REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 67, No. 1, 2024, Pages 339–395
Published online: June 25, 2024
https://doi.org/10.33044/revuma.2296

TWO CONSTRUCTIONS OF BIALGEBROIDS AND THEIR RELATIONS

YUDAI OTSUTO

ABSTRACT. We generalize the construction of face algebras by Hayashi and obtain a left bialgebroid $\mathfrak{A}(w)$. There are some relations between the left bialgebroid $\mathfrak{A}(w)$ and the generalized Shibukawa–Takeuchi left bialgebroid A_σ.

1. Introduction

The quantum Yang–Baxter equation (QYBE for short) [1, 13, 25] plays an important role in the study of bialgebras. Coquasitriangular bialgebras give birth to solutions to the QYBE. On the other hand, Faddeev, Reshetikhin, and Takhtajan [8] introduced a construction of coquasitriangular bialgebras using solutions to the QYBE. This construction has been generalized with the development of the study of the QYBE and bialgebras.

The quantum dynamical Yang–Baxter equation (QDYBE for short), a generalization of the QYBE, was introduced by Gervais and Neveu [9]. Dynamical R-matrices, solutions to the QDYBE, give birth to $\mathfrak{h}$-bialgebroids introduced by Etingof and Varchenko [7]. If the dynamical R-matrix satisfies a certain condition, called rigidity, this $\mathfrak{h}$-bialgebroid has an antipode and is called an $\mathfrak{h}$-Hopf algebroid.

A set-theoretical analogue of the QDYBE is the dynamical Yang–Baxter map (DYBM for short) introduced by Shibukawa [19, 20]. The DYBM is a generalization of the Yang–Baxter map [6, 24] suggested by Drinfel'd [5]. Shibukawa and Takeuchi studied a construction of left bialgebroids by using DYBMs in [22, 21]. A left bialgebroid A_σ is obtained by this construction. If the solution σ satisfies rigidity, then A_σ becomes a Hopf algebroid with a bijective antipode. The notion of left bialgebroids (Takeuchi's $\times_R$-bialgebras) was introduced in [23]. This is a generalization of the bialgebra using a non-commutative base algebra R. Its comultiplication and counit are (R, R)-bimodule homomorphisms. Schauenburg [17] proposed a Hopf algebraic structure on the left bialgebroid without an antipode, called a $\times_R$-Hopf algebra. As a special case of the $\times_R$-Hopf algebra, Böhm and Szláchanyi [4] introduced the Hopf algebroid, which has a bijective antipode.

2020 *Mathematics Subject Classification.* Primary 16T05, 16T10, 16T20; Secondary 05C20, 05C25.

Key words and phrases. Left bialgebroids, Hopf algebroids, weak bialgebras, weak Hopf algebras, Hopf envelopes.

On the other hand, Hayashi [10] introduced the notion of face algebras. In [11], a coquasitriangular face algebra $\mathfrak{A}(w)$ was constructed using a solution w to the quiver-theoretical QYBE. In addition, if a coquasitriangular face algebra is closurable, this face algebra has a Hopf closure, which is a Hopf face algebra satisfying a certain universal property. Hayashi [11] constructed this Hopf closure by using the double cross product and the localization of the face algebra. Later (Hopf) face algebras were integrated to weak bialgebras (weak Hopf algebras) by Böhm, Nill, and Szláchanyi [3]. Bennoun and Pfeiffer mentioned the Hopf closure (they called it the Hopf envelope) of the coquasitriangular weak bialgebra in [2]. Schauenburg [18] showed that a weak bialgebra (weak Hopf algebra) is a left bialgebroid ($\times_R$-Hopf algebra) whose base algebra is Frobenius-separable. Conversely, a left bialgebroid ($\times_R$-Hopf algebra) becomes a weak bialgebra (weak Hopf algebra) if its base algebra is Frobenius-separable.

There is an interesting relation between Hayashi's construction of face algebras and Shibukawa–Takeuchi's construction of left bialgebroids. If the parameter set Λ of a DYBM σ is finite, the left bialgebroid A_σ becomes a weak bialgebra since the base algebra $\overline{M}$ consisting of maps from Λ to a field $\mathbb{K}$ is a Frobenius-separable algebra over $\mathbb{K}$. Matsumoto and Shimizu [12] showed that the DYBM σ gives birth to a solution w_σ to the quiver-theoretical QYBE and gave a weak bialgebra homomorphism ϕ from $\mathfrak{A}(w_\sigma)$ to A_σ.

Shibukawa and the author generalized the construction of [22, 21] to get a left bialgebroid (Hopf algebroid) A_σ that is not a weak bialgebra (weak Hopf algebra) from a DYBM with a finite parameter set Λ. In [16], we generalized $\overline{M}$ to an arbitrary $\mathbb{K}$-algebra L.

It is natural to try to get a left bialgebroid $\mathfrak{A}(w_\sigma)$ corresponding to the generalized left bialgebroid A_σ.

The purpose of this paper is to discuss relations of two constructions of left bialgebroids by extending Matsumoto-Shimizu's homomorphism. Let R be an arbitrary $\mathbb{K}$-algebra, and we denote by M the $\mathbb{K}$-algebra consisting of maps from Λ to R. We try to generalize Hayashi's construction to gain a left bialgebroid $\mathfrak{A}(w_\sigma)$ corresponding to the generalized A_σ with the base algebra $L = M$ and construct a left bialgebroid homomorphism Φ from $\mathfrak{A}(w_\sigma)$ to A_σ. In order to be able to investigate properties of this Φ easily, we specify a necessary and sufficient condition for bijectivity. This condition is very useful because it is unnecessary to see elements in A_σ. If M is a Frobenius-separable $\mathbb{K}$-algebra, the left bialgebroid A_σ in [16] is a weak bialgebra. We also show that this A_σ becomes the Hopf closure of $\mathfrak{A}(w_\sigma)$ through Φ if the collection σ of elements in M is rigid. We expect that this conclusion will pioneer the study of the Hopf closure of a left bialgebroid with the non-commutative base algebra.

In [15], we construct a left bialgebroid $\mathfrak{U}(w)$ generated by two kinds of elements $\left\{\mathbf{e}\begin{bmatrix}p\\q\end{bmatrix}\right\}$ and $\left\{\widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix}\right\}$ (in this paper, we only use one kind of generators $\left\{\mathbf{e}\begin{bmatrix}p\\q\end{bmatrix}\right\}$ to generalize $\mathfrak{A}(w)$ in [11]). The left bialgebroid A_σ with the base algebra M induces $\mathfrak{U}(w_\sigma) := \mathfrak{U}(w)$ and we can construct a left bialgebroid homomorphism

$\Phi'\colon \mathfrak{U}(w_\sigma) \to A_\sigma$. This Φ' is always bijective and it is equivalent to saying that $\mathfrak{U}(w_\sigma)$ and A_σ become Hopf algebroids.

The paper is organized as follows. In Section 2, we review relations between left bialgebroids (Hopf algebroids) and weak bialgebras (weak Hopf algebras) following [4] and [18]. In Section 3, we recall a left bialgebroid (Hopf algebroid) A_σ with the base algebra M from [16] and introduce a left bialgebroid $\mathfrak{A}(w)$ as a generalization of [11]. The weak bialgebra $\mathfrak{A}(w)$ as in [11] has the base algebra $\overline{M}$ as a left bialgebroid. We generalize $\overline{M}$ to M like the above left bialgebroid A_σ. If the family w of elements in R is rigid, this $\mathfrak{A}(w)$ becomes a Hopf algebroid. In Section 4, we induce a left bialgebroid $\mathfrak{A}(w_\sigma) := \mathfrak{A}(w)$ by using the setting of the left bialgebroid A_σ and construct a left bialgebroid homomorphism Φ from $\mathfrak{A}(w_\sigma)$ to A_σ. As a point of difference between [12] and this paper, we do not use DYBMs to construct these $\mathfrak{A}(w_\sigma)$ and Φ. We also give an example of the left bialgebroids A_σ, $\mathfrak{A}(w_\sigma)$ and Φ not using the DYBM. This Φ is bijective if and only if the family w satisfies a part of the condition for rigidity (see Theorem 4.5). In Section 5, if the $\mathbb{K}$-algebra R is a Frobenius-separable $\mathbb{K}$-algebra and A_σ is a weak Hopf algebra, then $\mathfrak{A}(w_\sigma)$, A_σ, and Φ satisfy a certain universal property, called the Hopf closure. For this purpose, we introduce the notion of the antipode as a generalization of the antipode in the weak Hopf algebra theory and of Hayashi's antipode with respect to the face algebra in [11]. We can characterize weak bialgebras and generalize Hopf envelopes in [2] by using these antipodes.

2. Preliminaries

In this section, we recall the notion of left bialgebroids (Hopf algebroids) and discuss relations with weak bialgebras (weak Hopf algebras). If the base algebra of a left bialgebroid is a Frobenius-separable algebra, the total algebra has a weak bialgebra structure. In addition, the total algebra is a weak Hopf algebra when the left bialgebroid becomes a Hopf algebroid. For more details, we refer the reader to [4] and [18].

Throughout this paper, we denote by $\mathbb{K}$ a field.

Definition 2.1. Let A and L be $\mathbb{K}$-algebras. A *left bialgebroid* (or *Takeuchi's $\times_L$-bialgebra*) $\mathcal{A}_L$ is a sextuplet $\mathcal{A}_L := (A, L, s_L, t_L, \Delta_L, \pi_L)$ satisfying the following conditions:

(1) The maps $s_L\colon L \to A$ and $t_L\colon L^{\mathrm{op}} \to A$ are $\mathbb{K}$-algebra homomorphisms and satisfy

$$s_L(l)t_L(l') = t_L(l')s_L(l) \quad (\forall l, l' \in L). \tag{2.1}$$

Here L^{op} means the opposite $\mathbb{K}$-algebra of L. These two homomorphisms make A an (L, L)-bimodule ${}_LA_L$ by the following left and right L-module structures ${}_LA$ and A_L:

$${}_LA\colon l \cdot a = s_L(l)a; \quad A_L\colon a \cdot l = t_L(l)a \quad (l \in L, a \in A). \tag{2.2}$$

(2) The triple $({}_LA_L, \Delta_L, \pi_L)$ is a comonoid in the category of (L, L)-bimodules such that

$$a_{[1]}t_L(l) \otimes a_{[2]} = a_{[1]} \otimes a_{[2]}s_L(l); \tag{2.3}$$
$$\Delta_L(1_A) = 1_A \otimes 1_A; \tag{2.4}$$
$$\Delta_L(ab) = \Delta_L(a)\Delta_L(b); \tag{2.5}$$
$$\pi_L(1_A) = 1_L; \tag{2.6}$$
$$\pi_L(as_L(\pi_L(b))) = \pi_L(ab) = \pi(at_L(\pi_L(b))) \tag{2.7}$$

for all $l \in L$ and $a, b \in A$. Here we write $\Delta_L(a) = a_{[1]} \otimes a_{[2]}$, known as *Sweedler's notation*. The right-hand side of (2.5) is well defined because of (2.3).

We write $\mathcal{A}_L = (A, L, s_L^A, t_L^A, \Delta_L^A, \pi_L^A)$ if there is a possibility of confusion. For a left bialgebroid $\mathcal{A}_L$, these $\mathbb{K}$-algebras A and L are called the *total algebra* and the *base algebra*, respectively.

Definition 2.2. Let $\mathcal{A}_L = (A, L, s_L, t_L, \Delta_L, \pi_L)$ and $\mathcal{A}'_{L'} = (A', L', s_{L'}, t_{L'}, \Delta_{L'}, \pi_{L'})$ be left bialgebroids. A pair of $\mathbb{K}$-algebra homomorphisms $(\Phi\colon A \to A', \phi\colon L \to L')$ is called a *left bialgebroid homomorphism* $\mathcal{A}_L \to \mathcal{A}'_{L'}$ if and only if

$$s_{L'} \circ \phi = \Phi \circ s_L; \tag{2.8}$$
$$t_{L'} \circ \phi = \Phi \circ t_L; \tag{2.9}$$
$$\pi_{L'} \circ \Phi = \phi \circ \pi_L; \tag{2.10}$$
$$\Delta_{L'} \circ \Phi = (\Phi \otimes \Phi) \circ \Delta_L. \tag{2.11}$$

The map $\Phi \otimes \Phi : A \otimes_L A \to A' \otimes_{L'} A'$ makes sense because of (2.8) and (2.9).

Let $\mathcal{A}_L := (A, L, s_L, t_L, \Delta_L, \pi_L)$ be a left bialgebroid and N a $\mathbb{K}$-algebra isomorphic to the opposite $\mathbb{K}$-algebra L^{op}. We suppose that A has a $\mathbb{K}$-algebra anti-automorphism S satisfying

$$S \circ t_L = s_L; \tag{2.12}$$
$$S(a_{[1]})a_{[2]} = (t_L \circ \pi_L \circ S)(a) \tag{2.13}$$

for all $a \in A$. The left-hand side of (2.13) makes sense because of (2.12). We fix a $\mathbb{K}$-algebra isomorphism $\omega\colon L^{\mathrm{op}} \to N$. Then A has left and right N-module structures ${}^NA$ and A^N through the following actions:

$$ {}^NA\colon n \cdot a = a(s_L \circ \omega^{-1})(n); \quad A^N : a \cdot n = a(S \circ s_L \circ \omega^{-1})(n) \quad (a \in A,\ n \in N). \tag{2.14}$$

By virtue of (2.12), these two actions (2.14) make A an (N, N)-bimodule. We can also define two $\mathbb{K}$-linear maps $S_{A\otimes_L A}$ and $S_{A\otimes_N A}$ by

$$S_{A\otimes_L A}\colon A \otimes_L A \ni a \otimes b \mapsto S(b) \otimes S(a) \in A \otimes_N A; \tag{2.15}$$
$$S_{A\otimes_N A}\colon A \otimes_N A \ni a \otimes b \mapsto S(b) \otimes S(a) \in A \otimes_L A. \tag{2.16}$$

Definition 2.3. Let $(\mathcal{A}_L, S)$ be a pair of a left bialgebroid $\mathcal{A}_L$ and a $\mathbb{K}$-algebra anti-automorphism $S\colon A \to A$ satisfying (2.12) and (2.13). Suppose that $S_{A\otimes_N A}$

has the inverse $S^{-1}_{A\otimes_N A}$. We say that the pair $(\mathcal{A}_L, S)$ is a *Hopf algebroid* if and only if

$$S_{A\otimes_L A}\circ\Delta_L\circ S^{-1} = S^{-1}_{A\otimes_N A}\circ\Delta_L\circ S; \tag{2.17}$$
$$(\Delta_L\otimes \mathrm{id}_A)\circ\Delta_N = (\mathrm{id}_A\otimes\Delta_N)\circ\Delta_L; \tag{2.18}$$
$$(\Delta_N\otimes \mathrm{id}_A)\circ\Delta_L = (\mathrm{id}_A\otimes\Delta_L)\circ\Delta_N. \tag{2.19}$$

Here we define $\Delta_N = S_{A\otimes_L A}\circ\Delta_L\circ S^{-1}$. The map S is called an *antipode*.

We next introduce the notion of weak bialgebras and weak Hopf algebras.

Definition 2.4. Let B be a $\mathbb{K}$-algebra endowed with a $\mathbb{K}$-coalgebra structure by $\Delta\colon B\to B\otimes_{\mathbb{K}} B$ and $\varepsilon\colon B\to\mathbb{K}$. We say that a triple $B := (B,\Delta,\varepsilon)$ is a *weak bialgebra* if the following conditions are satisfied:

$$\Delta(ab) = \Delta(a)\Delta(b); \tag{2.20}$$
$$(\Delta(1)\otimes 1)(1\otimes\Delta(1)) = 1_{(1)}\otimes 1_{(2)}\otimes 1_{(3)} = (1\otimes\Delta(1))(\Delta(1)\otimes 1); \tag{2.21}$$
$$\varepsilon(ab_{(1)})\varepsilon(b_{(2)}c) = \varepsilon(abc) = \varepsilon(ab_{(2)})\varepsilon(b_{(1)}c) \tag{2.22}$$

for all $a, b, c\in B$. Here we write simply $1 = 1_B$ and use Sweedler's notation, which is written by

$$\Delta(a) = a_{(1)}\otimes a_{(2)} \quad\text{and}\quad ((\Delta\otimes\mathrm{id}_B)\circ\Delta)(a) = a_{(1)}\otimes a_{(2)}\otimes a_{(3)} = ((\mathrm{id}_B\otimes\Delta)\circ\Delta)(a).$$

In order to avoid ambiguity, we write $\Delta_B = \Delta$ and $\varepsilon_B = \varepsilon$ as needed. The biopposite weak bialgebra B^{bop} of B can be defined similar to the ordinary bialgebra.

Let B' be a weak bialgebra. A $\mathbb{K}$-linear map $f\colon B\to B'$ is called a *weak bialgebra homomorphism* if f is a $\mathbb{K}$-algebra and $\mathbb{K}$-coalgebra homomorphism.

We introduce two maps ε_s, $\varepsilon_t\colon B\to B$ defined by

$$\begin{aligned}\varepsilon_s(a) &= 1_{(1)}\varepsilon(a1_{(2)});\\ \varepsilon_t(a) &= \varepsilon(1_{(1)}a)1_{(2)} \quad (a\in B).\end{aligned} \tag{2.23}$$

These maps ε_s and ε_t are respectively called the *source counital map* and the *target counital map.*

Lemma 2.5. *The maps ε_s and ε_t satisfy*

$$\begin{aligned}&\varepsilon_s\circ\varepsilon_s = \varepsilon_s,\quad \varepsilon_t\circ\varepsilon_t = \varepsilon_t;\\ &\varepsilon_s(1_B) = \varepsilon_t(1_B) = 1_B.\end{aligned} \tag{2.24}$$

Lemma 2.6. *For an arbitrary element a in a weak bialgebra B,*

$$1_{(1)}\varepsilon_s(a1_{(2)}) = \varepsilon_s(a),\quad \varepsilon_t(1_{(1)}a)1_{(2)} = \varepsilon_t(a);$$
$$\Delta(\varepsilon_s(a)) = 1_{(1)}\otimes\varepsilon_s(a)1_{(2)} = 1_{(1)}\otimes 1_{(2)}\varepsilon_s(a); \tag{2.25}$$
$$\Delta(\varepsilon_t(a)) = \varepsilon_t(a)1_{(1)}\otimes 1_{(2)} = 1_{(1)}\varepsilon_t(a)\otimes 1_{(2)}; \tag{2.26}$$
$$a_{(1)}\otimes\varepsilon_s(a_{(2)}) = a1_{(1)}\otimes\varepsilon_s(1_{(2)}),\quad \varepsilon_t(a_{(1)})\otimes a_{(2)} = \varepsilon_t(1_{(1)})\otimes 1_{(2)}a; \tag{2.27}$$
$$\varepsilon_s(a_{(1)})\otimes a_{(2)} = 1_{(1)}\otimes a1_{(2)},\quad a_{(1)}\otimes\varepsilon_t(a_{(2)}) = 1_{(1)}a\otimes 1_{(2)}. \tag{2.28}$$

Lemma 2.7. *We denote by B a weak bialgebra. For any $a, b \in B$,*

$$\varepsilon_s(a)\varepsilon_t(b) = \varepsilon_t(b)\varepsilon_s(a); \tag{2.29}$$

$$\varepsilon(ab) = \varepsilon(\varepsilon_s(a)b), \quad \varepsilon(ab) = \varepsilon(a\varepsilon_t(b)); \tag{2.30}$$

$$\varepsilon_s(ab) = \varepsilon_s(\varepsilon_s(a)b), \quad \varepsilon_t(ab) = \varepsilon_t(a\varepsilon_t(b)); \tag{2.31}$$

$$\varepsilon_s(a)b = b_{(1)}\varepsilon_s(ab_{(2)}), \quad a\varepsilon_t(b) = \varepsilon_t(a_{(1)}b)a_{(2)}; \tag{2.32}$$

$$\varepsilon_s(a)\varepsilon_s(b) = \varepsilon_s(a\varepsilon_s(b)), \quad \varepsilon_t(a)\varepsilon_t(b) = \varepsilon_t(\varepsilon_t(a)b.$$

Definition 2.8. A weak bialgebra H with an $\mathbb{K}$-linear map $S\colon H \to H$ is called a *weak Hopf algebra* if and only if

$$S(h_{(1)})h_{(2)} = \varepsilon_s(h);$$
$$h_{(1)}S(h_{(2)}) = \varepsilon_t(h);$$
$$S(h_{(1)})h_{(2)}S(h_{(3)}) = S(h)$$

are satisfied for all $h \in H$. This S, also called an *antipode*, is unique if it exists.

If there is a possibility of confusion, we write S^{WHA} for the antipode of a weak Hopf algebra and S^{HAD} for an antipode of a Hopf algebroid.

Let us recall the notion of Frobenius-separable $\mathbb{K}$-algebras to discuss relations between the left bialgebroid and the weak bialgebra. A *Frobenius-separable* $\mathbb{K}$-*algebra* is a $\mathbb{K}$-algebra L equipped with a $\mathbb{K}$-linear map $\psi\colon L \to \mathbb{K}$ and an element $e^{(1)} \otimes e^{(2)} \in L \otimes_{\mathbb{K}} L$ such that

$$l = \psi(le^{(1)})e^{(2)} = e^{(1)}\psi(e^{(2)}l), \quad e^{(1)}e^{(2)} = 1_L \quad (\forall l \in L).$$

This pair $(\psi, e^{(1)} \otimes e^{(2)})$ is called an *idempotent Frobenius system.*

Proposition 2.9 (See [18, Theorem 5.5])**.** *Let $\mathcal{A}_L = (A, L, s_L, t_L, \Delta_L, \pi_L)$ be a left bialgebroid. If the base algebra L is a Frobenius-separable $\mathbb{K}$-algebra with an idempotent Frobenius system $(\psi, e^{(1)} \otimes e^{(2)})$, then the total algebra A has the following weak bialgebra structure (A, Δ, ε):*

$$\Delta(a) = t_L(e^{(1)})a_{[1]} \otimes s_L(e^{(2)})a_{[2]}; \tag{2.33}$$

$$\varepsilon(a) = (\psi \circ \pi_L)(a) \quad (a \in A). \tag{2.34}$$

Under the conditions of Proposition 2.9, we suppose that the left bialgebroid $\mathcal{A}_L$ has an antipode S^{HAD}. Then it is important to discuss whether the weak bialgebra A becomes a weak Hopf algebra or not. Schauenburg [18] solved this problem when $\mathcal{A}_L$ is a $\times_L$-Hopf algebra, which is a generalization of the Hopf algebroid. We briefly sketch a special case of Corollary 6.2 in [18].

For the total algebra A of a Hopf algebroid $(\mathcal{A}_L, S^{\mathrm{HAD}})$, we can define another left N-module structure ${}_N A$ by

$${}_N A\colon n \cdot a = (S^{\mathrm{HAD}} \circ s_L \circ \omega^{-1})(n)a \quad (a \in A,\, n \in N).$$

Then the tensor product $A \otimes_N A$ has two meanings depending on left actions ${}^N A$ and ${}_N A$. In order to avoid misunderstandings, we specify these actions by $A^N \otimes^N A$ and $A^N \otimes_N A$. For example, the tensor product $A \otimes_N A$ in (2.15) and (2.16) stands for $A^N \otimes^N A$.

Proposition 2.10 (See [4, Proposition 4.2 (iv)]). *Let $(\mathcal{A}_L, S^{\mathrm{HAD}})$ be a Hopf algebroid. Then the following $\mathbb{K}$-linear map α is bijective with the inverse α^{-1}:*

$$\alpha\colon A^N \otimes_N A \ni a \otimes b \mapsto a_{[1]} \otimes a_{[2]}b \in A \otimes_L A;$$
$$\alpha^{-1}\colon A \otimes_L A \ni a \otimes b \mapsto a^{[1]} \otimes S^{\mathrm{HAD}}(a^{[2]})b \in A^N \otimes_N A.$$

These maps make sense by virtue of (2.12)*. Here we write $\Delta_N(a) = a^{[1]} \otimes a^{[2]}$.*

Proposition 2.11 (See [18, Corollary 6.2]). *Let $\mathcal{A}_L = (A, L, s_L, t_L, \Delta_L, \pi_L)$ be a left bialgebroid satisfying the conditions of Proposition 2.9. If $(\mathcal{A}_L, S^{\mathrm{HAD}})$ is a Hopf algebroid, then the total algebra A becomes a weak Hopf algebra whose antipode S^{WHA} is defined by*

$$S^{\mathrm{WHA}}(a) = \varepsilon_s(a^{[1]})S^{\mathrm{HAD}}(a^{[2]}) \quad (a \in A).$$

This S^{WHA} makes sense because of α^{-1} and the following $\mathbb{K}$-linear map:

$$\beta\colon A^N \otimes_N A \ni a \otimes b \mapsto \varepsilon_s(a)b \in A.$$

This β is well defined because of (2.20) *and* (2.32)*.*

3. Two left bialgebroids $\mathfrak{A}(w)$ and A_σ

3.1. **Summary of left bialgebroid A_σ.** In this subsection, we recall a left bialgebroid A_σ. For more details, we refer the reader to [16]. This is a generalization of [21].

Let R be a $\mathbb{K}$-algebra and Λ a non-empty finite set. Let G denote a subgroup of the opposite group of the symmetric group on the set Λ. We can define a right group action of this group G on the set Λ:$\lambda\alpha = \alpha(\lambda)$ ($\lambda \in \Lambda$, $\alpha \in G$). We denote by M the $\mathbb{K}$-algebra consisting of maps from Λ to R. For any $\alpha \in G$, the map $T_\alpha : M \to M$ is defined by

$$T_\alpha(f)(\lambda) = f(\lambda\alpha) \quad (f \in M, \lambda \in \Lambda).$$

The map T_α ($\alpha \in G$) is a $\mathbb{K}$-algebra homomorphism such that $T_\alpha \circ T_{\alpha^{-1}} = \mathrm{id}_M$. Let deg be a map from a finite set X to the group G. Define

$$\Lambda X := (M \otimes_{\mathbb{K}} M^{\mathrm{op}}) \sqcup \{L_{ab} \mid a, b \in X\} \sqcup \{(L^{-1})_{ab} \mid a, b \in X\}.$$

Let $\sigma^{ab}_{cd} \in M$ be a collection of elements for $a, b, c, d \in X$, and we denote by $\mathbb{K}\langle\Lambda X\rangle$ the free $\mathbb{K}$-algebra generated by the set ΛX. The symbol I_σ means the two-sided ideal of $\mathbb{K}\langle\Lambda X\rangle$ whose generators are:

(1) $\xi + \xi' - (\xi + \xi'),\ c\xi - (c\xi),\ \xi\xi' - (\xi\xi') \quad (\forall c \in \mathbb{K}, \forall \xi, \xi' \in M \otimes_{\mathbb{K}} M^{\mathrm{op}})$.
Here the notation $\xi + \xi'$ stands for the addition in the algebra $\mathbb{K}\langle\Lambda X\rangle$, while the notation $(\xi + \xi')(\in \Lambda X)$ is that of the algebra $M \otimes_{\mathbb{K}} M^{\mathrm{op}}$. The other two generators for the scalar multiplication and multiplication are similar.

