REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 65, No. 1, 2023, Pages 33–46
Published online: March 10, 2023
https://doi.org/10.33044/revuma.2305

ON THE ZEROS OF UNIVARIATE E-POLYNOMIALS

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Abstract. We consider two problems concerning real zeros of univariate E-polynomials. First, we prove an explicit upper bound for the absolute values of the zeroes of an E-polynomial defined by polynomials with integer coefficients that improves the bounds known up to now. On the other hand, we extend the classical Budan–Fourier theorem for real polynomials to E-polynomials. This result gives, in particular, an upper bound for the number of real zeroes of an E-polynomial. We show this bound is sharp for particular families of these functions, which proves that a conjecture by D. Richardson is false.

1. Introduction

In the '70s, Khovanskii introduced a class of real analytic functions, called Pfaffian functions, that includes polynomials, exponentials, logarithms, and trigonometric functions in bounded intervals, among others (see [7]). A fundamental result proved by Khovanskii (see [8]) states that a system of n equations given by Pfaffian functions in n variables defined over a domain $\mathcal{U} \subset \mathbb{R}^n$ has a finite number of non-degenerate zeros in $\mathcal{U}$ and that the number of these zeros can be bounded explicitly in terms of parameters associated to the system.

Among the most elementary Pfaffian functions (besides the polynomials) we can find the E-polynomials, which are functions of the kind $F(x, e^{h(x)})$ with $F \in \mathbb{R}[X, Y]$ and $h \in \mathbb{R}[X]$. The interest in these functions goes back to Tarski's foundational work [13], where he posed the decidability problem for the first order theory of the reals extended with exponentiation. Although E-polynomials may seem simple functions, there are fundamental questions concerning them that have not been completely answered. In this paper we deal with two of these questions.

First, we consider the so-called *'last root' problem* for E-polynomials. This problem was posed in [14] as follows: *Consider non-zero real E-polynomials of "bounded complexity". Is there an intelligible function of the parameters on which*

2020 *Mathematics Subject Classification.* 14P15, 30C15, 26C10.

Key words and phrases. E-polynomials, problem of the last root, number of zeros, Budan–Fourier theorem.

Partially supported by the following grants: PIP 11220130100527CO (CONICET) and UBACYT 20020190100116BA (Universidad de Buenos Aires).

the E-polynomial depends, which bounds the absolute value of its real roots? A positive answer to this question was given in [15], where the existence of such a bound was proved for general exponential terms. However, even though the bound is deduced by an inductive argument with a computable number of iterations, it is not an explicit function. Later, in [9], the author presented an algorithm computing upper bounds for the absolute values of the real zeros of E-polynomials of the type $F(x, e^x)$, but no complexity estimate is given for the algorithm and, therefore, these bounds cannot be computed explicitly as a function of syntactic parameters associated with F. More generally, in [10], an algorithm for the same task is given for functions of the kind $F(x, \text{trans}(x))$ with $\text{trans}(x) = e^x$, $\ln(x)$ or $\arctan(x)$, motivated by its application to the design of numerical algorithms that approximate zeros of these functions. In this case, when $\text{trans}(x) = e^x$, further computations following the algorithm would enable one to deduce an a priori bound. The first explicit upper bound for the absolute value of the real zeros of an E-polynomial $f(x) = F(x, e^{h(x)})$ defined from polynomials F and h with integer coefficients was given recently in [1].

Here, in Section 3, we prove a new explicit upper bound for the absolute value of the real zeros of an E-polynomial defined by polynomials with integer coefficients in terms of the degrees and the heights of the polynomials involved, which improves the previous bounds. We also exhibit families of examples showing that the dependence of the bound on the parameters considered is unavoidable.

The second question about E-polynomials we consider in this paper concerns bounds for the number of real zeros of an E-polynomial. For real univariate polynomials, classical results such as the Descartes rule of signs or, more generally, the Budan–Fourier theorem ([4, 6]; see also [3, Theorem 2.46]) enable us to obtain upper bounds for the number of real zeros of the polynomial in an interval by simply counting the number of variations in sign of finite sequences of real numbers. In Section 4, we use the notion of pseudo-derivative of an E-polynomial already introduced in [10] to generalize the classical Budan–Fourier theorem to the context of E-polynomials. Another generalization of this theorem can be found in [5].

As a consequence of our generalization of the Budan–Fourier theorem, we deduce that an upper bound for the number of real zeros of an E-polynomial counting multiplicities can be obtained by simply considering the degrees of the polynomials involved in its definition and the signs of certain coefficients. Finally, for the particular case of E-polynomials of the form $f(x) = F(x, e^x)$, we prove a sharp upper bound for their number of zeros. In Section 5, we deduce from our previous arguments that if $\deg_X(F) = n$ and $\deg_Y(F) = m$, then $f(x) = F(x, e^x)$ has at most $N = (n+1)(m+1)-1$ real roots and we show the existence of E-polynomials of this kind with N real roots. This also provides a negative answer to a conjecture on the number of roots of an E-polynomial raised in [12].

2. Preliminaries

A real univariate *E-polynomial* (which we will simply call an E-polynomial) is a function of the type $f(x) = F(x, e^{h(x)})$, where $F \in \mathbb{R}[X, Y]$ and $h \in \mathbb{R}[X]$. In this case, we will say that f is the E-polynomial defined from the polynomials $F \in \mathbb{R}[X,Y]$ and $h \in \mathbb{R}[X]$.

For a polynomial $F \in \mathbb{R}[X,Y]$, we will write $\deg(F)$ for its total degree, and $\deg_Y(F)$ (respectively, $\deg_X(F)$) for its degree as a polynomial in $\mathbb{R}[X][Y]$ (respectively, in $\mathbb{R}[Y][X]$). We will use the same notation $\deg(h)$ for the degree of a univariate polynomial $h \in \mathbb{R}[X]$.

