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THE EXT-ALGEBRA OF THE BRAUER TREE ALGEBRA ASSOCIATED TO A LINE

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Abstract. We compute the Ext-algebra of the Brauer tree algebra associated to a line with no exceptional vertex.

Introduction

This note provides a detailed computation of the Ext-algebra for a very specific finite dimensional algebra, namely a Brauer tree algebra associated to a line, with no exceptional vertex. Such algebras appear for example as the principal p-block of the symmetric group $\mathfrak{S}_p$, and in a different context, as blocks of the Verlinde categories Ver_{p^2} studied by Benson and Etingof [2] (our computation is actually motivated by [2, Conj. 1.3]).

Let us emphasise that Ext-algebras for more general biserial algebras were explicitly computed by Green, Schroll, Snashall, and Taillefer [4], but under some assumption on the multiplicity of the vertices, assumption which is not satisfied for the simple example treated in this note. Other general results relying on Auslander–Reiten theory were obtained by Antipov and Generalov [1] and Brown [3]. However, we did not manage to use their work to get an explicit description in our case. Nevertheless, the simple structure of the projective indecomposable modules for the line allows a straightforward approach using explicit projective resolutions of simple modules. The Poincaré series for the Ext-algebra is given in Proposition 2.2 and its structure as a path algebra with relations is given in Proposition 3.2.

1. Notation

Let $\mathbb{F}$ be a field, and A be a self-injective finite dimensional $\mathbb{F}$-algebra. All A-modules will be assumed to be finitely generated. Given an A-module M, we denote by $\Omega(M)$ the kernel of a projective cover $P \twoheadrightarrow M$. Up to isomorphism it does not depend on the cover. We then define inductively $\Omega^n(M) = \Omega(\Omega^{n-1}(M))$ for $n \geq 2$.

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To compute the extension groups between simple modules we will use the property that

$$\operatorname{Ext}_A^n(M,S) \simeq \operatorname{Hom}_A(\Omega^n(M),S)$$

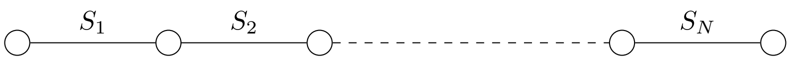
for any simple A-module S and any $n \geq 1$.

For computing the algebra structure on the various Ext-groups it will be convenient to work in the homotopy category $\mathsf{Ho}(A)$ of the complexes of finitely generated A-modules. If S (resp. S') is a simple A-module, and $P_\bullet \to S$ (resp. $P'_\bullet \to S'$) is a projective resolution, then

$$\operatorname{Ext}_A^n(S,S') \simeq \operatorname{Hom}_{\mathsf{Ho}(A)}(P_\bullet, P'_\bullet[n])$$

with the Yoneda product being given by the composition of maps in $\mathsf{Ho}(A)$.

Assume now that A is an $\mathbb{F}$-algebra associated to the following Brauer tree with $N+1$ vertices:



Here, unlike in [4], we assume that there is no exceptional vertex. The edges are labelled by the simple A-modules $S_1,\ldots,S_N$. We will denote by $P_1,\ldots,P_N$ the corresponding indecomposable projective A-modules. The head and socle of P_i are isomorphic to S_i and $\operatorname{rad}(P_i)/S_i \simeq S_{i-1} \oplus S_{i+1}$ with the convention that $S_0 = S_{N+1} = 0$.

Given $1 \leq i \leq N-1$, we fix non-zero maps $f_i : P_i \longrightarrow P_{i+1}$ and $f_i^* : P_{i+1} \longrightarrow P_i$ such that $f_i^* \circ f_i + f_{i-1} \circ f_{i-1}^* = 0$ for all $2 \leq i \leq N-1$. This is possible since $f_i^* \circ f_i$ and $f_{i-1} \circ f_{i-1}^*$ are two non-zero elements of the Jacobson radical of $\operatorname{End}(P_i)$, which is isomorphic to $\mathbb{F}$. It follows that the algebra A is Morita equivalent to the path algebra of the quiver

$$\begin{tikzcd} P_1 \arrow[r, bend left, "f_1"] & P_2 \arrow[l, bend left, "f_1^*"] \arrow[r, bend left, "f_2"] & P_3 \arrow[l, bend left, "f_2^*"] & \cdots & P_{N-2} \arrow[r, bend left, "f_{N-2}"] & P_{N-1} \arrow[l, bend left, "f_{N-2}^*"] \arrow[r, bend left, "f_{N-1}"] & P_N \arrow[l, bend left, "f_{N-1}^*"] \end{tikzcd}$$

subject to the relations $f_i^* \circ f_i + f_{i-1} \circ f_{i-1}^* = 0$ for all $2 \leq i \leq N-1$.

