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A COMPACT MANIFOLD WITH INFINITE-DIMENSIONAL CO-INVARIANT COHOMOLOGY

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Abstract. Let M be a smooth manifold. When Γ is a group acting on M by diffeomorphisms, one can define the Γ-co-invariant cohomology of M to be the cohomology of the complex $\Omega_c(M)_\Gamma = \mathrm{span}\{\omega - \gamma^*\omega : \omega \in \Omega_c(M),\, \gamma \in \Gamma\}$. For a Lie algebra $\mathcal{G}$ acting on the manifold M, one defines the cohomology of $\mathcal{G}$-divergence forms to be the cohomology of the complex $\mathcal{C}_{\mathcal{G}}(M) = \mathrm{span}\{L_X\omega : \omega \in \Omega_c(M),\, X \in \mathcal{G}\}$. In this short paper we present a situation where these two cohomologies are infinite dimensional on a compact manifold.

1. Introduction

The concept of co-invariant cohomology was introduced in [1]. In basic terms, it is the cohomology of a subcomplex of the de Rham complex generated by the action of a group on a smooth manifold. The authors showed that, under nice enough hypotheses on the nature of the action, there is an interplay between the de Rham cohomology of the manifold, the cohomology of invariant forms and the co-invariant cohomology, and this relationship can be exhibited either by vector space decompositions or through long exact sequences depending on the case of study ([1, Theorems 1.1 and 1.3]). Among the various consequences that can be derived from this inspection, it is evident that the dimension of the de Rham cohomology has some control over the dimension of the co-invariant cohomology, and in most cases presented in [1] the latter is finite whenever the former is also finite. This occurs for instance in the case of a finite action on a compact manifold or more generally in the case of an isometric action on a compact oriented Riemannian manifold, and this fact holds as well for a non-compact manifold as long as one requires the action to be free and properly discontinuous with compact orbit space. A concept closely related to co-invariant cohomology is the cohomology of divergence forms, which is defined by means of a Lie algebra action on a smooth manifold and was introduced by A. Abouqateb [2]. In the course of his study, the author gave many examples where the cohomology of divergence forms is finite-dimensional.

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The goal of this paper is to show that this phenomenon heavily depends on the nature of the action in play, and that without underlying hypotheses, co-invariant cohomology and cohomology of divergence forms are not generally well behaved. This is illustrated by an example of a vector field action on a smooth compact manifold giving rise to infinite-dimensional cohomology of divergence forms and whose discrete flow induces an infinite-dimensional co-invariant cohomology as opposed to the de Rham cohomology of the manifold. This shows in particular that many results obtained in [1] and [2] cannot be easily generalized and brings into perspective the necessity to look for finer finiteness conditions of co-invariant cohomology in a future study which would put the present paper in a broader context.

The general outline of the paper is as follows: In Section 2, we briefly recall the notions of co-invariant forms and divergence forms, then we define an homomorphism of the de Rham complex that is induced by a complete vector field on the manifold, and which maps divergence forms relative to the action of the vector field onto the complex of co-invariant differential forms associated to its discrete flow (see (2.1) and Proposition 2.1). Section 3 is concerned with the setting on which our cohomology computations will take place; it comprises a smooth compact manifold, the 3-dimensional hyperbolic torus, which can be obtained as the quotient of a solvable Lie group by a uniform lattice (the construction given here is that of A. El Kacimi in [3]); the Lie algebra action considered is by means of a left-invariant vector field. We then use a number of results to prove Theorem 3.5, which states that the operator defined in (2.1) is an isomorphism between the complex of divergence forms and the complex of co-invariant forms, hence allowing us to only consider the cohomology of co-invariant forms for computation. Finally, Section 4 is dedicated to the main computation, in which we prove that the discrete flow of the vector field in question on the hyperbolic torus gives infinite-dimensional co-invariant cohomology.

