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# THE ISOLATION OF THE FIRST EIGENVALUE FOR A DIRICHLET EIGENVALUE PROBLEM INVOLVING THE FINSLER $p$-LAPLACIAN AND A NONLOCAL TERM

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Abstract. We analyse the isolation of the first eigenvalue for an eigenvalue problem involving the Finsler $p$-Laplace operator and a nonlocal term on a bounded domain subject to the homogeneous Dirichlet boundary condition.

## 1. Introduction: statement of the problem and motivation

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with smooth boundary denoted by $\partial\Omega$. Let $H$ be a Finsler norm (i.e., $H : \mathbb{R}^N \to [0,\infty)$ is a convex function of class $C^2\left(\mathbb{R}^N \setminus \{0\}\right)$, even and homogeneous of degree 1) such that $H^2$ is strongly convex (i.e., the Hessian matrix $D^2[H^2](\xi)$ is positive definite for $\xi \in \mathbb{R}^N \setminus \{0\}$). Define $J : \mathbb{R}^N \to \mathbb{R}^N$ by

$$J(\xi) = H(\xi)\,\nabla H(\xi) \quad \forall\, \xi \in \mathbb{R}^N.$$

The goal of this paper is to analyse the isolation of the first (the lowest) eigenvalue of the following eigenvalue problem:

$$\begin{cases} -Q_{pq}u(x) = \lambda\left(\displaystyle\int_\Omega |u(y)|^p\,dy\right)^{q-1} |u(x)|^{p-2}u(x) & \text{for } x \in \Omega,\\ u(x) = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $p$, $q \in (1,\infty)$ are real numbers, $\lambda$ is a real parameter, and

$$Q_{pq}u := \operatorname{div}\left(H(\nabla u)^{pq-2}J(\nabla u)\right)$$

stands for the Finsler $pq$-Laplace operator.

**Definition 1.1.** We say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of problem (1.1) if there exists $u \in W_0^{1,pq}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega H(\nabla u)^{pq-2}\,J(\nabla u)\cdot\nabla\varphi\,\mathrm{d}x = \lambda\left(\int_\Omega |u|^p\,\mathrm{d}x\right)^{q-1}\int_\Omega |u|^{p-2}u\varphi\,\mathrm{d}x \tag{1.2}$$

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for all $\varphi \in W_0^{1,pq}(\Omega)$. Furthermore, $u$ from the above relation will be called an *eigenfunction* corresponding to the eigenvalue $\lambda$.

For all $p, q \in (1, \infty)$, we define

$$\lambda_1(p,q) := \inf_{u \in W_0^{1,pq}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega H(\nabla u)^{pq} \, \mathrm{d}x}{\left(\displaystyle\int_\Omega |u|^p \, \mathrm{d}x\right)^q}. \tag{1.3}$$

In the particular case when $q = 1$, problem (1.1) becomes the eigenvalue problem

$$\begin{cases} -Q_p u(x) = \lambda |u(x)|^{p-2} u(x) & \text{for } x \in \Omega, \\ u(x) = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{1.4}$$

which was investigated by Belloni, Ferone & Kawohl in [3]. In particular, they showed that the minimum of the Rayleigh quotient which can be associated to this eigenvalue problem, i.e.,

$$\lambda_1(p,1) := \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega H(\nabla u)^{p} \, \mathrm{d}x}{\displaystyle\int_\Omega |u|^p \, \mathrm{d}x},$$

gives the lowest eigenvalue of problem (1.4), and its minimizers are corresponding eigenfunctions of $\lambda_1(p,1)$ that do not change sign in $\Omega$. Note that if $H$ is the Euclidean norm in $\mathbb{R}^N$, i.e., $H(x) = |x|$, then the Finsler $p$-Laplace operator $Q_p$ becomes the classical $p$-Laplace operator, i.e., $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$, and problem (1.4) becomes the celebrated eigenvalue problem for the $p$-Laplace operator with homogeneous Dirichlet boundary condition:

$$\begin{cases} -\Delta_p u(x) = \lambda |u(x)|^{p-2} u(x) & \text{for } x \in \Omega, \\ u(x) = 0 & \text{for } x \in \partial\Omega. \end{cases} \tag{1.5}$$

We recall that the isolation of the first eigenvalue of problem (1.5) was investigated by Anane [1], Diaz & Saa [6], Lindqvist [11] and Lê [10]. These works inspired the study from this paper and stand at the base of the proof of our main result, which is given by the following theorem.

**Theorem 1.2.** *The number $\lambda_1(p,q)$ given by relation* (1.3) *is positive and represents the first eigenvalue of problem* (1.1)*. Furthermore, $\lambda_1(p,q)$ is an isolated eigenvalue of problem* (1.1)*.*

**Remark 1.3.** Note that the facts that $\lambda_1(p,q)$ is positive and represents the first eigenvalue of problem (1.1) were already obtained in our previous works [8, 9]. Moreover, in [8, Lemma 3 & Theorem 1] we proved that the eigenvalue $\lambda_1(p,q)$ is simple in the sense that the ratio between any two corresponding eigenfunctions is a nontrivial constant.