2. Ext-groups

Given $1 \leq i \leq j \leq N$ with $i-j$ even, there is, up to isomorphism, a unique non-projective indecomposable module ${}^i\mathsf{X}^j$ such that

- $\operatorname{rad}({}^i\mathsf{X}^j) = S_{i+1} \oplus S_{i+3} \oplus \cdots \oplus S_{j-1}$,
- $\operatorname{hd}({}^i\mathsf{X}^j) = S_i \oplus S_{i+2} \oplus \cdots \oplus S_j$.

In particular we have ${}^i\mathsf{X}^i = S_i$. The structure of ${}^i\mathsf{X}^j$ can be represented by the following diagram:

$$ {}^i\mathsf{X}^j = \begin{array}{ccccccccccc} S_i & & S_{i+2} & & S_{i+4} & \cdots & S_{j-2} & & S_j \\ & \diagdown\;\diagup & & \diagdown\;\diagup & & \cdots & & \diagdown\;\diagup & \\ & S_{i+1} & & S_{i+3} & & \cdots & & S_{j-1} & \end{array}$$

Similarly, we denote by ${}_i\mathsf{X}_j$ the unique indecomposable module with the following structure:

$$
{}_i\mathsf{X}_j = \begin{array}{ccccccccccc}
 & S_{i+1} & & S_{i+3} & & \cdots & & S_{j-1} & \\
/ & & \backslash \quad / & & \backslash & \cdots & / & & \backslash \\
S_i & & S_{i+2} & & S_{i+4} & \cdots & S_{j-2} & & S_j
\end{array}
$$

Note that ${}_i\mathsf{X}_i = S_i = {}^i\mathsf{X}^i$. Finally, in the case where $i-j$ is odd we define the modules ${}_i\mathsf{X}^j$ and ${}^i\mathsf{X}_j$ as the indecomposable modules with the following respective structures:

$$
{}_i\mathsf{X}^j = \begin{array}{cccccccc}
 & S_{i+1} & & S_{i+3} & & \cdots & & S_j \\
/ & & \backslash \quad / & & \backslash & \cdots & / & \\
S_i & & S_{i+2} & & S_{i+4} & \cdots & S_{j-1} &
\end{array}
$$

$$
{}^i\mathsf{X}_j = \begin{array}{cccccccc}
S_i & & S_{i+2} & & S_{i+4} & \cdots & S_{j-1} & \\
 & \backslash \quad / & & \backslash \quad / & & \cdots & & \backslash \\
 & S_{i+1} & & S_{i+3} & & \cdots & & S_j
\end{array}
$$

For convenience we will extend the notation ${}^i\mathsf{X}^j$, ${}_i\mathsf{X}_j$, ${}_i\mathsf{X}^j$, and ${}^i\mathsf{X}_j$ to any integers $i, j \in \mathbb{Z}$ (with the suitable parity condition on $i-j$) so that the following relations hold:

$$
{}^i\mathsf{X} = {}_{1-i}\mathsf{X}, \qquad {}^i\mathsf{X}^j = {}_j\mathsf{X}_i, \qquad {}^{i\pm 2N}\mathsf{X} = {}^i\mathsf{X}. \tag{2.1}
$$

Note that this also implies $\mathsf{X}^j = \mathsf{X}_{1-j}$, $\mathsf{X}^{j\pm 2N} = \mathsf{X}^j$, and ${}^i\mathsf{X}_j = {}_j\mathsf{X}^i$.