2. Preliminaries

Let M be a smooth n-dimensional manifold and let us denote by $\mathrm{Diff}(M)$ the group of diffeomorphisms of M and by $\mathcal{X}(M)$ the Lie algebra of smooth vector fields on M. Let $\rho : \Gamma \longrightarrow \mathrm{Diff}(M)$ be an action of a group Γ on M by diffeomorphisms. For an r-form ω on M and element $\gamma \in \Gamma$, we denote by $\gamma^*\omega$ the pull-back of ω by the diffeomorphism $\rho(\gamma) : M \longrightarrow M$. Let $\Omega_c(M) = \oplus_p \Omega_c^p(M)$ denote the de Rham complex of forms with compact support on M and put

$$\Omega_c^p(M)_\rho := \mathrm{span}\{\omega - \gamma^*\omega : \gamma \in \Gamma,\ \omega \in \Omega_c^p(M)\}.$$

Any element of $\Omega_c^p(M)_\rho$ is called a *ρ-co-invariant* or just a (Γ-)*co-invariant* when there is no ambiguity. The graded vector space $\Omega_c(M)_\rho := \oplus_p \Omega_c^p(M)_\rho$ is a differential subcomplex of the de Rham complex $\Omega_c(M)$, called the *complex of co-invariant differential forms on M*. When M is compact this complex is simply denoted by $\Omega(M)_\rho$. In the case where $\rho : \mathbb{Z} \longrightarrow M$ is the action induced by a diffeomorphism $\gamma : M \longrightarrow M$, i.e., $\rho(n) := \gamma^n$, then we get that

$$\Omega_c^p(M) = \{\omega - \gamma^*\omega : \omega \in \Omega_c^p(M)\}.$$

Let $\tau : \mathcal{G} \longrightarrow \mathcal{X}(M)$ be a Lie algebra homomorphism and set $\hat{X} := \tau(X)$ for any $X \in \mathcal{G}$; then define

$$\mathcal{C}^p_\tau(M) := \operatorname{span}\{L_{\hat{X}}\omega : X \in \mathcal{G},\ \omega \in \Omega^p_c(M)\}.$$

Any element of $\mathcal{C}^p_\tau(M)$ is called a *τ-divergence p-form* or simply *$\mathcal{G}$-divergence form.* The graded vector space $\mathcal{C}_\tau(M) := \oplus_p \mathcal{C}^p_\tau(M)$ is a differential subcomplex of the de Rham complex. If X is any vector field on M, with corresponding Lie algebra homomorphism $\tau : \mathbb{R} \longrightarrow \mathcal{X}(M)$, $\tau(1) := X$, then

$$\mathcal{C}^p_\tau(M) = \operatorname{span}\{L_X\omega : \omega \in \Omega^p_c(M)\}.$$

In what follows, $X \in \mathcal{X}(M)$ is a complete vector field and $\Phi : M \times [0,1] \longrightarrow M$ is the flow Φ^X of the vector field X restricted to $M \times [0,1]$. We define the linear operator $I : \Omega(M) \longrightarrow \Omega(M)$ by the expression

$$I(\eta) := \fint_0^1 \Phi^*\eta \wedge \mathrm{pr}_2^*(ds), \tag{2.1}$$

where

$$\fint_0^1 : \Omega^*(M \times [0,1]) \longrightarrow \Omega^{*-1}(M)$$

is the fiberwise integration operator of the trivial bundle $M \times [0,1] \xrightarrow{\mathrm{pr}_1} M$ (see [4]) and ds is the usual volume form on $[0,1]$.

Let $\tau : \mathbb{R} \longrightarrow \mathcal{X}(M)$ be the Lie algebra homomorphism induced by X and let $\rho : \mathbb{Z} \longrightarrow \mathrm{Diff}(M)$ be the discrete flow of X, i.e., the group action given by $\rho(n) := \Phi^X_n$.

Proposition 2.1. *The operator $I : \Omega(M) \longrightarrow \Omega(M)$ defined by* (2.1) *is a differential complex homomorphism, i.e., $I \circ d = -d \circ I$. Moreover, $I(\mathcal{C}_\tau(M)) \subset \Omega_c(M)_\rho$ and the restriction of $I : \mathcal{C}_\tau(M) \longrightarrow \Omega_c(M)_\rho$ is surjective.*

Proof. Let $\eta \in \Omega(M)$ and let $\iota_s : M \longrightarrow M \times \{s\} \hookrightarrow M \times [0,1]$ be the natural inclusion; then using the Stokes formula for fiberwise integration we get

$$I(d\eta) = \fint_0^1 \Phi^*(d\eta) \wedge \mathrm{pr}_2^*(ds) = \fint_0^1 d(\Phi^*\eta \wedge \mathrm{pr}_2^*(ds)) = -dI(\eta) + \left[\iota_s^*(\Phi^*\eta \wedge \mathrm{pr}_2^* ds)\right]_0^1,$$

and since $\iota_s^*\mathrm{pr}_2^*(ds) = 0$ we get $I(d\eta) = -dI(\eta)$. For the second claim we start by showing that $I(\eta)$ has compact support whenever η does. Indeed, assume $\eta \in \Omega_c(M)$ and set $K := \operatorname{supp}(\eta)$. Next consider the map

$$f : M \times \mathbb{R} \longrightarrow M, \quad (x,s) \mapsto \Phi_s^{-1}(x) := \Phi(x,-s).$$