(2) $\displaystyle\sum_{c\in X} L_{ac}(L^{-1})_{cb} - \delta_{a,b}\emptyset,\ \sum_{c\in X}(L^{-1})_{ac}L_{cb} - \delta_{a,b}\emptyset \quad (\forall a, b \in X)$.
Here $\delta_{a,b} \in \mathbb{K}$ $(a, b \in X)$ means Kronecker's delta symbol and $\emptyset$ means the empty word.

(3) $(T_{\deg(a)}(f)\otimes 1_M)L_{ab} - L_{ab}(f\otimes 1_M)$,
$(1_M\otimes T_{\deg(b)}(f))L_{ab} - L_{ab}(1_M\otimes f)$,
$(f\otimes 1_M)(L^{-1})_{ab} - (L^{-1})_{ab}(T_{\deg(b)}(f)\otimes 1_M)$,
$(1_M\otimes f)(L^{-1})_{ab} - (L^{-1})_{ab}(1_M\otimes T_{\deg(a)}(f)) \quad (\forall f\in M,\ \forall a,b\in X)$.

(4) $\displaystyle\sum_{x,y\in X}\left(\sigma^{xy}_{ac}\otimes 1_M\right)L_{yd}L_{xb} - \sum_{x,y\in X}\left(1_M\otimes \sigma^{bd}_{xy}\right)L_{cy}L_{ax} \quad (\forall a,b,c,d\in X)$.

(5) $\emptyset - 1_M\otimes 1_M$.

Theorem 3.1 (See also [16, Theorem 2.1])**.** *If the following conditions are satisfied, then the quotient $A_\sigma := \mathbb{K}\langle \Lambda X\rangle / I_\sigma$ is a left bialgebroid whose base algebra is M.*

$$\begin{cases} \sigma^{ab}_{cd}(\lambda)\in Z(R) \quad (\forall \lambda\in\Lambda,\ \forall a,b,c,d\in X); \\ \lambda\deg(d)\deg(b)\neq \lambda\deg(c)\deg(a) \Rightarrow \sigma^{bd}_{ac}(\lambda)=0. \end{cases} \tag{3.1}$$

Here $Z(R)$ is the center of R.

Proof. It is sufficient to show that the condition (3.1) implies

$$\sigma^{bd}_{ac}(T_{\deg(d)}\circ T_{\deg(b)})(f) = (T_{\deg(c)}\circ T_{\deg(a)})(f)\sigma^{bd}_{ac}$$

for any $f\in M$ and $a,b,c,d\in X$ (see (11) in [16]).

We suppose that $\lambda\deg(d)\deg(b) = \lambda\deg(c)\deg(a)$. By using the first condition in (3.1),

$$\begin{aligned}(\sigma^{bd}_{ac}(T_{\deg(d)}\circ T_{\deg(b)})(f))(\lambda) &= \sigma^{bd}_{ac}(\lambda)(T_{\deg(d)}\circ T_{\deg(b)})(f))(\lambda)\\ &= \sigma^{bd}_{ac}(\lambda)f(\lambda\deg(d)\deg(b))\\ &= f(\lambda\deg(d)\deg(b))\sigma^{bd}_{ac}(\lambda)\\ &= f(\lambda\deg(c)\deg(a))\sigma^{bd}_{ac}(\lambda)\\ &= (T_{\deg(c)}\circ T_{\deg(a)})(f)(\lambda)\sigma^{bd}_{ac}(\lambda)\\ &= ((T_{\deg(c)}\circ T_{\deg(a)})(f)\sigma^{bd}_{ac})(\lambda).\end{aligned}$$

If $\lambda\deg(d)\deg(b)\neq\lambda\deg(c)\deg(a)$, then the second condition in (3.1) induces that

$$\begin{aligned}(\sigma^{bd}_{ac}(T_{\deg(d)}\circ T_{\deg(b)})(f))(\lambda) &= f(\lambda\deg(d)\deg(b))\sigma^{bd}_{ac}(\lambda)\\ &= 0\\ &= f(\lambda\deg(c)\deg(a))\sigma^{bd}_{ac}(\lambda)\\ &= ((T_{\deg(c)}\circ T_{\deg(a)})(f)\sigma^{bd}_{ac})(\lambda).\end{aligned}$$

Therefore the left bialgebroid A_σ can be constructed by using the condition (3.1). □

From now on, we will introduce s_M, t_M, Δ_M, and π_M in order to construct the left bialgebroid A_σ.

The maps $s_M\colon M \to A_\sigma$ and $t_M\colon M^{\mathrm{op}} \to A_\sigma$ are defined by

$$s_M(f) = f \otimes 1_M + I_\sigma;$$
$$t_M(f) = 1_M \otimes f + I_\sigma \quad (f \in M).$$

These are $\mathbb{K}$-algebra homomorphisms and satisfy (2.1). Thus A_σ is an (M,M)-bimodule via (2.2).

Let I_2 denote the right ideal of $A_\sigma \otimes_{\mathbb{K}} A_\sigma$ whose generators are $t_M(f) \otimes 1_{A_\sigma} - 1_{A_\sigma} \otimes s_M(f)$ ($\forall f \in M$). The $\mathbb{K}$-algebra homomorphism $\overline{\Delta}\colon \mathbb{K}\langle \Lambda X\rangle \to A_\sigma \otimes_{\mathbb{K}} A_\sigma$ is defined by

$$\overline{\Delta}(\xi) = s_M \otimes t_M(\xi) \quad (\xi \in M \otimes_{\mathbb{K}} M^{\mathrm{op}});$$
$$\overline{\Delta}(L_{ab}) = \sum_{c\in X} L_{ac} + I_\sigma \otimes L_{cb} + I_\sigma \quad (a,b \in X);$$
$$\overline{\Delta}((L^{-1})_{ab}) = \sum_{c\in X} (L^{-1})_{cb} + I_\sigma \otimes (L^{-1})_{ac} + I_\sigma.$$

This map $\overline{\Delta}$ satisfies $\overline{\Delta}(I_\sigma) \subset I_2$. Thus the $\mathbb{K}$-linear map $\tilde{\Delta}(\alpha + I_\sigma) = \overline{\Delta}(\alpha) + I_2$ ($\alpha \in \mathbb{K}\langle \Lambda X\rangle$) is well defined. Since $A_\sigma \otimes_{\mathbb{K}} A_\sigma / I_2 \cong A_\sigma \otimes_M A_\sigma$ as $\mathbb{K}$-vector spaces, we can induce the $\mathbb{K}$-linear map $\Delta_M\colon A_\sigma \to A_\sigma \otimes_M A_\sigma$ from the map $\tilde{\Delta}$. This Δ_M is an (M,M)-bimodule homomorphism.

The next task is to define the map $\pi_M\colon A_\sigma \to M$. The $\mathbb{K}$-algebra homomorphism $\overline{\chi}\colon \mathbb{K}\langle \Lambda X\rangle \to \mathrm{End}_{\mathbb{K}}(M)$ is defined by

$$\overline{\chi}(f \otimes g)(h) = fhg \quad (f,g,h \in M);$$
$$\overline{\chi}(L_{ab}) = \delta_{a,b} T_{\deg(a)};$$
$$\overline{\chi}((L^{-1})_{ab}) = \delta_{a,b} T_{\deg(a)^{-1}} \quad (a,b \in X).$$

Because $\overline{\chi}(I_\sigma) = \{0\}$ is satisfied, the map $\chi(\alpha + I_\sigma) = \overline{\chi}(\alpha)$ ($\alpha \in \mathbb{K}\langle \Lambda X\rangle$) makes sense and is a $\mathbb{K}$-algebra homomorphism. We define the map π_M by

$$\pi_M\colon A_\sigma \ni a \mapsto \chi(a)(1_M) \in M. \tag{3.2}$$

This π_M is an (M,M)-bimodule homomorphism.

The triplet $(A_\sigma, \Delta_M, \pi_M)$ is a comonoid in the tensor category of (M,M)-bimodules. Since the maps Δ_M and π_M satisfy the conditions (2.3)–(2.7), the sextuplet $(A_\sigma, M, s_M, t_M, \Delta_M, \pi_M)$ is a left bialgebroid.

Let $\sigma = \{\sigma^{ab}_{cd}\}_{a,b,c,d\in X}$. This left bialgebroid A_σ has a Hopf algebroid structure if σ satisfies a certain condition, called rigidity.

Definition 3.2 (See [16, Definition 4.2])**.** The family $\sigma = \{\sigma^{ab}_{cd}\}_{a,b,c,d\in X}$ is called *rigid* if and only if, for any $a,b \in X$, there exist elements $x_{ab}, y_{ab} \in A_\sigma$ satisfying the following conditions:

$$\begin{aligned}\sum_{c\in X}((L^{-1})_{cb}+I_\sigma)x_{ac}&=\delta_{a,b}1_{A_\sigma};\\ \sum_{c\in X}x_{cb}((L^{-1})_{ac}+I_\sigma)&=\delta_{a,b}1_{A_\sigma};\\ \sum_{c\in X}(L_{cb}+I_\sigma)y_{ac}&=\delta_{a,b}1_{A_\sigma};\\ \sum_{c\in X}y_{cb}(L_{ac}+I_\sigma)&=\delta_{a,b}1_{A_\sigma}.\end{aligned} \tag{3.3}$$

Proposition 3.3 (See [16, Proposition 4.1]). *The following are equivalent:*

1. σ *is rigid.*
2. *There exists a unique* $\mathbb{K}$*-algebra anti-automorphism* $S\colon A_\sigma\to A_\sigma$ *satisfying*

$$\begin{cases}S(f\otimes g+I_\sigma)=g\otimes f+I_\sigma & (f,g\in M);\\ S(L_{ab}+I_\sigma)=(L^{-1})_{ab}+I_\sigma & (a,b\in X).\end{cases}$$

Proposition 3.4 (See [16, Proposition 4.2]). *If* σ *is rigid, then the pair* (A_σ,S) *is a Hopf algebroid for* $N=M^{\mathrm{op}}$ *and* $\omega=\mathrm{id}_M$*.*

3.2. **Left bialgebroid** $\mathfrak{A}(w)$**.** In this subsection, we introduce a left bialgebroid $\mathfrak{A}(w)$ as a generalization of [11]. Similar to the above left bialgebroid A_σ, this $\mathfrak{A}(w)$ has a Hopf algebroid structure if the collection w of elements in R satisfies rigidity (see also [15]).

Definition 3.5. Let Λ be a non-empty set. A set Q endowed with two maps $\mathfrak{s},\mathfrak{t}\colon Q\to\Lambda$ is said to be a *quiver over* Λ. These maps $\mathfrak{s}$ and $\mathfrak{t}$ are respectively called the *source map* and the *target map*. For a non-negative integer m, we define the *fiber product* $Q^{(m)}$ by

$$\begin{cases}Q^{(0)}:=\Lambda,\\ Q^{(1)}:=Q,\\ Q^{(m)}:=\{q=(q_1,\dots,q_m)\in Q^m\mid \mathfrak{t}(q_i)=\mathfrak{s}(q_{i+1})\,\forall i\in\{1,\dots,m-1\}\} & (m>1).\end{cases}$$

The set $Q^{(m)}$ $(m>0)$ is a quiver over Λ with $\mathfrak{s}(q)=\mathfrak{s}(q_1)$, $\mathfrak{t}(q)=\mathfrak{t}(q_m)$. $Q^{(0)}$ is also a quiver over Λ by $\mathfrak{s}=\mathfrak{t}=\mathrm{id}_\Lambda$.

Let Λ be a non-empty finite set, and Q a finite quiver over Λ. We denote by $\mathfrak{G}(Q)$ the linear span of the symbols $\mathbf{e}\begin{bmatrix}p\\q\end{bmatrix}$ $(p,q\in Q^{(m)},\ m\in\mathbb{Z}_{\geq 0})$:

$$\mathfrak{G}(Q):=\bigoplus_{p,q\in Q^{(m)},\,m\in\mathbb{Z}_{\geq0}}\mathbb{K}\,\mathbf{e}\begin{bmatrix}p\\q\end{bmatrix}.$$

This $\mathfrak{G}(Q)$ is a $\mathbb{K}$-algebra by the following multiplication:

$$\mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}\mathbf{e}\begin{bmatrix} p' \\ q' \end{bmatrix} = \delta_{\mathfrak{t}(p),\mathfrak{s}(p')}\delta_{\mathfrak{t}(q),\mathfrak{s}(q')}\mathbf{e}\begin{bmatrix} pp' \\ qq' \end{bmatrix};$$

$$1_{\mathfrak{G}(Q)} = \sum_{\lambda,\mu\in\Lambda}\mathbf{e}\begin{bmatrix} \lambda \\ \mu \end{bmatrix}$$

for $p, q \in Q^{(m)}$, $p', q' \in Q^{(n)}$, and $m, n \in \mathbb{Z}_{\geq 0}$. Here $\delta_{\lambda,\mu} \in \mathbb{K}$ $(\lambda, \mu \in \Lambda)$ means Kronecker's delta symbol. For a $\mathbb{K}$-algebra R, let $\mathbf{w}\left[\begin{smallmatrix} & a & \\ c & & b \\ & d & \end{smallmatrix}\right] \in R$ be a collection of elements for $(a, b), (c, d) \in Q^{(2)}$. We write $\mathfrak{I}_{\mathbf{w}}$ for the two-sided ideal of the $\mathbb{K}$-algebra $\mathfrak{H}(Q) := R \otimes_{\mathbb{K}} R^{\mathrm{op}} \otimes_{\mathbb{K}} \mathfrak{G}(Q)$ whose generators are

$$\sum_{(x,y)\in Q^{(2)}} \mathbf{w}\left[\begin{smallmatrix} & x & \\ a & & y \\ & b & \end{smallmatrix}\right] \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} x \\ c \end{bmatrix}\mathbf{e}\begin{bmatrix} y \\ d \end{bmatrix}$$
$$- \sum_{(x,y)\in Q^{(2)}} 1_R \otimes \mathbf{w}\left[\begin{smallmatrix} & c & \\ x & & d \\ & y & \end{smallmatrix}\right] \otimes \mathbf{e}\begin{bmatrix} a \\ x \end{bmatrix}\mathbf{e}\begin{bmatrix} b \\ y \end{bmatrix} \quad (\forall (a, b), (c, d) \in Q^{(2)}). \tag{3.4}$$

We define $\mathfrak{A}(w)$ by the quotient $\mathfrak{A}(w) := \mathfrak{H}(Q)/\mathfrak{I}_{\mathbf{w}}$.

Theorem 3.6. *If the following conditions are satisfied, then $\mathfrak{A}(w)$ is a left bialgebroid whose base algebra is M:*

$$\begin{cases} \mathbf{w}\left[\begin{smallmatrix} & a & \\ c & & b \\ & d & \end{smallmatrix}\right] \in Z(R) \quad (\forall (a, b), (c, d) \in Q^{(2)}); \\ \mathfrak{s}(a) \neq \mathfrak{s}(c) \text{ or } \mathfrak{t}(b) \neq \mathfrak{t}(d) \Rightarrow \mathbf{w}\left[\begin{smallmatrix} & a & \\ c & & b \\ & d & \end{smallmatrix}\right] = 0. \end{cases} \tag{3.5}$$

We split the proof of Theorem 3.6 into multiple steps.

The maps $s_M : M \to \mathfrak{A}(w)$ and $t_M : M^{\mathrm{op}} \to \mathfrak{A}(w)$ are defined by

$$s_M(f) = \sum_{\lambda,\mu\in\Lambda} f(\lambda) \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \lambda \\ \mu \end{bmatrix} + \mathfrak{I}_{\mathbf{w}};$$

$$t_M(f) = \sum_{\lambda,\mu\in\Lambda} 1_R \otimes f(\lambda) \otimes \mathbf{e}\begin{bmatrix} \mu \\ \lambda \end{bmatrix} + \mathfrak{I}_{\mathbf{w}} \quad (f \in M).$$

These maps are $\mathbb{K}$-algebra homomorphisms satisfying (2.1). As a result, $\mathfrak{A}(w)$ is an (M, M)-bimodule by the actions (2.2).

Let $\mathfrak{I}_2$ denote the right ideal of $\mathfrak{A}(w) \otimes_{\mathbb{K}} \mathfrak{A}(w)$ whose generators are $t_M(f) \otimes 1_{\mathfrak{A}(w)} - 1_{\mathfrak{A}(w)} \otimes s_M(f)$ $(\forall f \in M)$. In order to construct the map Δ_M, we define the $\mathbb{K}$-linear map $\overline{\nabla} : \mathfrak{H}(Q) \to \mathfrak{A}(w) \otimes_{\mathbb{K}} \mathfrak{A}(w)$ by

$$\overline{\nabla}\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}\Big) = \sum_{u\in Q^{(m)}} \Big(r \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes \Big(1_R \otimes r' \otimes \mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)$$

for $r, r' \in R$, $p, q \in Q^{(m)}$, and $m \in \mathbb{Z}_{\geq 0}$. The following proposition plays an important role in order to construct Δ_M. In the proof, we use a map $r_\sharp \in M$ $(\forall r \in R)$ defined by $r_\sharp(\lambda) = r$ $(\lambda \in \Lambda)$.

Proposition 3.7. *The map $\overline{\nabla}$ is multiplicative and $\overline{\nabla}(\mathfrak{I}_{\mathbf{w}}) \subset \mathfrak{I}_2$.*

Proof. We first show that $\overline{\nabla}$ preserves the multiplication of the $\mathbb{K}$-algebra $\mathfrak{H}(Q)$. Let m and n be non-negative integers. For all $r, r', s, s' \in R$, $p, q \in Q^{(m)}$, p', and $q' \in Q^{(n)}$,

$$\begin{aligned}
&\overline{\nabla}\Big(\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}\Big)\Big(s \otimes s' \otimes \mathbf{e}\begin{bmatrix} p' \\ q' \end{bmatrix}\Big)\Big) \\
&\quad = \delta_{\mathfrak{t}(p),\mathfrak{s}(p')}\delta_{\mathfrak{t}(q),\mathfrak{s}(q')} \sum_{y \in Q^{(m+n)}} \Big(rs \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} pp' \\ y \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&\qquad\quad \otimes \Big(1_R \otimes s'r' \otimes \mathbf{e}\begin{bmatrix} y \\ qq' \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&\quad = \sum_{\substack{u \in Q^{(m)} \\ v \in Q^{(n)}}} \Big(rs \otimes 1_R \otimes \delta_{\mathfrak{t}(p),\mathfrak{s}(p')}\delta_{\mathfrak{t}(u),\mathfrak{s}(v)}\mathbf{e}\begin{bmatrix} pp' \\ uv \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&\qquad\quad \otimes \Big(1_R \otimes s'r' \otimes \delta_{\mathfrak{t}(q),\mathfrak{s}(q')}\mathbf{e}\begin{bmatrix} uv \\ qq' \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&\quad = \Bigg(\sum_{u \in Q^{(m)}} \Big(r \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes \Big(1_R \otimes r' \otimes \mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Bigg) \\
&\qquad\quad \times \Bigg(\sum_{v \in Q^{(n)}} \Big(s \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p' \\ v \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes \Big(1_R \otimes s' \otimes \mathbf{e}\begin{bmatrix} v \\ q' \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Bigg) \\
&\quad = \overline{\nabla}\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}\Big)\overline{\nabla}\Big(s \otimes s' \otimes \mathbf{e}\begin{bmatrix} p' \\ q' \end{bmatrix}\Big).
\end{aligned}$$

Let us check that $\overline{\nabla}(\alpha)\beta \in \mathfrak{I}_2$ for any $\alpha \in \mathfrak{H}(Q)$ and $\beta \in \mathfrak{I}_2$. Since $\overline{\nabla}$ is a $\mathbb{K}$-linear map, we have only to prove this fact when $\alpha = r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}$ and $\beta = t_M(f) \otimes 1_{\mathfrak{A}(w)} - 1_{\mathfrak{A}(w)} \otimes s_M(f)$ $(f \in M)$:

$$\begin{aligned}
\overline{\nabla}(\alpha)\beta = &\sum_{\substack{u \in Q^{(m)} \\ \lambda,\mu,\tau,\nu \in \Lambda}} \Big(\Big(r \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes \Big(1_R \otimes r' \otimes \mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big) \\
&\quad \times \Big(\Big(1_R \otimes f(\lambda) \otimes \mathbf{e}\begin{bmatrix} \mu \\ \lambda \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes 1_{\mathfrak{A}(w)} \\
&\quad - 1_{\mathfrak{A}(w)} \otimes \Big(f(\tau) \otimes 1_{\mathfrak{A}(w)} \otimes \mathbf{e}\begin{bmatrix} \tau \\ \nu \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big) \\
= &\sum_{u \in Q^{(m)}} \Big(r \otimes f(\mathfrak{t}(u)) \otimes \mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes \Big(1_R \otimes r' \otimes \mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&\quad - \Big(r \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes \Big(f(\mathfrak{t}(u)) \otimes r' \otimes \mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{u\in Q^{(m)}\\ \lambda,\mu,\tau,\nu\in\Lambda}} \left(\left(1_R \otimes f(\mathfrak{t}(u)) \otimes \mathbf{e}\begin{bmatrix}\mu\\ \lambda\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \otimes 1_{\mathfrak{A}(w)}\right.\\
&\qquad \left. - 1_{\mathfrak{A}(w)} \otimes \left(f(\mathfrak{t}(u)) \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\tau\\ \nu\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right)\right)\\
&\qquad \times \left(\left(r \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}p\\ u\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \otimes \left(1_R \otimes r' \otimes \mathbf{e}\begin{bmatrix}u\\ q\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right)\right)\\
&= \sum_{u\in Q^{(m)}} \left(t_M(f(\mathfrak{t}(u))_\sharp) \otimes 1_{\mathfrak{A}(w)} - 1_{\mathfrak{A}(w)} \otimes s_M(f(\mathfrak{t}(u))_\sharp)\right)\\
&\qquad \times \left(\left(r \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}p\\ u\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \otimes \left(1_R \otimes r' \otimes \mathbf{e}\begin{bmatrix}u\\ q\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right)\right) \in \mathfrak{I}_2.
\end{aligned}
$$

In order to complete the proof, we need to show that $\overline{\nabla}(\gamma) \in \mathfrak{I}_2$ for an arbitrary generator γ as in (3.4). For any $(a,b),(c,d) \in Q^{(2)}$, we can induce the following equality by using the definition of $\mathfrak{I}_{\mathbf{w}}$:

$$
\begin{aligned}
&\overline{\nabla}\left(\sum_{(x,y)\in Q^{(2)}} \mathbf{w}\left[{}^{a}\,{}^{x}_{b}\,{}^{y}\right] \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}x\\ c\end{bmatrix}\mathbf{e}\begin{bmatrix}y\\ d\end{bmatrix} - \sum_{(x,y)\in Q^{(2)}} 1_R \otimes \mathbf{w}\left[{}^{x}\,{}^{c}_{y}\,{}^{d}\right] \otimes \mathbf{e}\begin{bmatrix}a\\ x\end{bmatrix}\mathbf{e}\begin{bmatrix}b\\ y\end{bmatrix}\right)\\
&= \sum_{(x,y),(u,v)\in Q^{(2)}} \left(\mathbf{w}\left[{}^{a}\,{}^{x}_{b}\,{}^{y}\right] \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}x\\ u\end{bmatrix}\mathbf{e}\begin{bmatrix}y\\ v\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \otimes \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}u\\ c\end{bmatrix}\mathbf{e}\begin{bmatrix}v\\ d\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right)\\
&\quad - \sum_{(x,y),(u,v)\in Q^{(2)}} \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}a\\ u\end{bmatrix}\mathbf{e}\begin{bmatrix}b\\ v\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right)\left(1_R \otimes \mathbf{w}\left[{}^{x}\,{}^{c}_{y}\,{}^{d}\right] \otimes \mathbf{e}\begin{bmatrix}u\\ x\end{bmatrix}\mathbf{e}\begin{bmatrix}v\\ y\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right)\\
&= \sum_{(x,y),(u,v)\in Q^{(2)}} \left(1_R \otimes \mathbf{w}\left[{}^{x}\,{}^{u}_{y}\,{}^{v}\right] \otimes \mathbf{e}\begin{bmatrix}a\\ x\end{bmatrix}\mathbf{e}\begin{bmatrix}b\\ y\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \otimes \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}u\\ c\end{bmatrix}\mathbf{e}\begin{bmatrix}v\\ d\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right)\\
&\quad - \sum_{(x,y),(u,v)\in Q^{(2)}} \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}a\\ x\end{bmatrix}\mathbf{e}\begin{bmatrix}b\\ y\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \otimes \left(\mathbf{w}\left[{}^{x}\,{}^{u}_{y}\,{}^{v}\right] \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}u\\ c\end{bmatrix}\mathbf{e}\begin{bmatrix}v\\ d\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right)\\
&= \sum_{(x,y),(u,v)\in Q^{(2)}} \left(t_M\left(\mathbf{w}\left[{}^{x}\,{}^{u}_{y}\,{}^{v}\right]_\sharp\right) \otimes 1_{\mathfrak{A}(w)} - 1_{\mathfrak{A}(w)} \otimes s_M\left(\mathbf{w}\left[{}^{x}\,{}^{u}_{y}\,{}^{v}\right]_\sharp\right)\right)\\
&\quad \times \left(\left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}a\\ x\end{bmatrix}\mathbf{e}\begin{bmatrix}b\\ y\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \otimes \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}u\\ c\end{bmatrix}\mathbf{e}\begin{bmatrix}v\\ d\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right)\right) \in \mathfrak{I}_2.
\end{aligned}
$$

Thus this proposition is proved. □

By this proposition, $\overline{\nabla}$ induces a $\mathbb{K}$-linear map $\tilde{\nabla}(\alpha + \mathfrak{I}_{\mathbf{w}}) = \overline{\nabla}(\alpha) + \mathfrak{I}_2$ ($\alpha \in \mathfrak{H}(Q)$). Since $\mathfrak{A}(w) \otimes_{\mathbb{K}} \mathfrak{A}(w)/\mathfrak{I}_2 \cong \mathfrak{A}(w) \otimes_M \mathfrak{A}(w)$ as $\mathbb{K}$-vector spaces, we can construct the $\mathbb{K}$-linear map $\Delta_M \colon \mathfrak{A}(w) \to \mathfrak{A}(w) \otimes_M \mathfrak{A}(w)$ from the map $\tilde{\nabla}$.