Also, for a (univariate or bivariate) polynomial G with *integer coefficients*, we define its *height*, which we will denote $H(G)$, as the maximum of the absolute values of its coefficients.

We will use the classical Cauchy bound for the size of roots of univariate polynomials (see, for instance, [11, Corollary 2.5.22]):

Lemma 2.1. *If $p(X) = \sum_{i=0}^n p_i X^i \in \mathbb{R}[X]$ is a polynomial of degree n and α is a root of p, then $|\alpha| < 1 + \max\left\{\frac{|p_i|}{|p_n|} : 0 \le i \le n-1\right\}$. In particular, for $p \in \mathbb{Z}[X]$, we have that $|\alpha| < 1 + H(p)$ for every root α of p.*

3. On the problem of the last root

In this section, we show a new upper bound for the absolute values of all the real zeros of an E-polynomial defined by polynomials with integer coefficients. The bound is given in terms of the degrees and heights of the polynomials involved.

Theorem 3.1. *Let $f(x) = F(x, e^{h(x)})$ be an E-polynomial defined by polynomials $F \in \mathbb{Z}[X,Y]$ and $h \in \mathbb{Z}[X]$ such that $\deg_X(F) = n$, $\deg_Y(F) = m \ge 1$, and $\deg(h) = \delta \ge 1$. Let H and $T \in \mathbb{Z}$ be upper bounds for the heights of F and h respectively. If $\alpha \in \mathbb{R}$ is such that $f(\alpha) = 0$, then*

$$|\alpha| < \max\left\{3H,\ 4T+1, \left(\frac{8n}{\delta}\ln(n)\right)^{1/\delta}\right\}.$$

Before proving the theorem, we state some auxiliary bounds for polynomials.

Lemma 3.2. *Let $p(X) = \sum_{i=0}^n c_i X^i \in \mathbb{Z}[X]$ be a polynomial of degree $n \ge 1$ and height at most Λ. Then:*

(a) *for every $\alpha \in \mathbb{C}$ such that $|\alpha| \ge 2$, we have that $|p(\alpha)| \le 2\Lambda|\alpha|^n - 1$;*
(b) *if $k \ge 1$, every $\alpha \in \mathbb{C}$ such that $|\alpha| \ge k\Lambda + 1$ satisfies $|p(\alpha)| > \left(1 - \frac{1}{k}\right)|\alpha|^n$.*

Proof. (a) If $|\alpha| \ge 2$, then

$$|p(\alpha)| \le \sum_{i=0}^n |c_i||\alpha|^i \le \Lambda|\alpha|^n + \Lambda\sum_{i=0}^{n-1}|\alpha|^i = \Lambda\left(|\alpha|^n + \frac{|\alpha|^n - 1}{|\alpha| - 1}\right)$$
$$\le \Lambda\left(|\alpha|^n + |\alpha|^n - 1\right) \le 2\Lambda|\alpha|^n - 1.$$

(b) For every $\alpha \in \mathbb{C}$ such that $|\alpha| \ne 1$, we have that

$$|p(\alpha)| \ge |c_n||\alpha|^n - \sum_{i=0}^{n-1}|c_i||\alpha|^i \ge |\alpha|^n - \Lambda\sum_{i=0}^{n-1}|\alpha|^i = |\alpha|^n - \Lambda\left(\frac{|\alpha|^n - 1}{|\alpha| - 1}\right).$$

If $|\alpha| \geq k\Lambda + 1$, then $\dfrac{\Lambda}{|\alpha|-1} \leq \dfrac{1}{k}$, and so

$$|p(\alpha)| \geq |\alpha|^n - \frac{1}{k}(|\alpha|^n - 1) > \Big(1 - \frac{1}{k}\Big)|\alpha|^n.$$

□

Now we can prove Theorem 3.1:

Proof. Assume $F(X,Y) = \sum_{i=0}^m a_i(X)\, Y^i \in \mathbb{Z}[X,Y]$, and $h(X) = \sum_{k=0}^{\delta} h_k X^k \in \mathbb{Z}[X]$.

If $\deg_X(F) = 0$, for $\alpha \in \mathbb{R}$ such that $f(\alpha) = 0$, Lindemann's theorem implies that $h(\alpha) = 0$, and therefore, using a well-known bound for the size of zeroes of polynomials (see, for instance, Lemma 2.1), we have that

$$|\alpha| < 1 + \max\left\{\left|\frac{h_i}{h_\delta}\right| : 0 \leq i \leq \delta - 1\right\} \leq 1 + T$$

and the theorem is true.

If $\deg_X(F) \geq 1$, we may assume without loss of generality that $a_0(X) \not\equiv 0$, because $e^{h(x)} \neq 0$ for all $x \in \mathbb{R}$. For every $0 \leq i \leq m$ such that $a_i(X) \not\equiv 0$, let $d_i := \deg(a_i)$.

Let $\alpha \in \mathbb{R}$ be such that $|\alpha| \geq \max\Big\{3H,\ 4T+1, \Big(\dfrac{8n}{\delta}\ln(n)\Big)^{1/\delta}\Big\}$.

As $|\alpha| \geq 3H \geq 2H + 1 \geq 2$, for every i such that $d_i \geq 1$:

- from Lemma 3.2 (a), it follows that $|a_i(\alpha)| \leq 2H|\alpha|^{d_i} - 1 \leq 2H\,|\alpha|^n - 1$;
- from Lemma 3.2 (b) applied for $k = 2$, it follows that $|a_i(\alpha)| > \frac{1}{2}\,|\alpha|^{d_i} \geq 1$;

that is,

$$1 \leq |a_i(\alpha)| \leq 2H\,|\alpha|^n - 1.$$

Note that these bounds also hold when $d_i = 0$, because in this case $1 \leq |a_i(\alpha)| \leq H \leq 2H\,|\alpha|^n - 1$.