Then f is continuous and therefore $L := f(K \times [0,1])$ is compact. For any $y \in M \backslash L$ and any $s \in [0,1]$ we get that $\Phi_s(y) \notin K$ and therefore $(\Phi^*\eta)_{(y,s)} = 0$; this implies that $I(\eta)_y = 0$. We conclude that $\operatorname{supp} I(\eta) \subset L$, i.e., $I(\eta) \in \Omega_c(M)$.

From the relation $\mathrm{T}_{(x,t)}\Phi(0,1) = X_{\Phi_t(x)}$ one gets that $\Phi^* \circ i_X = i_{(0,\frac{\partial}{\partial s})} \circ \Phi^*$ and therefore $\Phi^* \circ L_X = L_{(0,\frac{\partial}{\partial s})} \circ \Phi^*$. Moreover, we have that

$$L_{(0,\frac{\partial}{\partial s})}\mathrm{pr}_2^*(ds) = 0 \quad \text{and} \quad \fint_0^1 \circ\, i_{(0,\frac{\partial}{\partial s})} = 0.$$

If we write $\eta = L_X\omega$ for some $\omega \in \Omega_c(M)$ then we get that

$$\begin{aligned}
I(L_X\omega) &= \fint_0^1 \Phi^*(L_X\omega) \wedge \mathrm{pr}_2^*(ds) \\
&= \fint_0^1 L_{(0,\frac{\partial}{\partial s})}(\Phi^*\omega) \wedge \mathrm{pr}_2^*(ds) \\
&= \fint_0^1 L_{(0,\frac{\partial}{\partial s})}(\Phi^*\omega \wedge \mathrm{pr}_2^*(ds)) \\
&= \fint_0^1 d \circ i_{(0,\frac{\partial}{\partial s})}(\Phi^*\omega \wedge \mathrm{pr}_2^*(ds)) + \fint_0^1 i_{(0,\frac{\partial}{\partial s})} d(\Phi^*\omega \wedge \mathrm{pr}_2^*(ds)) \\
&= d\left(\fint_0^1 i_{(0,\frac{\partial}{\partial s})}(\Phi^*\omega \wedge \mathrm{pr}_2^*(ds))\right) + [\iota_s^* i_{(0,\frac{\partial}{\partial s})}(\Phi^*\omega \wedge \mathrm{pr}_2^*(ds))]_0^1 \\
&= [\Phi_s^*\omega]_0^1 \\
&= \Phi_1^*\omega - \Phi_0^*\omega \\
&= \Phi_1^*\omega - \omega.
\end{aligned}$$

It follows that $I(\mathcal{C}_\tau(M)) \subset \Omega_c(M)_\rho$. This also shows that $I : \mathcal{C}_\tau(M) \longrightarrow \Omega_c(M)_\rho$ is surjective. $\square$

Remark 2.2. Note that Φ^X-invariant forms on M are fixed by I, i.e., if $\omega \in \Omega(M)$ is such that $L_X\omega = 0$ then $I(\omega) = \omega$.

3. The hyperbolic torus

Consider $A \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(A) > 2$. It is easy to check that $A = PDP^{-1}$ for some $P \in \mathrm{GL}(2,\mathbb{R})$ and $D = \mathrm{diag}(\lambda, \lambda^{-1})$. Clearly $\lambda > 0$ and $\lambda \neq 1$. Hence it makes sense to set $D^t = \mathrm{diag}(\lambda^t, \lambda^{-t})$ and define $A^t = PD^tP^{-1}$ for any $t \in \mathbb{R}$. Next we define the Lie group homomorphism

$$\varphi : \mathbb{R} \longrightarrow \mathrm{Aut}(\mathbb{R}^2), \quad t \mapsto A^t.$$

The hyperbolic torus $\mathbb{T}_A^3$ is the smooth manifold defined as the quotient $\Gamma_3 \backslash G_3$ where $G_3 := \mathbb{R}^2 \rtimes_\varphi \mathbb{R}$ and $\Gamma_3 := \mathbb{Z}^2 \rtimes_\varphi \mathbb{Z}$. The natural projection $\mathbb{R}^2 \rtimes_\varphi \mathbb{R} \xrightarrow{p} \mathbb{R}$ induces a fiber bundle structure $\mathbb{T}_A^3 \xrightarrow{p} \mathbb{S}^1$ with fiber type $\mathbb{T}^2$ and $p[x,y,t] = [t]$; in particular, $\mathbb{T}_A^3$ is a compact manifold.