The next task is to construct the map $\pi_M \colon \mathfrak{A}(w) \to M$. By using the map $\delta_\lambda \in M$ ($\forall \lambda \in \Lambda$) defined by $\delta_\lambda(\mu) = \delta_{\lambda,\mu}$ ($\mu \in \Lambda$), we define the $\mathbb{K}$-linear map $\overline{\zeta} \colon \mathfrak{H}(Q) \to \mathrm{End}_{\mathbb{K}}(M)$ as follows:

$$
\overline{\zeta}\left(r \otimes r' \otimes \mathbf{e}\begin{bmatrix}p\\ q\end{bmatrix}\right)(f) = \delta_{p,q}\left(r f(\mathfrak{t}(q)) r'\right)_\sharp \delta_{\mathfrak{s}(q)} \quad (f \in M).
$$

Proposition 3.8. *The map* $\overline{\zeta}$ *is a* $\mathbb{K}$*-algebra homomorphism and* $\overline{\zeta}(\mathfrak{I}_{\mathbf{w}}) = \{0\}$.

Proof. We first check that $\overline{\zeta}$ is a $\mathbb{K}$-algebra homomorphism. For all $r, r', s, s' \in R$, $p, q \in Q^{(m)}$, $p', q' \in Q^{(n)}$, and $f \in M$, we have

$$\begin{aligned}
&\overline{\zeta}\Big(\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}\Big)\Big(s \otimes s' \otimes \mathbf{e}\begin{bmatrix} p' \\ q' \end{bmatrix}\Big)\Big)(f) \\
&\quad = \delta_{\mathfrak{t}(p),\mathfrak{s}(p')}\delta_{\mathfrak{t}(q),\mathfrak{s}(q')}\overline{\zeta}\Big(rs \otimes s'r' \otimes \mathbf{e}\begin{bmatrix} pp' \\ qq' \end{bmatrix}\Big)(f) \\
&\quad = \delta_{\mathfrak{t}(p),\mathfrak{s}(p')}\delta_{\mathfrak{t}(q),\mathfrak{s}(q')}\delta_{p,q}\delta_{p',q'}\Big(rsf(\mathfrak{t}(q'))s'r'\Big)_{\sharp}\delta_{\mathfrak{s}(q)},
\end{aligned}$$

and

$$\begin{aligned}
&\Big(\overline{\zeta}\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}\Big)\overline{\zeta}\Big(s \otimes s' \otimes \mathbf{e}\begin{bmatrix} p' \\ q' \end{bmatrix}\Big)\Big)(f) \\
&\quad = \delta_{p',q'}\overline{\zeta}\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}\Big)\Big((sf(\mathfrak{t}(q'))s')_{\sharp}\delta_{\mathfrak{s}(q')}\Big) \\
&\quad = \delta_{p,q}\delta_{p',q'}\Big(r((sf(\mathfrak{t}(q'))s')_{\sharp}\delta_{\mathfrak{s}(q')}\Big)\Big(\mathfrak{t}(q))r'\Big)_{\sharp}\delta_{\mathfrak{s}(q)} \\
&\quad = \delta_{\mathfrak{t}(q),\mathfrak{s}(q')}\delta_{p,q}\delta_{p',q'}\Big(rsf(\mathfrak{t}(q'))s'r'\Big)_{\sharp}\delta_{\mathfrak{s}(q)}.
\end{aligned}$$

If $p = q$, $p' = q'$, and $\mathfrak{t}(q) = \mathfrak{s}(q')$, then we can deduce that $\mathfrak{t}(p) = \mathfrak{s}(p')$. Thus this $\overline{\zeta}$ is a $\mathbb{K}$-algebra homomorphism.

We next prove that $\overline{\zeta}(\mathfrak{I}_{\mathbf{w}}) = \{0\}$. Since the map $\overline{\zeta}$ is a $\mathbb{K}$-algebra homomorphism, it suffices to show that $\overline{\zeta}(\gamma) = 0$ for any generator γ as in (3.4). We denote by f an arbitrary element in M. By using the first condition in (3.5),

$$\begin{aligned}
&\overline{\zeta}\Bigg(\sum_{(x,y)\in Q^{(2)}} \mathbf{w}\left[\begin{smallmatrix} & x & \\ a & & y \\ & b & \end{smallmatrix}\right] \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} x \\ c \end{bmatrix}\mathbf{e}\begin{bmatrix} y \\ d \end{bmatrix} - \sum_{(x,y)\in Q^{(2)}} 1_R \otimes \mathbf{w}\left[\begin{smallmatrix} & c & \\ x & & d \\ & y & \end{smallmatrix}\right] \otimes \mathbf{e}\begin{bmatrix} a \\ x \end{bmatrix}\mathbf{e}\begin{bmatrix} b \\ y \end{bmatrix}\Bigg)(f) \\
&\quad = \Big(\mathbf{w}\left[\begin{smallmatrix} & c & \\ a & & d \\ & b & \end{smallmatrix}\right] f(\mathfrak{t}(d))\Big)_{\sharp}\delta_{\mathfrak{s}(c)} - \Big(f(\mathfrak{t}(b))\mathbf{w}\left[\begin{smallmatrix} & c & \\ a & & d \\ & b & \end{smallmatrix}\right]\Big)_{\sharp}\delta_{\mathfrak{s}(a)} \\
&\quad = \Big(\mathbf{w}\left[\begin{smallmatrix} & c & \\ a & & d \\ & b & \end{smallmatrix}\right] f(\mathfrak{t}(d))\Big)_{\sharp}\delta_{\mathfrak{s}(c)} - \Big(\mathbf{w}\left[\begin{smallmatrix} & c & \\ a & & d \\ & b & \end{smallmatrix}\right] f(\mathfrak{t}(b))\Big)_{\sharp}\delta_{\mathfrak{s}(a)}
\end{aligned}$$

for all (a, b) and $(c, d) \in Q^{(2)}$. If $\mathbf{w}\left[\begin{smallmatrix} & c & \\ a & & d \\ & b & \end{smallmatrix}\right] \neq 0$, then $\mathfrak{s}(a) = \mathfrak{s}(c)$ and $\mathfrak{t}(b) = \mathfrak{t}(d)$ are satisfied because of the second condition in (3.5). This completes the proof. □

As a result of this proposition, the map $\zeta(\alpha + \mathfrak{I}_{\mathbf{w}}) = \overline{\zeta}(\alpha)$ $(\alpha \in \mathfrak{H}(Q))$ is a well-defined $\mathbb{K}$-algebra homomorphism. We define the map π_M by

$$\pi_M \colon \mathfrak{A}(w) \ni a \mapsto \zeta(a)(1_M) \in M.$$

Proposition 3.9. *The triplet* $(\mathfrak{A}(w), \Delta_M, \pi_M)$ *is a comonoid in the tensor category of* (M, M)*-bimodules.*

Proof. We first check that the map Δ_M is an (M, M)-bimodule homomorphism. Let $\otimes_M$ denote the tensor product of $\mathfrak{A}(w) \otimes_M \mathfrak{A}(w)$. For any $f, g \in M$, $r, r' \in R$, $p, q \in Q^{(m)}$ and $m \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}
&f \cdot \Delta_M\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \cdot g \\
&= \sum_{u \in Q^{(m)}} \Big(s_M(f)\Big(r \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_w\Big)\Big) \otimes_M \Big(t_M(g)\Big(1_R \otimes r' \otimes \mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_w\Big)\Big) \\
&= \sum_{u \in Q^{(m)}} \Big(f(\mathfrak{s}(p))r \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_w\Big) \otimes_M \Big(1_R \otimes r'g(\mathfrak{s}(q)) \otimes \mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_w\Big) \\
&= \Delta_M\Big(f(\mathfrak{s}(p))r \otimes r'g(\mathfrak{s}(q)) \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_w\Big) \\
&= \Delta_M\Big(f \cdot \Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \cdot g\Big).
\end{aligned}$$

We next prove that the map π_M is an (M, M)-bimodule homomorphism. For any $f, g \in M$, $r, r' \in R$, $p, q \in Q^{(m)}$ and $m \in \mathbb{Z}_{\geq 0}$,

$$f \cdot \pi_M\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \cdot g = \delta_{p,q} f(rr')_\sharp g \delta_{\mathfrak{s}(q)}.$$

On the other hand,

$$\begin{aligned}
\pi_M\Big(f \cdot (r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \cdot g) &= \pi_M\Big(f(\mathfrak{s}(p))r \otimes r'g(\mathfrak{s}(q)) \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_w\Big) \\
&= \delta_{p,q}\Big(f(\mathfrak{s}(p))rr'g(\mathfrak{s}(q)\Big))_\sharp \delta_{\mathfrak{s}(q)}.
\end{aligned}$$

Suppose $p = q$. For all $\lambda \in \Lambda$,

$$\begin{aligned}
((f(\mathfrak{s}(p))rr'g(\mathfrak{s}(q)))_\sharp \delta_{\mathfrak{s}(q)})(\lambda) &= f(\mathfrak{s}(p))rr'g(\mathfrak{s}(p))\delta_{\mathfrak{s}(p),\lambda} \\
&= f(\lambda)rr'g(\lambda)\delta_{\mathfrak{s}(p),\lambda} \\
&= (f(rr')_\sharp g \delta_{\mathfrak{s}(p)})(\lambda).
\end{aligned}$$

Thus π_M is an (M, M)-bimodule homomorphism.

Let us prove that $(\mathfrak{A}(w), \Delta_M, \pi_M)$ is a comonoid in the tensor category of (M, M)-bimodules. We have

$$\begin{aligned}
&\Big(\Big(\Delta_M \otimes \mathrm{id}_{\mathfrak{A}(\mathrm{w})}\Big) \circ \Delta_M\Big)\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&\qquad = \sum_{u \in Q^{(m)}} \Big(\sum_{v \in Q^{(m)}} \Big(r \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ v \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes_M \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} v \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big) \\
&\qquad\qquad \otimes_M \Big(1_R \otimes r' \otimes \mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&\qquad = \sum_{v \in Q^{(m)}} \Big(r \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ v \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes_M \Big(\sum_{u \in Q^{(m)}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} v \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&\qquad\qquad \otimes_M \Big(1_R \otimes r' \otimes \mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big) \\
&\qquad = \Big((\mathrm{id}_{\mathfrak{A}(\mathrm{w})} \otimes \Delta_M) \circ \Delta_M\Big)\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big).
\end{aligned}$$

We write $a = r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}$. By using Sweedler's notation $\Delta_M(a) = a_{[1]} \otimes a_{[2]}$, we get

$$\begin{aligned}
(s_M \circ \pi_M)(a_{[1]})a_{[2]} &= \sum_{\substack{u \in Q^{(m)} \\ \lambda, \mu \in \Lambda}} \delta_{p,u}\Big((r_\sharp \delta_{\mathfrak{s}(u)})(\lambda) \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \lambda \\ \mu \end{bmatrix}\Big)\Big(1_R \otimes r' \otimes \mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix}\Big) + \mathfrak{I}_{\mathbf{w}} \\
&= \sum_{u \in Q^{(m)}} \delta_{p,u}\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix}\Big) + \mathfrak{I}_{\mathbf{w}} \\
&= r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}.
\end{aligned}$$

The proof for $(t_M \circ \pi_M)(a_{[2]})a_{[1]} = a$ is similar. This is the desired conclusion. □

Proposition 3.10. *The maps Δ_M and π_M satisfy the conditions* (2.3)–(2.7).

Proof. We first show (2.3). For any $r, r' \in R$, $p, q \in Q^{(m)}$, and $m \in \mathbb{Z}_{\geq 0}$, we write $a = r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}$. Let f be an arbitrary element in M. We can evaluate that

$$\begin{aligned}
&a_{[1]} t_M(f) \otimes_M a_{[2]} \\
&\quad = \sum_{u \in Q^{(m)}} \Big(r \otimes f(\mathfrak{t}(u)) \otimes \begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes_M \Big(1_R \otimes r' \otimes \begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&\quad = \sum_{u \in Q^{(m)}} t_M(f(\mathfrak{t}(u))_\sharp)\Big(r \otimes 1_R \otimes \begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes_M \Big(1_R \otimes r' \otimes \begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&\quad = \sum_{u \in Q^{(m)}} \Big(r \otimes 1_R \otimes \begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes_M s_M(f(\mathfrak{t}(u))_\sharp)\Big(1_R \otimes r' \otimes \begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)
\end{aligned}$$

$$= \sum_{u\in Q^{(m)}} \left(r \otimes 1_R \otimes \begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \otimes_M \left(f(\mathfrak{t}(u)) \otimes r' \otimes \begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right)$$
$$= a_{[1]} \otimes_M a_{[2]} s_M(f).$$

Therefore (2.3) is satisfied.

We next prove (2.4). Let $\otimes_{\mathbb{K}}$ be the tensor product of $\mathfrak{A}(w) \otimes_{\mathbb{K}} \mathfrak{A}(w)$ and $\otimes_M$ that of $\mathfrak{A}(w) \otimes_M \mathfrak{A}(w)$. For any $\lambda \in \Lambda$,

$$\begin{aligned}
&\mathfrak{I}_2 \ni t_M(\delta_\lambda) \otimes_{\mathbb{K}} 1_{\mathfrak{A}(w)} - 1_{\mathfrak{A}(w)} \otimes_{\mathbb{K}} s_M(\delta_\lambda) \\
&= \sum_{\mu,\tau,\nu\in\Lambda} \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \mu \\ \lambda \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \otimes_{\mathbb{K}} \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \tau \\ \nu \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \\
&\quad - \sum_{\eta,\theta,\upsilon\in\Lambda} \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \eta \\ \theta \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \otimes_{\mathbb{K}} \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \lambda \\ \upsilon \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \\
&= \sum_{\mu,\nu\in\Lambda} \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \mu \\ \lambda \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \otimes_{\mathbb{K}} \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \lambda \\ \nu \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \\
&\quad - \sum_{\eta,\upsilon\in\Lambda} \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \eta \\ \lambda \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \otimes_{\mathbb{K}} \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \lambda \\ \upsilon \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \\
&\quad + \sum_{\substack{\mu,\tau,\nu\in\Lambda \\ \lambda\neq\tau}} \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \mu \\ \lambda \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \otimes_{\mathbb{K}} \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \tau \\ \nu \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \\
&\quad - \sum_{\substack{\eta,\theta,\upsilon\in\Lambda \\ \lambda\neq\theta}} \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \eta \\ \theta \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \otimes_{\mathbb{K}} \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \lambda \\ \upsilon \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \\
&= \sum_{\substack{\mu,\tau,\nu\in\Lambda \\ \lambda\neq\tau}} \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \mu \\ \lambda \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \otimes_{\mathbb{K}} \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \tau \\ \nu \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \\
&\quad - \sum_{\substack{\eta,\theta,\upsilon\in\Lambda \\ \lambda\neq\theta}} \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \eta \\ \theta \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \otimes_{\mathbb{K}} \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \lambda \\ \upsilon \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right).
\end{aligned}$$

Because $\mathfrak{I}_2$ is a right ideal,

$$\begin{aligned}
&\mathfrak{I}_2 \ni \sum_{\gamma\in\Lambda} \Big(t_M(\delta_\lambda) \otimes_{\mathbb{K}} 1_{\mathfrak{A}(w)} - 1_{\mathfrak{A}(w)} \otimes_{\mathbb{K}} s_M(\delta_\lambda)\Big)\left(\left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \gamma \\ \lambda \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \otimes_{\mathbb{K}} 1_{\mathfrak{A}(w)}\right) \\
&= \sum_{\substack{\mu,\tau,\nu,\gamma\in\Lambda \\ \lambda\neq\tau}} \left(1_R \otimes 1_R \otimes \delta_{\mu,\gamma}\mathbf{e}\begin{bmatrix} \mu \\ \lambda \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \otimes_{\mathbb{K}} \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \tau \\ \nu \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \\
&\quad - \sum_{\substack{\eta,\theta,\upsilon,\gamma\in\Lambda \\ \lambda\neq\theta}} \left(1_R \otimes 1_R \otimes \delta_{\gamma,\eta}\delta_{\lambda,\theta}\mathbf{e}\begin{bmatrix} \eta \\ \theta \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right) \otimes_{\mathbb{K}} \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \lambda \\ \upsilon \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right)
\end{aligned}$$

$$= \sum_{\substack{\mu,\tau,\nu\in\Lambda \\ \lambda\neq\tau}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\mu\\ \lambda\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes_{\mathbb{K}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\tau\\ \nu\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big).$$

Since $\mathfrak{A}(w) \otimes_{\mathbb{K}} \mathfrak{A}(w)/\mathfrak{I}_2 \cong \mathfrak{A}(w) \otimes_M \mathfrak{A}(w)$ as $\mathbb{K}$-vector spaces, we can deduce that

$$\sum_{\substack{\mu,\tau,\nu\in\Lambda \\ \lambda\neq\tau}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\mu\\ \lambda\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes_M \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\tau\\ \nu\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) = 0. \tag{3.6}$$

By using the above conclusion, the left-hand side of (2.4) can be evaluated as follows:

$$\begin{aligned}\Delta_M(1_{\mathfrak{A}(w)}) &= \sum_{\lambda,\mu,\tau\in\Lambda} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\lambda\\ \tau\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes_M \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\tau\\ \mu\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\ &\quad + \sum_{\substack{\lambda,\mu,\tau,\nu\in\Lambda \\ \tau\neq\nu}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\lambda\\ \tau\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes_M \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\nu\\ \mu\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\ &= \sum_{\lambda,\mu,\tau,\nu\in\Lambda} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\lambda\\ \tau\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes_M \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\nu\\ \mu\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big).\end{aligned}$$

Thus (2.4) is proved.

The proof for (2.5) is similar to that of multiplicativity of the map $\overline{\nabla}$.

Let us prove (2.6). Because $1_{\mathfrak{G}(Q)} = \sum_{\lambda,\mu\in\Lambda} \mathbf{e}\left[\begin{smallmatrix}\lambda\\ \mu\end{smallmatrix}\right]$, we have

$$\begin{aligned}\pi_M(1_{\mathfrak{A}(w)}) &= \sum_{\lambda,\mu\in\Lambda} \delta_{\lambda,\mu}\delta_\mu \\ &= \sum_{\lambda\in\Lambda} \delta_\lambda = 1_M.\end{aligned}$$

Finally, we give the proof of (2.7). Because ζ is a $\mathbb{K}$-algebra homomorphism, it is sufficient to prove that $\zeta(a)(1_M) = \zeta(s_M(\pi_M(a)))(1_M) = \zeta(t_M(\pi_M(a)))(1_M)$ for all $a \in \mathfrak{A}(w)$. Let $r, r' \in R$, $p, q \in Q^{(m)}$, and $m \in \mathbb{Z}_{\geq 0}$. We can evaluate that

$$\begin{aligned}&\zeta\Big(s_M\Big(\pi_M\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix}p\\ q\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big)\Big)(1_M) \\ &\qquad = \sum_{\lambda\in\Lambda} \delta_{p,q}\zeta\Big(rr' \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\mathfrak{s}(q)\\ \lambda\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)(1_M) \\ &\qquad = \delta_{p,q}(rr')_\sharp \delta_{\mathfrak{s}(q)} \\ &\qquad = \zeta\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix}p\\ q\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)(1_M).\end{aligned}$$

The proof for $\zeta\Big(t_M\Big(\pi_M\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix}p\\ q\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big)\Big)(1_M) = \zeta\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix}p\\ q\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)(1_M)$ is similar. Thus we conclude (2.7). This completes the proof. □

The sextuplet $(\mathfrak{A}(w), M, s_M, t_M, \Delta_M, \pi_M)$ is therefore a left bialgebroid by the above propositions.

We next introduce a condition under which this $\mathfrak{A}(w)$ becomes a Hopf algebroid.

Definition 3.11. The family $w = \left\{ \mathbf{w}\left[c \begin{smallmatrix} a \\ \\ d \end{smallmatrix} b \right] \right\}_{(a,b),(c,d)\in Q^{(2)}}$ is said to be *rigid* if and only if, for any $m \in \mathbb{Z}_{\geq 0}$, p, and $q \in Q^{(m)}$, there exist elements $X_{p,q}, Y_{p,q} \in \mathfrak{A}(w)$ satisfying the following conditions:

$$\sum_{u\in Q^{(m)}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) X_{u,q} = \delta_{p,q} s_M(\delta_{\mathfrak{s}(p)}); \tag{3.7}$$

$$\sum_{u\in Q^{(m)}} X_{p,u}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) = \delta_{p,q} t_M(\delta_{\mathfrak{t}(p)}); \tag{3.8}$$

$$\sum_{u,v\in Q^{(m)}} X_{p,u}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u \\ v \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) X_{v,q} = X_{p,q}; \tag{3.9}$$

$$\sum_{u\in Q^{(m)}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) Y_{p,u} = \delta_{p,q} t_M(\delta_{\mathfrak{s}(p)}); \tag{3.10}$$

$$\sum_{u\in Q^{(m)}} Y_{u,q}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) = \delta_{p,q} s_M(\delta_{\mathfrak{t}(p)}); \tag{3.11}$$

$$\sum_{u,v\in Q^{(m)}} Y_{u,q}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} v \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) Y_{p,v} = Y_{p,q}. \tag{3.12}$$

Definition 3.11 entails that

$$\begin{gathered} X_{p,q} s_M(\delta_{\mathfrak{s}(q)}) = X_{p,q} = t_M(\delta_{\mathfrak{t}(p)}) X_{p,q}; \\ s_M(\delta_{\mathfrak{t}(q)}) Y_{p,q} = Y_{p,q} = Y_{p,q} t_M(\delta_{\mathfrak{s}(p)}) \end{gathered} \tag{3.13}$$

for any $m \in \mathbb{Z}_{\geq 0}$, $p, q \in Q^{(m)}$. We use these identities frequently.

Proposition 3.12. *If w is rigid, then the elements $X_{p,q}$ and $Y_{p,q}$ ($m \in \mathbb{Z}_{\geq 0}$, $p, q \in Q^{(m)}$) are unique.*

Proof. We give the proof only for the uniqueness of $X_{p,q}$. Suppose that $X_{p,q}$ and $X'_{p,q} \in \mathfrak{A}(w)$ satisfy (3.7)–(3.9). We have

$$\begin{aligned} X_{p,q} &\underset{(3.13)}{=} X_{p,q} s_M(\delta_{\mathfrak{s}(q)}) \\ &= \sum_{u\in Q^{(m)}} \delta_{u,q} X_{p,u} s_M(\delta_{\mathfrak{s}(u)}) \\ &\underset{(3.7)}{=} \sum_{u,v\in Q^{(m)}} X_{p,u}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u \\ v \end{bmatrix}\Big) X'_{v,q} \\ &\underset{(3.8)}{=} \sum_{v\in Q^{(m)}} \delta_{p,v} t_M(\delta_{\mathfrak{t}(p)}) X'_{v,q} \end{aligned}$$

$$
\begin{aligned}
&= t_M(\delta_{\mathfrak{t}(p)})X'_{p,q} \\
&\underset{(3.13)}{=} X'_{p,q}.
\end{aligned}
$$

This completes the proof. □

We can construct a $\mathbb{K}$-algebra anti-automorphism $S\colon \mathfrak{A}(w) \to \mathfrak{A}(w)$ by using these elements $X_{p,q}$ and $Y_{p,q}$. As a result, $\mathfrak{A}(w)$ has a Hopf algebroid structure if w is rigid.

Theorem 3.13. *If w is rigid, then $\mathfrak{A}(w)$ has a Hopf algebroid structure for $N = M^{\mathrm{op}}$ and $\omega = \mathrm{id}_M$.*

Proof. We first construct a $\mathbb{K}$-algebra anti-automorphism $S\colon \mathfrak{A}(w) \to \mathfrak{A}(w)$. The $\mathbb{K}$-linear map $\overline{S}\colon \mathfrak{H}(Q) \to \mathfrak{A}(w)$ is defined by

$$
\overline{S}\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}\Big) = \Big(r' \otimes r \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_w\Big)X_{p,q}
$$

for any $r, r' \in R$, $m \in \mathbb{Z}_{\geq 0}$, p, and $q \in Q^{(m)}$. Let us check that this $\overline{S}$ is a $\mathbb{K}$-algebra anti-homomorphism. For the proof, the following lemma plays an important role:

Lemma 3.14. *For any $r, r' \in R$, $m, n \in \mathbb{Z}_{\geq 0}$, $p, q \in Q^{(m)}$, $p', q' \in Q^{(n)}$,*

$$
\begin{aligned}
X_{p,q}(r \otimes r' \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}) &= (r \otimes r' \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}})X_{p,q}; && (3.14)\\
X_{p',q'}X_{p,q} &= \delta_{\mathfrak{t}(p),\mathfrak{s}(p')}\delta_{\mathfrak{t}(q),\mathfrak{s}(q')}X_{pp',qq'}; && (3.15)\\
Y_{p,q}(r \otimes r' \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}) &= (r \otimes r' \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}})Y_{p,q};\\
Y_{p',q'}Y_{p,q} &= \delta_{\mathfrak{t}(p),\mathfrak{s}(p')}\delta_{\mathfrak{t}(q),\mathfrak{s}(q')}Y_{pp',qq'}.
\end{aligned}
$$

We postpone the proof of Lemma 3.14 until the end of this section.