## 2. Proof of the main result

In order to prove the main result of the paper we recall some auxiliary results. Before presenting their statements and proofs we point out the simple observation that since any two norms are equivalent on $\mathbb{R}^N$, there exist two positive constants $a$ and $b$ such that

$$aH(\eta)| \leq |\eta| \leq bH(\eta) \quad \forall\, \eta \in \mathbb{R}^N. \tag{2.1}$$

We also recall a simple but useful relation which can be found, for example, in [2]:

$$\langle \nabla H(\xi), \xi\rangle = H(\xi) \quad \forall\, \xi \in \mathbb{R}^N. \tag{2.2}$$

**Remark 2.1.** We point out that arguments similar to those used by Tolksdorf [12] (see also [10]) can be used in order to show that the eigenfunctions corresponding to any eigenvalue $\lambda$ of problem (1.1) belong to $C^{1,\alpha}(\overline{\Omega})$.

**Lemma 2.2** (See [8, Lemma 1])**.** *$\lambda_1(p,q)$ is positive.*

**Lemma 2.3** (See [8, Lemmas 2 & 3] or [9, Theorems 3.1 & 3.2])**.** *For each $p, q \in (1,\infty)$, there exists a function $u_1 \in W_0^{1,pq}(\Omega) \setminus \{0\}$ such that $\int_\Omega H(\nabla u_1)^{pq}\,\mathrm{d}x = \lambda_1(p,q)$ and $\int_\Omega |u_1|^p\,\mathrm{d}x = 1$. Moreover, $\lambda_1(p,q)$ is an eigenvalue of problem* (1.1)*, and $u_1$ represents its associated eigenfunction. Furthermore, $u_1$ does not change sign in $\Omega$.*

In order to go further, for each $u \in W_0^{1,pq}(\Omega)$ we define

$$u_\pm(x) := \max\{\pm u(x), 0\} \quad \forall\, x \in \Omega.$$

By [7, Lemma 7.6] we know that $u_+, u_- \in W_0^{1,p}(\Omega)$, and

$$\nabla u_+ = \begin{cases} 0 & \text{if } [u \leq 0], \\ \nabla u & \text{if } [u > 0] \end{cases} \qquad \text{and} \qquad \nabla u_- = \begin{cases} 0 & \text{if } [u \geq 0], \\ \nabla u & \text{if } [u < 0]. \end{cases}$$

We propose to prove the following result.

**Proposition 2.4.** *Let $v$ be an eigenfunction corresponding to an eigenvalue $\lambda > \lambda_1(p,q)$ of problem* (1.1)*. Then $v_\pm \not\equiv 0$.*

In order to prove the above proposition for each $u, v \in W^{1,pq}(\Omega)$, we define

$$\langle -Q_{pq}u, v\rangle := \int_\Omega H(\nabla u)^{pq-2} J(\nabla u)\cdot \nabla v\,\mathrm{d}x.$$

Define also

$$G := \{(u,v) \in W^{1,pq}(\Omega) \times W^{1,pq}(\Omega) : u, v \geq 0,\ u, v \in L^\infty(\Omega)\}.$$

For each $\epsilon > 0$ and $(u,v) \in G$, we consider

$$I_\epsilon(u,v) := \left\langle -Q_{pq}u, \frac{(u+\epsilon)^p - (v+\epsilon)^p}{(u+\epsilon)^{p-1}}\right\rangle - \left\langle -Q_{pq}v, \frac{(u+\epsilon)^p - (v+\epsilon)^p}{(v+\epsilon)^{p-1}}\right\rangle.$$

Set $u_\epsilon := u+\epsilon$ and $v_\epsilon := v+\epsilon$. It is clear that $(u_\epsilon, v_\epsilon) \in G$. We compute $\left\langle -Q_{pq}u, \frac{u_\epsilon^p - v_\epsilon^p}{u_\epsilon^{p-1}} \right\rangle$ and $\left\langle -Q_{pq}v, \frac{u_\epsilon^p - v_\epsilon^p}{v_\epsilon^{p-1}} \right\rangle$. First, direct computations based on relation (2.2) yield