Lemma 2.1. *Let $i, j \in \mathbb{Z}$ with $i-j$ even. Then*

$$
\Omega({}^i\mathsf{X}^j) \simeq {}^{i-1}\mathsf{X}^{j+1}.
$$

Proof. Since ${}^i\mathsf{X}^j \simeq {}^{i\pm 2N}\mathsf{X}^{j\pm 2N}$, we can assume that both i and j are in $\{-N+1, \dots, N\}$. If $i \leq 0$ then $1-i \in \{1, \dots, N\}$, but $1-(i-1) = (1-i)+1$. Similarly, if $j \leq 0$ then $1-j \in \{1, \dots, N\}$, but $1-(j+1) = (1-j)-1$. Therefore using the relations (2.1) it is enough to prove that for $1 \leq k \leq l \leq N$ we have the following isomorphisms:

$$
\Omega({}^k\mathsf{X}^l) \simeq {}^{k-1}\mathsf{X}^{l+1}, \quad \Omega({}_k\mathsf{X}^l) \simeq {}_{k+1}\mathsf{X}^{l+1}, \quad \Omega({}^k\mathsf{X}_l) \simeq {}^{k-1}\mathsf{X}_{l-1}, \quad \Omega({}_k\mathsf{X}_l) \simeq {}_{k+1}\mathsf{X}_{l-1}.
$$

We only consider the first one; the others are similar. If $1 \leq k \leq l \leq N$, a projective cover of ${}^k\mathsf{X}^l$ is given by $P_k \oplus P_{k+2} \oplus \cdots \oplus P_l \twoheadrightarrow {}^k\mathsf{X}^l$, whose kernel equals ${}^{k-1}\mathsf{X}^{l+1}$. Note that this holds even when $k=1$ since ${}^0\mathsf{X}^{l+1} = {}_1\mathsf{X}^{l+1}$ or when $l = N$ since ${}^{k-1}\mathsf{X}^{N+1} = {}^{k-1}\mathsf{X}^{-N+1} = {}^{k-1}\mathsf{X}_N$. □

We deduce from Lemma 2.1 that for any simple module S_i and for all $k \geq 0$ we have

$$
\Omega^k(S_i) = \Omega^k({}^i\mathsf{X}^i) \simeq {}^{i-k}\mathsf{X}^{i+k}
$$

as A-modules. Consequently, we have

$$\operatorname{Ext}_A^k(S_i, S_j) = \begin{cases} \mathbb{F} & \text{if } S_j \text{ appears in the head of } {}_{i-k}\mathsf{X}^{i+k}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

From this description one can compute explicitly the Poincaré series of the Ext-groups.

Proposition 2.2. *Given $1 \leq i, j \leq N$, the Poincaré series of $\operatorname{Ext}_A^\bullet(S_i, S_j)$ is given by*

$$\sum_{k \geq 0} \dim_{\mathbb{F}} \operatorname{Ext}_A^k(S_i, S_j) t^k = \frac{Q_{i,j}(t) + t^{2N-1} Q_{i,j}(t^{-1})}{1 - t^{2N}},$$

where $Q_{i,j}(t) = t^{|j-i|} + t^{|j-i|+2} + \cdots + t^{N-1-|N+1-j-i|}$.

Proof. First observe that

$$\Omega^N(S_i) = {}_{i-N}\mathsf{X}^{i+N} = {}_{1+N-i}\mathsf{X}_{1-N-i} = {}_{1+N-i}\mathsf{X}_{1+N-i} = S_{N+1-i}.$$

Then for all $k \geq 0$ we have $\operatorname{Ext}_A^k(S_i, S_j) = \operatorname{Ext}_A^k(S_{N+1-i}, S_{N+1-j})$. Moreover, $Q_{N+1-i,N+1-j} = Q_{i,j} = Q_{j,i}$ so it is enough to prove the lemma under the assumption that $i \leq j$.

Now, assume that $i \leq j$ and let $k \in \{0, \ldots, N-1\}$. If $i + j \leq N + 1$, the simple module S_j appears in the head of ${}_{i-k}\mathsf{X}^{i+k}$ if and only if $k = j-i, j-i+2, \ldots, j+i-2$. The limit cases are indeed ${}_{2i-j}\mathsf{X}^j$ for $k = j - i$ and ${}_{2-j}\mathsf{X}^{2i+j-2} = {}_{j-1}\mathsf{X}^{2i+j-2}$ for $k = j + i - 2$. Note that if $j - i \leq k \leq i + j - 2$ then $j \leq i + k$ and $j \leq 2N - i - k$, so S_j appears in the head of ${}_{i-k}\mathsf{X}^{i+k} = {}_{i-k}\mathsf{X}_{2N-i-k+1}$ whenever k has the suitable parity. If $i + j > N + 1$, one must ensure that $j \leq 2N - i - k$, and therefore S_j appears in the head of ${}_{i-k}\mathsf{X}^{i+k}$ if and only if $k = j - i, j - i + 2, \ldots, 2N - i - j$. Consequently, using the description of the Ext-groups given in (2.2) we have

$$\begin{aligned} \sum_{k=0}^{N-1} \dim_{\mathbb{F}} \operatorname{Ext}_A^k(S_i, S_j) t^k &= t^{j-i} + t^{j-i+2} + \cdots + t^{N-1-|N+1-j-i|} \\ &= t^{|j-i|} + t^{|j-i|+2} + \cdots + t^{N-1-|N+1-j-i|} \\ &= Q_{i,j}(t). \end{aligned} \tag{2.3}$$