If $(1,a)$ and $(1,b)$ are the eigenvectors of A respectively associated to the eigenvalues λ and λ^{-1} then

$$v = (1,a,0), \quad w = (1,b,0) \quad \text{and} \quad e = (0,0,-\log(\lambda)^{-1})$$

form a basis of $\mathfrak{g}_3 = \mathrm{Lie}(G_3)$, and we can check that

$$[v,w]_{\mathfrak{g}_3} = 0, \quad [e,v]_{\mathfrak{g}_3} = -v \quad \text{and} \quad [e,w]_{\mathfrak{g}_3} = w.$$

Denote by X, Y and Z the left invariant vector fields on $\mathbb{R}^2 \rtimes_\varphi \mathbb{R}$ associated to v, w and e respectively; then $\{X, Y, Z\}$ defines a parallelism on $\mathbb{T}_3^A$. A direct calculation leads to

$$X = \lambda^t \left(\frac{\partial}{\partial x} + a\frac{\partial}{\partial y}\right), \quad Y = \lambda^{-t}\left(\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) \quad \text{and} \quad Z = -\log(\lambda)^{-1}\frac{\partial}{\partial t}. \tag{3.1}$$

Now denote by α, β and θ the dual forms associated to X, Y and Z respectively. It is clear that the vector fields X and Y of $\mathbb{T}_A^3$ are tangent to the fibers of the fiber bundle $\mathbb{T}_A^3 \xrightarrow{p} \mathbb{S}^1$, and that $\theta = -(\log\lambda)p^*(\sigma)$, where σ is the invariant volume form on $\mathbb{S}^1$ satisfying $\int_{\mathbb{S}^1}\sigma = 1$.

The conditions $A \in \mathrm{SL}(2, \mathbb{Z})$ and $\mathrm{tr}(A) > 2$ imply that λ is irrational, hence so is a. Therefore, the orbits of the linear vector field $v = (1, a) = \frac{\partial}{\partial x} + a\frac{\partial}{\partial y}$ on the torus $\mathbb{T}^2$ are dense. So if $f \in \mathcal{C}^\infty(\mathbb{T}^2)$ satisfies $v(f) = 0$, it is constant on the orbits of v and, by continuity, it is also constant on $\mathbb{T}^2$.

Now consider the vector field $X = \lambda^t(v, 0)$ on $\mathbb{T}_A^3$, i.e., given by (3.1). Its restriction to each fiber $F_t := p^{-1}(t)$ of the fibration $\mathbb{T}^2 \longrightarrow \mathbb{T}_A^3 \xrightarrow{p} \mathbb{S}^1$ is a linear vector field whose direction is the vector v. From the preceding remark, any function $f \in \mathcal{C}^\infty(\mathbb{T}_A^3)$ satisfying $X(f) = 0$ is constant on the fibers F_t. Thus $f = p^*(\bar{f})$ for some $\bar{f} \in \mathcal{C}^\infty(\mathbb{S}^1)$. Summarizing:

Proposition 3.1. *The orbits of the vector field X defined in* (3.1) *are dense in the fibers of the fiber bundle $\mathbb{T}_A^3 \xrightarrow{p} \mathbb{S}^1$. In particular, for any $f \in \mathcal{C}^\infty(\mathbb{T}_A^3)$, $X(f) = 0$ is equivalent to $f = p^*(\phi)$ for some $\phi \in \mathcal{C}^\infty(\mathbb{S}^1)$.* □

The following lemma is of central importance for the development of this paragraph and for the computations of the next section.

Lemma 3.2. *Let $f \in \mathcal{C}^\infty(\mathbb{T}_A^3)$. Then for every $s \in \mathbb{R}$ we have the formula*

$$Z\big((\Phi_s^X)^*(f)\big) = -s(\Phi_s^X)^*\big(X(f)\big) + (\Phi_s^X)^*\big(Z(f)\big).$$

In particular, $Z(\gamma^* f) = -X(\gamma^* f) + \gamma^*(Z(f))$ and $i_Z \circ \gamma^* = -\gamma^* \circ i_X + \gamma^* \circ i_Z$, where $\gamma := \Phi_1^X$.