$$\begin{aligned}\left\langle -Q_{pq}u, \frac{u_\epsilon^p - v_\epsilon^p}{u_\epsilon^{p-1}} \right\rangle &= \int_\Omega H(\nabla u)^{pq-2} J(\nabla u)\cdot \nabla \left(\frac{u_\epsilon^p - v_\epsilon^p}{u_\epsilon^{p-1}}\right) \mathrm{d}x \\ &= \int_\Omega H(\nabla u)^{pq}\, \mathrm{d}x + (p-1)\int_\Omega \left(\frac{v_\epsilon}{u_\epsilon}\right)^p H(\nabla u)^{pq}\, \mathrm{d}x \\ &\quad - p\int_\Omega \left(\frac{v_\epsilon}{u_\epsilon}\right)^{p-1} H(\nabla u)^{pq-2} J(\nabla u)\cdot \nabla v\, \mathrm{d}x.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\left\langle -Q_{pq}v, \frac{u_\epsilon^p - v_\epsilon^p}{v_\epsilon^{p-1}} \right\rangle &= -\left\langle -Q_{pq}v, \frac{v_\epsilon^p - u_\epsilon^p}{v_\epsilon^{p-1}} \right\rangle \\ &= -\int_\Omega H(\nabla v)^{pq}\, \mathrm{d}x - (p-1)\int_\Omega \left(\frac{u_\epsilon}{v_\epsilon}\right)^p H(\nabla v)^{pq}\, \mathrm{d}x \\ &\quad + p\int_\Omega \left(\frac{u_\epsilon}{v_\epsilon}\right)^{p-1} H(\nabla v)^{pq-2} J(\nabla v)\cdot \nabla u\, \mathrm{d}x.\end{aligned}$$

The following two results are based on the ideas by Diaz & Saa in [6].

**Lemma 2.5.** *Let* $K : L^q(\Omega) \to \mathbb{R}\cup\{+\infty\}$,

$$K(\varphi) = \begin{cases} \dfrac{1}{q}\displaystyle\int_\Omega H\left(\nabla \varphi^{\frac{1}{p}}\right)^{pq} \mathrm{d}x & \text{if } \varphi \in D(K), \\ +\infty & \text{otherwise,}\end{cases}$$

*where* $D(K) := \left\{\varphi^{\frac{1}{p}} \in W^{1,pq}(\Omega),\ \varphi \geq 0\right\}$. *Then* $K \not\equiv +\infty$ *and* $K$ *is convex.*

*Proof.* Let $u \in W_0^{pq}(\Omega)$ be a positive eigenfunction corresponding to the eigenvalue $\lambda_1(p,q)$. By Lemma 2.3 we can assume that $\int_\Omega H(\nabla u)^{pq}\, \mathrm{d}x = \lambda_1(p,q)$ and $\int_\Omega |u|^p\, \mathrm{d}x = 1$. Then, for $\varphi = u^p \in D(K)$, we have $K(u^p) = \dfrac{1}{q}\lambda_1(p,q)$. Thus, $K \not\equiv +\infty$.

In order to show the convexity of $K$ we consider $\alpha_1, \alpha_2 \in D(K)$. We have to show that

$$K(t\alpha_1 + (1-t)\alpha_2) \leq tK(\alpha_1) + (1-t)K(\alpha_2) \quad \forall\, t\in[0,1].$$

Let $t \in [0,1]$ be arbitrary but fixed. Define

$$\beta_1 := \alpha_1^{\frac{1}{p}}, \quad \beta_2 := \alpha_2^{\frac{1}{p}}, \quad \beta_3 := (t\alpha_1 + (1-t)\alpha_2)^{\frac{1}{p}}.$$

Simple computations yield

$$\nabla\beta_1 = \frac{1}{p}\beta_1^{1-p}\nabla\alpha_1, \quad \nabla\beta_2 = \frac{1}{p}\beta_2^{1-p}\nabla\alpha_2, \quad \nabla\beta_3 = \frac{1}{p}\beta_3^{1-p}\left(t\nabla\alpha_1 + (1-t)\nabla\alpha_2\right).$$

Consequently, we have that

$$\begin{aligned}\beta_3^{p-1}\nabla\beta_3 &= \frac{1}{p}\left(t\nabla\alpha_1 + (1-t)\nabla\alpha_2\right)\\ &= \frac{1}{p}\left(tp\beta_1^{p-1}\nabla\beta_1 + (1-t)p\beta_2^{p-1}\nabla\beta_2\right)\\ &= t\beta_1^{p-1}\nabla\beta_1 + (1-t)\beta_2^{p-1}\nabla\beta_2.\end{aligned}$$

It follows that

$$H(\beta_3^{p-1}\nabla\beta_3) = H\left(t\beta_1^{p-1}\nabla\beta_1 + (1-t)\beta_2^{p-1}\nabla\beta_2\right).$$

Taking into account that $H$ is homogeneous of degree 1 and convex we get

$$\begin{aligned}&\beta_3^{p-1}H(\nabla\beta_3) \leq t\beta_1^{p-1}H(\nabla\beta_1) + (1-t)\beta_2^{p-1}H(\nabla\beta_2)\\ &\quad= \left(t^{\frac{p-1}{p}}\beta_1^{p-1}\right)\left(t^{\frac{1}{p}}H(\nabla\beta_1)\right) + \left((1-t)^{\frac{p-1}{p}}\beta_2^{p-1}\right)\left((1-t)^{\frac{1}{p}}H(\nabla\beta_2)\right).\end{aligned} \tag{2.3}$$