Then, using again Lemma 2.1 for the polynomial $F(\alpha, Y) \in \mathbb{R}[Y]$, we conclude that, if $F(\alpha,\beta) = 0$, then

$$|\beta| < 1 + \max_i\left\{\frac{|a_i(\alpha)|}{|a_m(\alpha)|}\right\} \leq 1 + \max_i\{2H\,|\alpha|^n - 1\} = 2H|\alpha|^n.$$

Note that if $F(\alpha,\beta) = 0$, then $\beta \neq 0$ since $a_0(\alpha) \neq 0$. In order to get a lower bound on $|\beta|$, we consider the polynomial $F^*(Y) := Y^m F(\alpha, 1/Y)$, which satisfies $F^*(1/\beta) = 0$. Since the coefficients of F^* are those of $F(\alpha, Y)$ in reverse order, the previous upper bound also holds for the roots of F^*; therefore, $|\beta| > (2H|\alpha|^n)^{-1}$.

In this way, we obtain that, if $|\alpha| \geq 3H$, for every zero $\beta \in \mathbb{R}$ of $F(\alpha, Y)$ the following inequalities hold:

$$(2H|\alpha|^n)^{-1} < |\beta| < 2H|\alpha|^n. \tag{3.1}$$

We will now show that if $|\alpha| \geq \max\Big\{3H,\ 4T+1, \big(\frac{8n}{\delta}\ln(n)\big)^{1/\delta}\Big\}$, then $\beta = e^{h(\alpha)}$ does not verify one of the previous inequalities and, therefore, α is not a zero of $f(x) = F(x, e^{h(x)})$.

Due to Lemma 3.2 (b) applied to the polynomial h with $k = 4$, we have that

$$|h(\alpha)| > \frac{3}{4}|\alpha|^\delta$$

for every $\alpha \in \mathbb{R}$ such that $|\alpha| \geq 4T + 1$, and so

$$e^{h(\alpha)} > e^{\frac{3}{4}|\alpha|^\delta} \quad \text{if } h(\alpha) > 0 \qquad \text{and} \qquad e^{h(\alpha)} < e^{-\frac{3}{4}|\alpha|^\delta} \quad \text{if } h(\alpha) < 0. \tag{3.2}$$

Then, because of (3.1) and (3.2), it suffices to show that $e^{\frac{3}{4}|\alpha|^\delta} \geq 2H|\alpha|^n$ or, equivalently, that

$$\frac{3}{4}|\alpha|^\delta \geq \ln(2H) + n\ln(|\alpha|). \tag{3.3}$$

Let us note first that, if $|\alpha| \geq 3H$, then

$$\frac{1}{4}|\alpha|^\delta \geq \frac{1}{4}|\alpha| \geq \frac{3}{4}H \geq \ln(2H). \tag{3.4}$$

On the other hand, if $n \geq 2$, for $|\alpha| \geq \left(\frac{8n}{\delta}\ln(n)\right)^{1/\delta}$, we have that

$$\frac{1}{2}|\alpha|^\delta > n\ln(|\alpha|), \tag{3.5}$$

since $c(t) = \frac{1}{2}t^\delta - n\ln(t)$ is a strictly increasing function in $\left(\left(\frac{2n}{\delta}\right)^{1/\delta}; +\infty\right)$ and

$$c\Big(\Big(\frac{8n}{\delta}\ln(n)\Big)^{1/\delta}\Big) = \frac{4n}{\delta}\ln(n) - \frac{n}{\delta}\ln\Big(\frac{8n}{\delta}\ln(n)\Big) = \frac{n}{\delta}\ln\Big(\frac{n^3\delta}{8\ln(n)}\Big) > 0$$

(note that $n^3\delta \geq n^3 > 8\ln(n)$ for $n \geq 2$). If $n = 1$ and $|\alpha| \geq 2$, we have that $\frac{1}{2}|\alpha| > \ln(|\alpha|)$; then, inequality (3.5) also holds in this case. Combining (3.4) and (3.5), we obtain (3.3). □

Example 3.3. Using the notation in Theorem 3.1, the following simple examples show that any bound for the absolute value of a zero of an E-polynomial must depend on H, T, and n.

(1) Let $f(x) = (x - H)e^x + x - H$. Then $F(X, Y) = (X - H)Y + X - H$ and $h(X) = X$. A zero of f is $\alpha = H$.
(2) Let $f(x) = e^{x-T} - 1$. Then $F(X, Y) = Y - 1$ and $h(X) = X - T$. A zero of f is $\alpha = T$.
(3) Let $f(x) = x^n e^{-x} - 1$ with $n \geq 3$. Then $F(X, Y) = X^nY - 1$ and $h(X) = -X$. As $f(n\ln(n)) = \ln^n(n) - 1 > 0$ for $n \geq 3$, and $\lim_{x\to+\infty} f(x) = -1 < 0$, we deduce that f has a zero $\alpha > n\ln(n)$.

4. Budan–Fourier theorem for E-polynomials

In this section, we will generalize the classical Budan–Fourier theorem for polynomials (see for example, [3, Theorem 2.46]) to the family of E-polynomials. This result provides an upper bound for the number of zeros of a polynomial in a real interval by counting the number of variations in sign of suitable sequences of real numbers.

For $\gamma = (\gamma_0, \dots, \gamma_D) \in \mathbb{R}^{D+1}$ with $\gamma_i \neq 0$ for every $0 \leq i \leq D$, the *number of variations in sign* of γ is the cardinality of the set $\{1 \leq i \leq D \mid \gamma_{i-1}\gamma_i < 0\}$, and for a tuple γ of arbitrary real numbers, the number of variations in sign of γ is the number of variations in sign of the tuple which is obtained from γ by removing its zero coordinates.