Using the relation $\Omega^N(S_i) = S_{N+1-i}$ we obtain

$$\begin{aligned} \sum_{k=0}^{2N-1} \dim_{\mathbb{F}} \operatorname{Ext}_A^k(S_i, S_j) t^k = & \sum_{k=0}^{N-1} \dim_{\mathbb{F}} \operatorname{Ext}_A^k(S_i, S_j) t^k \\ & + t^N \sum_{k=0}^{N-1} \dim_{\mathbb{F}} \operatorname{Ext}_A^k(S_{N+1-i}, S_j) t^k, \end{aligned}$$

which by (2.3) equals $Q_{i,j}(t) + Q_{N+1-i,j}(t)$. Since $Q_{N+1-i,j}(t) = t^{N-1} Q_{i,j}(t^{-1})$, we finally get

$$\sum_{k=0}^{2N-1} \dim_{\mathbb{F}} \operatorname{Ext}_A^k(S_i, S_j) t^k = Q_{i,j}(t) + t^{N-1} Q_{i,j}(t^{-1}),$$

and we conclude using the fact that $\mathrm{Ext}_A^{k+2N}(S_i, S_j) = \mathrm{Ext}_A^k(S_i, S_j)$. □

3. Algebra structure

We denote by $\mathsf{E}(A)$ the Ext-algebra of A, that is, the graded algebra

$$\mathsf{E}(A) := \bigoplus_{1\le i,j\le N} \mathrm{Ext}_A^\bullet(S_i, S_j)$$

endowed with the Yoneda product. We will give in Proposition 3.2 a description of $\mathsf{E}(A)$ as the path algebra of a quiver with relations.

3.1. **Generation.** Let $1 \le i, j \le N$ and let $k \ge 1$. Assume that there is a non-zero map between $\Omega^k S_i$ and S_j; therefore S_j appears in the head of $\Omega^k S_i \simeq {}^{i-k}\mathsf{X}^{i+k}$. If $k \ge N$, any map between $\Omega^k S_i$ and S_j factors through the (unique up to a scalar) isomorphism $\Omega^N S_{N+1-j} \xrightarrow{\sim} S_j$. If $0 < k < N$, one can use the relations (2.1) to see that the module ${}^{i-k}\mathsf{X}^{i+k}$ is not simple. It follows from its structure that at least one of S_{j-1} and S_{j+1} appears in the socle. Consequently, any map between $\Omega^k S_i$ and S_j will factor through a map $\Omega S_{j-1} \longrightarrow S_j$ (if S_{j-1} appears in the socle of ${}^{i-k}\mathsf{X}^{i+k}$) or $\Omega S_{j+1} \longrightarrow S_j$ (if S_{j+1} appears in the socle of ${}^{i-k}\mathsf{X}^{i+k}$). This shows that $\mathsf{E}(A)$ is generated in degrees 1 and N as a left module over itself, hence as an algebra.

3.2. **Minimal resolution.** Recall from §1 that we have chosen non-zero maps $f_i : P_i \longrightarrow P_{i+1}$ and $f_i^* : P_{i+1} \longrightarrow P_i$ such that $f_i^* \circ f_i + f_{i-1} \circ f_{i-1}^* = 0$ for all $2 \le i \le N-1$. Given $1 \le i \le j \le N$ with $j-i$ even we denote by ${}_iP_j$ the following projective A-module:

$${}_iP_j := P_i \oplus P_{i+2} \oplus \cdots \oplus P_{j-2} \oplus P_j.$$

For $1 \le i < j \le N$ with $j-i$ even we let $d_{i,j} : {}_iP_j \longrightarrow {}_{i+1}P_{j-1}$ be the morphism of A-modules corresponding to the following matrix:

$$d_{i,j} = \begin{bmatrix} f_i & f_{i+1}^* & 0 & \cdots & \cdots & 0 \\ 0 & f_{i+2} & f_{i+3}^* & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & f_{j-2} & f_{j-1}^* \end{bmatrix}.$$