Recall now the classical Cauchy–Schwarz inequality which holds true for each $p>1$, namely

$$a_1a_2 + b_1b_2 \leq \left(a_1^{\frac{p}{p-1}} + b_1^{\frac{p}{p-1}}\right)^{(p-1)/p}\left(a_2^p + b_2^p\right)^{1/p} \quad \forall\, a_1,\, a_2,\, b_1,\, b_2 > 0.$$

Taking

$$\begin{aligned}a_1 &:= t^{\frac{p-1}{p}}\beta_1^{p-1}, & b_1 &:= (1-t)^{\frac{p-1}{p}}\beta_2^{p-1},\\ a_2 &:= t^{\frac{1}{p}}H(\nabla\beta_1), & b_2 &:= (1-t)^{\frac{1}{p}}H(\nabla\beta_2)\end{aligned}$$

in the above inequality we deduce that

$$\begin{aligned}&\left(t^{\frac{p-1}{p}}\beta_1^{p-1}\right)\left(t^{\frac{1}{p}}H(\nabla\beta_1)\right) + \left((1-t)^{\frac{p-1}{p}}\beta_2^{p-1}\right)\left((1-t)^{\frac{1}{p}}H(\nabla\beta_2)\right)\\ &\qquad\leq \left(t\beta_1^p + (1-t)\beta_2^p\right)^{(p-1)/p}\left(tH(\nabla\beta_1)^p + (1-t)H(\nabla\beta_2)^p\right)^{1/p}.\end{aligned} \tag{2.4}$$

Combining (2.3) with (2.4) and taking into account that $(t\beta_1^p + (1-t)\beta_2^p)^{(p-1)/p} = (t\alpha_1 + (1-t)\alpha_2)^{(p-1)/p} = \beta_3^{p-1}$ we find

$$\beta_3^{p-1}H(\nabla\beta_3) \leq \beta_3^{p-1}\left(tH(\nabla\beta_1)^p + (1-t)H(\nabla\beta_2)^p\right)^{1/p}.$$

From the last inequality, using the fact that the function $(0,\infty)\ni t\to t^q$ (with $q>1$) is convex we deduce via Jensen's inequality that

$$\begin{aligned}H(\nabla\beta_3)^{pq} &\leq [tH(\nabla\beta_1)^p + (1-t)H(\nabla\beta_2)^p]^q\\ &\leq tH(\nabla\beta_1)^{pq} + (1-t)H(\nabla\beta_2)^{pq}.\end{aligned}$$

Integrating over $\Omega$ and multiplying with $\frac{1}{q}$ we get the conclusion of the lemma. □

**Lemma 2.6.** *For each $\epsilon>0$, we have*

$$I_\epsilon(u,v) = \langle K'(u_\epsilon^p), u_\epsilon^p - v_\epsilon^p\rangle - \langle K'(v_\epsilon^p), u_\epsilon^p - v_\epsilon^p\rangle \quad \forall\, u,\, v \in G,$$

*where $u_\epsilon := u+\epsilon$, $v_\epsilon := v+\epsilon$, and $\langle K'(\varphi),\xi\rangle$ stands for the Gâteaux derivative of $K$ at $\varphi$ in the direction $\xi$ when $\varphi$, $\xi\in D(K)$. Moreover,*

$$I_\epsilon(u,v)\geq 0 \quad \forall\, u,\, v\in G.$$

*Proof.* Let $\varphi$, $\xi\in D(K)$ be arbitrary but fixed such that $\varphi>0$ in $\overline{\Omega}$. Define $f:\mathbb{R}\to\mathbb{R}$ by

$$f(t)=\frac{1}{q}\int_\Omega H\left(\nabla(\varphi+t\xi)^{\frac{1}{p}}\right)^{pq}\,\mathrm{d}x \quad \forall\, t\in\mathbb{R}.$$

Clearly, $f\in C^1(\mathbb{R})$ and

$$f'(0)=\lim_{t\to 0}\frac{f(t)-f(0)}{t}=\lim_{t\to 0}\frac{K(\varphi+t\xi)-K(\varphi)}{t}=\langle K'(\varphi),\xi\rangle.$$

On the other hand, simple computations yield

$$\begin{aligned} f'(t)&=\int_\Omega H\left(\nabla(\varphi+t\xi)^{\frac{1}{p}}\right)^{pq-2} J\left(\nabla(\varphi+t\xi)^{\frac{1}{p}}\right)\frac{\partial}{\partial t}\left[(\varphi+t\xi)^{\frac{1-p}{p}}(\nabla\varphi+t\nabla\xi)\right]\mathrm{d}x\\ &=\int_\Omega H\left(\nabla(\varphi+t\xi)^{\frac{1}{p}}\right)^{pq-2} J\left(\nabla(\varphi+t\xi)^{\frac{1}{p}}\right)\\ &\qquad\times\left[\frac{1-p}{p}(\varphi+t\xi)^{\frac{1}{p}-2}\xi(\nabla\varphi+t\nabla\xi)+(\varphi+t\xi)^{\frac{1-p}{p}}\nabla\xi\right]\mathrm{d}x. \end{aligned}$$