We start by introducing some definitions and notation in our setting of E-polynomials.

Recall that, for $c \in \mathbb{R}$ and a non-zero analytic function f, the multiplicity of c as a zero of f, which we will denote with $\operatorname{mult}(c, f)$, is defined as $\operatorname{mult}(c, f) = \min\{\mu \in \mathbb{Z}_{\geq 0} \mid f^{(\mu)}(c) \neq 0\}$.

For an E-polynomial $f(x) = F(x, e^{h(x)})$ with $F \in \mathbb{R}[X, Y]$ and $h \in \mathbb{R}[X]$, the derivative of f is the E-polynomial $f'(x) = \widetilde{F}(x, e^{h(x)})$, where

$$\widetilde{F}(X, Y) = \frac{\partial F}{\partial X}(X, Y) + h'(X)Y\frac{\partial F}{\partial Y}(X, Y).$$

Note that $\deg_Y(\widetilde{F}) = \deg_Y(F)$ and $\deg_X(\widetilde{F}) \leq \deg_X(F) + \deg(h) - 1$.

The sequence of successive derivatives $(f^{(i)})_{i \in \mathbb{Z}_{\geq 0}}$ of an arbitrary E-polynomial is not finite; so, in order to establish a result similar to the Budan–Fourier theorem in this context we will consider an alternative construction which keeps the main properties of the derivation.

Definition 4.1 (see [10] and [2])**.** Let $f(x) = F(x, e^{h(x)})$ with $F \in \mathbb{R}[X, Y]$, $F \neq 0$. The *pseudo-degree* of f is defined as

$$\operatorname{pdeg}(f) = \begin{cases} (\deg_Y(F), \deg_X(F(X, 0))) & \text{if } F(X, 0) \neq 0; \\ (\deg_Y(F), 0) & \text{if } F(X, 0) = 0. \end{cases}$$

We define the *pseudo-derivative* of f as

$$\operatorname{pder}(f)(x) = \begin{cases} e^{-kh(x)} f'(x) & \text{if } f'(x) \neq 0,\ Y^k \mid \widetilde{F}(X, Y), \text{ and } Y^{k+1} \nmid \widetilde{F}(X, Y); \\ 0 & \text{if } f'(x) = 0. \end{cases}$$

For an E-polynomial f, we write $\operatorname{pder}^{(0)}(f) = f$ and, for every $i \in \mathbb{N}$, $\operatorname{pder}^{(i)}(f)$ for the ith successive pseudo-derivative of f, that is, $\operatorname{pder}^{(i)}(f) = \operatorname{pder}(\operatorname{pder}^{(i-1)}(f))$.

Lemma 4.2. *Let f be an E-polynomial.*

(1) *If $f \notin \mathbb{R}$, $\operatorname{pdeg}(\operatorname{pder}(f)) <_{\text{lex}} \operatorname{pdeg}(f)$, where $<_{\text{lex}}$ denotes the lexicographic order.*
(2) *For every $x \in \mathbb{R}$, $\operatorname{sg}(\operatorname{pder}(f)(x)) = \operatorname{sg}(f'(x))$.*
(3) *For every $c \in \mathbb{R}$ such that $f(c) = 0$, $\operatorname{mult}(c, \operatorname{pder}(f)) = \operatorname{mult}(c, f) - 1$. Moreover, for every $c \in \mathbb{R}$, $\operatorname{mult}(c, f) = \min\{\mu \in \mathbb{Z}_{\geq 0} \mid \operatorname{pder}^{(\mu)}(f)(c) \neq 0\}$.*

Proof. (1) It follows straightforwardly from the definitions of pder and pdeg.
(2) The equality of signs is a consequence of the fact that, if $f' \neq 0$, then $\operatorname{pder}(f)(x) = e^{-kh(x)} f'(x)$ for a certain $k \in \mathbb{Z}_{\geq 0}$.

(3) If $f(c) = 0$, we have that $\operatorname{mult}(c, \operatorname{pder}(f)) = \operatorname{mult}(c, f') = \operatorname{mult}(f, c) - 1$, where the first equality follows from the fact that pder and f' differ by an exponential factor.

To prove the second statement, we proceed by induction on the multiplicity $\nu \geq 0$. If $c \in \mathbb{R}$ has multiplicity $\nu = 0$ as a zero of f, we have that $\operatorname{pder}^{(0)}(f)(c) = f(c) \neq 0$, and the equality holds.

Assume $\nu \geq 1$ and let $c \in \mathbb{R}$ be a zero of multiplicity ν of f. Then $f(c) = 0$ and $\operatorname{mult}(c, \operatorname{pder}(f)) = \operatorname{mult}(f, c) - 1 = \nu - 1$. By the inductive assumption applied to $\operatorname{pder}(f)$, we have that $\min\{\mu \in \mathbb{Z}_{\geq 0} \mid \operatorname{pder}^{(\mu)}(\operatorname{pder}(f))(c) \neq 0\} = \nu - 1$. Since $\operatorname{pder}^{(0)}(f) = f$ and $\operatorname{pder}^{(\mu)}(f) = \operatorname{pder}^{(\mu-1)}(\operatorname{pder}(f))$ for $\mu \geq 1$, it follows that $\min\{\mu \in \mathbb{Z}_{\geq 0} \mid \operatorname{pder}^{(\mu)}(f)(c) \neq 0\} = \nu$.

□

By Lemma 4.2 (1), we have that $\{\operatorname{pder}^{(i)}(f) \mid i \in \mathbb{Z}_{\geq 0}\}$ is a finite set, since $<_{\text{lex}}$ in $(\mathbb{Z}_{\geq 0})^2$ is a well-ordering. Let $D = \min\{i \mid \operatorname{pder}^{(i+1)}(f) = 0\}$.