The definition of ${}_iP_j$ extends to any integers $i, j \in \mathbb{Z}$ with the convention that

$${}_iP_j = {}_{j+1}P_{i-1}, \quad {}_iP_{-j} = {}_iP_j, \quad {}_iP_{j\pm 2N} = {}_iP_j. \tag{3.1}$$

Note that these relations imply ${}_{1-i}P_j = {}_{1+i}P_j$ and ${}_{i\pm 2N}P_j = {}_iP_j$. Furthermore, the definition of $d_{i,j}$ extends naturally to any pair i, j if we set in addition

$$d_{i,i} = (-1)^i f_i^* \circ f_i = (-1)^{i-1} f_{i-1} \circ f_{i-1}^*,$$

a map from ${}_iP_i = P_i$ to ${}_{i+1}P_{i-1} = P_i$. With this notation one checks that for all $k > 0$ the image of the map $d_{i-k,i+k} : {}_{i-k}P_{i+k} \longrightarrow {}_{i-k+1}P_{i+k-1}$ is isomorphic to ${}^{i-k}\mathsf{X}^{i+k} \simeq \Omega^k(S_i)$ and its kernel to ${}^{i-k-1}\mathsf{X}^{i+k+1} \simeq \Omega^{k+1}(S_i)$, so the bounded above complex

$$R_i := \cdots \xrightarrow{d_{i-k-1,i+k+1}} {}_{i-k}P_{i+k} \xrightarrow{d_{i-k,i+k}} \cdots \xrightarrow{d_{i-2,i+2}} {}_{i-1}P_{i+1} \xrightarrow{d_{i-1,i+1}} P_i \longrightarrow 0$$

forms a minimal projective resolution of S_i.

3.3. **Generators and relations.** We have seen in Section 3.1 that the Ext-algebra is generated in degrees 1 and N. Here we will construct explicit generators using the minimal resolutions defined above.

We start by defining a map $z_i \in \mathrm{Hom}_{\mathsf{Ho}(A)}(R_i, R_{i+1}[1])$ for any $1 \leq i \leq N-1$. Let k be a positive integer. If $k \notin N\mathbb{Z}$, the projective modules ${}_{i-k}P_{i+k}$ and ${}_{i+1-(k-1)}P_{i+1+(k-1)} = {}_{i-k+2}P_{i+k}$ have at least one indecomposable summand in common and we can consider the map $Z_{i,k} : {}_{i-k}P_{i+k} \longrightarrow {}_{i-k+2}P_{i+k}$ given by the identity map on the common factors, followed by multiplication by $(-1)^k$. If $k \in N + 2N\mathbb{Z}$, then from the relations (3.1) we have

$${}_{i-k}P_{i+k} = {}_{i-N}P_{i+N} = {}_{i+N+1}P_{i-N-1} = {}_{-i-N+1}P_{-i+N+1} = P_{N+1-i}$$

and

$${}_{i-k+2}P_{i+k} = {}_{i-N+2}P_{i+N} = {}_{N-i}P_{-N-i} = P_{N-i}.$$

In that case we set $Z_{i,k} := (-1)^i f^*_{N-i}$. If $k \in 2N\mathbb{Z}$ then ${}_{i-k}P_{i+k} = P_i$, ${}_{i-k+2}P_{i+k} = {}_{i+2}P_i = P_{i+1}$ and we set $Z_{i,k} := (-1)^i f_i$. If $k \geq 0$ we set $Z_{i,k} := 0$. Then the family of morphisms of A-modules $Z_i := (Z_{i,k})_{k\in\mathbb{Z}}$ defines a morphism of complexes of A-modules from R_i to $R_{i+1}[1]$ and we denote by z_i its image in $\mathsf{Ho}(A)$. Note that z_i is non-zero; indeed, the composition of Z_i with the natural map $R_{i+1}[1] \longrightarrow S_{i+1}[1]$ is already not null-homotopic since $\mathrm{Ext}^1_A(S_i, S_{i+1}) \neq 0$.

Similarly, we define a map $Z_i^* : R_{i+1} \longrightarrow R_i[1]$ by exchanging the roles of f and f^*. More precisely, we consider in that case $Z^*_{i,-N} := (-1)^i f_{N-i}$ and $Z^*_{i,-2N} := (-1)^i f_i^*$. We denote by z_i^* the image of Z_i^* in $\mathsf{Ho}(A)$.