Then, we have

$$\langle K'(\varphi),\xi\rangle=f'(0)=\int_\Omega H\left(\nabla\varphi^{\frac{1}{p}}\right)^{pq-2} J\left(\nabla\varphi^{\frac{1}{p}}\right)\left[(1-p)\frac{\xi}{\varphi}\nabla\varphi^{\frac{1}{p}}+\varphi^{\frac{1-p}{p}}\nabla\xi\right]\mathrm{d}x.$$

Observe that $\langle K'(\varphi),\xi\rangle$ is linear with respect to $\xi$ and consequently we can allow $\xi$ to change sign in $\Omega$. Let now $(u,v)\in G$. Taking $\varphi=u_\epsilon^p$ and $\xi=u_\epsilon^p-v_\epsilon^p$ we have

$$\begin{aligned} &\langle K'(u_\epsilon^p),u_\epsilon^p-v_\epsilon^p\rangle\\ &\quad=\int_\Omega H\left(\nabla u_\epsilon\right)^{pq-2} J\left(\nabla u_\epsilon\right)\left[(1-p)\frac{u_\epsilon^p-v_\epsilon^p}{u_\epsilon^p}\nabla u_\epsilon+u_\epsilon^{1-p}\nabla(u_\epsilon^p-v_\epsilon^p)\right]\mathrm{d}x\\ &\quad=\int_\Omega H(\nabla u)^{pq}\,\mathrm{d}x+(p-1)\int_\Omega\left(\frac{v_\epsilon}{u_\epsilon}\right)^p H(\nabla u)^{pq}\,\mathrm{d}x\\ &\qquad\quad-p\int_\Omega\left(\frac{v_\epsilon}{u_\epsilon}\right)^{p-1} H(\nabla u)^{pq-2}J(\nabla u)\cdot\nabla v\,\mathrm{d}x\\ &\quad=\left\langle -Q_{pq}u,\frac{u_\epsilon^p-v_\epsilon^p}{u_\epsilon^{p-1}}\right\rangle. \end{aligned}$$

Similarly, taking $\varphi = v_\epsilon^p$ and $\xi = u_\epsilon^p - v_\epsilon^p$ we have

$$\begin{aligned}
&\langle K'(v_\epsilon^p), u_\epsilon^p - v_\epsilon^p\rangle \\
&\quad = \int_\Omega H(\nabla v_\epsilon)^{pq-2} J(\nabla v_\epsilon)\left[(1-p)\frac{u_\epsilon^p - v_\epsilon^p}{v_\epsilon^p}\nabla v_\epsilon + v_\epsilon^{1-p}\nabla(u_\epsilon^p - v_\epsilon^p)\right] \mathrm{d}x \\
&\quad = -\int_\Omega H(\nabla v)^{pq}\,\mathrm{d}x - (p-1)\int_\Omega \left(\frac{u_\epsilon}{v_\epsilon}\right)^p H(\nabla v)^{pq}\,\mathrm{d}x \\
&\qquad + p\int_\Omega \left(\frac{u_\epsilon}{v_\epsilon}\right)^{p-1} H(\nabla v)^{pq-2} J(\nabla v)\cdot\nabla u\,\mathrm{d}x \\
&\quad = -\left\langle -Q_{pq}v, \frac{v_\epsilon^p - u_\epsilon^p}{v_\epsilon^{p-1}}\right\rangle = \left\langle -Q_{pq}v, \frac{u_\epsilon^p - v_\epsilon^p}{v_\epsilon^{p-1}}\right\rangle.
\end{aligned}$$

Now, using the convexity of $K$ given by Lemma 2.5 we have

$$\begin{aligned}
K(\varphi) &\geq K(\xi) + \langle K'(\xi), \varphi - \xi\rangle \quad \forall\,\varphi,\,\xi \in D(K), \\
K(\xi) &\geq K(\varphi) + \langle K'(\varphi), \xi - \varphi\rangle \quad \forall\,\varphi,\,\xi \in D(K).
\end{aligned}$$

Adding the above two relations we find

$$\langle K'(\xi), \xi - \varphi\rangle - \langle K'(\varphi), \xi - \varphi\rangle \geq 0 \quad \forall\,\varphi,\,\xi \in D(K).$$

For each $\epsilon > 0$, letting $\xi = u_\epsilon^p$ and $\varphi = v_\epsilon^p$, the above pieces of information lead to the conclusion of the lemma. □

We are now ready to prove Proposition 2.4.

