alpha\big)(Z,T)\right). \end{aligned} \tag{6.13}$$

Now condition (6.12) yields that the first seven lines of (6.13) *vanish* and therefore we have the equality

$$\begin{aligned} 0 &= \big(\overline{\nabla}_{(U+V+Z+T)}\alpha\big)\left((U+V+Z+T),(U+V+Z+T)\right)\\ &= 6\left[\big(\overline{\nabla}_U\alpha\big)(V,Z) + \big(\overline{\nabla}_U\alpha\big)(V,T) + \big(\overline{\nabla}_U\alpha\big)(Z,T) + \big(\overline{\nabla}_V\alpha\big)(Z,T)\right]. \end{aligned}$$

But condition (6.4) *allows us to apply Lemma 5.1, and by* (6.3) *we clearly have*

$$\big(\overline{\nabla}_V\alpha\big)(Z,T)=0.$$

Then, for our vector $(U+V+Z+T) \in Co\big(\widehat{X}_{E_0}[M]\big)$ we have

$$\begin{aligned} 0 &= \big(\overline{\nabla}_{(U+V+Z+T)}\alpha\big)\left((U+V+Z+T),(U+V+Z+T)\right)\\ &= 6\left[\big(\overline{\nabla}_U\alpha\big)(V,Z) + \big(\overline{\nabla}_U\alpha\big)(V,T) + \big(\overline{\nabla}_U\alpha\big)(Z,T)\right]. \end{aligned}$$

Therefore, $(U+V+Z+T)$ has the property that

$$\left[\big(\overline{\nabla}_U\alpha\big)(V,Z) + \big(\overline{\nabla}_U\alpha\big)(V,T) + \big(\overline{\nabla}_U\alpha\big)(Z,T)\right]=0. \tag{6.14}$$

Now we associate to the vector $(U+V+Z+T)$ the segment

$$\eta(s) = (sU+V+Z+T) \qquad \text{for } s \in [0,1] \subset \mathbb{R},$$

which *joins* the vector $(U+V+Z+T) = (X+Y+Z+T)$, which by hypothesis is in $Co\big(\widehat{X}_{E_0}[M]\big)$, to the vector $(V+Z+T)$, which also belongs to $Co\big(\widehat{X}_{E_0}[M]\big)$, by (6.3).

We observe now that the whole segment $\eta(s)$ is contained in $Co\big(\widehat{X}_{E_0}[M]\big)$ because

$$\begin{aligned}
&\left(\overline{\nabla}_{((sU)+V+Z+T)}\alpha\right)\left(((sU)+V+Z+T),((sU)+V+Z+T)\right)\\
&\qquad = 6\left[\left(\overline{\nabla}_{(sU)}\alpha\right)(V,Z) + \left(\overline{\nabla}_{(sU)}\alpha\right)(V,T) + \left(\overline{\nabla}_{(sU)}\alpha\right)(Z,T)\right]\\
&\qquad = (s)6\left[\left(\overline{\nabla}_{U}\alpha\right)(V,Z) + \left(\overline{\nabla}_{U}\alpha\right)(V,T) + \left(\overline{\nabla}_{U}\alpha\right)(Z,T)\right] = 0
\end{aligned}$$

by (6.14). So, for each $s \in [0,1]$, $\eta(s) \in Co\big(\widehat{X}_{E_0}[M]\big)$.

Remark 6.4. It is important to observe that conditions (6.3), (6.4), (6.5), (6.12), and Lemma 6.3 also hold if we replace U with (sU) in all of them.

Then we have shown that *if the vector $(X+Y+Z+T) \in Co\big(\widehat{X}_{E_0}[M]\big)$ exists, then it can be joined to the point $(V+Z+T) \in Co\big(\widehat{X}_{E_0}[M]\big)$ by a segment which is totally contained in $Co\big(\widehat{X}_{E_0}[M]\big)$.*

There is now a small point to be made here, as Corollary 5.3 cannot be applied to the vector $(V+Z+T)$ since $V \in \Delta^{\perp}$ and this is not an eigenspace. However, it follows from Lemma 6.1 and Corollary 6.2 that

$$(rV+Z+T) \in Co\big(\widehat{X}_{E_0}[M]\big) \qquad \text{for } r \in [0,1] \subset \mathbb{R}$$

because $(rV) \in \Delta^{\perp}$ for all $r \in [0,1] \subset \mathbb{R}$; then the vector $(V+Z+T)$ can be joined to one of the form $(Y+Z) \in Co\big(\widehat{X}_{E_0}[M]\big)$ *inside* $Co\big(\widehat{X}_{E_0}[M]\big)$.