Proof. For any $(x, y, t) \in \mathbb{R}^3$, a straightforward computation gives that

$$\begin{aligned}
Z\big((\Phi_s^X)^*(f)\big)(x,y,t) &= -\frac{1}{\log\lambda} d(f \circ \Phi_s^X)_{(x,y,t)}(0,0,1) \\
&= -\frac{1}{\log\lambda}\frac{d}{du}_{|u=0}(f \circ \Phi_s^X)(x,y,t+u) \\
&= -\frac{1}{\log\lambda}\frac{d}{du}_{|u=0} f(s\lambda^{t+u} + x, as\lambda^{t+u} + y, t+u) \\
&= -\frac{1}{\log\lambda}(df)_{\Phi_s^X(x,y,t)}(s\log(\lambda)\lambda^t, as\log(\lambda)\lambda^t, 1) \\
&= -s(df)_{\Phi_s^X(x,y,t)}(\lambda^t, a\lambda^t, 0) - \frac{1}{\log\lambda}(df)_{\Phi_s^X(x,y,t)}(0,0,1) \\
&= -s(X(f) \circ \Phi_s^X)(x,y,t) + (Z(f) \circ \Phi_s^X)(x,y,t).
\end{aligned}$$
□

Given a compact smooth manifold M, we set, for any $f \in \mathcal{C}^\infty(M)$,

$$\|f\|_\infty := \sup_{x\in M} |f(x)|;$$

it is straightforward to check that $\|\cdot\|_\infty$ defines a vector space norm on $\mathcal{C}^\infty(M)$; moreover, for any $g \in \mathrm{Diff}(M)$, the composition $\mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$, $f \mapsto f \circ g$ is an isometry relative to $\|\cdot\|_\infty$, i.e., $\|f \circ g\|_\infty = \|f\|_\infty$.

Corollary 3.3. *Let $f \in \mathcal{C}^\infty(\mathbb{T}^3_A)$ be such that $f = \gamma^* f$ with $\gamma := \Phi^X_1$. Then $X(f) = 0$ and consequently $f = p^*\psi$ with $\psi \in \mathcal{C}^\infty(\mathbb{S}^1)$.*

Proof. Since $f = \gamma^* f$, we get that $f = (\gamma^n)^*(f)$ for every $n \in \mathbb{Z}$; thus the preceding lemma gives that

$$Z(f) = -nX(f) + (\gamma^n)^*(Z(f)).$$

Consequently we obtain that, for every $n \in \mathbb{Z}$,

$$|X(f)| \leq \frac{1}{n}\big(\|Z(f)\|_\infty + \|(\gamma^n)^*(Z(f))\|_\infty\big) \leq \frac{2}{n}\|Z(f)\|_\infty,$$

which leads to $X(f) = 0$. □

Corollary 3.4. *Let $f \in \mathcal{C}^\infty(\mathbb{T}^3_A)$ be such that $f - \gamma^* f = p^*\phi$ for some $\phi \in \mathcal{C}^\infty(\mathbb{S}^1)$. Then $f = p^*\psi$ with $\psi \in \mathcal{C}^\infty(\mathbb{S}^1)$.*

Proof. In view of Corollary 3.3, it suffices to show that $\phi = 0$ to prove this result. Using that $p^*\phi$ is γ-invariant, we can prove inductively that $f = (\gamma^n)^* f + np^*\phi$ for any $n \in \mathbb{Z}$, and thus, for $n \neq 0$,

$$|p^*\phi| \leq \frac{1}{n}(\|f\|_\infty + \|(\gamma^n)^* f\|_\infty) \leq \frac{2}{n}\|f\|_\infty,$$

which leads to $\phi = 0$. □

In what follows we set $M := \mathbb{T}^3_A$. It is straightforward to check that

$$d\alpha = -\alpha \wedge \theta, \quad d\beta = \beta \wedge \theta, \quad d\theta = 0,$$

and that $L_X\alpha = -\theta$ and $L_X\beta = L_X\theta = 0$, thus $L_X(\alpha \wedge \beta \wedge \theta) = 0$.