*Proof of Proposition 2.4.* Let $u$ be a positive eigenfunction corresponding to the principal eigenvalue $\lambda_1(p,q)$ such that

$$\int_\Omega H(\nabla u)^{pq}\,\mathrm{d}x = 1 \quad \text{and} \quad \left(\int_\Omega u^p\,\mathrm{d}x\right)^q = \frac{1}{\lambda_1(p,q)}.$$

The existence of such a function $u$ follows from Lemma 2.3. Let $\lambda > \lambda_1(p,q)$ be another fixed eigenvalue of problem (1.1). We assume by contradiction that $\lambda$ has an eigenfunction $v$ which does not change sign in $\Omega$. Assume for instance that $v > 0$ in $\Omega$, and by a rescaling argument we may also assume that $\int_\Omega H(\nabla v)^{pq}\,\mathrm{d}x = 1$ and $\left(\int_\Omega |v|^p\,\mathrm{d}x\right)^q = \lambda^{-1}$. Taking into account that $u$ and $v$ are smooth enough (see Remark 2.1) by Lemma 2.6 we know that, for each $\epsilon > 0$, $I_\epsilon(u,v) \geq 0$, and consequently,

$$\liminf_{\epsilon\to 0^+} I_\epsilon(u,v) \geq 0. \tag{2.5}$$

On the other hand, by relation (1.2) we have

$$\begin{aligned}
I_\epsilon(u,v) &= \left\langle -Q_{pq}u, \frac{u_\epsilon^p - v_\epsilon^p}{u_\epsilon^{p-1}}\right\rangle - \left\langle -Q_{pq}v, \frac{u_\epsilon^p - v_\epsilon^p}{v_\epsilon^{p-1}}\right\rangle \\
&= \lambda_1(p,q)\frac{1}{\lambda_1(p,q)^{\frac{q-1}{q}}}\int_\Omega u^{p-1}\frac{(u_\epsilon^p - v_\epsilon^p)}{u_\epsilon^{p-1}}\,\mathrm{d}x - \lambda\frac{1}{\lambda^{\frac{q-1}{q}}}\int_\Omega v^{p-1}\frac{(u_\epsilon^p - v_\epsilon^p)}{v_\epsilon^{p-1}}\,\mathrm{d}x.
\end{aligned}$$

Thus, letting $\epsilon \to 0^+$ and using Lebesgue's dominated convergence theorem we obtain

$$\begin{aligned}\limsup_{\epsilon\to 0^+} I_\epsilon(u,v) &= \limsup_{\epsilon\to 0^+}\left\{\lambda_1(p,q)^{\frac{1}{q}}\int_\Omega u^{p-1}\frac{(u_\epsilon^p - v_\epsilon^p)}{u_\epsilon^{p-1}}\,\mathrm{d}x\right.\\ &\qquad\left. -\lambda^{\frac{1}{q}}\int_\Omega v^{p-1}\frac{(u_\epsilon^p - v_\epsilon^p)}{v_\epsilon^{p-1}}\,\mathrm{d}x\right\}\\ &= \left(\lambda_1(p,q)^{\frac{1}{q}} - \lambda^{\frac{1}{q}}\right)\left(\frac{1}{\lambda_1(p,q)^{\frac{1}{q}}} - \frac{1}{\lambda^{\frac{1}{q}}}\right) < 0,\end{aligned}$$

a contradiction with (2.5). Hence, $v$ must change sign in $\Omega$. The proof of Proposition 2.4 is complete. □

**Proposition 2.7.** *Let $\lambda > \lambda_1(p,q)$ be an eigenvalue of problem* (1.1)*. Then, there exists a corresponding eigenfunction $\eta$ such that*

$$\int_\Omega H(\nabla\eta)^{pq}\,\mathrm{d}x = 1 \quad \text{and} \quad \left(\int_\Omega |\eta|^p\,\mathrm{d}x\right)^q = \lambda^{-1},$$

*and*

$$|\Omega_\eta^-| \geq \left(\lambda 2^{q-1}C^{pq}b^{pq}\right)^{1/(1-q)},$$

*where $\Omega_\eta^-$ is the set $\{x\in\Omega : \eta(x) < 0\}$, $C = C(p,q,N,\Omega)$ is the constant given by Poincaré's inequality, and $b$ is given by relation* (2.1)*.*

*Proof.* First, note that the first part of the conclusion holds true via a rescaling argument. Next, note that from Proposition 2.4 we know that any eigenfunction corresponding to $\lambda > \lambda_1(p,q)$ changes sign over $\Omega$. We observe that if $u$ is an eigenfunction corresponding to eigenvalue $\lambda$, then $-u$ is also an eigenfunction corresponding to eigenvalue $\lambda$. Testing in (1.2) with $\varphi = u_-$ we find that

$$\int_\Omega H\left(\nabla u_-\right)^{pq}\,\mathrm{d}x = \lambda\left(\int_\Omega u_+^p\,\mathrm{d}x + \int_\Omega u_-^p\,\mathrm{d}x\right)^{q-1}\int_\Omega u_-^p\,\mathrm{d}x.$$