Notation 4.3. For an E-polynomial f, we denote $\operatorname{PDer}(f) = (\operatorname{pder}^{(i)}(f))_{0 \leq i \leq D}$. Given $a, b \in \mathbb{R}$, $a < b$, we write $V(\operatorname{PDer}(f), a, b)$ for the number of variations in sign of the sequence $(\operatorname{pder}^{(0)}(f)(a), \dots, \operatorname{pder}^{(D)}(f)(a))$ minus the number of variations in sign of the sequence $(\operatorname{pder}^{(0)}(f)(b), \dots, \operatorname{pder}^{(D)}(f)(b))$.

Lemma 4.4. *Let $f(x) = F(x, e^{h(x)})$ be a non-constant E-polynomial defined by $F \in \mathbb{R}[X, Y]$ and $h \in \mathbb{R}[X]$. Let $I = [a, b]$ and $c \in (a, b)$ be such that the functions* $\operatorname{pder}^{(i)}$ *do not vanish in I except possibly at c for $i = 0, \dots, D$. Then $V\big(\operatorname{PDer}(f), c, b\big) = 0$ and $V\big(\operatorname{PDer}(f), a, c\big) - \operatorname{mult}(c, f)$ is a non-negative even integer.*

Proof. We will prove it by induction on $\operatorname{pdeg}(f) = (m, n_0)$, considering the lexicographic order.

If $m = 0$, f is a univariate polynomial and in this case the result is true (see [3, Theorem 2.46]).

Let f be an E-polynomial such that $\operatorname{pdeg}(f) = (m, n_0)$, with $m > 0$, and assume the result holds for every E-polynomial with pseudo-degree smaller than (m, n_0). Let $\mu := \operatorname{mult}(c, f) \geq 0$. As f is non-constant, $\operatorname{pdeg}(\operatorname{pder}(f)) <_{\text{lex}} \operatorname{pdeg}(f)$. Then, by the inductive hypothesis applied to $\operatorname{pder}(f)$, if $\mu' := \operatorname{mult}(c, \operatorname{pder}(f)) \geq 0$, we have that

- $V\left(\operatorname{PDer}(\operatorname{pder}(f)), c, b\right) = 0$,
- $V\big(\operatorname{PDer}(\operatorname{pder}(f)), a, c\big) - \mu' = 2j$ for some $j \in \mathbb{Z}_{\geq 0}$.

We will now relate the number of variations in sign in $\operatorname{PDer}(f)$ with those in $\operatorname{PDer}(\operatorname{pder}(f))$, taking into account that $\operatorname{PDer}(f) = (f, \operatorname{PDer}(\operatorname{pder}(f)))$ and the following facts:

(a) For $i = 0, \dots, D$, the sign of $\operatorname{pder}^{(i)}(f)$ is constant and non-zero in each of the intervals $[a, c)$ and $(c, b]$. In particular, if $\operatorname{pder}^{(i)}(f)(c) \neq 0$ then $\operatorname{sg}\big(\operatorname{pder}^{(i)}(f)\big)$ is constant and non-zero in $[a, b]$.

(b) If $\mathrm{pder}^{(i)}(f)(c)=0$, for $i<D$, then:

$$\begin{cases} \mathrm{sg}\left(\mathrm{pder}^{(i)}(f)\right) = -\,\mathrm{sg}\left(\mathrm{pder}^{(i+1)}(f)\right) & \text{in } [a,c), \\ \mathrm{sg}\left(\mathrm{pder}^{(i)}(f)\right) = \mathrm{sg}\left(\mathrm{pder}^{(i+1)}(f)\right) & \text{in } (c,b]. \end{cases}$$

This follows from the fact that $\mathrm{sg}\left(\mathrm{pder}^{(i+1)}(f)\right)$ is constant and non-zero in $[a,c)$ and in $(c,b]$. Since $\mathrm{pder}^{(i+1)}(f)$ has the same sign as the derivative of $\mathrm{pder}^{(i)}(f)$ (see Lemma 4.2 (2)), this implies that $\mathrm{pder}^{(i)}(f)$ is strictly increasing (resp., decreasing) in $[a,c)$ and in $(c,b]$ if $\mathrm{pder}^{(i+1)}(f)$ is positive (resp., negative) in those intervals.

Assume first that $f(c)=0$. Then, $\mu\geq 1$ and $\mu'=\mu-1$. Using (b) for $i=0$, we have that $\mathrm{sg}(f(a))=-\,\mathrm{sg}(\mathrm{pder}(f)(a))$ and $\mathrm{sg}(f(b))=\mathrm{sg}(\mathrm{pder}(f)(b))$, that is, for $\sigma_1,\sigma_2\in\{1,-1\}$, the signs of f and $\mathrm{pder}(f)$ at a,c and b are as follows:

	a	c	b
f	$-\sigma_1$	0	σ_2
$\mathrm{pder}(f)$	σ_1		σ_2

Therefore $V\big(\mathrm{PDer}(f),c,b\big)=V\big(\mathrm{PDer}(\mathrm{pder}(f)),c,b\big)=0$ and $V\big(\mathrm{PDer}(f),a,c\big)=1+V\big(\mathrm{PDer}(\mathrm{pder}(f)),a,c\big)$, and so $V\big(\mathrm{PDer}(f),a,c\big)-\mu=1+\mu'+2j-\mu=2j$, as we wanted to prove.