We have then the following theorem.

Theorem 6.5. *$Co\big(\widehat{X}_{E_0}[M]\big)$ is connected by arcs.*

Now, if we had more than one connected component in $\widehat{X}_{E_0}[M]$, then we should have more that one connected component in $Co\big(\widehat{X}_{E_0}[M]\big)$ because $0 \notin Co\big(\widehat{X}_{E_0}[M]\big)$. *Then we have that $\widehat{X}_{E_0}[M]$ is connected by arcs.* The point E_0 is determined by our choice of $x_0 \in M_+$, and this choice was arbitrary, so we have proved Theorem 1.1.

7. Appendix

Here we prove first formula (7.2) for the shape operator used above.

We keep the previous notation, in particular $H(E_0) = \frac{1}{4}\nabla F(E_0)$. Let us take $U \in T_{E_0}(M)$; we have $A_{H(E_0)}(U) = \left(-\frac{1}{4}\right)\left(\nabla^E_U(\nabla F)\right)$, where ∇^E is the Euclidean covariant derivative in $\mathbb{R}^{2l}$. Let $\gamma(s)$ be a curve in M with $\gamma(0) = E_0$, parametrized by arc length. By recalling (3.3) and evaluating on $\gamma(s)$ we have to compute

$$\left(\nabla^E_U(\nabla F)\right)(E) = \frac{d}{ds}\bigg|_{s=0} (\nabla F(\gamma(s)))$$

for

$$\nabla F(\gamma(s)) = 4\gamma(s) - 8\sum_{i=0}^{m} \langle P_i(\gamma(s)), \gamma(s)\rangle P_i(\gamma(s)). \tag{7.1}$$

We have $\frac{d}{ds}\big|_{s=0}\gamma(s) = \gamma'(0) = U$, and also

$$\begin{aligned}\frac{d}{ds}\bigg|_{s} &\langle P_i(\gamma(s)), \gamma(s)\rangle P_i(\gamma(s)) \\ &= 2\langle P_i(\gamma(s)), \gamma'(s)\rangle P_i(\gamma(s)) + \langle P_i(\gamma(s)), \gamma(s)\rangle P_i(\gamma'(s)),\end{aligned}$$

which evaluating at $s = 0$ yields

$$\frac{d}{ds}\bigg|_{s=0} \langle P_i(\gamma(s)), \gamma(s)\rangle P_i(\gamma(s)) = 2\langle P_i(E_0), U\rangle P_i(E_0) + \langle P_i(E_0), E_0\rangle P_i(U).$$

Hence

$$\frac{d}{ds}\bigg|_{s=0} \nabla F(\gamma(s)) = 4U - 8\sum_{i=0}^{m}\left(2\langle P_i(E_0), U\rangle P_i(E_0) + \langle P_i(E_0), E_0\rangle P_i(U)\right),$$

which dividing by 4 clearly yields

$$\begin{aligned}A_{H(E_0)}(U) = (-U) &+ 2\sum_{i=0}^{m} 2\langle P_i(E_0), U\rangle P_i(E_0) \\ &+ 2\sum_{i=0}^{m}\langle P_i(E_0), E_0\rangle P_i(U).\end{aligned} \tag{7.2}$$

Now we establish formula (7.4), which is the objective of this Appendix.