Next, if $\int_\Omega u_-^p\,\mathrm{d}x \geq \int_\Omega u_+^p\,\mathrm{d}x$, we get

$$\int_\Omega H\left(\nabla u_-\right)^{pq}\,\mathrm{d}x \leq \lambda 2^{q-1}\left(\int_\Omega u_-^p\,\mathrm{d}x\right)^q. \tag{2.6}$$

Using Hölder's inequality, Poincaré's inequality (Brezis [5, Corollary 9.19]) and relation (2.1) we get the existence of a positive constant $C = C(p,q,N,\Omega)$ such that

$$\left(\int_\Omega u_-^p\,\mathrm{d}x\right)^q \leq |\Omega_\eta^-|^{q-1}\int_\Omega u_-^{pq}\,\mathrm{d}x \leq |\Omega_\eta^-|^{q-1}C^{pq}b^{pq}\int_\Omega H\left(\nabla u_-\right)^{pq}\,\mathrm{d}x. \tag{2.7}$$

Combining (2.6) and (2.7) we deduce that

$$|\Omega_\eta^-| \geq \left(\lambda 2^{q-1}C^{pq}b^{pq}\right)^{1/(1-q)}.$$

If $\int_\Omega u_-^p\,\mathrm{d}x \leq \int_\Omega u_+^p\,\mathrm{d}x$, then we consider $v = -u$ to be the eigenfunction corresponding to the eigenvalue $\lambda$. Since $v_- = u_+$ and $v_+ = u_-$, we obtain $\int_\Omega v_-^p\,\mathrm{d}x \geq$

$\int_\Omega v_+^p \,\mathrm{d}x$. Then, testing in (1.2) with $\varphi = v_-$ and then repeating the above arguments we find again the desired estimation of $|\Omega_\eta^-|$. Thus, the proof of Proposition 2.7 is complete. □

We are now ready to prove Theorem 1.2.

*Proof of Theorem 1.2.* Let $\lambda$ be an eigenvalue of problem (1.1). Let $v$ be an eigenfunction corresponding to the eigenvalue $\lambda$ such that

$$\int_\Omega H(\nabla v)^{pq}\,\mathrm{d}x = 1 \quad \text{and} \quad \left(\int_\Omega |v|^p\,\mathrm{d}x\right)^q = \lambda^{-1}.$$

Then it is clear that

$$\int_\Omega H(\nabla v)^{pq}\,\mathrm{d}x = \lambda\left(\int_\Omega |v|^p\,\mathrm{d}x\right)^q,$$

and since

$$\int_\Omega H(\nabla v)^{pq}\,\mathrm{d}x \geq \lambda_1(p,q)\left(\int_\Omega |v|^p\,\mathrm{d}x\right)^q$$

and $v \neq 0$, we deduce that $\lambda \geq \lambda_1(p,q)$. It follows that $\lambda_1(p,q)$ is isolated to the left. Assume by contradiction that it is not isolated to the right. Then, there exists a sequence of eigenvalues $\{\lambda_n\}_n$ of problem (1.1) such that $\lambda_n > \lambda_1(p,q)$ for each $n$ and

$$\lim_{n\to\infty} \lambda_n = \lambda_1(p,q).$$

Let $\{u_n\}_n$ be a sequence of corresponding eigenfunctions given by Proposition 2.7 such that

$$\int_\Omega H(\nabla u_n)^{pq}\,\mathrm{d}x = 1 \quad \text{and} \quad \left(\int_\Omega |u_n|^p\,\mathrm{d}x\right)^q = \lambda_n{}^{-1},$$

and

$$|\Omega_{u_n}^-| \geq \left(\lambda_n 2^{q-1} C^{pq} b^{pq}\right)^{1/(1-q)}. \tag{2.8}$$

Since $\int_\Omega H(\nabla u_n)^{pq}\,\mathrm{d}x = 1$ for each $n$, by relation (2.1) we deduce that $\{u_n\}_n$ is bounded in $W_0^{1,pq}(\Omega)$. This implies the existence of a subsequence of $\{u_n\}_n$, still denoted by $\{u_n\}_n$, that converges weakly in $W_0^{1,pq}(\Omega)$ to a function $u \in W_0^{1,pq}(\Omega)$. Since $W_0^{1,pq}(\Omega)$ is compactly embedded in $L^p(\Omega)$, we deduce that $\{u_n\}_n$ converges strongly to $u$ in $L^p(\Omega)$ and a.e. in $\Omega$. Moreover, since $\left(\int_\Omega |u_n|^p\,\mathrm{d}x\right)^q = \lambda_n{}^{-1}$ for each integer $n$, and $\lim_{n\to\infty}\lambda_n = \lambda_1(p,q)$, it follows that $\left(\int_\Omega |u|^p\,\mathrm{d}x\right)^q = \lambda_1(p,q)$, which ensures that $u \neq 0$.