Now, if $f(c)\neq 0$ (that is, $\mu=0$), f has a non-zero constant sign σ_0 in $[a,b]$. We consider separately the cases when $\mathrm{pder}(f)(c)\neq 0$ and $\mathrm{pder}(f)(c)=0$:

- If $\mathrm{pder}(f)(c)\neq 0$ (that is, $\mu'=0$), then $\mathrm{pder}(f)$ has a non-zero constant sign σ_1 in $[a,b]$. Then:

	a	c	b
f	σ_0	σ_0	σ_0
$\mathrm{pder}(f)$	σ_1	σ_1	σ_1

  and we conclude that $V\big(\mathrm{PDer}(f),c,b\big)=V\big(\mathrm{PDer}(\mathrm{pder}(f)),c,b\big)=0$ and $V\big(\mathrm{PDer}(f),a,c\big)=V\big(\mathrm{PDer}(\mathrm{pder}(f)),a,c\big)=2j+\mu'\geq 0$ is an even integer.
- If $\mu'\geq 1$, $\mathrm{pder}^{(i)}(f)(c)=0$ for $i=1,\dots,\mu'$ (see Lemma 4.2 (3)). Remark (a) implies that $\mathrm{pder}^{(\mu'+1)}(f)$ has a non-zero constant sign σ_1 in $[a,b]$. In addition, according to (b), we have that $\mathrm{sg}\left(\mathrm{pder}^{(i)}(f)(b)\right)=\mathrm{sg}\left(\mathrm{pder}^{(\mu'+1)}(f)(b)\right)$ for every $i=1,\dots,\mu'$, and that the signs of $\mathrm{pder}^{(i)}(f)(a)$ alternate for $i=1,\dots,\mu'+1$.
  - If μ' is odd, it follows that $\mathrm{sg}\left(\mathrm{pder}^{(\mu'+1)}(f)(a)\right)=-\,\mathrm{sg}\left(\mathrm{pder}(f)(a)\right)$. Summarizing:

	a	c	b
f	σ_0	σ_0	σ_0
$\mathrm{pder}(f)$	$-\sigma_1$	0	σ_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathrm{pder}^{(\mu')}(f)$	$-\sigma_1$	0	σ_1
$\mathrm{pder}^{(\mu'+1)}(f)$	σ_1	σ_1	σ_1

Thus, if $\sigma_1 = \sigma_0$, $V\big(\mathrm{PDer}(f), c, b\big) = V\big(\mathrm{PDer}(\mathrm{pder}(f)), c, b\big) = 0$ and $V\big(\mathrm{PDer}(f), a, c\big) = V\big(\mathrm{PDer}(\mathrm{pder}(f)), a, c\big) + 1 = 2j + \mu' + 1 \geq 0$ is an even integer. On the other hand, if $\sigma_1 = -\sigma_0$, then $V\big(\mathrm{PDer}(f), c, b\big) = 1 - 1 + V\big(\mathrm{PDer}(\mathrm{pder}(f)), c, b\big) = 0$ and $V\big(\mathrm{PDer}(f), a, c\big) = -1 + V\big(\mathrm{PDer}(\mathrm{pder}(f)), a, c\big) = -1 + 2j + \mu' \geq 0$ is an even integer.

– If μ' is even, we have that $\mathrm{sg}\left(\mathrm{pder}^{(\mu'+1)}(f)(a)\right) = \mathrm{sg}\left(\mathrm{pder}(f)(a)\right)$, and so the situation is as follows:

	a	c	b
f	σ_0	σ_0	σ_0
$\mathrm{pder}(f)$	σ_1	0	σ_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathrm{pder}^{(\mu')}(f)$	$-\sigma_1$	0	σ_1
$\mathrm{pder}^{(\mu'+1)}(f)$	σ_1	σ_1	σ_1

Then, $V\big(\mathrm{PDer}(f), c, b\big) = V\big(\mathrm{PDer}(\mathrm{pder}(f)), c, b\big) = 0$ and $V\big(\mathrm{PDer}(f), a, c\big) = V\big(\mathrm{PDer}(\mathrm{pder}(f)), a, c\big) = 2j + \mu' \geq 0$ is an even integer.

$\square$

Now we can generalize the Budan–Fourier theorem to E-polynomials:

Theorem 4.5. *Let f be an E-polynomial, $I = (a, b]$ an interval, and $N \geq 0$ the number of zeros of f in I counted with multiplicities. Then $V(\mathrm{PDer}(f), a, b) - N$ is a non-negative even integer.*

Proof. Suppose $c_1 < \cdots < c_r$ are all the zeros in (a, b) of the functions in $\mathrm{PDer}(f)$. Let $c_0 = a$, $c_{r+1} = b$ and $\mu_i = \mathrm{mult}(c_i, f)$ for $i = 1, \ldots, r+1$. Let $d_i \in (c_i, c_{i+1})$ for each $i = 0, \ldots, r$. By Lemma 4.4, for every $i = 0, \ldots, r$, $V(\mathrm{PDer}(f), c_i, d_i) = 0$ and $V(\mathrm{PDer}(f), d_i, c_{i+1}) - \mu_{i+1} = 2j_i \geq 0$ for a non-negative integer j_i. As $\{x \in (c_i, d_i] \mid f(x) = 0\} = \emptyset$ and μ_{i+1} is the number of zeros of f in $(d_i, c_{i+1}]$

counted with multiplicities, for every $i = 0, \dots, r$, we conclude that

$$N = \sum_{i=0}^{r} \mu_{i+1} = \sum_{i=0}^{r} V(\mathrm{PDer}(f), c_i, d_i) + V(\mathrm{PDer}(f), d_i, c_{i+1}) - 2j_i$$
$$= V(\mathrm{PDer}(f), a, b) - 2j$$

for a non-negative integer j. □

For an E-polynomial $f(x) = F(x, e^{h(x)})$ defined by polynomials $F \in \mathbb{Z}[X,Y]$ and $h \in \mathbb{Z}[X]$ with *integer* coefficients, Theorem 3.1 provides us with an explicit bounded interval containing all the roots of $f(x)$ and so, applying the previous result to this interval, we can obtain an upper bound on the number of real zeros of f.