Assume now that $1 \leq i \leq N$. The modules

$${}_{i-k}P_{i+k} \quad \text{and} \quad {}_{(N+1-i)-(k-N)}P_{(N+1-i)+(k-N)}$$

are equal, which means that starting from the degree $-N$ the terms of the complexes R_i and $R_{N+1-i}[N]$ coincide. In addition, the differentials only differ by $(-1)^N$. We denote by $Y_i : R_i \longrightarrow R_{N+1-i}[N]$ the natural projection between R_i and its obvious truncation at degrees $\leq -N$, followed by the multiplication by $(-1)^{Nk}$ in each degree k. We will write y_i for its image in $\mathsf{Ho}(A)$. Again, y_i is non-zero since $\mathrm{Ext}^N_A(S_i, S_{N+1-i}) \neq 0$.

Lemma 3.1. *The following relations hold in* $\mathrm{End}^\bullet_{\mathsf{Ho}(A)}(\bigoplus R_i)$*:*

(a) $z_1^*[1] \circ z_1 = 0$, $z_{N-1}[1] \circ z^*_{N-1} = 0$*;*
(b) $z_i[1] \circ z_i^* = z^*_{i+1}[1] \circ z_{i+1}$ *for all* $i = 1, \ldots, N-2$*;*
(c) $y_{i+1}[1] \circ z_i = z^*_{N-i}[N] \circ y_i$ *for all* $i = 1, \ldots, N-1$*;*
(d) $y_i[1] \circ z_i^* = z_{N-i}[N] \circ y_{i+1}$ *for all* $i = 1, \ldots, N-1$.

Proof. If $N = 1$, there are no relations to check. Note that in that case the algebra A is isomorphic to $\mathbb{F}[t]/(t^2)$. It is a Koszul algebra whose dual is isomorphic to $\mathbb{F}[t]$. Therefore we assume $N \geq 2$. The relations in (a) follow from the fact that $\mathrm{Ext}^2_A(S_1, S_1) = \mathrm{Ext}^2_A(S_N, S_N) = 0$, which is for example a consequence of Proposition 2.2.

To show (c), we observe that the morphism of complexes $Z_i : R_i \longrightarrow R_{i+1}[1]$ defined above coincides with $Z^*_{N-i}[N] : R_{N+1-i}[N] \longrightarrow R_{N-i}[N+1]$ in degrees less than $-N$. Since Y_i and Y_{i+1} are just obvious truncations with suitable signs we actually have $Y_{i+1}[1] \circ Z_i = Z^*_{N-i}[N] \circ Y_i$. The relation (d) is obtained by a similar argument.

We now consider (b). The morphisms of complexes $Z_i[1] \circ Z^*_i$ and $Z^*_{i+1}[1] \circ Z_{i+1}$ coincide at every degree k except when k is congruent to 0 or -1 modulo N. Let us first look in detail at the degrees $-N$ and $-N-1$. The map $Z_i[1] \circ Z^*_i$ is as follows:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & P_{N-1-i} \oplus P_{N+1-i} \arrow[r, "{\begin{bmatrix} f_{N-1-i} & f^*_{N-i} \end{bmatrix}}"] \arrow[d, "{\begin{bmatrix} 0 & (-1)^{N+1} \end{bmatrix}}"] & P_{N-i} \arrow[r, "(-1)^{N-i} f^*_{N-i} \circ f_{N-i}"] \arrow[d, "(-1)^i f_{N-i}"] & P_{N-i} \arrow[d, "{\begin{bmatrix} (-1)^{N-1} \\ 0 \end{bmatrix}}"] \\
P_{N-i} \oplus P_{N+2-i} \arrow[r, "{-\begin{bmatrix} f_{N-i} & f^*_{N+1-i} \end{bmatrix}}"] \arrow[d, "{\begin{bmatrix} (-1)^{N+1} & 0 \end{bmatrix}}"] & P_{N+1-i} \arrow[r, "(-1)^{N-i+1} f_{N-i} \circ f^*_{N-i}"] \arrow[d, "(-1)^i f^*_{N-i}"] & P_{N+1-i} \arrow[r, "{-\begin{bmatrix} f^*_{N-i} \\ f_{N+1-i} \end{bmatrix}}"] \arrow[d, "{\begin{bmatrix} 0 \\ (-1)^{N-1} \end{bmatrix}}"] & P_{N-i} \oplus P_{N+2-i} \arrow[d] \\
P_{N-i} \arrow[r, "(-1)^{N-i} f^*_{N-i} \circ f_{N-i}"] & P_{N-i} \arrow[r, "{\begin{bmatrix} f^*_{N-1-i} \\ f_{N-i} \end{bmatrix}}"] & P_{N-1-i} \oplus P_{N+1-i} \arrow[r] & \cdots
\end{tikzcd}$$