Let $\tau : \mathbb{R} \longrightarrow \mathcal{X}(M)$ be the Lie algebra homomorphism corresponding to the vector field X and let $\rho : \mathbb{Z} \longrightarrow \mathrm{Diff}(M)$ be the discrete action generated by $\gamma := \Phi^X_1$, where Φ^X is the flow of X, that is, $\rho(n)(x) = \Phi^X_n(x)$ for any $n \in \mathbb{Z}$.

Theorem 3.5. *The homomorphism $I : \mathcal{C}_\tau(M) \longrightarrow \Omega(M)_\rho$ defined in* (2.1) *is an isomorphism.*

Proof. In view of Proposition 2.1 it only remains to prove that I is injective. Choose $\eta \in \mathcal{C}_\tau(M)$ and write $\eta = L_X\omega$ for some $\omega \in \Omega(M)$. Assume that $I(\eta) = 0$; in view of the previous computation this is equivalent to $\omega = \gamma^*\omega$.

If $\eta \in \mathcal{C}^0_\tau(M)$ then Corollary 3.3 gives that $\omega = p^*\phi$ for some $\phi \in \mathcal{C}^\infty(\mathbb{S}^1)$, thus $\eta = 0$. On the other hand, if $\eta \in \mathcal{C}^3_\tau(M)$ we can write $\eta = X(f)\alpha \wedge \beta \wedge \theta$ for some $f \in \mathcal{C}^\infty(M)$ satisfying $f = \gamma^* f$ and so by Corollary 3.3 we get $\eta = 0$.

Now for $\eta \in \mathcal{C}^1_\tau(M)$ we can write

$$\omega = f\alpha + g\beta + h\theta, \quad f, g, h \in \mathcal{C}^\infty(M).$$

Applying I to $L_X\alpha = -\theta$ leads to $\gamma^*\alpha = \alpha - \theta$. Moreover, since θ and β are Φ^X-invariant, we get that $\beta = \gamma^*\beta$ and $\theta = \gamma^*\theta$, thus

$$\gamma^*\omega = (\gamma^* f)\alpha + (\gamma^* g)\beta + (\gamma^* h - \gamma^* f)\theta,$$

and hence the relation $\omega = \gamma^*\omega$ is equivalent to $f = \gamma^* f$, $g = \gamma^* g$ and $h = \gamma^* h - f$. Corollary 3.3 shows that $X(g) = X(f) = 0$, hence $f \in p^*(\mathcal{C}^\infty(\mathbb{S}^1))$, and thus Corollary 3.4 applied to $h - \gamma^* h = f$ gives that $f = 0$ and $h = \gamma^* h$, which implies that $X(h) = 0$ in view of Corollary 3.3. Now using that $L_X\beta = 0$ and $L_X\theta = 0$ it follows that $\eta = L_X\omega = 0$. Finally, let $\eta \in \mathcal{C}^2_\tau(M)$ and write

$$\omega = f\alpha\wedge\beta + g\alpha\wedge\theta + h\beta\wedge\theta, \quad f, g, h \in \mathcal{C}^\infty(M).$$

Then using $\gamma^*\alpha = \alpha - \theta$ we obtain that

$$\gamma^*\omega = (\gamma^* f)\alpha\wedge\beta + (\gamma^* g)\alpha\wedge\theta + (\gamma^* h + \gamma^* f)\beta\wedge\theta,$$

and so $\omega = \gamma^*\omega$ is equivalent in this case to $f = \gamma^* f$, $g = \gamma^* g$ and $h = \gamma^* h + \gamma^* f$. As before, this leads to $f = 0$, $X(g) = 0$ and $X(h) = 0$, and so

$$\eta = L_X\omega = L_X(g\alpha\wedge\theta + h\beta\wedge\theta) = gL_X(\alpha\wedge\theta) = -g\theta\wedge\theta = 0.$$

Thus $I : \mathcal{C}_\tau(M) \longrightarrow \Omega(M)_\rho$ is an isomorphism. □

This result gives in particular that $\mathrm{H}(\mathcal{C}_\tau(M)) \simeq \mathrm{H}(\Omega(M)_\rho)$ and therefore we only need to compute the cohomology of ρ-co-invariant forms in this case.