Lemma 3.14 implies that

$$
\begin{aligned}
&\overline{S}\Big(\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}\Big)\Big(s \otimes s' \otimes \mathbf{e}\begin{bmatrix} p' \\ q' \end{bmatrix}\Big)\Big)\\
&\quad = \delta_{\mathfrak{t}(p),\mathfrak{s}(p')}\delta_{\mathfrak{t}(q),\mathfrak{s}(q')}(s'r' \otimes rs \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}})X_{pp',qq'}\\
&\quad \underset{(3.15)}{=} (s' \otimes s \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}})(r' \otimes r \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}})X_{p',q'}X_{p,q}\\
&\quad \underset{(3.14)}{=} (s' \otimes s \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}})X_{p',q'}(r' \otimes r \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}})X_{p,q}\\
&\quad = \overline{S}\Big(s \otimes s' \otimes \mathbf{e}\begin{bmatrix} p' \\ q' \end{bmatrix}\Big)\overline{S}\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}\Big)
\end{aligned}
$$

for any $r, r', s, s' \in R$, $m, n \in \mathbb{Z}_{\geq 0}$, $p, q \in Q^{(m)}$, p', and $q' \in Q^{(n)}$. Since it is easy to see that

$$
X_{\lambda,\mu} = Y_{\lambda,\mu} = 1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \mu \\ \lambda \end{bmatrix} + \mathfrak{I}_{\mathbf{w}} \tag{3.16}
$$

for any $\lambda, \mu \in \Lambda$, we have

$$\begin{aligned}\overline{S}(1_{\mathfrak{A}(w)}) &= \sum_{\lambda,\mu\in\Lambda} \overline{S}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\\ &= \sum_{\lambda,\mu\in\Lambda} X_{\lambda,\mu}\\ &= \sum_{\lambda,\mu\in\Lambda} 1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\mu\\ \lambda\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\\ &= 1_{\mathfrak{A}(w)}.\end{aligned}$$

Therefore $\overline{S}$ is a $\mathbb{K}$-algebra anti-homomorphism. We show that $\overline{S}(\mathfrak{I}_{\mathbf{w}}) = \{0\}$. Since the map $\overline{S}$ is a $\mathbb{K}$-algebra anti-homomorphism, it is sufficient to prove that $\overline{S}(\alpha) = 0$ for every generator α in $\mathfrak{I}_{\mathbf{w}}$. We define a collection of elements $\underline{\mathbf{w}}\left[\begin{smallmatrix} & a & \\ c & & b\\ & d & \end{smallmatrix}\right]$ $(a, b, c, d \in Q)$ in R as follows:

$$\underline{\mathbf{w}}\left[\begin{smallmatrix} & a & \\ c & & b\\ & d & \end{smallmatrix}\right] = \begin{cases} \mathbf{w}\left[\begin{smallmatrix} & a & \\ c & & b\\ & d & \end{smallmatrix}\right], & (a,b),(c,d) \in Q^{(2)};\\ 0, & \text{otherwise.}\end{cases}$$

For any x', y', x'', and $y'' \in Q$,

$$\begin{aligned}0 = &\sum_{a,b,c,d\in Q} X_{y'',b}X_{x'',a}\\ &\times \Bigg(\sum_{x,y\in Q} \underline{\mathbf{w}}\left[\begin{smallmatrix} & x & \\ a & & y\\ & b & \end{smallmatrix}\right] \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}x\\ c\end{bmatrix}\mathbf{e}\begin{bmatrix}y\\ d\end{bmatrix} - 1_R \otimes \underline{\mathbf{w}}\left[\begin{smallmatrix} & c & \\ x & & d\\ & y & \end{smallmatrix}\right] \otimes \mathbf{e}\begin{bmatrix}a\\ x\end{bmatrix}\mathbf{e}\begin{bmatrix}b\\ y\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Bigg)\\ &\times X_{d,y'}X_{c,x'}\\ \underset{(3.7),(3.8)}{=} &\sum_{a,b,c,x\in Q} \delta_{\mathfrak{t}(x),\mathfrak{s}(y')}\Big(\underline{\mathbf{w}}\left[\begin{smallmatrix} & x & \\ a & & y'\\ & b & \end{smallmatrix}\right] \otimes 1_R \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}\Big)X_{y'',b}X_{x'',a}\\ &\quad\times \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}x\\ c\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)X_{c,x'}\\ &- \sum_{b,c,d,y\in Q} \delta_{\mathfrak{t}(x''),\mathfrak{s}(y)}\Big(1_R \otimes \underline{\mathbf{w}}\left[\begin{smallmatrix} & c & \\ x'' & & d\\ & y & \end{smallmatrix}\right] \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}\Big)X_{y'',b}\\ &\quad\times \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}b\\ y\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)X_{d,y'}X_{c,x'}\\ \underset{(3.7),(3.8)}{=} &\sum_{\substack{a,b\in Q\\ \lambda\in\Lambda}} \delta_{\mathfrak{t}(x'),\mathfrak{s}(y')}\Big(\underline{\mathbf{w}}\left[\begin{smallmatrix} & x' & \\ a & & y'\\ & b & \end{smallmatrix}\right] \otimes 1_R \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}\Big)X_{y'',b}X_{x'',a}\\ &\quad\times \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\mathfrak{s}(x')\\ \lambda\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\\ &- \sum_{\substack{c,d\in Q\\ \lambda\in\Lambda}} \delta_{\mathfrak{t}(x''),\mathfrak{s}(y'')}\Big(1_R \otimes \underline{\mathbf{w}}\left[\begin{smallmatrix} & c & \\ x'' & & d\\ & y'' & \end{smallmatrix}\right] \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}\Big)\end{aligned}$$

$$\times\Big(1_R\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}\lambda\\ \mathfrak{t}(y'')\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)X_{d,y'}X_{c,x'}.$$

By using (3.15) and (3.16), we have

$$\begin{aligned}X_{x'',a}\Big(\sum_{\lambda\in\Lambda}1_R\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}\mathfrak{s}(x')\\ \lambda\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)&=\sum_{\lambda\in\Lambda}X_{x'',a}X_{\lambda,\mathfrak{s}(x')}\\&=\sum_{\lambda\in\Lambda}\delta_{\mathfrak{s}(x''),\lambda}\delta_{\mathfrak{s}(a),\mathfrak{s}(x')}X_{x'',a}\\&=\delta_{\mathfrak{s}(a),\mathfrak{s}(x')}X_{x'',a}\end{aligned}$$

for any $a\in Q$. We can similarly prove that

$$\Big(1_R\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}\lambda\\ \mathfrak{t}(y'')\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)X_{d,y'}=\delta_{\mathfrak{t}(y''),\mathfrak{t}(d)}X_{d,y'}\quad(\forall\, d\in Q).$$

Thus, by using the condition (3.5),

$$\begin{aligned}0&=\sum_{a,b\in Q}\delta_{\mathfrak{t}(x'),\mathfrak{s}(y')}\delta_{\mathfrak{s}(a),\mathfrak{s}(x')}\Big(\underline{\mathbf{w}}\left[\begin{smallmatrix}&x'&\\a&&y'\\&b&\end{smallmatrix}\right]\otimes 1_R\otimes 1_{\mathfrak{G}(Q)}+\mathfrak{I}_{\mathbf{w}}\Big)X_{y'',b}X_{x'',a}\\&\quad-\sum_{c,d\in Q}\delta_{\mathfrak{t}(x''),\mathfrak{s}(y'')}\delta_{\mathfrak{t}(y''),\mathfrak{t}(d)}\Big(1_R\otimes\underline{\mathbf{w}}\left[\begin{smallmatrix}&c&\\x''&&d\\&y''&\end{smallmatrix}\right]\otimes 1_{\mathfrak{G}(Q)}+\mathfrak{I}_{\mathbf{w}}\Big)X_{d,y'}X_{c,x'}\\&=\sum_{(a,b)\in Q^{(2)}}\Big(\mathbf{w}\left[\begin{smallmatrix}&x'&\\a&&y'\\&b&\end{smallmatrix}\right]\otimes 1_R\otimes 1_{\mathfrak{G}(Q)}+\mathfrak{I}_{\mathbf{w}}\Big)X_{y'',b}X_{x'',a}\\&\quad-\sum_{(c,d)\in Q^{(2)}}\Big(1_R\otimes\mathbf{w}\left[\begin{smallmatrix}&c&\\x''&&d\\&y''&\end{smallmatrix}\right]\otimes 1_{\mathfrak{G}(Q)}+\mathfrak{I}_{\mathbf{w}}\Big)X_{d,y'}X_{c,x'}\\&=\overline{S}\Bigg(\sum_{(a,b)\in Q^{(2)}}1_R\otimes\mathbf{w}\left[\begin{smallmatrix}&x'&\\a&&y'\\&b&\end{smallmatrix}\right]\otimes\mathbf{e}\begin{bmatrix}x''\\a\end{bmatrix}\mathbf{e}\begin{bmatrix}y''\\b\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\\&\quad-\sum_{(c,d)\in Q^{(2)}}\mathbf{w}\left[\begin{smallmatrix}&c&\\x''&&d\\&y''&\end{smallmatrix}\right]\otimes 1_R\otimes\mathbf{e}\begin{bmatrix}c\\x'\end{bmatrix}\mathbf{e}\begin{bmatrix}d\\y'\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Bigg).\end{aligned}$$

This fact implies that $\overline{S}(\alpha)=\{0\}$ for any generator α in $\mathfrak{I}_w$. Therefore we can define a $\mathbb{K}$-algebra anti-homomorphism $S(\beta+\mathfrak{I}_{\mathbf{w}})=\overline{S}(\beta)$ $(\beta\in\mathfrak{H}(Q))$. Similarly, another $\mathbb{K}$-algebra anti-homomorphism $S'\colon\mathfrak{A}(w)\to\mathfrak{A}(w)$ can be defined such that

$$S'\Big(r\otimes r'\otimes\mathbf{e}\begin{bmatrix}p\\q\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)=(r'\otimes r\otimes 1_{\mathfrak{G}(Q)}+\mathfrak{I}_w)Y_{p,q}$$

for any $r, r' \in R$, $m \in \mathbb{Z}_{\geq 0}$, p, and $q \in Q^{(m)}$. Let us show that $S' \circ S = S \circ S' = \mathrm{id}_{\mathfrak{A}(w)}$. We can calculate that

$$\begin{aligned}
S'(X_{p,q}) &\underset{(3.13)}{=} \sum_{\substack{u \in Q^{(m)} \\ \lambda \in \Lambda}} \delta_{u,q} S'\Big(X_{p,u}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \mathfrak{s}(u) \\ \lambda \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big) \\
&\underset{(3.16)}{=} \sum_{\substack{u \in Q^{(m)} \\ \lambda \in \Lambda}} \delta_{u,q}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \lambda \\ \mathfrak{s}(u) \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) S'(X_{p,u}) \\
&\underset{(3.10)}{=} \sum_{u,v \in Q^{(m)}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} v \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) Y_{u,v} S'(X_{p,u}) \\
&= \sum_{u,v \in Q^{(m)}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} v \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) S'\Big(X_{p,u}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u \\ v \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big) \\
&\underset{(3.8)}{=} \sum_{\lambda \in \Lambda} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) S'\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \lambda \\ \mathfrak{t}(p) \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&\underset{(3.16)}{=} \sum_{\lambda \in \Lambda} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \mathfrak{t}(p) \\ \lambda \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&= 1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}
\end{aligned}$$

for all $p, q \in Q^{(m)}$. Thus,

$$\begin{aligned}
(S' \circ S)\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) &= S'(X_{p,q})(r \otimes r' \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}) \\
&= r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}
\end{aligned}$$

for any $r, r' \in R$, $m \in \mathbb{Z}_{\geq 0}$, p, and $q \in Q^{(m)}$. Proving that $S \circ S' = \mathrm{id}_{\mathfrak{A}(w)}$ is similar. Therefore the map S is a $\mathbb{K}$-algebra anti-automorphism.

We next prove that this S gives $\mathfrak{A}(w)$ a Hopf algebroid structure if w is rigid. Let $N = M^{\mathrm{op}}$ and $\omega = \mathrm{id}_{\mathfrak{A}(w)}$. We assume the lemma below for the moment.

Lemma 3.15. *Let $X_{p,q}$ and $Y_{p,q}$ ($p, q \in Q^{(m)}$, $m \in \mathbb{Z}_{\geq 0}$) be elements in $\mathfrak{A}(w)$ satisfying* (3.7)–(3.12). *Then the following identities hold:*

$$\Delta_M(X_{p,q}) = \sum_{u \in Q^{(m)}} X_{u,q} \otimes X_{p,u}; \tag{3.17}$$

$$\Delta_M(Y_{p,q}) = \sum_{u \in Q^{(m)}} Y_{u,q} \otimes Y_{p,u}; \tag{3.18}$$

$$\pi_M(X_{p,q}) = \delta_{p,q}\delta_{\mathfrak{t}(p)}. \tag{3.19}$$

For the proof of (2.12), we have

$$\begin{aligned}(S\circ t_M)(f) &= \sum_{\lambda,\mu\in\Lambda} S\Big(1_R\otimes f(\lambda)\otimes \mathbf{e}\begin{bmatrix}\mu\\ \lambda\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)\\ &= \sum_{\lambda,\mu\in\Lambda}(f(\lambda)\otimes 1_R\otimes 1_{\mathfrak{G}(Q)}+\mathfrak{I}_{\mathbf{w}})X_{\mu,\lambda}\\ &\underset{(3.16)}{=} \sum_{\lambda,\mu\in\Lambda} f(\lambda)\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\\ &= s_M(f).\end{aligned}$$

Here the symbol f means an arbitrary element in M. Let us check that the identity (2.13) holds. We write $a = r\otimes r'\otimes \mathbf{e}\begin{bmatrix}p\\ q\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}$ for all $r, r'\in R$, $m\in\mathbb{Z}_{\geq 0}$, p, and $q\in Q^{(m)}$. The left-hand side of (2.13) satisfies

$$\begin{aligned}S(a_{[1]})a_{[2]} &= \sum_{u\in Q^{(m)}}\Big(1_R\otimes r\otimes 1_{\mathfrak{G}(Q)}+\mathfrak{I}_{\mathbf{w}}\Big)X_{p,u}\Big(1_R\otimes r'\otimes \mathbf{e}\begin{bmatrix}u\\ q\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)\\ &\underset{(3.14)}{=} \sum_{u\in Q^{(m)}}\Big(1_R\otimes r'r\otimes 1_{\mathfrak{G}(Q)}+\mathfrak{I}_{\mathbf{w}}\Big)X_{p,u}\Big(1_R\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}u\\ q\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)\\ &\underset{(3.8)}{=} \delta_{p,q}\sum_{\lambda\in\Lambda} 1_R\otimes r'r\otimes \mathbf{e}\begin{bmatrix}\lambda\\ \mathfrak{t}(p)\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}.\end{aligned}$$

By using (3.19), the right-hand side of (2.13) can be calculated as follows:

$$\begin{aligned}(t_M\circ\pi_M\circ S)(a) &= \delta_{p,q}t_M(\zeta(r'\otimes r\otimes 1_{\mathfrak{G}(Q)}+\mathfrak{I}_{\mathbf{w}})(\delta_{\mathfrak{t}(p)}))\\ &= \delta_{p,q}t_M((r'r)_{\sharp}\delta_{\mathfrak{t}(p)})\\ &= \delta_{p,q}\sum_{\lambda\in\Lambda} 1_R\otimes r'r\otimes \mathbf{e}\begin{bmatrix}\lambda\\ \mathfrak{t}(p)\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}.\end{aligned}$$

Thus (2.13) is proved. Let $\otimes_M$ be the tensor product of $\mathfrak{A}(w)\otimes_M\mathfrak{A}(w)$ and $\otimes_N$ that of $\mathfrak{A}(w)\otimes_N\mathfrak{A}(w)$. It is easy to show that there exists the inverse $S^{-1}_{\mathfrak{A}(w)\otimes_N\mathfrak{A}(w)}$ of $S_{\mathfrak{A}(w)\otimes_N\mathfrak{A}(w)}$. $S^{-1}_{\mathfrak{A}(w)\otimes_N\mathfrak{A}(w)}$ satisfies

$$S^{-1}_{\mathfrak{A}(w)\otimes_N\mathfrak{A}(w)}(a\otimes_M b) = S^{-1}(b)\otimes_N S^{-1}(a)$$

for all $a, b \in \mathfrak{A}(w)$. We will check (2.17). By using (3.16) and (3.17),

$$
\begin{aligned}
&(S_{\mathfrak{A}(w)\otimes_M\mathfrak{A}(w)} \circ \Delta_M \circ S^{-1})\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&= \sum_{\substack{\lambda,\mu,\tau\in\Lambda \\ u\in Q^{(m)}}} S_{\mathfrak{A}(w)\otimes_M\mathfrak{A}(w)}\Big(\Big(\Big(r' \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \lambda \\ \tau \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)Y_{u,q}\Big) \\
&\qquad \otimes_M \Big(\Big(1_R \otimes r \otimes \mathbf{e}\begin{bmatrix} \tau \\ \mu \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)Y_{p,u}\Big)\Big) \\
&= \sum_{\substack{\lambda,\mu,\tau\in\Lambda \\ u\in Q^{(m)}}} \Big(\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big(r \otimes 1_R \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}\Big)X_{\tau,\mu}\Big) \\
&\qquad \otimes_N \Big(\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big(1_R \otimes r' \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}\Big)X_{\lambda,\tau}\Big) \\
&= \sum_{\substack{\lambda,\mu,\tau\in\Lambda \\ u\in Q^{(m)}}} \Big(r \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \mu \\ \tau \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&\qquad \otimes_N \Big(1_R \otimes r' \otimes \mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \tau \\ \lambda \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&= \sum_{u\in Q^{(m)}} \Big(r \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes_N \Big(1_R \otimes r' \otimes \mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)
\end{aligned}
$$

for all $r, r' \in R$, $m \in \mathbb{Z}_{\geq 0}$, p, and $q \in Q^{(m)}$. Similarly, the identities (3.16) and (3.17) imply that

$$
\begin{aligned}
&\Big(S^{-1}_{\mathfrak{A}(w)\otimes_N\mathfrak{A}(w)} \circ \Delta_M \circ S\Big)\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&= \sum_{\substack{\lambda,\mu,\tau\in\Lambda \\ u\in Q^{(m)}}} S^{-1}_{\mathfrak{A}(w)\otimes_N\mathfrak{A}(w)}\Big(\Big(\Big(r' \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \lambda \\ \tau \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)X_{u,q}\Big) \\
&\qquad \otimes_M \Big(\Big(1_R \otimes r \otimes \mathbf{e}\begin{bmatrix} \tau \\ \mu \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)X_{p,u}\Big)\Big) \\
&= \sum_{\substack{\lambda,\mu,\tau\in\Lambda \\ u\in Q^{(m)}}} \Big(\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big(r \otimes 1_R \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}\Big)Y_{\tau,\mu}\Big) \\
&\qquad \otimes_N \Big(\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big(1_R \otimes r' \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}\Big)Y_{\lambda,\tau}\Big) \\
&= \sum_{\substack{\lambda,\mu,\tau\in\Lambda \\ u\in Q^{(m)}}} \Big(r \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}})\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \mu \\ \tau \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)
\end{aligned}
$$

$$\otimes_N \Big(1_R \otimes r' \otimes \mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \tau \\ \lambda \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)$$
$$= \sum_{u \in Q^{(m)}} \Big(r \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes_N \Big(1_R \otimes r' \otimes \mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big).$$

Thus (2.17) is proved.

We can check (2.18) and (2.19) similarly to the proof of Proposition 3.9. This is the desired conclusion. ☐

Proof of Lemma 3.14. We give the proof only for (3.14) and (3.15).

Let us show (3.14). For any $r, r' \in R$, $m \in \mathbb{Z}_{\geq 0}$, $p, q \in Q^{(m)}$, we have

$$\begin{aligned}
&X_{p,q}(r \otimes r' \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}) \\
&\underset{(3.13)}{=} X_{p,q}(r \otimes r' \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}) s_M(\delta_{\mathfrak{s}(q)}) \\
&= \sum_{u \in Q^{(m)}} \delta_{u,q} X_{p,u}(r \otimes r' \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}) s_M(\delta_{\mathfrak{s}(q)}) \\
&\underset{(3.7)}{=} \sum_{u,v \in Q^{(m)}} X_{p,u}(r \otimes r' \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}})\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u \\ v \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) X_{v,q} \\
&= \sum_{u,v \in Q^{(m)}} X_{p,u}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u \\ v \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)(r \otimes r' \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}) X_{v,q} \\
&\underset{(3.8)}{=} \sum_{v \in Q^{(m)}} \delta_{p,v} t_M(\delta_{\mathfrak{t}(p)})(r \otimes r' \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}) X_{v,q} \\
&= t_M(\delta_{\mathfrak{t}(p)})(r \otimes r' \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}) X_{p,q} \\
&\underset{(3.13)}{=} (r \otimes r' \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}) X_{p,q}.
\end{aligned}$$

We next prove (3.15). For any $m, n \in \mathbb{Z}_{\geq 0}$, let $p, q \in Q^{(m)}$ and $p', q' \in Q^{(n)}$. If $\mathfrak{t}(p) = \mathfrak{s}(p')$ and $\mathfrak{t}(q) = \mathfrak{s}(q')$, then

$$\begin{aligned}
&\sum_{\substack{u,v \in Q^{(m)} \\ u',v' \in Q^{(n)}}} \delta_{\mathfrak{t}(v),\mathfrak{s}(v')} X_{p',u'} X_{p,u}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u \\ v \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&\quad \times \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u' \\ v' \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) X_{vv',qq'} \\
&\underset{(3.8)}{=} \sum_{\substack{u',v' \in Q^{(n)} \\ \lambda \in \Lambda}} \delta_{\mathfrak{t}(p),\mathfrak{s}(v')} X_{p',u'}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \lambda \\ \mathfrak{t}(p) \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&\quad \times \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u' \\ v' \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) X_{pv',qq'} \\
&= \sum_{u',v' \in Q^{(n)}} \delta_{\mathfrak{t}(p),\mathfrak{s}(v')} X_{p',u'}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u' \\ v' \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) X_{pv',qq'}
\end{aligned}$$

$$\underset{(3.8)}{=} \delta_{\mathfrak{t}(p),\mathfrak{s}(p')} t_M(\delta_{\mathfrak{t}(p')}) X_{pp',qq'}$$
$$\underset{(3.13)}{=} X_{pp',qq'}.$$

On the other hand,

$$\sum_{\substack{u,v\in Q^{(m)}\\ u',v'\in Q^{(n)}}} \delta_{\mathfrak{t}(v),\mathfrak{s}(v')} X_{p',u'} X_{p,u} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u \\ v \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)$$
$$\times \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u' \\ v' \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) X_{vv',qq'}$$
$$= \sum_{\substack{u,v\in Q^{(m)}\\ u',v'\in Q^{(n)}}} \delta_{\mathfrak{t}(u),\mathfrak{s}(u')} \delta_{\mathfrak{t}(v),\mathfrak{s}(v')} X_{p',u'} X_{p,u} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} uu' \\ vv' \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) X_{vv',qq'}$$
$$\underset{(3.7)}{=} X_{p',q'} X_{p,q} s_M(\delta_{\mathfrak{s}(q)})$$
$$\underset{(3.13)}{=} X_{p',q'} X_{p,q}.$$

If $\mathfrak{t}(p) \neq \mathfrak{s}(p')$, then it follows that

$$\sum_{\substack{u,v\in Q^{(m)}\\ u',v'\in Q^{(n)}}} X_{p',u'} X_{p,u} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u \\ v \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u' \\ v' \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) X_{v',q'} X_{v,q}$$
$$\underset{(3.8)}{=} \sum_{\substack{u',v'\in Q^{(n)}\\ \lambda\in\Lambda}} X_{p',u'} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \lambda \\ \mathfrak{t}(p) \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u' \\ v' \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) X_{v',q'} X_{p,q}$$
$$= \sum_{u',v'\in Q^{(n)}} \delta_{\mathfrak{t}(p),\mathfrak{s}(v')} X_{p',u'} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u' \\ v' \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) X_{v',q'} X_{p,q}$$
$$\underset{(3.8)}{=} \delta_{\mathfrak{t}(p),\mathfrak{s}(p')} t_M(\delta_{\mathfrak{t}(p')}) X_{p',q'} X_{p,q}$$
$$\underset{(3.13)}{=} \delta_{\mathfrak{t}(p),\mathfrak{s}(p')} X_{p',q'} X_{p,q}$$
$$= 0.$$

In addition, we can calculate that

$$\sum_{\substack{u,v\in Q^{(m)}\\ u',v'\in Q^{(n)}}} X_{p',u'} X_{p,u} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u \\ v \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u' \\ v' \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) X_{v',q'} X_{v,q}$$
$$\underset{(3.7)}{=} \sum_{u,v\in Q^{(m)}} X_{p',q'} X_{p,u} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u \\ v \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) s_M(\delta_{\mathfrak{s}(q')}) X_{v,q}$$
$$\underset{(3.8)}{=} X_{p',q'} t_M(\delta_{\mathfrak{t}(p)}) s_M(\delta_{\mathfrak{s}(q')}) X_{p,q}$$

$$\underset{(2.1)}{=} X_{p',q'} s_M(\delta_{\mathfrak{s}(q')}) t_M(\delta_{\mathfrak{t}(p)}) X_{p,q}$$
$$\underset{(3.13)}{=} X_{p',q'} X_{p,q}.$$

We can similarly prove that $X_{p',q'} X_{p,q} = \delta_{\mathfrak{t}(q),\mathfrak{s}(q')} X_{p',q'} X_{p,q} = 0$ if $\mathfrak{t}(q) \neq \mathfrak{s}(q')$. This completes the proof. □

Proof of Lemma 3.15. We first prove (3.17). By using (3.6),

$$\begin{aligned}
\mathfrak{I}_2 \ni & \Bigg(\sum_{\substack{\lambda,\mu,\tau,\nu \in \Lambda \\ \mu \neq \tau}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \lambda \\ \mu \end{bmatrix} + \mathfrak{I}_{\mathbf{w}} \Big) \otimes_{\mathbb{K}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \tau \\ \nu \end{bmatrix} + \mathfrak{I}_{\mathbf{w}} \Big) \Bigg) \\
& \times \Big(\sum_{\eta \in \Lambda} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \gamma \\ \eta \end{bmatrix} + \mathfrak{I}_{\mathbf{w}} \Big) \otimes_{\mathbb{K}} 1_{\mathfrak{A}(w)} \Big) \\
& \qquad = \sum_{\substack{\mu,\tau,\nu \in \Lambda \\ \mu \neq \tau}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \gamma \\ \mu \end{bmatrix} + \mathfrak{I}_{\mathbf{w}} \Big) \otimes_{\mathbb{K}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \tau \\ \nu \end{bmatrix} + \mathfrak{I}_{\mathbf{w}} \Big)
\end{aligned}$$

for all $\gamma \in \Lambda$. Thus we can calculate that

$$\begin{aligned}
& \sum_{u,v,z \in Q^{(m)}} \Big(\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}} \Big) X_{z,q} \Big) \otimes_{\mathbb{K}} \Big(\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} u \\ v \end{bmatrix} + \mathfrak{I}_{\mathbf{w}} \Big) X_{v,z} \Big) + \mathfrak{I}_2 \\
& \underset{(3.7)}{=} \sum_{u \in Q^{(m)}} \Big(\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}} \Big) X_{u,q} \Big) \otimes_{\mathbb{K}} s_M(\delta_{\mathfrak{s}(u)}) + \mathfrak{I}_2 \\
& = \sum_{u \in Q^{(m)}} \Big(t_M(\delta_{\mathfrak{s}(u)}) \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}} \Big) X_{u,q} \Big) \otimes_{\mathbb{K}} 1_{\mathfrak{A}(w)} + \mathfrak{I}_2 \\
& = \sum_{u \in Q^{(m)}} \Big(\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_{\mathbf{w}} \Big) X_{u,q} \Big) \otimes_{\mathbb{K}} 1_{\mathfrak{A}(w)} + \mathfrak{I}_2 \\
& \underset{(3.7)}{=} \delta_{p,q} s_M(\delta_{\mathfrak{s}(p)}) \otimes_{\mathbb{K}} 1_{\mathfrak{A}(w)} + \mathfrak{I}_2 \\
& = \delta_{p,q} \sum_{\lambda,\mu \in \Lambda} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \mathfrak{s}(p) \\ \lambda \end{bmatrix} + \mathfrak{I}_{\mathbf{w}} \Big) \otimes_{\mathbb{K}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \lambda \\ \mu \end{bmatrix} + \mathfrak{I}_{\mathbf{w}} \Big) \\
& \quad + \delta_{p,q} \sum_{\substack{\lambda,\tau,\mu \in \Lambda \\ \lambda \neq \tau}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \mathfrak{s}(p) \\ \lambda \end{bmatrix} + \mathfrak{I}_{\mathbf{w}} \Big) \otimes_{\mathbb{K}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \tau \\ \mu \end{bmatrix} + \mathfrak{I}_{\mathbf{w}} \Big) + \mathfrak{I}_2 \\
& = \delta_{p,q} \sum_{\lambda,\mu \in \Lambda} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \mathfrak{s}(p) \\ \lambda \end{bmatrix} + \mathfrak{I}_{\mathbf{w}} \Big) \otimes_{\mathbb{K}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \lambda \\ \mu \end{bmatrix} + \mathfrak{I}_{\mathbf{w}} \Big) + \mathfrak{I}_2 \\
& = \delta_{p,q} \sum_{\lambda \in \Lambda} \overline{\nabla} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \mathfrak{s}(p) \\ \lambda \end{bmatrix} \Big) + \mathfrak{I}_2
\end{aligned}$$

for any $m \in \mathbb{Z}_{\geq 0}$, p, and $q \in Q^{(m)}$. We fix $\overline{X}_{p,q} \in \mathfrak{H}(Q)$ such that $X_{p,q} = \overline{X}_{p,q} + \mathfrak{I}_{\mathbf{w}} \in \mathfrak{A}(w)$. By using (3.13), we can deduce that

$$\overline{X}_{p,q} - \sum_{\lambda \in \Lambda} \overline{X}_{p,q}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\mathfrak{s}(q)\\ \lambda\end{bmatrix}\Big) \in \mathfrak{I}_{\mathbf{w}}.$$

We define an element $\alpha_v \in \mathfrak{I}_{\mathbf{w}}$ $(m \in \mathbb{Z}_{\geq 0}, v \in Q^{(m)})$ as follows:

$$\alpha_v = \sum_{u \in Q^{(m)}} \overline{X}_{p,u}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}u\\ v\end{bmatrix}\Big) - \delta_{p,v}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\lambda\\ \mathfrak{t}(p)\end{bmatrix}\Big).$$

As a result,

$$\begin{aligned}
\overline{\nabla}(\overline{X}_{p,q}) + \mathfrak{I}_2 &\underset{\text{Prop. 3.7}}{=} \sum_{\lambda \in \Lambda} \overline{\nabla}\Big(\overline{X}_{p,q}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\mathfrak{s}(q)\\ \lambda\end{bmatrix}\Big)\Big) + \mathfrak{I}_2\\
&= \sum_{\substack{\lambda \in \Lambda\\ u \in Q^{(m)}}} \delta_{u,q}\overline{\nabla}\Big(\overline{X}_{p,u}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\mathfrak{s}(u)\\ \lambda\end{bmatrix}\Big)\Big) + \mathfrak{I}_2\\
&\underset{\overline{\nabla}(\mathfrak{H}(Q))\mathfrak{I}_2 \in \mathfrak{I}_2}{=} \sum_{u,v,z \in Q^{(m)}} \overline{\nabla}\Big(\overline{X}_{p,u}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}u\\ v\end{bmatrix}\Big)\Big)(X_{z,q} \otimes_{\mathbb{K}} X_{v,z}) + \mathfrak{I}_2\\
&\underset{(3.8)}{=} \sum_{\substack{\lambda \in \Lambda\\ v,z \in Q^{(m)}}} \delta_{p,v}\overline{\nabla}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\lambda\\ \mathfrak{t}(p)\end{bmatrix} + \alpha_v\Big)(X_{z,q} \otimes_{\mathbb{K}} X_{v,z}) + \mathfrak{I}_2\\
&\underset{(3.16)}{=} \sum_{\substack{\lambda,\mu \in \Lambda\\ u \in Q^{(m)}}} X_{\mu,\lambda}X_{u,q} \otimes_{\mathbb{K}} X_{\mathfrak{t}(p),\mu}X_{p,u} + \mathfrak{I}_2\\
&\underset{(3.15)}{=} \sum_{u \in Q^{(m)}} X_{u,q} \otimes_{\mathbb{K}} X_{p,u} + \mathfrak{I}_2.
\end{aligned}$$

Since $\mathfrak{A}(w) \otimes_{\mathbb{K}} \mathfrak{A}(w)/\mathfrak{I}_2 \cong \mathfrak{A}(w) \otimes_M \mathfrak{A}(w)$ as $\mathbb{K}$-vector spaces, we can induce (3.17). It is similar to show (3.18).