whereas the map $Z^*_{i+1}[1] \circ Z_{i+1}$ corresponds to the following composition:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & P_{N-1-i} \oplus P_{N+1-i} \arrow[r, "{\begin{bmatrix} f_{N-1-i} & f^*_{N-i} \end{bmatrix}}"] \arrow[d, "{\begin{bmatrix} (-1)^{N+1} & 0 \end{bmatrix}}"] & P_{N-i} \arrow[r, "(-1)^{N-i} f^*_{N-i} \circ f_{N-i}"] \arrow[d, "(-1)^{i+1} f^*_{N-1-i}"] & P_{N-i} \arrow[d, "{\begin{bmatrix} 0 \\ (-1)^{N-1} \end{bmatrix}}"] \\
P_{N-2-i} \oplus P_{N-i} \arrow[r, "{-\begin{bmatrix} f_{N-2-i} & f^*_{N-1-i} \end{bmatrix}}"] \arrow[d, "{\begin{bmatrix} 0 & (-1)^{N+1} \end{bmatrix}}"] & P_{N-1-i} \arrow[r, "(-1)^{N-i} f^*_{N-1-i} \circ f_{N-1-i}"] \arrow[d, "(-1)^{i+1} f_{N-1-i}"] & P_{N-1-i} \arrow[r, "{-\begin{bmatrix} f^*_{N-2-i} \\ f_{N-1-i} \end{bmatrix}}"] \arrow[d, "{\begin{bmatrix} (-1)^{N-1} \\ 0 \end{bmatrix}}"] & P_{N-2-i} \oplus P_{N-i} \arrow[d] \\
P_{N-i} \arrow[r, "(-1)^{N-i} f^*_{N-i} \circ f_{N-i}"] & P_{N-i} \arrow[r, "{\begin{bmatrix} f^*_{N-1-i} \\ f_{N-i} \end{bmatrix}}"] & P_{N-1-i} \oplus P_{N+1-i} \arrow[r] & \cdots
\end{tikzcd}$$

We deduce that at the degrees $-N$ and $-N-1$ the map $Z_i[1]\circ Z_i^* - Z_{i+1}^*[1]\circ Z_{i+1}$ is given by

$$\begin{tikzcd}
P_{N-1-i}\oplus P_{N+1-i} \arrow[r, "{\begin{bmatrix} f_{N-1-i} & f_{N-i}^* \end{bmatrix}}"] \arrow[d, "{(-1)^{N+1-i}\begin{bmatrix} f_{N-1-i} & f_{N-i}^* \end{bmatrix}}"'] & P_{N-i} \arrow[d, "{(-1)^{N+1-i}\begin{bmatrix} f_{N-1-i}^* \\ f_{N-i} \end{bmatrix}}"] \\
P_{N-i} \arrow[r, "{\begin{bmatrix} f_{N-1-i}^* \\ f_{N-i} \end{bmatrix}}"] & P_{N-1-i}\oplus P_{N+1-i}
\end{tikzcd}$$

A similar picture holds at the degrees $-2N$ and $-2N-1$:

$$\begin{tikzcd}
P_i\oplus P_{i+2} \arrow[r, "{\begin{bmatrix} f_i & f_{i+1}^* \end{bmatrix}}"] \arrow[d, "{(-1)^{i+1}\begin{bmatrix} f_i & f_{i+1}^* \end{bmatrix}}"'] & P_{i+1} \arrow[d, "{(-1)^{i+1}\begin{bmatrix} f_i^* \\ f_{i+1} \end{bmatrix}}"] \\
P_{i+1} \arrow[r, "{\begin{bmatrix} f_i^* \\ f_{i+1} \end{bmatrix}}"] & P_i\oplus P_{i+2}
\end{tikzcd}$$