4. Cohomology computation

We now have all the necessary ingredients to perform our computation. Let M denote the hyperbolic torus $\mathbb{T}^3_A$ defined in the previous section with $A \in \mathrm{SL}(2,\mathbb{Z})$ such that $\mathrm{tr}(A) > 2$, and let $X, Y, Z \in \mathcal{X}(M)$ be the vector fields defined in (3.1) with respective dual 1-forms α, β and θ. Define the action $\rho : \mathbb{Z} \longrightarrow \mathrm{Diff}(M)$ to be the discrete flow of the vector field X with $\gamma := \rho(1)$. The main goal is to prove that the first and second co-invariant cohomology groups are infinite dimensional; however, we shall compute the whole cohomology in order to get a global picture.

Calculating $\mathrm{H}^0(\Omega(M)_\rho)$**.** Choose $f \in \Omega^0(M)_\rho$ such that $df = 0$. Then f is a constant function equal to $g - \gamma^* g$ for some $g \in \mathcal{C}^\infty(M)$. Consequently we obtain that

$$\int_M f\alpha\wedge\beta\wedge\theta = \int_M (g - \gamma^* g)\alpha\wedge\beta\wedge\theta = \int_M g\alpha\wedge\beta\wedge\theta - \int_M \gamma^*(g\alpha\wedge\beta\wedge\theta) = 0.$$

Thus $f = 0$, and we conclude that $\mathrm{H}^0(\Omega(M)_\rho) = 0$.

Calculating $\mathrm{H}^1(\Omega(M)_\rho)$**.** We prove that $\mathrm{H}^1(\Omega(M)_\rho)$ is infinite dimensional. In order to do so, we prove that the map $p^* : \Omega^1(\mathbb{S}^1) \longrightarrow \mathrm{H}^1(\Omega(M)_\rho)$ is well defined and injective, or equivalently we can show that $p^*(\Omega^1(\mathbb{S}^1)) \subset \mathrm{Z}^1(\Omega(M)_\rho)$ and $p^*(\Omega^1(\mathbb{S}^1)) \cap \mathrm{B}^1(\Omega(M)_\rho) = 0$.

An element $\eta \in p^*(\Omega^1(\mathbb{S}^1))$ can always be written as $\eta = p^*(\phi)\theta$, where $\phi \in \mathcal{C}^\infty(\mathbb{S}^1)$. Since $L_X\theta = 0$ and $L_X\alpha = -\theta$, by applying I to $L_X\alpha$ we get that $\theta = \alpha - \gamma^*\alpha$; therefore

$$\eta = p^*(\phi)\theta = p^*(\phi)\alpha - \gamma^*(p^*(\phi)\alpha).$$

Moreover, observe that $d\eta = 0$, hence we deduce that $p^*(\Omega^1(\mathbb{S}^1)) \subset \mathrm{Z}^1(\Omega(M)_\rho)$. Now suppose $\eta = d(g - \gamma^*g)$, then clearly $X(g-\gamma^*g) = 0$ and $Z(g-\gamma^*g) = p^*(\phi)$; thus, according to Proposition 3.1, $g - \gamma^*g = p^*\psi$ for some $\psi \in \mathcal{C}^\infty(\mathbb{S}^1)$. Using Corollary 3.4 we obtain that $g - \gamma^*g = 0$ and so $\eta = 0$. Thus $p^*(\Omega^1(\mathbb{S}^1)) \cap \mathrm{B}^1(\Omega(M)_\mathbb{Z}) = 0$.

Calculating $\mathrm{H}^2(\Omega(M)_\rho)$. We will show that $p^*(\Omega^1(\mathbb{S}^1)) \wedge \beta \subset \mathrm{H}^2(\Omega(M)_\rho)$. To do this, we fix a 2-form $\eta = p^*(\phi)\theta \wedge \beta$ such that $\phi \in \mathcal{C}^\infty(\mathbb{S}^1)$. We can easily check that $d\eta = 0$; moreover, from the previous calculations and the fact that $L_X\beta = 0$ we get that $\beta = \gamma^*\beta$, and therefore

$$p^*(\phi)\theta \wedge \beta = (p^*\phi\alpha \wedge \beta) - \gamma^*(p^*(\phi)\alpha \wedge \beta).$$