Finally, let us check (3.19). By using (3.13), we have

$$\begin{aligned}
\pi_M(X_{p,q}) &= \sum_{\lambda \in \Lambda} \pi_M\Big(X_{p,q}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\mathfrak{s}(q)\\ \lambda\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big)\\
&= \sum_{\lambda \in \Lambda} \delta_{\mathfrak{s}(q),\lambda}\zeta(X_{p,q})(\delta_\lambda)\\
&= \zeta(X_{p,q})(\delta_{\mathfrak{s}(q)})
\end{aligned}$$

for any $m \in \mathbb{Z}_{\geq 0}$, p, and $q \in Q^{(m)}$. On the other hand, we have

$$\begin{aligned}
\sum_{u \in Q^{(m)}} \pi_M\Big(X_{p,u}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}u\\ q\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big) &= \sum_{u \in Q^m} \delta_{u,q}\zeta(X_{p,u})(\delta_{\mathfrak{s}(q)})\\
&= \zeta(X_{p,q})(\delta_{\mathfrak{s}(q)}).
\end{aligned}$$

Since π_M is an (M,M)-bimodule homomorphism, we can conclude that

$$\begin{aligned}\pi_M(X_{p,q}) &= \sum_{u\in Q^{(m)}} \pi_M\Big(X_{p,u}\Big(1_R\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}u\\ q\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)\Big)\\ &\underset{(3.8)}{=} \delta_{p,q}\pi_M(t_M(\delta_{\mathfrak{t}(p)}))\\ &= \delta_{p,q}\delta_{\mathfrak{t}(p)}.\end{aligned}$$

This is the desired conclusion. □

4. Left bialgebroid homomorphism Φ

In this section, we induce a left bialgebroid $\mathfrak{A}(w_\sigma)$ as in Subsection 3.2 from the settings of the left bialgebroid A_σ in Subsection 3.1, and construct a left bialgebroid homomorphism Φ from $\mathfrak{A}(w_\sigma)$ to A_σ. This is a generalization of [12]. Moreover, we can also show that the bijectivity of Φ is equivalent to a part of the condition of rigidity for w (see also [15]).

Let A_σ be a left bialgebroid as in Subsection 3.1 and $\sigma^{ab}_{cd}\in M$ $(a,b,c,d\in X)$ satisfying the condition (3.1). We define a quiver Q over Λ by

$$Q := \Lambda\times X,\quad \mathfrak{s}(\lambda,x)=\lambda,\quad \mathfrak{t}(\lambda,x)=\lambda\deg(x)\qquad (\lambda\in\Lambda,\ x\in X) \tag{4.1}$$

and set

$$\mathbf{w}\left[\begin{smallmatrix} & (\lambda,a) & \\ (\mu,c) & & (\lambda',b) \\ & (\mu',d) & \end{smallmatrix}\right] = \delta_{\lambda,\mu}\sigma^{ba}_{dc}(\lambda) \tag{4.2}$$

for all $((\lambda,a),(\lambda',b)),((\mu,c),(\mu',d))\in Q^{(2)}$.

Proposition 4.1. *The definition* (4.2) *satisfies the condition* (3.5)*.*

Proof. Let $((\lambda,a),(\lambda',b)),((\mu,c),(\mu',d))\in Q^{(2)}$. That $\mathbf{w}\left[\begin{smallmatrix} & (\lambda,a) & \\ (\mu,c) & & (\lambda',b) \\ & (\mu',d) & \end{smallmatrix}\right]\in Z(R)$ is clear because of $\sigma^{ba}_{dc}(\lambda)\in Z(R)$.

We next prove that $\mathbf{w}\left[\begin{smallmatrix} & (\lambda,a) & \\ (\mu,c) & & (\lambda',b) \\ & (\mu',d) & \end{smallmatrix}\right]=0$ if $\mathfrak{s}(\lambda,a)\neq\mathfrak{s}(\mu,c)$ or $\mathfrak{t}(\lambda',b)\neq \mathfrak{t}(\mu',d)$. It follows from (4.2) that $\mathbf{w}\left[\begin{smallmatrix} & (\lambda,a) & \\ (\mu,c) & & (\lambda',b) \\ & (\mu',d) & \end{smallmatrix}\right]=0$ if $\mathfrak{s}(\lambda,a)\neq\mathfrak{s}(\mu,c)$.

Suppose that $\mathfrak{t}(\lambda',b)\neq\mathfrak{t}(\mu',d)$. By the definition of the fiber product of the quiver Q, we have

$$\mathfrak{t}(\lambda',b)=\lambda\deg(a)\deg(b)\quad\text{and}\quad \mathfrak{t}(\mu',d)=\mu\deg(c)\deg(d).$$

If $\lambda=\mu$, then $\mathbf{w}\left[\begin{smallmatrix} & (\lambda,a) & \\ (\mu,c) & & (\lambda',b) \\ & (\mu',d) & \end{smallmatrix}\right]=0$ because $\sigma^{ba}_{dc}(\lambda)=0$. This completes the proof. □

Therefore we can construct the left bialgebroid $\mathfrak{A}(w_\sigma):=\mathfrak{A}(w)$ as in Subsection 3.2.

Theorem 4.2. *Let $r, r' \in R$, $m \in \mathbb{Z}_{\geq 0}$, $p = ((\lambda_1, x_1), \dots, (\lambda_m, x_m))$, and $q = ((\mu_1, y_1), \dots, (\mu_m, y_m)) \in Q^{(m)}$. We define the $\mathbb{K}$-linear map $\overline{\Phi}\colon \mathfrak{H}(Q) \to A_\sigma$ by*

$$\overline{\Phi}\left(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}\right) = (r_\sharp \otimes r'_\sharp)(\delta_{\mathfrak{s}(p)} \otimes \delta_{\mathfrak{s}(q)}) L_{x_1 y_1} \cdots L_{x_m y_m} + I_\sigma.$$

Then $\overline{\Phi}$ induces a left bialgebroid homomorphism $(\Phi\colon \mathfrak{A}(w_\sigma) \to A_\sigma, \mathrm{id}_M)$.

Proof. We first prove that $\overline{\Phi}(\mathfrak{I}_{w_\sigma}) = \{0\}$. Since the map $\overline{\Phi}$ is a $\mathbb{K}$-algebra homomorphism, we only need to prove that $\overline{\Phi}(\alpha) = 0$ for every generator α as in (3.4). For any $((\mu, a), (\mu', b))$, $((\nu, c), (\nu', d)) \in Q^{(2)}$, we have

$$\begin{aligned}
\sum_{((\lambda,x),(\lambda',y)) \in Q^{(2)}} & \overline{\Phi}\left(\mathbf{w}\begin{bmatrix} & {\scriptstyle(\lambda,x)} & \\ {\scriptstyle(\mu,a)} & & {\scriptstyle(\lambda',y)} \\ & {\scriptstyle(\mu',b)} & \end{bmatrix} \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} (\lambda, x) \\ (\nu, c) \end{bmatrix} \mathbf{e}\begin{bmatrix} (\lambda', y) \\ (\nu', d) \end{bmatrix}\right) \\
&= \sum_{\lambda \in \Lambda, x, y \in X} (\delta_{\lambda,\mu} \sigma_{ba}^{yx}(\lambda)_\sharp \otimes 1_M)(\delta_\lambda \otimes \delta_\nu) L_{xc} L_{yd} + I_\sigma \\
&= \sum_{x,y \in X} (\sigma_{ba}^{yx} \otimes 1_M)(\delta_\mu \otimes \delta_\nu) L_{xc} L_{yd} + I_\sigma \\
&= (\delta_\mu \otimes \delta_\nu) \sum_{x,y \in X} (\sigma_{ba}^{yx} \otimes 1_M) L_{xc} L_{yd} + I_\sigma,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{((\lambda,x),(\lambda',y)) \in Q^{(2)}} & \overline{\Phi}\left(1_R \otimes \mathbf{w}\begin{bmatrix} & {\scriptstyle(\nu,c)} & \\ {\scriptstyle(\lambda,x)} & & {\scriptstyle(\nu',d)} \\ & {\scriptstyle(\lambda',y)} & \end{bmatrix} \otimes \mathbf{e}\begin{bmatrix} (\mu, a) \\ (\lambda, x) \end{bmatrix} \mathbf{e}\begin{bmatrix} (\mu', b) \\ (\lambda', y) \end{bmatrix}\right) \\
&= \sum_{\lambda \in \Lambda, x, y \in X} (1_M \otimes \delta_{\nu,\lambda} \sigma_{yx}^{dc}(\nu)_\sharp)(\delta_\mu \otimes \delta_\lambda) L_{ax} L_{by} + I_\sigma \\
&= \sum_{x,y \in X} (1_M \otimes \sigma_{yx}^{dc})(\delta_\mu \otimes \delta_\nu) L_{ax} L_{by} + I_\sigma \\
&= (\delta_\mu \otimes \delta_\nu) \sum_{x,y \in X} (1_M \otimes \sigma_{yx}^{dc}) L_{ax} L_{by} + I_\sigma.
\end{aligned}$$

Hence $\overline{\Phi}$ induces a $\mathbb{K}$-algebra homomorphism $\Phi\colon \mathfrak{A}(w_\sigma) \to A_\sigma$.

Let us prove that the pair of $\mathbb{K}$-algebra homomorphisms (Φ, id_M) satisfies (2.8)–(2.11). We can prove (2.8) as follows:

$$\begin{aligned}
(\Phi \circ s_M^{\mathfrak{A}(w_\sigma)})(f) &= \sum_{\lambda,\mu \in \Lambda} (f(\lambda) \otimes 1_M)(\delta_\lambda \otimes \delta_\mu) + I_\sigma \\
&= \sum_{\lambda \in \Lambda} f(\lambda)\delta_\lambda \otimes 1_M + I_\sigma \\
&= f \otimes 1_M + I_\sigma = s_M^{A_\sigma}(f).
\end{aligned}$$

The proof of (2.9) is similar to that of (2.8). We next prove (2.10). Since $T_{\deg(a)}$ is a $\mathbb{K}$-algebra homomorphism for all $a \in X$, the left-hand side of (2.10) satisfies that

$$\begin{aligned}(\pi_M^{A_\sigma} \circ \Phi)\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\ &= \chi((r_\sharp \otimes r'_\sharp)(\delta_{\lambda_1} \otimes \delta_{\mu_1}) L_{x_1 y_1} \cdots L_{x_m y_m} + I_\sigma)(1_M) \\ &= \delta_{x_1,y_1} \cdots \delta_{x_m,y_m} (rr')_\sharp \delta_{\lambda_1} \delta_{\mu_1}.\end{aligned}$$

For any $i \in \{1, \dots, m-1\}$, we can induce that $\lambda_{i+1} = \lambda_i \deg(x_i)$ and $\mu_{i+1} = \mu_i \deg(y_i)$. This fact implies that

$$\begin{aligned}\pi_M^{\mathfrak{A}(w_\sigma)}\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) &= \delta_{\lambda_1,\mu_1} \delta_{x_1,y_1} \cdots \delta_{x_m,y_m} (rr')_\sharp \delta_{\mu_1} \\ &= \begin{cases} (rr')_\sharp \delta_{\lambda_1}, & p = q; \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

We conclude (2.10) because of the above calculation. Finally, we give the proof of (2.11). Let $\otimes_{\mathbb{K}}$ be the tensor product of $A_\sigma \otimes_{\mathbb{K}} A_\sigma$ and $\otimes_M$ that of $A_\sigma \otimes_M A_\sigma$. We have

$$\begin{aligned}(\Delta_M^{A_\sigma} \circ \Phi)\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\ = \sum_{z_1, \dots, z_m \in X} ((r_\sharp \delta_{\lambda_1} \otimes 1_M) L_{x_1 z_1} \cdots L_{x_m z_m} + I_\sigma) \\ \otimes_M ((1_M \otimes r'_\sharp \delta_{\mu_1}) L_{z_1 y_1} \cdots L_{z_m y_m} + I_\sigma), \\ \big((\Phi \otimes \Phi) \circ \Delta_M^{\mathfrak{A}(w_\sigma)}\big)\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\ = \sum_{\substack{\tau \in \Lambda \\ z_1, \dots, z_m \in X}} ((r_\sharp \delta_{\lambda_1} \otimes \delta_\tau) L_{x_1 z_1} \cdots L_{x_m z_m} + I_\sigma) \\ \otimes_M ((\delta_\tau \otimes r'_\sharp \delta_{\mu_1}) L_{z_1 y_1} \cdots L_{z_m y_m} + I_\sigma).\end{aligned}$$

We can check the following, whose proof is similar to that of (2.4) in Proposition 3.10:

$$\begin{aligned}I_2 \ni (t_M^{A_\sigma}(\delta_\lambda) \otimes_{\mathbb{K}} 1_{A_\sigma} - 1_{A_\sigma} \otimes_{\mathbb{K}} s_M^{A_\sigma}(\delta_\lambda))((1_M \otimes \delta_\lambda + I_\sigma) \otimes_{\mathbb{K}} 1_{A_\sigma}) \\ = \sum_{\substack{\mu \in \Lambda \\ \lambda \neq \mu}} (1_M \otimes \delta_\lambda + I_\sigma) \otimes_{\mathbb{K}} (\delta_\mu \otimes 1_M + I_\sigma)\end{aligned}$$

for any $\lambda \in \Lambda$. Since $A_\sigma \otimes_{\mathbb{K}} A_\sigma / I_2 \cong A_\sigma \otimes_M A_\sigma$ as $\mathbb{K}$-vector spaces, (2.11) is proved. □

We introduce an example of σ satisfying (3.1). This example is not induced by dynamical Yang–Baxter maps.

Example 4.3. Let $\Lambda := \mathbb{Z}/2\mathbb{Z}$ and $X := \mathbb{Z}/2\mathbb{Z}$. We will denote by G the opposite group of the symmetric group on the set Λ. For $a \in \mathbb{Z}/2\mathbb{Z}$, $\deg(a)(\lambda) = a + \lambda$

($\lambda \in \Lambda = \mathbb{Z}/2\mathbb{Z}$). The map $\sigma_i : \Lambda \times X \times X \to X \times X$ ($i = 1, 2$) is defined by the following table:

(λ, a, b)	$\sigma_1(\lambda, a, b)$	$\sigma_2(\lambda, a, b)$	(λ, a, b)	$\sigma_1(\lambda, a, b)$	$\sigma_2(\lambda, a, b)$
$(0,0,0)$	$(0,0)$	$(1,1)$	$(1,0,0)$	$(1,1)$	$(0,0)$
$(0,0,1)$	$(0,1)$	$(1,0)$	$(1,0,1)$	$(1,0)$	$(0,1)$
$(0,1,0)$	$(0,1)$	$(1,0)$	$(1,1,0)$	$(1,0)$	$(0,1)$
$(0,1,1)$	$(0,0)$	$(1,1)$	$(1,1,1)$	$(1,1)$	$(0,0)$

Table 1. The definition of σ_i

We denote by R a $\mathbb{K}$-algebra. For any $i = 1, 2$ and $a, b, c, d \in X$, the map ${}^i\sigma^{ab}_{cd} \in M$ is defined by

$$ {}^i\sigma^{ab}_{cd}(\lambda) = \begin{cases} 1_R, & \sigma_i(\lambda, a, b) = (c, d); \\ 0, & \text{otherwise.} \end{cases} $$

For any $i = 1, 2$, the maps deg and ${}^i\sigma^{ab}_{cd}$ ($a, b, c, d \in X$) satisfy the condition (3.1). Thus we can construct the left bialgebroids A_σ, $\mathfrak{A}(w_\sigma)$, and the left bialgebroid homomorphism Φ for each $i = 1, 2$.

Remark 4.4. (1) This map σ_i ($i = 1, 2$) is not a dynamical Yang–Baxter map because the map $\overline{\sigma}_i \colon \Lambda \times X \times X \ni (\lambda, x, y) \mapsto \sigma_i(\lambda, y, x) \in X \times X$ does not satisfy (2.1) in [19]. Thus this example is a new one that we cannot construct by using the way of [16] and [21].

(2) The family ${}^i\sigma = \{{}^i\sigma^{ab}_{cd}\}_{a,b,c,d \in X}$ in Example 4.3 is rigid. Thus, this A_σ is a Hopf algebroid whose antipode $S : A_\sigma \to A_\sigma$ satisfies $S((L^{-1})_{ab} + I_\sigma) = L_{ab} + I_\sigma$ ($a, b \in X$).

By using a part of the condition of rigidity for w, we can specify a necessary and sufficient condition for Φ to be invertible.

Theorem 4.5. *The following conditions are equivalent:*

1. Φ *is bijective.*
2. *For any* $m \in \mathbb{Z}_{\geq 0}$, p, *and* $q \in Q^{(m)}$, *there exist elements* $X_{p,q} \in \mathfrak{A}(w)$ *satisfying* (3.7)–(3.9).

Proof. We first assume the condition (1). For any $m \in \mathbb{Z}_{\geq 0}$, p, and $q \in Q^{(m)}$, we set

$$ X_{p,q} = \Phi^{-1}((L^{-1})_{x_m y_m} \cdots (L^{-1})_{x_1 y_1}(\delta_{\mathfrak{s}(q)} \otimes \delta_{\mathfrak{s}(p)}) + I_\sigma). $$

Here $p=((\lambda_1,x_1),\ldots,(\lambda_m,x_m))$ and $q=((\mu_1,y_1),\ldots,(\mu_m,y_m))$. Let us show that this $X_{p,q}$ satisfies (3.8). It is easy to show that

$$\sum_{\lambda,\mu\in\Lambda} f(\lambda)\otimes g(\mu)\otimes \mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}=\Phi^{-1}(f\otimes g+I_\sigma),$$
$$1_R\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}p\\ q\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}=\Phi^{-1}((\delta_{\mathfrak{s}(p)}\otimes\delta_{\mathfrak{s}(q)})L_{x_1y_1}\cdots L_{x_my_m}+I_\sigma)$$

for any $f,g\in M$, $m\in\mathbb{Z}_{\geq 0}$, p, and $q\in Q^{(m)}$. Since $\lambda_{i+1}=\lambda_i\deg(x_i)$ and $\mu_{i+1}=\mu_i\deg(y_i)$ for any $i\in\{1,\ldots,m-1\}$,

$$\begin{aligned}
&\sum_{u\in Q^{(m)}} X_{p,u}\Big(1_R\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}u\\ q\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)\\
&\quad=\sum_{\substack{\tau\in\Lambda\\ z_1,\ldots,z_m\in X}}\Phi^{-1}((L^{-1})_{x_mz_m}\cdots(L^{-1})_{x_1z_1}(\delta_\tau\otimes\delta_{\lambda_1}\delta_{\mu_1})L_{z_1y_1}\cdots L_{z_my_m}+I_\sigma)\\
&\quad=\delta_{\lambda_1,\mu_1}\delta_{x_1,y_1}\cdots\delta_{x_m,y_m}\Phi^{-1}(1_M\otimes\delta_{\mathfrak{t}(p)}+I_\sigma)\\
&\quad=\delta_{p,q}t_M(\delta_{\mathfrak{t}(p)}).
\end{aligned}$$

Here we use the generators (2) and (3) in I_σ to induce the second equality. Thus (3.8) is proved. It is similar to show (3.7) and (3.9).