Furthermore, in the general case, since every E-polynomial has finitely many zeros, we may define the sign of an E-polynomial at $+\infty$ as the sign it takes when evaluated at a sufficiently large real number and, similarly, we may define its sign at $-\infty$. Using these definitions, Theorem 4.5 also holds for $a = -\infty$ or $b = +\infty$.

We point out that the signs of an E-polynomial $f(x) = F(x, e^{h(x)})$ at $-\infty$ or $+\infty$ can be easily determined from the signs of coefficients of the polynomials F and h. Without loss of generality, assume $F(X,Y) = \sum_{j=0}^{m} a_j(X)Y^j$ with $a_m \neq 0$ and $a_0 \neq 0$, and $h(X) = \sum_{i=0}^{\delta} h_i X^i$; then,

$$\mathrm{sg}(f, +\infty) = \begin{cases} \mathrm{sg}(\mathrm{lc}(a_m)) & \text{if } h_\delta > 0, \\ \mathrm{sg}(\mathrm{lc}(a_0)) & \text{if } h_\delta < 0. \end{cases}$$

and

$$\mathrm{sg}(f, -\infty) = \begin{cases} (-1)^{\deg(a_m)} \mathrm{sg}(\mathrm{lc}(a_m)) & \text{if } (-1)^\delta h_\delta > 0, \\ (-1)^{\deg(a_0)} \mathrm{sg}(\mathrm{lc}(a_0)) & \text{if } (-1)^\delta h_\delta < 0. \end{cases}$$

Thus, Theorem 4.5 enables us to obtain an upper bound for the number of zeros of an E-polynomial in the spirit of Descartes' rule of signs.

Example 4.6. Let $f(x) = (6x-1)e^{2x} - (8x+1)e^x - 1$. The family of pseudo-derivatives of f is $\mathrm{PDer}(f) = (\mathrm{pder}^{(i)}(f))_{0 \le i \le 4}$, where

$$\begin{aligned} \mathrm{pder}^{(0)}(f)(x) &= (6x-1)e^{2x} - (8x+1)e^x - 1, \\ \mathrm{pder}^{(1)}(f)(x) &= (12x+4)e^x - (8x+9), \\ \mathrm{pder}^{(2)}(f)(x) &= (12x+16)e^x - 8, \\ \mathrm{pder}^{(3)}(f)(x) &= 12x + 28, \\ \mathrm{pder}^{(4)}(f)(x) &= 12. \end{aligned}$$

The lists of signs of $\mathrm{PDer}(f)$ at $-\infty$ and $+\infty$ are $(-1, 1, -1, -1, 1)$ and $(1, 1, 1, 1, 1)$, respectively. Then, $V(\mathrm{PDer}(f), -\infty, +\infty) = 3$, which is the actual number of real zeros of f.

Theorem 4.5 implies, in particular:

Corollary 4.7. *The number of real zeros of a non-zero E-polynomial f (counting multiplicities) is at most $D = \min\{i \in \mathbb{Z}_{\geq 0} \mid \mathrm{pder}^{(i+1)}(f) = 0\}$.*

Since the length of the sequence of non-zero pseudo-derivatives of an E-polynomial can be estimated in terms of the degrees of the polynomials defining it, we deduce an explicit upper bound for the number of real zeros of an E-polynomial: if $f(x) = F(x, e^{h(x)})$, with $F \neq 0$ and $\deg(F) = d$, following the proof of [2, Lemma 28], we can obtain $D \leq \sum_{k=1}^{d}(k+1)\delta^k$. For $\delta \geq 2$, this bound is not sharp, since upper bounds depending polynomially on the degrees are known (see [8], [1, Corollary 17]). On the other hand, for $\delta = 1$, the bound $\frac{(d+1)(d+2)}{2} - 1$ we obtain for the *total* number of real zeros of an E-polynomial improves both Khovanskii's upper bound for the number of *non-degenerate* roots and [1, Corollary 17].

5. On the number of zeros of a class of E-polynomials

In [12], the question about a sharp bound for the number of zeros of an E-polynomial of the form $f(x) = F(x, e^x)$ with $F(X,Y) \in \mathbb{R}[X,Y]$ is posed and the author conjectures that, if $\deg_X(F) = n$ and $\deg_Y(F) = m$, an upper bound for the number of zeros is $n + m$. This section is devoted to the analysis of this conjecture, both for E-polynomials defined from polynomials with real coefficients and for the sub-class of those defined from polynomials with integer coefficients.

First recall that, with the previous notation, $\mathrm{pdeg}(f) = (m, n_0)$ with $n_0 \leq n$ and, assuming $n, m \in \mathbb{N}$, we have that $\mathrm{pdeg}(\mathrm{pder}(f)) = (m', n_0')$ where, either $m' = m$ and $n_0' \leq n-1$ or $m' \leq m-1$ and $n_0' \leq n$. Then, we can estimate the length of the sequence $\mathrm{PDer}(f) = (\mathrm{pder}^{(i)}(f))_{0 \leq i \leq D}$ in terms of the degrees n and m as follows: $D + 1 \leq (n+1)(m+1)$. By Corollary 4.7, we deduce:

Proposition 5.1. *Let $f(x) = F(x, e^x)$, where $F(X,Y) \in \mathbb{R}[X,Y]$ is a non-zero polynomial with $\deg_X(F) = n$ and $\deg_Y(F) = m$. Then, f has at most $N := (n+1)(m+1) - 1$ real zeros counting multiplicities.*

We will now prove that the bound in the previous proposition can be attained, thus providing a negative answer to the conjecture in [12], not only for arbitrary E-polynomials but also for the subclass of E-polynomials defined from polynomials with integer coefficients. Our result also shows that Khovanskii's upper bound on the number of non-degenerate roots for Pfaffian functions is sharp for the considered family of E-polynomials.