Hence $p^*(\Omega^1(\mathbb{S}^1)) \wedge \beta \subset \mathrm{Z}^2(\Omega(M)_\rho)$. Now assume that $\eta = d(\omega - \gamma^*\omega)$; the expression of η along with the fact that $\theta(X) = \theta(Y) = 0$ gives that $i_Yi_X(\eta) = 0$. Furthermore, it is clear that $i_X \circ \gamma^* = \gamma^* \circ i_X$, and since $[X, Y] = 0$ we get that $i_Y \circ \gamma^* = \gamma^* \circ i_Y$ as well. By combining these facts we get that

$$i_Yi_X(d\omega) = \gamma^*(i_Yi_Xd\omega).$$

Hence, according to Corollary 3.3, we can write $i_Xi_Yd\omega = p^*\psi$ for some $\psi \in \mathcal{C}^\infty(\mathbb{S}^1)$. On the other hand, by applying Lemma 3.2 to $i_Yi_Z\eta = p^*\phi$, we obtain that

$$p^*\phi - i_Yi_X(d\omega) = \gamma^*(i_Zi_Yd\omega) - i_Zi_Yd\omega.$$

It follows from these remarks that $p^*(\phi - \psi) = \gamma^*(i_Zi_Yd\omega) - i_Zi_Yd\omega$; thus by Corollary 3.4 we get $p^*(\phi - \psi) = 0$, and so we deduce that $p^*\phi = p^*\psi = i_Yi_X(d\omega)$. Now if we write $\omega = f\alpha + g\beta + h\theta$, then we get that $p^*\phi = X(g) - Y(f)$. Moreover, from $X(p^*\phi) = Y(p^*\phi) = 0$ we get that, for every $s \in \mathbb{R}$,

$$\begin{aligned}
s^2p^*\phi &= \int_0^s \int_0^s (\Phi_t^X)^*(\Phi_u^Y)^*(p^*\phi)\,du\,dt \\
&= \int_0^s \int_0^s (\Phi_t^X)^*(\Phi_u^Y)^*(X(g))\,du\,dt - \int_0^s \int_0^s (\Phi_t^X)^*(\Phi_u^Y)^*(Y(f))\,du\,dt \\
&= \int_0^s (\Phi_t^X)^*X\left(\int_0^s (\Phi_u^Y)^*(g)\,du\right)dt - \int_0^s (\Phi_u^Y)^*Y\left(\int_0^s (\Phi_t^X)^*(f)\,dt\right)du \\
&= (\Phi_s^X)^*\left(\int_0^s (\Phi_u^Y)^*(g)\,du\right) - \int_0^s (\Phi_u^Y)^*(g)\,du - (\Phi_s^Y)^*\left(\int_0^s (\Phi_t^X)^*(f)\,dt\right) \\
&\quad + \int_0^s (\Phi_t^X)^*(f)\,dt.
\end{aligned}$$

It follows that

$$s^2|p^*\phi| \leq 2\left\|\int_0^s (\Phi_u^Y)^*(g)\,du\right\|_\infty + 2\left\|\int_0^s (\Phi_t^X)^*(f)\,dt\right\|_\infty \leq 2|s|(\|g\|_\infty + \|f\|_\infty).$$

Hence $|p^*\phi| \leq \dfrac{2}{|s|}(\|g\|_\infty + \|f\|_\infty) \underset{s\to+\infty}{\longrightarrow} 0$.

We conclude that $\eta = 0$ and $p^*(\Omega^1(\mathbb{S}^1)) \wedge \beta \cap \mathrm{B}^2(\Omega(M)_\rho) = 0$; in particular, this proves that $\mathrm{H}^2(\Omega(M)_\rho)$ is infinite dimensional.

Calculating $\mathrm{H}^3(\Omega(M)_\rho)$**.** The elements of $\Omega^3(M)_\rho$ are of the form

$$(f - \gamma^* f)\alpha \wedge \beta \wedge \theta$$

for some $f \in \mathcal{C}^\infty(M)$. Put

$$c = \frac{\int_M f\alpha \wedge \beta \wedge \theta}{\int_M \alpha \wedge \beta \wedge \theta};$$

then

$$\int_M (f - c)\alpha \wedge \beta \wedge \theta = 0.$$

Thus $(f-c)\alpha\wedge\beta\wedge\theta = d\omega$, and since $L_X(\alpha\wedge\beta\wedge\theta) = 0$, we get $(\gamma^* f - c)\alpha\wedge\beta\wedge\theta = d(\gamma^*\omega)$ and therefore it follows that

$$(f - \gamma^* f)\alpha \wedge \beta \wedge \theta = d(\omega - \gamma^*\omega),$$

i.e., $\mathrm{H}^3(\Omega(M)_\rho) = 0$.

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