We have the Cartan–Münzner polynomial F of M and we have to compute the polynomial defining the planar normal sections. In order to do this we use the formula proved in [3], that is,

$$\Gamma(X) = (-X)\left\langle \nabla^E_{\gamma'(s)}(\nabla F(\gamma(s))), \gamma'(s)\right\rangle. \tag{7.3}$$

Let $E_0 \in M = F^{-1}(0)$ and let $\gamma(s)$ be *a normal section* of M at the point E_0. Then γ is a curve in M, parametrized by arc length, such that $\gamma(0) = E_0$, $\gamma'(0) = X$, $\|X\| = 1$, and $\nabla_X(\gamma'(s)) = 0$. We use formula (7.3).

We have to evaluate $\nabla F(X)$ on $\gamma(s)$, that is, again (7.1), and compute

$$\nabla^E_{\gamma'(s)}(\nabla F(\gamma(s))) = \frac{d}{ds}(\nabla F(\gamma(s))).$$

Using that the operators P_k are symmetric, we may write

$$\nabla^E_{\gamma'(s)} \left(\nabla F\left(\gamma(s)\right)\right)$$
$$= 4\gamma'(s) - 8\sum_{k=0}^{m} 2\left\langle P_k\left(\gamma'\right), \gamma\right\rangle P_k(\gamma) - 8\sum_{k=0}^{m} \left\langle P_k(\gamma), \gamma\right\rangle P_k\left(\gamma'\right),$$

and computing the inner product with γ', we get

$$\left\langle \nabla^E_{\gamma'(s)} \left(\nabla F\left(\gamma(s)\right)\right), \gamma'\left(s\right) \right\rangle$$
$$= 4 - 8\sum_{k=0}^{m} 2\left\langle P_k(\gamma), \gamma'\right\rangle^2 - 8\sum_{k=0}^{m} \left\langle P_k(\gamma), \gamma\right\rangle \left\langle P_k\left(\gamma'\right), \gamma'\right\rangle.$$

Now $(-X)\left\langle \nabla^E_{\gamma'(s)} \left(\nabla F\left(\gamma(s)\right)\right), \gamma'(s)\right\rangle$ is just the derivative of the inner product with respect to the parameter s of γ. We have

$$\left.\frac{d\left\langle \nabla^E_{\gamma'(s)} \left(\nabla F\left(\gamma(s)\right)\right), \gamma'(s)\right\rangle}{ds}\right|_s$$
$$= -8\sum_{k=0}^{m} 4\left\langle P_k(\gamma), \gamma'\right\rangle \left(\left\langle P_k\left(\gamma'\right), \gamma'\right\rangle + \left\langle P_k(\gamma), \nabla_{\gamma'}\left(\gamma'\right)\right\rangle\right)$$
$$- 8\sum_{k=0}^{m} 2\left\langle P_k(\gamma), \gamma'\right\rangle \left\langle P_k\left(\gamma'\right), \gamma'\right\rangle$$
$$- 8\sum_{k=0}^{m} 2\left\langle P_k(\gamma), \gamma\right\rangle \left\langle P_k\left(\nabla_{\gamma'}\left(\gamma'\right)\right), \gamma'\right\rangle.$$

Evaluating now at $s = 0$, we have $\gamma(0) = E_0$, $\gamma'(0) = X$, and also $\nabla_X\left(\gamma'\right) = \nabla_{\gamma'}\left(\gamma'\right) = 0$. Then we obtain

$$\left.\frac{d\left\langle \nabla^E_{\gamma'(s)} \left(\nabla F\left(\gamma(s)\right)\right), \gamma'(s)\right\rangle}{ds}\right|_{s=0} = -48\sum_{k=0}^{m} \left\langle P_k\left(E\right), X\right\rangle \left\langle P_k(X), X\right\rangle.$$

Then, finally, our polynomial $\Gamma(X)$ can be taken (eliminating the factor -48) as

$$\Gamma(X) = \sum_{k=0}^{m} \left\langle P_k\left(E\right), X\right\rangle \left\langle P_k(X), X\right\rangle. \tag{7.4}$$

Therefore the condition for $X \in T_{E_0}(M)$ to generate a planar normal section is $\Gamma(X) = 0$.

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Cristián U. Sánchez
CIEM–CONICET and Fa.M.A.F., Universidad Nacional de Córdoba, Medina Allende s/n, Ciudad Universitaria, 5000 Córdoba, Argentina
csanchez@famaf.unc.edu.ar

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