We suppose the condition (2). Let Θ denote the $\mathbb{K}$-algebra homomorphism defined by

$$\Theta\colon M\otimes_{\mathbb{K}} M^{\mathrm{op}}\ni f\otimes g\mapsto \sum_{\lambda,\mu\in\Lambda} f(\lambda)\otimes g(\mu)\otimes\mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\in\mathfrak{A}(w_\sigma).$$

By using this Θ, we can define the $\mathbb{K}$-algebra homomorphism $\overline{\Phi'}\colon\mathbb{K}\langle\Lambda X\rangle\to\mathfrak{A}(w_\sigma)$ as follows:

$$\begin{aligned}
\overline{\Phi'}(\xi)&=\Theta(\xi)\quad(\xi\in M\otimes_{\mathbb{K}} M^{\mathrm{op}});\\
\overline{\Phi'}(L_{ab})&=\sum_{\lambda,\mu\in\Lambda}1_R\otimes 1_R\otimes\mathbf{e}\begin{bmatrix}(\lambda,a)\\ (\mu,b)\end{bmatrix}+\mathfrak{I}_{\mathbf{w}};\\
\overline{\Phi'}((L^{-1})_{ab})&=\sum_{\lambda,\mu\in\Lambda}X_{(\lambda,a),(\mu,b)}\quad(a,b\in X).
\end{aligned}$$

We prove that $\overline{\Phi'}(\mathfrak{I}_{\mathbf{w}})=\{0\}$. It suffices to show that

$$\overline{\Phi'}(\alpha)=0 \tag{4.3}$$

for every generator in I_σ. Because Θ is a $\mathbb{K}$-algebra homomorphism, it is easy to prove (4.3) for any generator (1) in I_σ. We next show (4.3) with respect to the generators (2) in I_σ. For any $a, b \in X$,

$$
\begin{aligned}
&\overline{\Phi'}\Big(\sum_{c\in X} L_{ac}(L^{-1})_{cb}\Big)\\
&= \sum_{\substack{c\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\lambda,a)\\(\mu,c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) X_{(\tau,c),(\nu,b)}\\
&\underset{(3.15)}{=} \sum_{\substack{c\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\lambda,a)\\(\mu,c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) X_{\tau\deg(c),\nu\deg(b)} X_{(\tau,c),(\nu,b)}\\
&\underset{(3.16)}{=} \sum_{\substack{c\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\lambda,a)\\(\mu,c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\nu\deg(b)\\ \tau\deg(c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) X_{(\tau,c),(\nu,b)}\\
&= \sum_{\substack{c\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}} \delta_{\lambda\deg(a),\nu\deg(b)}\delta_{\mu,\tau}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\lambda,a)\\(\mu,c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) X_{(\tau,c),(\nu,b)}\\
&= \sum_{\substack{c\in X\\ \lambda,\mu\in\Lambda}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\lambda,a)\\(\mu,c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) X_{(\mu,c),(\lambda\deg(a)\deg(b)^{-1},b)}\\
&\underset{(3.7)}{=} \delta_{a,b} \sum_{\lambda,\mu\in\Lambda} \delta_{\lambda,\lambda\deg(a)\deg(b)^{-1}} \Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\\
&= \overline{\Phi'}(\delta_{a,b}\emptyset).
\end{aligned}
$$

The proof for $\overline{\Phi'}(\sum_{c\in X}(L^{-1})_{ac}L_{cb} - \delta_{a,b}\emptyset) = 0$ $(a, b \in X)$ is similar. Let us check that any generator as in (3) satisfies (4.3). For all $f \in M$, a, and $b \in X$,

$$
\begin{aligned}
&\overline{\Phi'}((f\otimes 1_M)(L^{-1})_{ab} - (L^{-1})_{ab}(T_{\deg(b)}(f)\otimes 1_M))\\
&\quad= \sum_{\lambda,\mu,\tau,\nu\in\Lambda} \Big(f(\lambda) \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) X_{(\tau,a),(\nu,b)}\\
&\qquad - \sum_{\gamma,\eta,\theta,\kappa\in\Lambda} X_{(\gamma,a),(\eta,b)}\Big(f(\theta\deg(b)) \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\theta\\ \kappa\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\\
&\quad\underset{(3.16)}{=} \sum_{\lambda,\mu,\tau,\nu\in\Lambda} (f(\lambda) \otimes 1_R \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}) X_{\mu,\lambda} X_{(\tau,a),(\nu,b)}\\
&\qquad - \sum_{\gamma,\eta,\theta,\kappa\in\Lambda} (f(\theta\deg(b)) \otimes 1_R \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}) X_{(\gamma,a),(\eta,b)} X_{\kappa,\theta}\\
&\quad\underset{(3.15)}{=} \sum_{\lambda,\mu,\tau,\nu\in\Lambda} \delta_{\tau\deg(a),\mu}\delta_{\nu\deg(b),\lambda}(f(\lambda) \otimes 1_R \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}) X_{(\tau,a),(\nu,b)}\\
&\qquad - \sum_{\gamma,\eta,\theta,\kappa\in\Lambda} \delta_{\gamma,\kappa}\delta_{\eta,\theta}(f(\theta\deg(b)) \otimes 1_R \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}) X_{(\gamma,a),(\eta,b)}
\end{aligned}
$$

$$= \sum_{\tau,\nu\in\Lambda} (f(\nu \deg(b)) \otimes 1_R \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}) X_{(\tau,a),(\nu,b)}$$
$$- \sum_{\gamma,\eta\in\Lambda} (f(\eta \deg(b)) \otimes 1_R \otimes 1_{\mathfrak{G}(Q)} + \mathfrak{I}_{\mathbf{w}}) X_{(\gamma,a),(\eta,b)}$$
$$= 0.$$

It is similar to show that the other three generators as in (3) satisfy (4.3). We will prove that the generators (4) in I_σ satisfy (4.3). By using the relations (3.4) and (4.2), we get

$$\overline{\Phi'}\left(\sum_{x,y\in X} (\sigma^{xy}_{ac} \otimes 1_M) L_{yd}L_{xb} - \sum_{x,y\in X} \left(1_M \otimes \sigma^{bd}_{xy}\right) L_{cy}L_{ax}\right)$$
$$= \sum_{\substack{\lambda.\mu\in\Lambda\\ x,y\in X}} \sigma^{xy}_{ac}(\lambda) \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}((\lambda,y),(\lambda\deg(y),x))\\((\mu,d),(\mu\deg(d),b))\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}$$
$$- \sum_{\substack{\tau.\nu\in\Lambda\\ x,y\in X}} 1_R \otimes \sigma^{bd}_{xy}(\nu) \otimes \mathbf{e}\begin{bmatrix}((\tau,c),(\tau\deg(c),a))\\((\nu,y),(\nu\deg(y),x))\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}$$
$$= \sum_{\substack{\lambda.\mu,\eta\in\Lambda\\ x,y\in X}} \mathbf{w}\begin{bmatrix} & {\scriptstyle(\lambda,y)} & \\ {\scriptstyle(\eta,c)} & & {\scriptstyle(\lambda\deg(y),x)} \\ & {\scriptstyle(\eta\deg(c),a)} & \end{bmatrix} \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}((\lambda,y),(\lambda\deg(y),x))\\((\mu,d),(\mu\deg(d),b))\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}$$
$$- \sum_{\substack{\tau.\nu,\theta\in\Lambda\\ x,y\in X}} 1_R \otimes \mathbf{w}\begin{bmatrix} & {\scriptstyle(\theta,d)} & \\ {\scriptstyle(\nu,y)} & & {\scriptstyle(\theta\deg(d),b)} \\ & {\scriptstyle(\nu\deg(y),x)} & \end{bmatrix} \otimes \mathbf{e}\begin{bmatrix}((\tau,c),(\tau\deg(c),a))\\((\nu,y),(\nu\deg(y),x))\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}$$
$$= 0$$

for all a, b, c, and $d \in X$. Since $1_{\mathfrak{G}(Q)} = \sum_{\lambda,\mu\in\Lambda} \mathbf{e}\left[\begin{smallmatrix}\lambda\\ \mu\end{smallmatrix}\right]$, it is obvious that the generator (5) satisfies (4.3). Thus we can conclude $\overline{\Phi'}(\mathfrak{I}_{\mathbf{w}}) = \{0\}$ and construct a well-defined $\mathbb{K}$-algebra homomorphism $\Phi'(\alpha + I_\sigma) = \overline{\Phi'}(\alpha)$ for all $\alpha \in \mathbb{K}\langle\Lambda X\rangle$. We finally show that this Φ' becomes the inverse of Φ. For any $r, r' \in R$, $m \in \mathbb{Z}_{\geq 0}$, $p = ((\lambda_1, x_1), \ldots, (\lambda_m, x_m))$, and $q = ((\mu_1, y_1), \ldots, (\mu_m, y_m)) \in Q^{(m)}$, we get

$$(\Phi' \circ \Phi)\left(r \otimes r' \otimes \mathbf{e}\begin{bmatrix}p\\q\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\right)$$
$$= \Phi'\left((r_\sharp \otimes r'_\sharp)(\delta_{\mathfrak{s}(p)} \otimes \delta_{\mathfrak{s}(q)}) L_{x_1y_1}\cdots L_{x_my_m} + I_\sigma\right)$$
$$= \sum_{\tau_1,\ldots,\tau_m,\nu_1,\ldots,\nu_m\in\Lambda} (r \otimes r' \otimes 1_{\mathfrak{G}(Q)})\left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\mathfrak{s}(p)\\ \mathfrak{s}(q)\end{bmatrix}\right)$$
$$\times \left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\tau_1,x_1)\\(\nu_1,y_1)\end{bmatrix}\right)\cdots\left(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\tau_m,x_m)\\(\nu_m,y_m)\end{bmatrix}\right) + \mathfrak{I}_{\mathbf{w}}$$
$$= r \otimes r' \otimes \mathbf{e}\begin{bmatrix}p\\q\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}.$$

Let us check that $\Phi \circ \Phi' = \mathrm{id}_{A_\sigma}$. Since the maps Φ and Φ' are $\mathbb{K}$-algebra homomorphisms, it is sufficient to show that $(\Phi \circ \Phi')(\alpha + I_\sigma) = \alpha + I_\sigma$. Here $\alpha \in \mathbb{K}\langle \Lambda X\rangle$ means that

$$\alpha = \begin{cases} f \otimes g \quad (\forall f, g \in M); \\ L_{ab}; \\ (L^{-1})_{ab} \quad (\forall a, b \in X). \end{cases}$$

Since $\sum_{\lambda\in\Lambda} f(\lambda)_\sharp \delta_\lambda = f$ for all $f \in M$,

$$\begin{aligned}(\Phi \circ \Phi')(f \otimes g + I_\sigma) &= \sum_{\lambda,\mu\in\Lambda} \Phi\Big(f(\lambda) \otimes g(\mu) \otimes \mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\ &= \sum_{\lambda,\mu\in\Lambda} (f(\lambda)_\sharp \otimes g(\mu)_\sharp)(\delta_\lambda \otimes \delta_\mu) + I_\sigma \\ &= f \otimes g + I_\sigma\end{aligned}$$

for any $f, g \in M$. Similarly, we can prove that $(\Phi \circ \Phi')(L_{ab} + I_\sigma) = L_{ab} + I_\sigma$ for all $a, b \in X$. Let $\alpha = (L^{-1})_{ab}$. We can calculate that

$$\begin{aligned}
&(\Phi \circ \Phi')((L^{-1})_{ab} + I_\sigma) \\
&\quad = \sum_{\lambda,\mu\in\Lambda} \Phi(X_{(\lambda,a),(\mu,b)}) \\
&\quad \underset{(3.13)}{=} \sum_{\lambda,\mu\in\Lambda} \Phi(X_{(\lambda,a),(\mu,b)} s_M(\delta_\mu)) \\
&\quad = \sum_{\substack{c\in X\\ \lambda,\mu,\tau\in\Lambda}} \delta_{a,c} \Phi\Big(X_{(\lambda,c),(\mu,b)}\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\mu\\ \tau\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big) \\
&\quad = \sum_{\substack{c,d\in X\\ \lambda,\mu\in\Lambda}} ((L^{-1})_{ad} L_{dc} + I_\sigma)\Phi(X_{(\lambda,c),(\mu,b)})(\delta_\mu \otimes 1_M + I_\sigma) \\
&\quad = \sum_{\substack{c,d\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}} ((L^{-1})_{ad} + I_\sigma)\Phi\Big(\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\tau,d)\\ (\nu,c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) X_{(\lambda,c),(\mu,b)}\Big) \\
&\qquad \times (\delta_\mu \otimes 1_M + I_\sigma) \\
&\quad = \sum_{\substack{c,d\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}} ((L^{-1})_{ad} + I_\sigma) \\
&\qquad \times \Phi\Big(\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\tau,d)\\ (\nu,c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\tau \deg(d)\\ \nu \deg(c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) X_{(\lambda,c),(\mu,b)}\Big) \\
&\qquad \times (\delta_\mu \otimes 1_M + I_\sigma) \\
&\underset{(3.15),(3.16)}{=} \sum_{\substack{c,d\in X\\ \lambda,\mu\in\Lambda}} ((L^{-1})_{ad} + I_\sigma)
\end{aligned}$$

$$
\begin{aligned}
&\times \Phi\Big(\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\mu \deg(b)\deg(d)^{-1}, d)\\ (\lambda, c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) X_{(\lambda,c),(\mu,b)}\Big)\\
&\times (\delta_\mu \otimes 1_M + I_\sigma)\\
\underset{(3.7)}{=} &\sum_{\substack{d\in X\\ \lambda\in\Lambda}} \delta_{\lambda \deg(b)\deg(d)^{-1},\lambda}\delta_{b,d}((L^{-1})_{ad} + I_\sigma)\\
&\times \Phi\left(s_M^{\mathfrak{A}(w_\sigma)}(\delta_{\lambda \deg(b)\deg(d)^{-1}})\right)(\delta_\lambda \otimes 1_M + I_\sigma)\\
\underset{\text{Thm. 4.2}}{=} &\sum_{\lambda\in\Lambda} (L^{-1})_{ab}(\delta_\lambda \otimes 1_M) + I_\sigma\\
= &(L^{-1})_{ab} + I_\sigma.
\end{aligned}
$$

Thus we can conclude that Φ' is the inverse of Φ. This completes the proof. □

Corollary 4.6. *w is rigid if and only if Φ is bijective and σ is rigid.*

Proof. Let us suppose that the family w is rigid. It is sufficient to prove that σ is rigid because of Theorem 4.5. For any $a, b \in X$, we define elements x_{ab} and $y_{ab} \in A_\sigma$ as follows:

$$
x_{ab} = \sum_{\lambda,\mu\in\Lambda} (\Phi\circ S)(X_{(\lambda,a),(\mu,b)});
$$
$$
y_{ab} = \sum_{\lambda,\mu\in\Lambda} \Phi(Y_{(\lambda,a),(\mu,b)}).
$$

We give the proof only for (3.3). Since $\sum_{\lambda,\mu\in\Lambda}\Phi(X_{(\lambda,a),(\mu,b)}) = (L^{-1})_{ab} + I_\sigma$ for $a, b \in X$, we can calculate that

$$
\begin{aligned}
&\sum_{c\in X} x_{cb}((L^{-1})_{ac} + I_\sigma)\\
&\quad= \sum_{\substack{c\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}} \Phi(S(X_{(\lambda,c),(\mu,b)})X_{(\tau,a),(\nu,c)})\\
&\quad= \sum_{\substack{c\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}} (\Phi\circ S)\Big(\Big(1_R\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}(\tau,a)\\(\nu,c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)X_{(\lambda,c),(\mu,b)}\Big)\\
&\quad= \sum_{\substack{c\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}} (\Phi\circ S)\Big(\Big(1_R\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}(\tau,a)\\(\nu,c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\\
&\qquad\qquad \times \Big(1_R\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}\tau\deg(a)\\ \nu\deg(c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)X_{(\lambda,c),(\mu,b)}\Big)\\
&\underset{(3.15),(3.16)}{=} \sum_{\substack{c\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}} \delta_{\lambda,\nu}\delta_{\mu\deg(b),\tau\deg(a)}(\Phi\circ S)\Big(\Big(1_R\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}(\tau,a)\\(\nu,c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)X_{(\lambda,c),(\mu,b)}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\substack{c\in X\\ \lambda,\mu\in\Lambda}} (\Phi\circ S)\Big(\Big(1_R\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}(\mu\deg(b)\deg(a)^{-1},a)\\ (\lambda,c)\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)X_{(\lambda,c),(\mu,b)}\Big)\\
&\underset{(3.7)}{=} \delta_{a,b}\sum_{\lambda,\mu\in\Lambda}\delta_{\lambda\deg(b)\deg(a)^{-1},\lambda}(\Phi\circ S)\Big(1_R\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}\lambda\deg(b)\deg(a)^{-1}\\ \mu\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)\\
&= \delta_{a,b}1_{A_\sigma}
\end{aligned}
$$

for all $a,b\in X$. The proofs for the other conditions are similar.

We assume that Φ has the inverse Φ^{-1} and σ is rigid. For any $m\in\mathbb{Z}_{\geq 0}$, $p=((\lambda_1,a_1),\dots,(\lambda_m,a_m))$, and $q=((\mu_1,b_1),\dots,(\mu_m,b_m))\in Q^{(m)}$, the elements $X_{p,q}$ and $Y_{p,q}$ are defined by

$$
\begin{aligned}
X_{p,q} &= \Phi^{-1}((L^{-1})_{a_mb_m}\cdots(L^{-1})_{a_1b_1}(\delta_{\mathfrak{s}(q)}\otimes\delta_{\mathfrak{s}(p)})+I_\sigma);\\
Y_{p,q} &= \Phi^{-1}(y_{a_mb_m}\cdots y_{a_1b_1}(\delta_{\mathfrak{s}(q)}\otimes\delta_{\mathfrak{s}(p)})+I_\sigma).
\end{aligned}
$$

We give the proof only for (3.11). By using the relations (2) and (3) of the generators in I_σ,

$$
\begin{aligned}
&\sum_{u\in Q^{(m)}} Y_{u,q}\Big(1_R\otimes 1_R\otimes\mathbf{e}\begin{bmatrix}p\\ u\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)\\
&= \sum_{\substack{\tau\in\Lambda\\ c_1,\dots,c_m\in X}}\Phi^{-1}(y_{c_mb_m}\cdots y_{c_1b_1}(\delta_{\mu_1}\delta_{\lambda_1}\otimes\delta_\tau)L_{a_1c_1}\cdots L_{a_mc_m}+I_\sigma)\\
&= \delta_{\lambda_1,\mu_1}\sum_{c_1,\dots,c_m\in X}\Phi^{-1}(y_{c_mb_m}\cdots y_{c_1b_1}L_{a_1c_1}\cdots L_{a_mc_m}(\delta_{\lambda_1\deg(a_1)\cdots\deg(a_m)}\otimes 1_M)+I_\sigma)\\
&= \delta_{\lambda_1,\mu_1}\delta_{a_1,b_1}\cdots\delta_{a_m,b_m}\Phi^{-1}(\delta_{\mathfrak{t}(p)}\otimes 1_M+I_\sigma)\\
&= \delta_{p,q}\sum_{\lambda\in\Lambda}1_R\otimes 1_R\otimes\mathbf{e}\begin{bmatrix}\mathfrak{t}(p)\\ \lambda\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\\
&= \delta_{p,q}s_M(\delta_{\mathfrak{t}(p)}).
\end{aligned}
$$

This is the desired conclusion. □

Remark 4.7. This homomorphism Φ is not always injective. Let $R=\mathbb{K}$ and $|\Lambda|=1$. Then M is isomorphic to $\mathbb{K}$ as $\mathbb{K}$-algebras and $G=\{\mathrm{id}_\Lambda\}$. For any $a,b\in X$, let $\alpha_{ab}(\neq 0)\in\mathbb{K}$ be an arbitrary element such that α_{11}^2, α_{22}^2, and $\alpha_{12}\alpha_{21}$ are mutually different. We set $X=\{1,2\}$ and define

$$
\sigma_{cd}^{ab}=\begin{cases}\alpha_{ab}, & c=b \text{ and } d=a;\\ 0, & \text{otherwise}\end{cases}
$$

for all $a,b,c,d\in X$. By using these settings and the relation (4.2), we can construct ordinary bialgebras A_σ and $\mathfrak{A}(w_\sigma)$ over $\mathbb{K}$.

Let $T_{ab} = 1_{\mathbb{K}} \otimes 1_{\mathbb{K}} \otimes \mathbf{e}\begin{bmatrix}(*,a)\\(*,b)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}$ for all $* \in \Lambda$ and $a, b \in X$. By the definition of $\mathfrak{I}_{\mathbf{w}}$,

$$\begin{aligned}
0 &= \sum_{x,y\in X} \mathbf{w}\left[\begin{smallmatrix} & x & \\ a & & y \\ & b & \end{smallmatrix}\right] T_{xc}T_{yd} - \sum_{x,y\in X} \mathbf{w}\left[\begin{smallmatrix} & c & \\ x & & d \\ & y & \end{smallmatrix}\right] T_{ax}T_{by} \\
&= \sum_{x,y\in X} \sigma_{ba}^{yx} T_{xc}T_{yd} - \sum_{x,y\in X} \sigma_{yx}^{dc} T_{ax}T_{by} \\
&= \sigma_{ba}^{ab} T_{bc}T_{ad} - \sigma_{cd}^{dc} T_{ad}T_{bc} \\
&= \alpha_{ab} T_{bc}T_{ad} - \alpha_{dc} T_{ad}T_{bc}
\end{aligned}$$

for any $a, b, c, d \in X$. This equality induces that

$$\begin{aligned}
&\alpha_{aa}T_{aa}T_{ab} - \alpha_{ba}T_{ab}T_{aa} = 0; \\
&\alpha_{aa}T_{ab}T_{aa} - \alpha_{ab}T_{aa}T_{ab} = 0 \quad (\forall a, b \in X,\ a \neq b).
\end{aligned}$$

Thus we can calculate that

$$\begin{aligned}
\alpha_{aa}(\alpha_{aa}T_{aa}T_{ab} - \alpha_{ba}T_{ab}T_{aa}) + \alpha_{ba}(\alpha_{aa}T_{ab}T_{aa} - \alpha_{ab}T_{aa}T_{ab}) \\
= (\alpha_{aa}^2 - \alpha_{ab}\alpha_{ba})T_{aa}T_{ab} \\
= 0.
\end{aligned}$$

Since $\alpha_{aa}^2 - \alpha_{ab}\alpha_{ba} \neq 0$, $T_{aa}T_{ab} = 0$ for any $a, b \in X$ $(a \neq b)$. We can similarly prove that

$$\begin{aligned}
&T_{aa}T_{ba} = T_{ab}T_{aa} = T_{ba}T_{aa} = 0; \\
&T_{ab}^2 = 0 \quad (\forall a, b \in X, a \neq b).
\end{aligned}$$

By using these facts,

$$\begin{aligned}
\Delta_M(T_{11}^2) &= T_{11}^2 \otimes T_{11}^2 + T_{11}T_{12} \otimes T_{11}T_{21} \\
&\quad + T_{12}T_{11} \otimes T_{21}T_{11} + T_{12}^2 \otimes T_{12}^2 \\
&= T_{11}^2 \otimes T_{11}^2.
\end{aligned}$$

In addition, because $\pi_M(T_{11}^2) = 1_{\mathbb{K}}$, we conclude that T_{11}^2 is a group-like element of $\mathfrak{A}(w_\sigma)$. Thus the localization $\mathfrak{A}(w_\sigma)[\langle T_{11}^2\rangle]$ with respect to the monoid $\langle T_{11}^2\rangle$ generated by T_{11}^2 can be constructed. Since $T_{11}^2 T_{21} = 0$, T_{21} is an element in the kernel of the canonical map $\iota\colon \mathfrak{A}(w_\sigma) \to \mathfrak{A}(w_\sigma)[\langle T_{11}^2\rangle]$.

Moreover,

$$1_{A_\sigma} = ((L^{-1})_{11})^2 L_{11}^2 + I_\sigma = L_{11}^2((L^{-1})_{11})^2 + I_\sigma$$

is satisfied because of [21, Lemma 4.2]. Therefore we can conclude that $\Phi(T_{21}) = 0$ and Φ is not injective (see also [11, Remark 8.1]).

5. Relations between A_σ, $\mathfrak{A}(w)$, and Φ

In this section, we show that $\mathfrak{A}(w)$, A_σ, and Φ satisfy a certain universal property in case the $\mathbb{K}$-algebra R is a Frobenius-separable $\mathbb{K}$-algebra. To this end, we characterize weak bialgebras (weak Hopf algebras) by generalizing the notion of the antipode S^{WHA} and Hayashi's antipode f^- in [11, Section 2].

We first recall the convolution product. For an arbitrary $\mathbb{K}$-coalgebra (C,Δ,ε) and $\mathbb{K}$-algebra A, the $\mathbb{K}$-vector space $\mathrm{Hom}_{\mathbb{K}}(C,A)$ becomes a $\mathbb{K}$-algebra by the following multiplication:

$$(f\star g)(c)=f(c_{(1)})g(c_{(2)})\quad (f,g\in \mathrm{Hom}_{\mathbb{K}}(C,A), c\in C);$$
$$1_{\mathrm{Hom}_{\mathbb{K}}(C,A)}(c)=\varepsilon(c)1_A.$$

This multiplication is called the *convolution product*.

Let A be a $\mathbb{K}$-algebra and e^+, e^-, and x^+ elements in A. An element $x^-\in A$ is called an (e^+,e^-)*-generalized inverse of* x^+ if the following conditions are satisfied:

$$x^\pm x^\mp = e^\mp,\quad x^\pm x^\mp x^\pm = x^\pm.$$

We can easily check that the (e^+,e^-)-generalized inverse of x^+ is unique if it exists.

Definition 5.1. Let H be a weak bialgebra, A a $\mathbb{K}$-algebra, and $f^+\colon H\to A$ a $\mathbb{K}$-linear map. A $\mathbb{K}$-linear map $f^-\colon H\to A$ is called an *antipode of* f^+ if f^- is the $(f^+\circ\varepsilon_s, f^+\circ\varepsilon_t)$-generalized inverse of f^+ with regard to the convolution product of $\mathrm{Hom}_{\mathbb{K}}(H,A)$.

The following lemmas are generalizations of [11, Lemma 2.1 and 2.2].

Lemma 5.2. *Let H be a weak bialgebra.*

(1) *This H is a weak Hopf algebra with the antipode S if and only if $S\in \mathrm{End}_{\mathbb{K}}(H)$ is the antipode of* id_H.
(2) *If H' is a weak Hopf algebra with the antipode S and $f^+\colon H\to H'$ is a weak bialgebra homomorphism, then $S\circ f^+$ is the antipode of f^+.*

Proof. We first prove (1). It is clear that H becomes a weak Hopf algebra whose antipode is S if $S\in\mathrm{End}_{\mathbb{K}}(H)$ is the antipode of id_H. Suppose that H is a weak Hopf algebra with the antipode S. We give the proof only for $\mathrm{id}_H\star S\star \mathrm{id}_H=\mathrm{id}_H$. By using (2.20),

$$\begin{aligned}h_{(1)}S(h_{(2)})h_{(3)} &= h_{(1)}\varepsilon_s(h_{(2)})\\ &= h_{(1)}1_{(1)}\varepsilon(h_{(2)}1_{(2)})\\ &= h\end{aligned}$$

for any $h\in H$. Hence the antipode of id_H is $S\in\mathrm{End}_{\mathbb{K}}(H)$.

Let us show (2). Since f^+ is a weak bialgebra homomorphism, we have

$$\begin{aligned}(\varepsilon_t\circ f^+)(h) &= \varepsilon_{H'}(1_{(1)}f^+(h))1_{(2)}\\ &= \varepsilon_{H'}(f^+(1_{(1)}h))f^+(1_{(2)})\\ &= \varepsilon_H(1_{(1)}h)f^+(1_{(2)})\\ &= (f^+\circ\varepsilon_t)(h)\end{aligned}$$

and

$$\begin{aligned}(f^+ \star (S \circ f^+))(h) &= f^+(h)_{(1)} S(f^+(h)_{(2)}) \\ &= (\varepsilon_t \circ f^+)(h) \\ &= (f^+ \circ \varepsilon_t)(h).\end{aligned}$$

Similarly, we can also prove that $\varepsilon_s \circ f^+ = f^+ \circ \varepsilon_s$ and $(S \circ f^+) \star f^+ = f^+ \circ \varepsilon_s$. The identity (2.20) induces

$$\begin{aligned}(f^+ \star (S \circ f^+) \star f^+)(h) &= f^+(h)_{(1)} \varepsilon_s(f^+(h)_{(2)}) \\ &= f^+(h)_{(1)} 1_{(1)} \varepsilon_{H'}(f^+(h)_{(2)} 1_{(2)}) \\ &= f^+(h)\end{aligned}$$

for all $h \in H$. The proof for $(S \circ f^+) \star f^+ \star (S \circ f^+) = S \circ f^+$ is similar. □

Lemma 5.3. *Let H be a weak bialgebra, A a $\mathbb{K}$-algebra, and $f^+ \colon H \to A$ a $\mathbb{K}$-algebra homomorphism.*

1. *If f^+ has the antipode f^-, then $f^- \colon H \to A^{\mathrm{op}}$ is a $\mathbb{K}$-algebra homomorphism.*
2. *In addition to the above situation (1), if A is a weak bialgebra and f^+ is a weak bialgebra homomorphism, then the antipode $f^- \colon H \to A^{\mathrm{bop}}$ is a weak bialgebra homomorphism.*

Proof. Let us first show (1). We define five maps P^+, P_1^-, P_2^-, $\mathcal{E}^+$, and $\mathcal{E}^- \in \mathrm{Hom}_{\mathbb{K}}(H \otimes_{\mathbb{K}} H, A)$ as

$$\begin{aligned}&P^+(g \otimes h) = f^+(g) f^+(h); \\ &P_1^-(g \otimes h) = f^-(gh); \quad P_2^-(g \otimes h) = f^-(h) f^-(g); \\ &\mathcal{E}^+(g \otimes h) = f^+ \circ \varepsilon_s(gh); \quad \mathcal{E}^-(g \otimes h) = f^+ \circ \varepsilon_t(gh) \quad (g, h \in H).\end{aligned}$$

Since the generalized inverse is unique, it suffices to prove that P_1^- and P_2^- are $(\mathcal{E}^+, \mathcal{E}^-)$-generalized inverses of P^+. It is easily seen that P_1^- is an $(\mathcal{E}^+, \mathcal{E}^-)$-generalized inverse of P^+ because the map f^+ has the antipode f^-. For all g and $h \in H$, we can calculate that

$$\begin{aligned}(P^+ \star P_2^-)(g \otimes h) &= f^+(g_{(1)}) f^+(h_{(1)}) f^-(h_{(2)}) f^-(g_{(2)}) \\ &= f^+(g_{(1)} \varepsilon_t(h)) f^-(g_{(2)}) \\ &\underset{(2.20)}{=} f^+(g_{(1)} 1_{(1)} \varepsilon_t(h)) f^-(g_{(2)} 1_{(2)}) \\ &\underset{(2.26)}{=} f^+(g_{(1)} \varepsilon_t(h)_{(1)}) f^-(g_{(2)} \varepsilon_t(h)_{(2)}) \\ &= (f^+ \circ \varepsilon_t)(g \varepsilon_t(h)) \\ &\underset{(2.31)}{=} (f^+ \circ \varepsilon_t)(gh) \\ &= \mathcal{E}^-(g \otimes h),\end{aligned}$$

and

$$\begin{aligned}(P^+ \star P_2^- \star P^+)(g \otimes h) &= f^+(g_{(1)})f^+(h_{(1)})f^-(h_{(2)})f^-(g_{(2)})f^+(g_{(3)})f^+(h_{(3)})\\ &= f^+(g_{(1)})(f^+ \circ \varepsilon_t)(h_{(1)})(f^+ \circ \varepsilon_s)(g_{(2)})f^+(h_{(2)})\\ &\underset{(2.29)}{=} f^+(g_{(1)})(f^+ \circ \varepsilon_s)(g_{(2)})(f^+ \circ \varepsilon_t)(h_{(1)})f^+(h_{(2)})\\ &= f^+(g)f^+(h)\\ &= P^+(g \otimes h).\end{aligned}$$

We can also prove that $P_2^- \star P^+ = \mathcal{E}^+$ and $P_2^- \star P^+ \star P_2^- = P_2^-$ by using (2.20), (2.25), (2.29), and (2.31). Hence f^- is multiplicative. In addition, this f^- preserves the unit. By using (2.21) and (2.24),

$$\begin{aligned}f^-(1_H) &= f^-(1_{(1)})f^+(1_{(2)}1'_{(1)})f^-(1'_{(2)})\\ &= (f^+ \circ \varepsilon_s)(1_H)(f^+ \circ \varepsilon_t)(1_H)\\ &= 1_A.\end{aligned}$$

Therefore $f^-\colon H \to A^{\mathrm{op}}$ is a $\mathbb{K}$-algebra homomorphism.