Proposition 5.2. *Let $n, m \in \mathbb{N}$. There exists a non-zero polynomial $F \in \mathbb{Z}[X,Y]$ with $\deg_X(F) = n$ and $\deg_Y(F) = m$ such that the E-polynomial $f(x) = F(x, e^x)$ has $N = (n+1)(m+1) - 1$ different non-degenerate real zeros.*

Proof. We start by proving the existence of a non-zero polynomial $G \in \mathbb{R}[X,Y]$ with $\deg_X(G) \leq n$ and $\deg_Y(G) \leq m$ such that the associated E-polynomial $g(x) = G(x, e^x)$ has $N = (n+1)(m+1) - 1$ different real zeros. From the upper bound in Proposition 5.1 for the number of real zeros of g counting multiplicities, it follows that $\deg_X(G) = n$, $\deg_Y(G) = m$, and all the real zeros of g are non-degenerate.

Then, we approximate the polynomial $G \in \mathbb{R}[X,Y]$ by a polynomial $\widehat{G} \in \mathbb{Q}[X,Y]$ with $\deg_X(\widehat{G}) = n$ and $\deg_Y(\widehat{G}) = m$ so that the E-polynomial defined by $\widehat{G}$ also has N different (non-degenerate) real zeros.

Let $\mathcal{G}$ be the polynomial in the variables $\boldsymbol{A} = \big(A_{kl}\big)_{0\le k\le n,\ 0\le l\le m}, X, Y$ defined by

$$\mathcal{G}(\boldsymbol{A}, X, Y) = \sum_{0\le k\le n,\ 0\le l\le m} A_{kl} X^k Y^l.$$

Choose N different real numbers $x_1 < \cdots < x_N$, and consider $P_1 = (x_1, e^{x_1}), \ldots$, $P_N = (x_N, e^{x_N})$ in $\mathbb{R}^2$. The polynomial

$$G(X,Y) = \sum_{0\le k\le n,\ 0\le l\le m} a_{kl} X^k Y^l$$

is non-zero and vanishes at the points $P_1, \ldots, P_N$ if and only if the vector of its coefficients $\boldsymbol{a} := (a_{kl})_{0\le k\le n,\ 0\le l\le m}$ is a non-trivial solution of the following system of linear equations in the unknowns $\boldsymbol{A}$:

$$\begin{cases} \mathcal{G}(\boldsymbol{A}, x_1, y_1) = 0 \\ \qquad\qquad\vdots \\ \mathcal{G}(\boldsymbol{A}, x_N, y_N) = 0. \end{cases}$$

It is clear that a non-trivial solution to this system of N equations exists since the number of unknowns is $(n+1)(m+1) = N+1$. Let $g(x) = G(x, e^x)$ be the E-polynomial obtained from a given non-trivial solution $\boldsymbol{a} \in \mathbb{R}^{N+1}$.

Now, in order to show that there exists a non-zero polynomial $\widehat{G}$ with $\deg_X(\widehat{G}) = n$, $\deg_Y(\widehat{G}) = m$ and *rational* coefficients that defines an E-polynomial with the same number of real zeros as g, proceed in the following way.

For $i = 1, \ldots, N$, consider mutually disjoint intervals $I_i = [\alpha_i, \beta_i]$ such that $\alpha_i < x_i < \beta_i$ and g' has no zero in I_i; in particular, as g is a continuous function, $g(\alpha_i) g(\beta_i) < 0$. Let

$$I = [\alpha_1, \beta_N],\ \varepsilon = \min_{1\le i\le N} \{|g(\alpha_i)|, |g(\beta_i)|\} > 0, \text{ and } M = \max\{1, |\alpha_1|, |\beta_N|\}.$$

For every $0 \le k \le n$, $0 \le l \le m$, let $b_{kl} \in \mathbb{Q} \setminus \{0\}$ be such that $|b_{kl} - a_{kl}| < \frac{\varepsilon}{2M^n e^{mM}(n+1)(m+1)}$, and

$$\widehat{G}(X,Y) := \sum_{0\le k\le n,\ 0\le l\le m} b_{kl} X^k Y^l \in \mathbb{Q}[X,Y].$$

For every $x \in I$,

$$\begin{aligned} \left|\widehat{G}(x, e^x) - G(x, e^x)\right| &\le \sum_{0\le k\le n,\ 0\le l\le m} |b_{kl} - a_{kl}|\, |x|^k\, e^{lx} \\ &< \sum_{0\le k\le n,\ 0\le l\le m} |b_{kl} - a_{kl}|\, M^n e^{mM} < \frac{\varepsilon}{2}. \end{aligned}$$

In particular, for every $x \in \{\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N\}$, we have that

$$\left|\widehat{G}(x, e^x) - G(x, e^x)\right| < \frac{\varepsilon}{2} \leq \frac{1}{2}|g(x)| = \frac{1}{2}\left|G(x, e^x)\right|.$$

We deduce that $\mathrm{sg}(\widehat{G}(x, e^x)) = \mathrm{sg}(G(x, e^x))$ for every $x \in \{\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N\}$. Then, $\widehat{G}(\alpha_i, e^{\alpha_i})\widehat{G}(\beta_i, e^{\beta_i}) < 0$ for $i = 1, \dots, N$ and, as a consequence, $\hat{g}(x) = \widehat{G}(x, e^x)$ has a zero in each interval I_i, for $i = 1, \dots, N$.

By clearing denominators in the coefficients of $\widehat{G}$, we obtain a polynomial F with integer coefficients satisfying the required conditions. □

Acknowledgments

The authors wish to thank the anonymous reviewer for their helpful suggestions and remarks.

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Received: December 16, 2020
Accepted: July 20, 2021