On the other hand, we recall that by relation (1.2) we have that, for each positive integer $n$, the following equality holds true:

$$\begin{aligned}\int_\Omega & H(\nabla u_n)^{pq-2} J(\nabla u_n)\cdot\nabla\varphi\,\mathrm{d}x \\ &= \lambda_n\left(\int_\Omega |u_n|^p\,\mathrm{d}x\right)^{q-1}\int_\Omega |u_n|^{p-2}u_n\varphi\,\mathrm{d}x \quad \forall\,\varphi\in W_0^{1,pq}(\Omega).\end{aligned} \tag{2.9}$$

Testing with $\varphi = u_n - u$ in the above relation we get

$$\int_\Omega H\left(\nabla u_n\right)^{pq-2} J\left(\nabla u_n\right) \cdot \left(\nabla u_n - \nabla u\right) \mathrm{d}x$$
$$= \lambda_n \left(\int_\Omega |u_n|^p \,\mathrm{d}x\right)^{q-1} \int_\Omega |u_n|^{p-2} u_n (u_n - u)\,\mathrm{d}x \quad \forall\, n \geq 1.$$

Since $u_n$ converges strongly to $u$ in $L^p(\Omega)$, the above pieces of information yield

$$\lim_{n\to\infty} \int_\Omega H\left(\nabla u_n\right)^{pq-2} J\left(\nabla u_n\right) \cdot \left(\nabla u_n - \nabla u\right) \mathrm{d}x = 0.$$

On the other hand, since $u_n$ converges weakly to $u$ in $W^{1,pq}(\Omega)$, we infer that

$$\lim_{n\to\infty} \int_\Omega H\left(\nabla u\right)^{pq-2} J\left(\nabla u\right) \cdot \left(\nabla u_n - \nabla u\right) \mathrm{d}x = 0.$$

Combining the last two relations we get

$$\lim_{n\to\infty} \int_\Omega \left(H\left(\nabla u_n\right)^{pq-2} J\left(\nabla u_n\right) - H\left(\nabla u\right)^{pq-2} J\left(\nabla u\right)\right) \cdot \left(\nabla u_n - \nabla u\right) \mathrm{d}x = 0,$$

which combined with a convexity argument implies that, actually, $u_n$ converges strongly to $u$ in $W^{1,pq}(\Omega)$. Thus, letting $n \to \infty$ in (2.9) we get

$$\int_\Omega H\left(\nabla u\right)^{pq-2} J\left(\nabla u\right) \cdot \nabla\varphi \,\mathrm{d}x$$
$$= \lambda_1(p,q) \left(\int_\Omega |u|^p \,\mathrm{d}x\right)^{q-1} \int_\Omega |u|^{p-2} u_n \varphi \,\mathrm{d}x \quad \forall\, \varphi \in W_0^{1,pq}\left(\Omega\right),$$

which means that the limit function $u$ is an eigenfunction corresponding to the eigenvalue $\lambda_1(p,q)$. By Lemma 2.3 we may assume that $u > 0$. Next, using Egorov's theorem we deduce that $u_n$ converges uniformly to $u$ on the exterior of an arbitrarily small subset of $\Omega$.

Let $\epsilon > 0$ be arbitrary but fixed and let $S \subset \Omega$ be a compact set such that $|\Omega \setminus S| < \epsilon$. Clearly, there exists a real number $\delta > 0$ (depending on $S$) such that $u(x) \geq \delta > 0$ for all $x \in S$. On the other hand, we know that $u_n(x)$ converges to $u(x)$ for a.e. $x \in \Omega$, and, consequently, we can construct $S$ as above such that $u_n$ converges uniformly to $u$ in $\Omega \setminus S$. Since $u > 0$ in $\Omega$, we find that $\Omega^-_{u_n} \subset \Omega \setminus S$ for each integer $n$ large enough. The above pieces of information and inequality (2.8) imply

$$\left(\lambda_n 2^{q-1} C^{pq} b^{pq}\right)^{-1/(1-q)} \leq |\Omega^-_{u_n}| \leq |\Omega \setminus S| < \epsilon$$

for each integer $n$ large enough. Letting $n \to \infty$ in the above relation we find

$$\left(\lambda_1(p,q) 2^{q-1} C^{pq} b^{pq}\right)^{-1/(1-q)} \leq \epsilon.$$

The above inequality should hold true for each $\epsilon > 0$, which, undoubtedly, leads to a contradiction with Lemma 2.2. Consequently, $\lambda_1(p,q)$ is isolated to the right. This concludes the proof of Theorem 1.2. □

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