We next prove (2). For this purpose, we assume the following lemma for the moment (see also [14, Lemmas B1 and B2]).

Lemma 5.4. *Let H and A be a weak bialgebra. If a weak bialgebra homomorphism $f^+\colon H \to A$ has the antipode f^-, the following conditions are satisfied for all $g, h \in H$:*

$$gh_{(1)} \otimes f^-(h_{(2)})f^+(h_{(3)}) = g_{(1)}h_{(1)} \otimes f^-(g_{(2)}h_{(2)})f^+(g_{(3)})f^+(h_{(3)}); \tag{5.1}$$

$$h_{(1)}g \otimes f^+(h_{(2)})f^-(h_{(3)}) = h_{(1)}g_{(1)} \otimes f^+(h_{(2)})f^+(g_{(2)})f^-(h_{(3)}g_{(3)}); \tag{5.2}$$

$$f^-(h_{(1)}) \otimes f^-(h_{(2)}) = f^-(h_{(1)})f^+(h_{(4)})f^-(h_{(5)}) \otimes f^-(h_{(2)})f^+(h_{(3)})f^-(h_{(6)}). \tag{5.3}$$

For the comultiplicativity of f^-, it is equivalent to show that

$$f^-(h_{(2)}) \otimes f^-(h_{(1)}) = f^-(h)_{(1)} \otimes f^-(h)_{(2)}$$

for all $h \in H$. We set

$$\begin{aligned}J &= (f^-(1_{(2)}) \otimes f^-(1_{(1)}))\Delta((f^+ \circ \varepsilon_t)(1_{(3)})),\\ \tilde{J} &= \Delta((f^+ \circ \varepsilon_s)(1_{(1)}))(f^-(1_{(3)}) \otimes f^-(1_{(2)})).\end{aligned}$$

For J and $\tilde{J}$, see [14, Proposition B4]. Let h be an arbitrary element in H. We show the following three formulas:

$$(f^-(h_{(2)}) \otimes f^-(h_{(1)}))J = J(f^-(h)_{(1)} \otimes f^-(h)_{(2)}); \tag{5.4}$$

$$J = \Delta(1_A) = \tilde{J}; \tag{5.5}$$

$$J\tilde{J} = f^-(1_{(2)}) \otimes f^-(1_{(1)}). \tag{5.6}$$

For (5.4), it follows that

$$\begin{aligned}
&(f^-(h_{(2)}) \otimes f^-(h_{(1)}))J \\
&\quad = (f^-(1_{(2)}h_{(2)}) \otimes f^-(1_{(1)}h_{(1)}))\Delta(f^+(1_{(3)})f^-(1_{(4)})) \\
&\quad \underset{(5.2)}{=} (f^-(1_{(2)}h_{(2)}) \otimes f^-(1_{(1)}h_{(1)}))\Delta(f^+(1_{(3)}h_{(3)})f^-(1_{(4)}h_{(4)})) \\
&\quad \underset{(2.20)}{=} (f^-(h_{(2)}) \otimes f^-(h_{(1)}))\Delta(f^+(h_{(3)})f^-(h_{(4)})) \\
&\quad = ((f^+ \circ \varepsilon_s)(h_{(2)}) \otimes f^-(h_{(1)})f^+(h_{(3)}))\Delta(f^-(h_{(4)})) \\
&\quad \underset{(2.27)}{=} ((f^+ \circ \varepsilon_s)(1_{(2)}) \otimes f^-(h_{(1)}1_{(1)})f^+(h_{(2)}))\Delta(f^-(h_{(3)})) \\
&\quad = ((f^+ \circ \varepsilon_s)(1_{(2)}) \otimes f^-(1_{(1)})(f^+ \circ \varepsilon_s)(h_{(1)}))\Delta(f^-(h_{(2)})) \\
&\quad \underset{(2.28)}{=} ((f^+ \circ \varepsilon_s)(1_{(2)}) \otimes f^-(1_{(1)})(f^+ \circ \varepsilon_s)(1'_{(1)}))\Delta(f^-(h1'_{(2)})) \\
&\quad = (f^-(1_{(2)})f^+(1_{(3)}) \otimes f^-(1'_{(1)}1_{(1)})f^+(1'_{(2)}))\Delta(f^-(h1'_{(3)})) \\
&\quad \underset{(5.1)}{=} (f^-(1_{(2)}1'_{(2)})f^+(1_{(3)}1'_{(3)}) \otimes f^-(1_{(1)}1'_{(1)})f^+(1_{(4)}))\Delta(f^-(h1_{(5)})) \\
&\quad \underset{(2.20)}{=} (f^-(1_{(2)}) \otimes f^-(1_{(1)}))\Delta(f^+(1_{(3)}))\Delta(f^-(1_{(4)}))\Delta(f^-(h)) \\
&\quad = J(f^-(h)_{(1)} \otimes f^-(h)_{(2)}).
\end{aligned}$$

Let us prove (5.5):

$$\begin{aligned}
J &= ((f^+ \circ \varepsilon_s)(1_{(2)}) \otimes f^-(1_{(1)})f^+(1_{(3)}))\Delta(f^-(1_{(4)})) \\
&\underset{(2.28)}{=} (f^+(1_{(1)}) \otimes f^-(1'_{(1)})f^+(1'_{(2)}1_{(2)}))\Delta(f^-(1'_{(3)})) \\
&= (f^+(1_{(1)}) \otimes (f^+ \circ \varepsilon_s)(1'_{(1)})f^+(1_{(2)}))\Delta(f^-(1'_{(2)})) \\
&\underset{(2.28)}{=} (f^+(1_{(1)}) \otimes f^+(1'_{(1)}1_{(2)}))\Delta(f^-(1'_{(2)})) \\
&\underset{(2.21)}{=} (f^+(1_{(1)}) \otimes f^+(1_{(2)}))\Delta(f^-(1_{(3)})) \\
&= \Delta((f^+ \circ \varepsilon_t)(1_H)) \\
&\underset{(2.24)}{=} \Delta(1_A).
\end{aligned}$$

Similarly, we can prove that $\tilde{J} = \Delta(1_A)$. For the proof of (5.6), we have

$$\begin{aligned}
J\tilde{J} &= J\Delta(f^-(1_{(1)}))\Delta(f^+(1_{(2)}))(f^-(1_{(4)}) \otimes f^-(1_{(3)})) \\
&= (f^-(1_{(2)}) \otimes f^-(1_{(1)}))J\Delta(f^+(1_{(3)}))(f^-(1_{(5)}) \otimes f^-(1_{(4)})) \\
&= (f^-(1_{(2)}1'_{(2)}) \otimes f^-(1_{(1)}1'_{(1)})) \\
&\qquad \times \Delta(f^+(1_{(3)})f^-(1_{(4)})f^+(1'_{(3)}))(f^-(1'_{(5)}) \otimes f^-(1'_{(4)})) \\
&\underset{(5.2)}{=} (f^-(1_{(2)}1'_{(2)}) \otimes f^-(1_{(1)}1'_{(1)})) \\
&\qquad \times \Delta(f^+(1_{(3)}1'_{(3)})f^-(1_{(4)}1'_{(4)})f^+(1'_{(5)}))(f^-(1'_{(7)}) \otimes f^-(1'_{(6)}))
\end{aligned}$$

$$\underset{(2.20)}{=} (f^-(1_{(2)}) \otimes f^-(1_{(1)}))\Delta(f^+(1_{(3)}))(f^-(1_{(5)}) \otimes f^-(1_{(4)}))$$
$$= f^-(1_{(2)})f^+(1_{(3)})f^-(1_{(6)}) \otimes f^-(1_{(1)})f^+(1_{(4)})f^-(1_{(5)})$$
$$\underset{(5.3)}{=} f^-(1_{(2)}) \otimes f^-(1_{(1)}).$$

It follows from (5.4)–(5.6) that $J = J\tilde{J} = f^-(1_{(2)}) \otimes f^-(1_{(1)})$. Therefore we can calculate that

$$\begin{aligned}(f^-(h)_{(1)} \otimes f^-(h)_{(2)}) &= \Delta(1_A)(f^-(h)_{(1)} \otimes f^-(h)_{(2)}) \\ &= J(f^-(h)_{(1)} \otimes f^-(h)_{(2)}) \\ &= (f^-(h_{(2)}) \otimes f^-(h_{(1)}))J \\ &= (f^-(h_{(2)}) \otimes f^-(h_{(1)}))(f^-(1_{(2)}) \otimes f^-(1_{(1)})) \\ &= (f^-(h_{(2)}) \otimes f^-(h_{(1)}))\end{aligned}$$

for any $h \in H$. Thus $f^-\colon H \to A^{\mathrm{bop}}$ preserves the comultiplication. By using (2.30), we can prove that f^- is counital:

$$\begin{aligned}(\varepsilon_A \circ f^-)(h) &= \varepsilon_A(f^-(h_{(1)})(f^+ \circ \varepsilon_t)(h_{(2)})) \\ &= \varepsilon_A(f^-(h_{(1)})f^+(h_{(2)})) \\ &= \varepsilon_A((\varepsilon_s \circ f^+)(h)) \\ &= (\varepsilon_A \circ f^+)(h) \\ &= \varepsilon_H(h)\end{aligned}$$

for all $h \in H$. This is the desired conclusion. □

Proof of Lemma 5.4. We first prove (5.1). For all $g, h \in H$,

$$\begin{aligned}gh_{(1)} \otimes f^-(h_{(2)})f^+(h_{(3)}) &= gh_{(1)} \otimes (f^+ \circ \varepsilon_s)(h_{(2)}) \\ &= gh1_{(1)} \otimes (f^+ \circ \varepsilon_s)(1_{(2)}) \\ &= g_{(1)}h_{(1)} \otimes (f^+ \circ \varepsilon_s)(g_{(2)}h_{(2)}) \\ &= g_{(1)}h_{(1)} \otimes f^-(g_{(2)}h_{(2)})f^+(g_{(3)})f^+(h_{(3)}).\end{aligned}$$

Here we use the identity (2.27). The proof for (5.2) is similar.

Let us evaluate (5.3). By using (5.1), (5.2), and Lemma 5.3 (1),

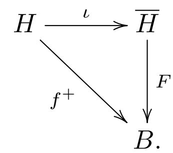
$$\begin{aligned}&f^-(h_{(1)}) \otimes f^-(h_{(2)}) \\ &\quad= f^-(h_{(1)})f^-(1_{(1)})f^+(1_{(2)})f^-(1_{(3)}) \otimes f^-(h_{(2)})f^+(h_{(3)})f^-(h_{(4)}) \\ &\quad= f^-(1_{(1)}h_{(1)})f^+(1_{(4)})f^-(1_{(5)}) \otimes f^-(1_{(2)}h_{(2)})f^+(1_{(3)}h_{(3)})f^-(h_{(4)}) \\ &\quad= f^-(1_{(1)}h_{(1)})f^+(1_{(4)}h_{(4)})f^-(1_{(5)}h_{(5)}) \otimes f^-(1_{(2)}h_{(2)})f^+(1_{(3)}h_{(3)})f^-(h_{(6)}) \\ &\quad= f^-(h_{(1)})f^+(h_{(4)})f^-(h_{(5)}) \otimes f^-(h_{(2)})f^+(h_{(3)})f^-(h_{(6)})\end{aligned}$$

for any $h \in H$. This completes the proof. □

The convolution product and the antipode f^- generalize the notion of the Hopf envelope in [2].

Definition 5.5. Let H be a weak bialgebra, $\overline{H}$ a weak Hopf algebra, and $\iota\colon H \to \overline{H}$ a weak bialgebra homomorphism. A *Hopf closure* of H is a pair as above $(\overline{H}, \iota)$ satisfying the following universal property:

> For any weak bialgebra B and any weak bialgebra homomorphism $f^{+}\colon H \to B$ with the antipode f^{-}, there exists a unique weak bialgebra homomorphism $F\colon \overline{H} \to B$ such that the following diagram is commutative:



We can deduce that $\overline{H}$ is unique up to isomorphism if it exists.

Remark 5.6. (1) In [2], the weak bialgebra B is always a weak Hopf algebra with the antipode S. Thus Definition 5.5 is a generalization of Definition 3.14 in [2] because $S \circ f^{+}$ gives the antipode of $f^{+} \in \mathrm{Hom}_{\mathbb{K}}(H, B)$.

(2) Let H be a face algebra. Hayashi [11] considered the construction of the Hopf closure $\overline{H}$ if H is coquasitriangular and closurable. Then this $\overline{H}$ satisfies Definition 5.5 replaced with the notion of face algebras, that is to say, $f^{+}\colon H \to B$ and $F\colon \overline{H} \to B$ are face algebra homomorphisms (see [11, Theorems 5.1, 8.2, and 8.3]).

Let Λ be a non-empty finite set and X a finite set. For a left bialgebroid A_σ as in Subsection 3.1, we suppose that the $\mathbb{K}$-algebra R is a Frobenius-separable $\mathbb{K}$-algebra with an idempotent Frobenius system $(\psi, e^{(1)} \otimes e^{(2)})$. For the $\mathbb{K}$-algebra M, we define a $\mathbb{K}$-linear map $\Psi\colon M \to \mathbb{K}$ and an element $E^{(1)} \otimes E^{(2)} \in M \otimes_{\mathbb{K}} M$ as follows:

$$\Psi(f) = \sum_{\lambda \in \Lambda} \psi(f(\lambda)) \quad (f \in M);$$
$$E^{(1)} \otimes E^{(2)} = \sum_{\lambda \in \Lambda} e_{\sharp}^{(1)} \delta_\lambda \otimes e_{\sharp}^{(2)} \delta_\lambda.$$

Proposition 5.7. *The pair $(\Psi, E^{(1)} \otimes E^{(2)})$ is an idempotent Frobenius system of M.*

Proof. For any $f \in M$ and $\tau \in \Lambda$,

$$\begin{aligned}
(\Psi(fE^{(1)})E^{(2)})(\tau) &= \sum_{\lambda,\mu \in \Lambda} (\psi((fe_{\sharp}^{(1)}\delta_\lambda)(\mu))e_{\sharp}^{(2)}\delta_\lambda)(\tau) \\
&= \sum_{\lambda \in \Lambda} (\psi(f(\lambda)e^{(1)})e_{\sharp}^{(2)}\delta_\lambda)(\tau) \\
&= \psi(f(\tau)e^{(1)})e^{(2)} \\
&= f(\tau).
\end{aligned}$$

The proof for $E^{(1)}\Psi(E^{(2)}f) = f$ is similar.

On the other hand,

$$\begin{aligned}(E^{(1)}E^{(2)})(\tau) &= \sum_{\lambda\in\Lambda}((e^{(1)}e^{(2)})_\sharp\delta_\lambda)(\tau)\\ &= \sum_{\lambda\in\Lambda}\delta_{\lambda,\tau}e^{(1)}e^{(2)}\\ &= e^{(1)}e^{(2)}\\ &= 1_R\end{aligned}$$

for any $\tau\in\Lambda$.

Hence this $(\Psi, E^{(1)}\otimes E^{(2)})$ is an idempotent Frobenius system of M. ☐

By virtue of Proposition 2.9 and 5.7, the left bialgebroid A_σ has a weak bialgebra structure. The quiver Q defined by (4.1) and elements $\mathbf{w}\left[{}_{(\mu,c)}\overset{(\lambda,a)}{\underset{(\mu',d)}{}}{}_{(\lambda',b)}\right]\in R$ $(((\lambda,a),(\lambda',b)),((\mu,c),(\mu',d))\in Q^{(2)})$ as in (4.2) give birth to a weak bialgebra $\mathfrak{A}(w_\sigma)$ and its homomorphism Φ as in Section 4. The symbols w, $w_{[1]c}$, and $w_{[2]c}$ stand for

$$w=\begin{cases}L_{ab}\\(L^{-1})_{ab}\end{cases}\quad(a,b\in X);$$

$$w_{[1]c}=\begin{cases}L_{ac} & (w=L_{ab});\\(L^{-1})_{cb} & (w=(L^{-1})_{ab});\end{cases}\qquad w_{[2]c}=\begin{cases}L_{cb} & (w=L_{ab});\\(L^{-1})_{ac} & (w=(L^{-1})_{ab})\end{cases}$$

for any $c\in X$. By using these notations, we get the explicit formulas of these weak bialgebras A_σ and $\mathfrak{A}(w_\sigma)$:

$$\Delta_{A_\sigma}((f\otimes g)+I_\sigma)=\sum_{\lambda\in\Lambda}\left(\left(f\otimes e_\sharp^{(1)}\delta_\lambda\right)+I_\sigma\right)\otimes\left(\left(e_\sharp^{(2)}\delta_\lambda\otimes g\right)+I_\sigma\right);$$

$$\Delta_{A_\sigma}(w+I_\sigma)=\sum_{\substack{\lambda\in\Lambda\\c\in X}}\left(\left(1_M\otimes e_\sharp^{(1)}\delta_\lambda\right)w_{[1]c}+I_\sigma\right)\otimes\left(\left(e_\sharp^{(2)}\delta_\lambda\otimes 1_M\right)w_{[2]c}+I_\sigma\right);$$

$$\varepsilon_{A_\sigma}((f\otimes g)w+I_\sigma)=\delta_{a,b}\sum_{\lambda\in\Lambda}\psi((fg)(\lambda))\quad(f,g\in M);$$

$$\Delta_{\mathfrak{A}(w_\sigma)}\left(r\otimes r'\otimes\mathbf{e}\begin{bmatrix}p\\q\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\right)=\sum_{u\in Q^{(m)}}\left(r\otimes e^{(1)}\otimes\mathbf{e}\begin{bmatrix}p\\u\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\right)$$
$$\otimes\left(e^{(2)}\otimes r'\otimes\mathbf{e}\begin{bmatrix}u\\q\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\right);$$

$$\varepsilon_{\mathfrak{A}(w_\sigma)}\left(r\otimes r'\otimes\mathbf{e}\begin{bmatrix}p\\q\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\right)=\delta_{p,q}\psi(rr')\quad(r,r'\in R,\ m\in\mathbb{Z}_{\geq 0},\ p,q\in Q^{(m)}).$$

Theorem 5.8. *If σ is rigid, the pair (A_σ,Φ) satisfies the following universal property:*

> *For any $\mathbb{K}$-algebra A and any $\mathbb{K}$-algebra homomorphism $f^+\colon\mathfrak{A}(w_\sigma)\to A$ with the antipode f^-, there exists a unique $\mathbb{K}$-algebra*

homomorphism $F\colon A_\sigma \to A$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{A}(w_\sigma) & \xrightarrow{\Phi} & A_\sigma \\ & \searrow^{f^+} & \downarrow F \\ & & A. \end{array}$$

If this $\mathbb{K}$*-algebra* A *has a weak bialgebra structure* (A, Δ, ε) *and* f^+ *is a weak bialgebra homomorphism, then so is* F*.*

Proof. We first show the existence of the $\mathbb{K}$-algebra homomorphism F. The $\mathbb{K}$-algebra homomorphism $\Upsilon\colon M \otimes_{\mathbb{K}} M^{\mathrm{op}} \to A$ is defined by

$$\Upsilon(g \otimes h) = \sum_{\lambda,\mu\in\Lambda} f^+\Big(g(\lambda) \otimes h(\mu) \otimes \mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \quad (g, h \in M).$$

By using this Υ, we define the $\mathbb{K}$-algebra homomorphism $\overline{F}\colon \mathbb{K}\langle \Lambda X\rangle \to A$ as follows:

$$\begin{aligned} \overline{F}(\xi) &= \Upsilon(\xi) \quad (\xi \in M \otimes_{\mathbb{K}} M^{\mathrm{op}}); \\ \overline{F}(L_{ab}) &= \sum_{\lambda,\mu\in\Lambda} f^+\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\lambda, a)\\ (\mu, b)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \quad (a, b \in X); \\ \overline{F}((L^{-1})_{ab}) &= \sum_{\lambda,\mu\in\Lambda} f^-\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\lambda, a)\\ (\mu, b)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big). \end{aligned}$$

We prove that $\overline{F}(I_\sigma) = \{0\}$. It suffices to check that

$$\overline{F}(\alpha) = 0 \tag{5.7}$$

for every generator α in I_σ. For any α as in the generators (1), the condition (5.7) is obviously satisfied since the map Υ is a $\mathbb{K}$-algebra homomorphism. We next prove that the generators (2) satisfy (5.7). Let r and r' be arbitrary elements in R. For any $m \in \mathbb{Z}_{\geq 0}$, p, and $q \in Q^{(m)}$, we get

$$\begin{aligned} &\varepsilon_s\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix}p\\ q\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\ &\quad= \sum_{\lambda,\mu,\tau\in\Lambda} \Big(1_R \otimes e^{(1)} \otimes \mathbf{e}\begin{bmatrix}\lambda\\ \tau\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\varepsilon\Big(\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix}p\\ q\end{bmatrix}\Big)\Big(e^{(2)} \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\tau\\ \mu\end{bmatrix}\Big) + \mathfrak{I}_{\mathbf{w}}\Big) \\ &\quad= \sum_{\lambda,\mu,\tau\in\Lambda} \Big(1_R \otimes e^{(1)} \otimes \mathbf{e}\begin{bmatrix}\lambda\\ \tau\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\varepsilon\Big(re^{(2)} \otimes r' \otimes \delta_{\mathfrak{t}(p),\tau}\delta_{\mathfrak{t}(q),\mu}\mathbf{e}\begin{bmatrix}p\\ q\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\ &\quad= \sum_{\lambda\in\Lambda} \Big(1_R \otimes e^{(1)} \otimes \mathbf{e}\begin{bmatrix}\lambda\\ \mathfrak{t}(p)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\varepsilon\Big(re^{(2)} \otimes r' \otimes \mathbf{e}\begin{bmatrix}p\\ q\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\ &\quad= \delta_{p,q} \sum_{\lambda\in\Lambda} \Big(1_R \otimes e^{(1)}\psi\big(re^{(2)}r'\big) \otimes \mathbf{e}\begin{bmatrix}\lambda\\ \mathfrak{t}(p)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big). \end{aligned}$$

We can also prove the following in a similar way:

$$\varepsilon_t\Big(r \otimes r' \otimes \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) = \delta_{p,q} \sum_{\lambda \in \Lambda} rr' \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \mathfrak{s}(q) \\ \lambda \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}.$$

For all $a, b \in X$,

$$\begin{aligned}
&\overline{F}\Big(\sum_{c \in X} (L^{-1})_{ac} L_{cb}\Big) \\
&\underset{(2.24)}{=} \sum_{\substack{c \in X \\ \lambda,\mu,\tau,\nu \in \Lambda}} f^-\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} (\lambda, a) \\ (\mu, c) \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\big(f^+ \circ \varepsilon_s\big)\big(1_{\mathfrak{A}(w_\sigma)}\big) \\
&\qquad \times f^+\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} (\tau, c) \\ (\nu, b) \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&= \sum_{\substack{c \in X \\ \lambda,\mu,\tau,\nu,\gamma,\eta,\kappa \in \Lambda}} f^-\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} (\lambda, a) \\ (\mu, c) \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&\qquad \times f^-\Big(1_R \otimes e^{(1)} \otimes \mathbf{e}\begin{bmatrix} \tau \\ \nu \end{bmatrix} + \mathfrak{I}_w\Big) f^+\Big(e^{(2)} \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \nu \\ \gamma \end{bmatrix} + \mathfrak{I}_w\Big) \\
&\qquad \times f^+\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} (\eta, c) \\ (\kappa, b) \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&\underset{\text{Lem. 5.3 (2)}}{=} \sum_{\substack{c \in X \\ \lambda,\mu,\tau \in \Lambda}} f^-\Big(1_R \otimes e^{(1)} \otimes \mathbf{e}\begin{bmatrix} (\lambda, a) \\ (\mu, c) \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) f^+\Big(e^{(2)} \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} (\mu, c) \\ (\tau, b) \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&= \sum_{\lambda,\mu \in \Lambda} (f^+ \circ \varepsilon_s)\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} (\lambda, a) \\ (\mu, b) \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&= \delta_{a,b} \sum_{\lambda,\mu,\tau \in \Lambda} \delta_{\lambda,\mu} f^+\Big(1_R \otimes e^{(1)}\psi(e^{(2)}) \otimes \mathbf{e}\begin{bmatrix} \tau \\ \lambda \deg(a) \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&= \delta_{a,b} \sum_{\lambda,\mu \in \Lambda} f^+\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \lambda \\ \mu \deg(a) \end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\
&= \overline{F}(\delta_{a,b}\emptyset).
\end{aligned}$$

Therefore $\overline{F}(\sum_{c \in X} (L^{-1})_{ac} L_{cb} - \delta_{a,b}\emptyset) = 0$ is satisfied for any $a, b \in X$. We can prove that $\overline{F}(\sum_{c \in X} L_{ac}(L^{-1})_{cb} - \delta_{a,b}\emptyset) = 0$ for all $a, b \in X$ in a similar way. Let us check that any generator α as in (3) satisfies (5.7). For $g \in M$, a, and $b \in X$,

$$\begin{aligned}
&\overline{F}((T_{\deg(a)}(g) \otimes 1_M)L_{ab} - L_{ab}(g \otimes 1_M)) \\
&\quad = \sum_{\lambda,\mu,\tau,\nu \in \Lambda} f^+\Big(\Big(g(\lambda \deg(a)) \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \lambda \\ \mu \end{bmatrix}\Big)\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} (\tau, a) \\ (\nu, b) \end{bmatrix}\Big) + \mathfrak{I}_{\mathbf{w}}\Big) \\
&\qquad - \sum_{\gamma,\eta,\theta,\kappa \in \Lambda} f^+\Big(\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} (\gamma, a) \\ (\eta, b) \end{bmatrix}\Big)\Big(g(\theta) \otimes 1_R \otimes \mathbf{e}\begin{bmatrix} \theta \\ \kappa \end{bmatrix}\Big) + \mathfrak{I}_{\mathbf{w}}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\lambda,\mu\in\Lambda} f^{+}\Big(g(\lambda\deg(a))\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}(\lambda,a)\\(\mu,b)\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)\\
&\qquad - \sum_{\gamma,\eta\in\Lambda} f^{+}\Big(g(\gamma\deg(a))\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}(\gamma,a)\\(\eta,b)\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)\\
&= 0.
\end{aligned}
$$

The proof of $\overline{F}((1_M\otimes T_{\deg(b)}(g))L_{ab}-L_{ab}(1_M\otimes g))=0$ $(\forall a,b\in X)$ is similar. To complete the proof of the other two generators as in (3), we use the following two identities (for the proof, see the calculation of $\varepsilon_s\Big(r\otimes r'\otimes \mathbf{e}\begin{bmatrix}p\\q\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)$ and $\varepsilon_t\Big(r\otimes r'\otimes \mathbf{e}\begin{bmatrix}p\\q\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)$):

$$
\sum_{\mu\in\Lambda}\varepsilon_s\Big(1_R\otimes r\otimes \mathbf{e}\begin{bmatrix}\mu\\\lambda\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)=\sum_{\mu\in\Lambda}1_R\otimes r\otimes \mathbf{e}\begin{bmatrix}\mu\\\lambda\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}; \tag{5.8}
$$

$$
\sum_{\mu\in\Lambda}\varepsilon_t\Big(r\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}\lambda\\\mu\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)=\sum_{\mu\in\Lambda}r\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}\lambda\\\mu\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\ (\forall r\in R,\lambda\in\Lambda). \tag{5.9}
$$

For any $g\in M$, a, and $b\in X$, we have

$$
\begin{aligned}
&\overline{F}((g\otimes 1_M)(L^{-1})_{ab})\\
&\quad= \sum_{\lambda,\mu,\tau,\nu\in\Lambda} f^{+}\Big(g(\lambda)\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}\lambda\\\mu\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)f^{-}\Big(1_R\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}(\tau,a)\\(\nu,b)\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)\\
&\quad\underset{(5.9)}{=} \sum_{\substack{c\in X\\ \lambda,\mu,\tau,\nu,\gamma\in\Lambda}} (f^{+}\circ\varepsilon_t)\Big(g(\lambda)\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}\lambda\\\mu\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)\\
&\qquad\qquad \times (f^{+}\circ\varepsilon_s)\Big(1_R\otimes e^{(1)}\otimes \mathbf{e}\begin{bmatrix}(\tau,a)\\(\gamma,c)\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)\\
&\qquad\qquad \times f^{-}\Big(e^{(2)}\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}(\gamma,c)\\(\nu,b)\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)\\
&\quad\underset{(2.29)}{=} \sum_{\substack{c\in X\\ \lambda,\mu,\tau,\nu,\gamma\in\Lambda}} (f^{+}\circ\varepsilon_s)\Big(1_R\otimes e^{(1)}\otimes \mathbf{e}\begin{bmatrix}(\tau,a)\\(\gamma,c)\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)\\
&\qquad\qquad \times (f^{+}\circ\varepsilon_t)\Big(g(\lambda)\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}\lambda\\\mu\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)\\
&\qquad\qquad \times f^{-}\Big(e^{(2)}\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}(\gamma,c)\\(\nu,b)\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)\\
&\quad= \sum_{\substack{c,d\in X\\ \lambda,\mu,\tau,\nu,\gamma,\eta,\theta\in\Lambda}} f^{-}\Big(1_R\otimes e^{(1)}\otimes \mathbf{e}\begin{bmatrix}(\lambda,a)\\(\mu,c)\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\times f^{+}\Big(e^{(2)} \otimes e^{(1)'} \otimes \mathbf{e}\begin{bmatrix}(\mu,c)\\(\tau,d)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\\
&\times f^{+}\Big(g(\nu) \otimes e^{(1)''} \otimes \mathbf{e}\begin{bmatrix}\nu\\\gamma\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) f^{-}\Big(e^{(2)''} \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\gamma\\\eta\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\\
&\times f^{-}\Big(e^{(2)'} \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\tau,d)\\(\theta,b)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
\underset{\text{Lem. } 5.3\,(2)}{=} &\sum_{\substack{c,d\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}} f^{-}\Big(1_R \otimes e^{(1)} \otimes \mathbf{e}\begin{bmatrix}(\lambda,a)\\(\mu,c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\\
&\times f^{+}\Big(e^{(2)} g(\mu \deg(c)) \otimes e^{(1)''} e^{(1)'} \otimes \mathbf{e}\begin{bmatrix}(\mu,c)\\(\tau,d)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\\
&\times f^{-}\Big(e^{(2)'} e^{(2)''} \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\tau,d)\\(\nu,b)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\\
= &\sum_{\substack{c,d\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}} f^{-}\Big(1_R \otimes e^{(1)} \otimes \mathbf{e}\begin{bmatrix}(\lambda,a)\\(\mu,c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\\
&\times f^{+}\Big(e^{(2)} g(\mu \deg(c)) \otimes e^{(1)'''} \psi\big(e^{(2)'''} e^{(1)''} e^{(1)'}\big) \otimes \mathbf{e}\begin{bmatrix}(\mu,c)\\(\tau,d)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\\
&\times f^{-}\Big(e^{(2)'} e^{(2)''} \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\tau,d)\\(\nu,b)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\\
= &\sum_{\substack{c,d\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}} f^{-}\Big(1_R \otimes e^{(1)} \otimes \mathbf{e}\begin{bmatrix}(\lambda,a)\\(\mu,c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\\
&\times f^{+}\Big(e^{(2)} g(\mu \deg(c)) \otimes e^{(1)'} \otimes \mathbf{e}\begin{bmatrix}(\mu,c)\\(\tau,d)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\\
&\times f^{-}\Big(e^{(2)'} \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\tau,d)\\(\nu,b)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\\
= &\sum_{\substack{c\in X\\ \lambda,\mu,\tau\in\Lambda}} f^{-}\Big(1_R \otimes e^{(1)} \otimes \mathbf{e}\begin{bmatrix}(\lambda,a)\\(\mu,c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\\
&\times (f^{+} \circ \varepsilon_t)\Big(e^{(2)} g(\mu \deg(c)) \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\mu,c)\\(\tau,b)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\\
= &\sum_{\substack{c\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}} \delta_{\mu,\tau}\delta_{b,c} f^{-}\Big(1_R \otimes e^{(1)} \otimes \mathbf{e}\begin{bmatrix}(\lambda,a)\\(\mu,c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\\
&\times f^{+}\Big(e^{(2)} g(\mu \deg(c)) \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}\tau\\\nu\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\\
= &\sum_{\lambda,\mu,\tau\in\Lambda} f^{-}\Big(1_R \otimes e^{(1)} \otimes \mathbf{e}\begin{bmatrix}(\lambda,a)\\(\mu,b)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad \times f^+\Big(e^{(2)}g(\mu\deg(b))\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}\mu\\ \tau\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)\\
&= \sum_{\lambda,\mu,\tau,\nu\in\Lambda} f^-\Big(1_R\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}(\lambda,a)\\ (\mu,b)\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)(f^+\circ\varepsilon_s)(1_{\mathfrak{A}(w_\sigma)})\\
&\qquad \times f^+\Big(g(\tau\deg(b))\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}\tau\\ \nu\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)\\
&\underset{(2.24)}{=} \overline{F}\big((L^{-1})_{ab}(T_{\deg(b)}(g)\otimes 1_M)\big).
\end{aligned}
$$

That $\overline{F}((1_M\otimes g)(L^{-1})_{ab}-(L^{-1})_{ab}(1_M\otimes T_{\deg(a)}(g)))=0$ $(\forall a,b\in X)$ is also induced by using Lemma 5.3 (2), (2.24), (2.29), and (5.8). We can show (5.7) for any generator α as in (4) similar to the proof of (4.3) in Theorem 4.5. It is easy to prove that the generator (5) satisfies (5.7) because of $1_{\mathfrak{A}(w_\sigma)}=\sum_{\lambda,\mu\in\Lambda}1_R\otimes 1_R\otimes \mathbf{e}\left[\begin{smallmatrix}\lambda\\ \mu\end{smallmatrix}\right]+\mathfrak{I}_{\mathbf{w}}$. Hence the $\mathbb{K}$-algebra homomorphism $F(\alpha+I_\sigma)=\overline{F}(\alpha)$ $(\alpha\in\mathbb{K}\langle\Lambda X\rangle)$ is well defined.

We next show that $f^+=F\circ\Phi$. Since these three maps f^+, F and Φ are $\mathbb{K}$-algebra homomorphisms, it is sufficient to prove that

$$
f^+\Big(r\otimes r'\otimes \mathbf{e}\begin{bmatrix}(\lambda,a)\\ (\mu,b)\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)=(F\circ\Phi)\Big(r\otimes r'\otimes \mathbf{e}\begin{bmatrix}(\lambda,a)\\ (\mu,b)\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)
$$

for all $r,r'\in R$, (λ,a), and $(\mu,b)\in Q$. We can evaluate that

$$
\begin{aligned}
&(F\circ\Phi)\Big(r\otimes r'\otimes \mathbf{e}\begin{bmatrix}(\lambda,a)\\ (\mu,b)\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)\\
&\quad = f^+\Big(r\otimes r'\otimes \mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)\Big(\sum_{\tau,\nu\in\Lambda} f^+\Big(1_R\otimes 1_R\otimes \mathbf{e}\begin{bmatrix}(\tau,a)\\ (\nu,b)\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big)\Big)\\
&\quad = f^+\Big(r\otimes r'\otimes \mathbf{e}\begin{bmatrix}(\lambda,a)\\ (\mu,b)\end{bmatrix}+\mathfrak{I}_{\mathbf{w}}\Big).
\end{aligned}
$$

We give the proof of the uniqueness of F. Let F' be a $\mathbb{K}$-algebra homomorphism such that $f^+=F'\circ\Phi$. We see at once that $F'(g\otimes h+I_\sigma)=F(g\otimes h+I_\sigma)$ and $F'(L_{ab}+I_\sigma)=F(L_{ab}+I_\sigma)$ for all $g,h\in M$, a, and $b\in X$. Let S^{WHA} denote the antipode of the weak Hopf algebra A_σ. We assume the following lemma for the moment.

Lemma 5.9. *We suppose that the family σ is rigid. For any $a,b\in X$,*

$$
\Delta_N(L_{ab}+I_\sigma)=\sum_{c\in X}(L_{ac}+I_\sigma)\otimes_N(L_{cb}+I_\sigma).
$$

For any $\alpha \in A_\sigma$,

$$\begin{aligned}\varepsilon_s(\alpha) &\underset{(2.23)}{=} 1_{(1)}\varepsilon(\alpha 1_{(2)}) \\ &\underset{(2.34)}{=} 1_{(1)}(\Psi \circ \pi_M)(\alpha 1_{(2)}) \\ &\underset{(2.33)}{=} t_M(E^{(1)})(\Psi \circ \pi_M)(\alpha s_M(E^{(2)})) \\ &= (\Psi \circ \pi_M)(\alpha(E^{(2)} \otimes 1_M + I_\sigma))(1_M \otimes E^{(1)} + I_\sigma).\end{aligned}$$

According to Proposition 2.11 and 5.7, Lemma 5.9, and the generators (3), S^{WHA} satisfies that

$$\begin{aligned}&S^{\mathrm{WHA}}(L_{ab} + I_\sigma) \\ &= \sum_{c\in X} \varepsilon_s(L_{ac} + I_\sigma) S^{\mathrm{HAD}}(L_{cb} + I_\sigma) \\ &= \sum_{c\in X} (\Psi \circ \pi_M)(L_{ac}(E^{(2)} \otimes 1_M) + I_\sigma)((1_M \otimes E^{(1)})(L^{-1})_{cb} + I_\sigma) \\ &= \sum_{c\in X} (\Psi \circ \pi_M)((T_{\deg(a)}(E^{(2)}) \otimes 1_M)L_{ac} + I_\sigma)((L^{-1})_{cb}(1_M \otimes T_{\deg(a)}(E^{(1)})) + I_\sigma) \\ &= \sum_{c\in X} \Psi\big((\chi(T_{\deg(a)}(E^{(2)}) \otimes 1_M + I_\sigma)\chi(L_{ac}))(1_M)\big) \\ &\qquad \times \big((L^{-1})_{cb}(1_M \otimes T_{\deg(a)}(E^{(1)})) + I_\sigma\big) \\ &\underset{(3.2)}{=} \sum_{c\in X} \delta_{a,c}\Psi(T_{\deg(a)}(E^{(2)}))((L^{-1})_{cb}(1_M \otimes T_{\deg(a)}(E^{(1)})) + I_\sigma) \\ &= \sum_{\lambda\in\Lambda} (L^{-1})_{ab}(1_M \otimes e_\sharp^{(1)}\delta_{\lambda\deg(a)^{-1}}\Psi(e_\sharp^{(2)}\delta_{\lambda\deg(a)^{-1}})) + I_\sigma \\ &= (L^{-1})_{ab}(1_M \otimes E^{(1)}\Psi(E^{(2)})) + I_\sigma \\ &= (L^{-1})_{ab} + I_\sigma\end{aligned}$$

for any $a, b \in X$. Therefore we compute that

$$\begin{aligned}F'((L^{-1})_{ab} + I_\sigma) &= (F' \circ S^{\mathrm{WHA}})(L_{ab} + I_\sigma) \\ &= \sum_{\lambda,\mu\in\Lambda} \big(F' \circ S^{\mathrm{WHA}} \circ \Phi\big)\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\lambda, a)\\(\mu, b)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big).\end{aligned}$$

Let us prove that the map $\tilde{f} := F' \circ S^{\mathrm{WHA}} \circ \Phi$ is the antipode of f^+. For all $\alpha \in \mathfrak{A}(w_\sigma)$,

$$\begin{aligned}(\tilde{f} \star f^+)(\alpha) &= F'(S^{\mathrm{WHA}}(\Phi(\alpha_{(1)}))\Phi(\alpha_{(2)})) \\ &= F'(S^{\mathrm{WHA}}(\Phi(\alpha)_{(1)})\Phi(\alpha)_{(2)}) \\ &= F'((\varepsilon_s \circ \Phi)(\alpha)) \\ &= (F' \circ \Phi \circ \varepsilon_s)(\alpha) \\ &= (f^+ \circ \varepsilon_s)(\alpha).\end{aligned}$$

The proof for $f^+ \star \tilde{f} = f^+ \circ \varepsilon_t$ and $\tilde{f} \star f^+ \star \tilde{f} = \tilde{f}$ is similar. Thus $\tilde{f}$ is the antipode of f^+. We can deduce that $\tilde{f} = f^-$ because of the uniqueness of the antipode. Hence $F' = F$ is satisfied.

Finally, we show that F is a weak bialgebra homomorphism if A is a weak bialgebra and f^+ is a weak bialgebra homomorphism. Let us prove that F is comultiplicative. Since Δ_{A_σ} and Δ_A satisfy (2.20), it suffices to check that $((F \otimes F) \circ \Delta_{A_\sigma})(\alpha + I_\sigma) = (\Delta_A \circ F)(\alpha + I_\sigma)$. Here,

$$\alpha = \begin{cases} g \otimes h \quad (\forall g, h \in M); \\ L_{ab}; \\ (L^{-1})_{ab} \quad (\forall a, b \in X). \end{cases}$$

If $\alpha = g \otimes h$ $(\forall g, h \in M)$, we get

$$\begin{aligned} &((F \otimes F) \circ \Delta_{A_\sigma})(g \otimes h + I_\sigma) \\ &\quad = \sum_{\lambda,\mu,\tau \in \Lambda} f^+\Big(g(\lambda) \otimes e^{(1)} \otimes \mathbf{e}\begin{bmatrix}\lambda \\ \tau\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes f^+\Big(e^{(2)} \otimes h(\mu) \otimes \mathbf{e}\begin{bmatrix}\tau \\ \mu\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\ &\quad = \sum_{\lambda,\mu \in \Lambda} (\Delta_A \circ f^+)\Big(g(\lambda) \otimes h(\mu) \otimes \mathbf{e}\begin{bmatrix}\lambda \\ \mu\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\ &\quad = (\Delta_A \circ F)(g \otimes h + I_\sigma). \end{aligned}$$

The proof for $\alpha = L_{ab}$ $(\forall a, b \in X)$ is similar. Let us suppose that $\alpha = (L^{-1})_{ab}$ for any a and $b \in X$. Since $f^- \colon \mathfrak{A}(w_\sigma) \to A^{\mathrm{bop}}$ is a weak bialgebra homomorphism (see Lemma 5.3 (2)), we can deduce that

$$\begin{aligned} &((F \otimes F) \circ \Delta_{A_\sigma})((L^{-1})_{ab} + I_\sigma) \\ &\quad = \sum_{\substack{c \in X \\ \lambda,\mu,\tau,\nu \in \Lambda}} \Delta_A(1_A) \\ &\qquad\qquad \times f^-\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\lambda, c) \\ (\mu, b)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes f^-\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\tau, a) \\ (\nu, c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\ &\quad = \sum_{\substack{c \in X \\ \lambda,\mu,\tau,\nu \in \Lambda}} \big((f^- \otimes f^-) \circ \Delta^{\mathrm{op}}_{\mathfrak{A}(w_\sigma)}\big)(1_{\mathfrak{A}(w_\sigma)}) \\ &\qquad\qquad \times f^-\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\lambda, c) \\ (\mu, b)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes f^-\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\tau, a) \\ (\nu, c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\ &\quad = \sum_{\lambda,\mu,\tau \in \Lambda} f^-\Big(e^{(2)} \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\tau, c) \\ (\mu, b)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \otimes f^-\Big(1_R \otimes e^{(1)} \otimes \mathbf{e}\begin{bmatrix}(\lambda, a) \\ (\tau, c)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\ &\quad = \sum_{\lambda,\mu \in \Lambda} (\Delta_A \circ f^-)\Big(1_R \otimes 1_R \otimes \mathbf{e}\begin{bmatrix}(\lambda, a) \\ (\mu, b)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big) \\ &\quad = (\Delta_A \circ F)((L^{-1})_{ab} + I_\sigma). \end{aligned}$$

Hence the map F is comultiplicative. We prove that F preserves the counit. Since the counit satisfies (2.22), it suffices to show that

$$(\varepsilon_A \circ F)((g \otimes h)L_{ab} + I_\sigma) = \varepsilon_{A_\sigma}((g \otimes h)L_{ab} + I_\sigma); \tag{5.10}$$

$$(\varepsilon_A \circ F)((g \otimes h)(L^{-1})_{ab} + I_\sigma) = \varepsilon_{A_\sigma}((g \otimes h)(L^{-1})_{ab} + I_\sigma) \tag{5.11}$$

for any $g, h \in M$, a, and $b \in X$. For (5.10), we can evaluate that

$$\begin{aligned}(\varepsilon_A \circ F)((g \otimes h)L_{ab} + I_\sigma) &= \sum_{\lambda,\mu\in\Lambda} \varepsilon_{\mathfrak{A}(w_\sigma)}\Big(g(\lambda) \otimes h(\mu) \otimes \mathbf{e}\begin{bmatrix}(\lambda, a)\\(\mu, b)\end{bmatrix} + \mathfrak{I}_{\mathbf{w}}\Big)\\ &= \sum_{\lambda,\mu\in\Lambda} \delta_{\lambda,\mu}\delta_{a,b}\psi(g(\lambda)h(\mu))\\ &= \delta_{a,b}\sum_{\lambda\in\Lambda}\psi((gh)(\lambda))\\ &= \varepsilon_{A_\sigma}((g \otimes h)L_{ab} + I_\sigma).\end{aligned}$$

We can prove (5.11) in a similar way. This completes the proof. □

Proof of Lemma 5.9. Let a and b be arbitrary elements in X. Since we can easily prove that $S^{-1}_{A_\sigma\otimes_N A_\sigma}(a \otimes_L b) = S^{-1}(b) \otimes_N S^{-1}(a)$ (see also [21, Theorem 3.9]), it follows that

$$\begin{aligned}\Delta_N(L_{ab} + I_\sigma) &= (S^{-1}_{A\otimes_N A} \circ \Delta_L \circ S)(L_{ab} + I_\sigma)\\ &= \sum_{c\in X} S^{-1}_{A\otimes_N A}((L^{-1})_{cb} + I_\sigma \otimes_M (L^{-1})_{ac} + I_\sigma)\\ &= \sum_{c\in X} L_{ac} + I_\sigma \otimes_N L_{cb} + I_\sigma.\end{aligned}$$

This is the desired conclusion. □

Corollary 5.10. *If σ is rigid, then the pair (A_σ, Φ) is the Hopf closure of $\mathfrak{A}(w_\sigma)$.*

Acknowledgment

The author is deeply grateful to Professor Youichi Shibukawa for his helpful advice.

References

[1] R. J. Baxter, Partition function of the eight-vertex lattice model, *Ann. Physics* **70** (1972), 193–228. DOI MR Zbl

[2] S. Bennoun and H. Pfeiffer, Weak bialgebras of fractions, *J. Algebra* **385** (2013), 145–163. DOI MR Zbl

[3] G. Böhm, F. Nill, and K. Szlachányi, Weak Hopf algebras. I. Integral theory and C^*-structure, *J. Algebra* **221** no. 2 (1999), 385–438. DOI MR Zbl

[4] G. Böhm and K. Szlachányi, Hopf algebroids with bijective antipodes: axioms, integrals, and duals, *J. Algebra* **274** no. 2 (2004), 708–750. DOI MR Zbl

[5] V. G. Drinfel′d, On some unsolved problems in quantum group theory, in *Quantum groups (Leningrad, 1990)*, Lecture Notes in Math. 1510, Springer, Berlin, 1992, pp. 1–8. DOI MR Zbl

[6] P. Etingof, T. Schedler, and A. Soloviev, Set-theoretical solutions to the quantum Yang-Baxter equation, *Duke Math. J.* **100** no. 2 (1999), 169–209. DOI MR Zbl

[7] P. Etingof and A. Varchenko, Solutions of the quantum dynamical Yang-Baxter equation and dynamical quantum groups, *Comm. Math. Phys.* **196** no. 3 (1998), 591–640. DOI MR Zbl

[8] L. D. Faddeev, N. Y. Reshetikhin, and L. A. Takhtajan, Quantization of Lie groups and Lie algebras, in *Algebraic analysis, Vol. I*, Academic Press, Boston, MA, 1988, pp. 129–139. DOI MR Zbl

[9] J.-L. Gervais and A. Neveu, Novel triangle relation and absence of tachyons in Liouville string field theory, *Nuclear Phys. B* **238** no. 1 (1984), 125–141. DOI MR

[10] T. Hayashi, Quantum group symmetry of partition functions of IRF models and its application to Jones' index theory, *Comm. Math. Phys.* **157** no. 2 (1993), 331–345. MR Zbl Available at http://projecteuclid.org/euclid.cmp/1104253942.

[11] T. Hayashi, Quantum groups and quantum semigroups, *J. Algebra* **204** no. 1 (1998), 225–254. DOI MR Zbl

[12] D. K. Matsumoto and K. Shimizu, Quiver-theoretical approach to dynamical Yang-Baxter maps, *J. Algebra* **507** (2018), 47–80. DOI MR Zbl

[13] J. B. McGuire, Study of exactly soluble one-dimensional N-body problems, *J. Math. Phys.* **5** (1964), 622–636. DOI MR Zbl

[14] F. Nill, *Axioms for weak bialgebras*, 1998. arXiv:math/9805104 [math.QA].

[15] Y. Otsuto, *Two constructions of Hopf algebroids based on the FRT construction and their relations*, Ph.D. thesis, Hokkaido University, 2022. DOI

[16] Y. Otsuto and Y. Shibukawa, FRT construction of Hopf algebroids, *Toyama Math. J.* **42** (2021), 51–72. MR Zbl

[17] P. Schauenburg, Duals and doubles of quantum groupoids ($\times_R$-Hopf algebras), in *New trends in Hopf algebra theory (La Falda, 1999)*, Contemp. Math. 267, Amer. Math. Soc., Providence, RI, 2000, pp. 273–299. DOI MR Zbl

[18] P. Schauenburg, Weak Hopf algebras and quantum groupoids, in *Noncommutative geometry and quantum groups (Warsaw, 2001)*, Banach Center Publ. 61, Polish Acad. Sci. Inst. Math., Warsaw, 2003, pp. 171–188. DOI MR Zbl

[19] Y. Shibukawa, Dynamical Yang-Baxter maps, *Int. Math. Res. Not.* no. 36 (2005), 2199–2221. DOI MR Zbl

[20] Y. Shibukawa, Dynamical Yang-Baxter maps with an invariance condition, *Publ. Res. Inst. Math. Sci.* **43** no. 4 (2007), 1157–1182. DOI MR Zbl

[21] Y. Shibukawa, Hopf algebroids and rigid tensor categories associated with dynamical Yang-Baxter maps, *J. Algebra* **449** (2016), 408–445. DOI MR Zbl

[22] Y. Shibukawa and M. Takeuchi, FRT construction for dynamical Yang-Baxter maps, *J. Algebra* **323** no. 6 (2010), 1698–1728. DOI MR Zbl

[23] M. Takeuchi, Groups of algebras over $A \otimes \overline{A}$, *J. Math. Soc. Japan* **29** no. 3 (1977), 459–492. DOI MR Zbl

[24] A. Veselov, Yang-Baxter maps: dynamical point of view, in *Combinatorial aspect of integrable systems*, MSJ Mem. 17, Math. Soc. Japan, Tokyo, 2007, pp. 145–167. DOI MR Zbl

[25] C. N. Yang, Some exact results for the many-body problem in one dimension with repulsive delta-function interaction, *Phys. Rev. Lett.* **19** (1967), 1312–1315. DOI MR Zbl

Yudai Otsuto
Independent researcher, Japan
yotsutom@gmail.com

Received: December 10, 2020
Accepted: May 24, 2022