Using the map $s: R_{i+1}\to R_{i+1}[1]$ defined by

$$s_k := \begin{cases} (-1)^{N+1-i}\mathrm{Id}_{P_{N-i}} & \text{if } -k\in N+2N\mathbb{N}, \\ (-1)^{i+1}\mathrm{Id}_{P_{i+1}} & \text{if } -k\in 2N+2N\mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

we see that $Z_i[1]\circ Z_i^* - Z_{i+1}^*[1]\circ Z_{i+1}$ is null-homotopic, which proves that $z_i[1]\circ z_i^* - z_{i+1}^*[1]\circ z_{i+1}$ is zero in $\mathrm{Hom}_{\mathsf{Ho}(A)}(R_{i+1}, R_{i+1}[2])$. $\square$

The next proposition shows that the relations given in Lemma 3.1 are actually enough to describe the Ext-algebra. We use here the concatenation of paths as opposed to the composition of arrows, which explains the discrepancy in the relations.

Proposition 3.2. *The* Ext-*algebra of A is isomorphic to the path algebra associated with the following quiver:*

$$\begin{tikzcd}
S_1 \arrow[r, bend left, "z_1"] \arrow[rrrrrr, bend left=40, "y_1"] & S_2 \arrow[l, bend left, "z_1^*"] \arrow[r, bend left, "z_2"] \arrow[rrrr, bend left=35, "y_2"] & S_3 \arrow[l, bend left, "z_2^*"] & \cdots \arrow[r, bend left, "y_3"] & S_{N-2} \arrow[l, bend left, "y_{N-2}"] \arrow[r, bend left, "z_{N-2}"] & S_{N-1} \arrow[l, bend left, "z_{N-2}^*"] \arrow[r, bend left, "z_{N-1}"] \arrow[llll, bend left=35, "y_{N-1}"] & S_N \arrow[l, bend left, "z_{N-1}^*"] \arrow[llllll, bend left=40, "y_N"]
\end{tikzcd}$$

with z_i's of degree 1 *and y_i's of degree N, subject to the relations*

(a) $z_1z_1^* = z_{N-1}^*z_{N-1} = 0$;
(b) $z_i^*z_i = z_{i+1}z_{i+1}^*$ *for all* $i = 1, \ldots, N-2$;
(c) $z_iy_{i+1} = y_iz_{N-i}^*$ *for all* $i = 1, \ldots, N-1$;
(d) $z_i^*y_i = y_{i+1}z_{N-i}$ *for all* $i = 1, \ldots, N-1$.

Proof. Let Q (resp. I) be the quiver (resp. the ideal generated by the set of relations) given in the proposition. Let $\Gamma = \mathbb{F}Q/I$ be the corresponding path algebra. By Section 3.1 and Lemma 3.1, the Ext-algebra $\mathsf{E}(A)$ of A is a quotient of Γ. To show that $\mathsf{E}(A) \simeq \Gamma$ it is enough to show that the graded dimension of Γ is smaller than that of $\mathsf{E}(A)$.

Let $1 \leq i, j \leq N$ and γ be a path between S_i and S_j in Q containing only z_l's and z_l^*'s. Let k be the length of γ. We have $k \geq |i-j|$, which is the length of the minimal path from S_i to S_j. Using the relations, there exist cycles γ_1 and γ_2 around S_i and S_j respectively such that

$$\gamma = \begin{cases} \gamma_1 z_i z_{i+1} \cdots z_{j-1} = z_i z_{i+1} \cdots z_{j-1}\gamma_2 & \text{if } i \leq j, \\ \gamma_1 z_{i-1}^* z_{i-2}^* \cdots z_j^* = z_{i-1}^* z_{i-2}^* \cdots z_j^* \gamma_2 & \text{otherwise.} \end{cases}$$

Maximal non-zero cycles starting and ending at S_i are either $z_{i-1}^*z_{i-2}^* \cdots z_1^*z_1z_2 \cdots z_{i-1}$ or $z_iz_{i+1} \cdots z_{N-1}z_{N-1}^* \cdots z_{i+1}^* \cdots z_i^*$ depending on whether S_i is closer to S_1 or S_N. Indeed, any longer cycle will involve $z_1z_1^*$ or $z_{N-1}^*z_{N-1}$, which are zero by (a). Therefore if $\deg(\gamma_1) > 2(i-1)$ or $\deg(\gamma_1) > 2(N-i)$ then $\gamma_1 = 0$. Using a similar argument for cycles around S_j we deduce that γ is zero whenever

$$k = \deg(\gamma) > |i-j| + 2\min(i-1, j-1, N-i, N-j),$$

which is equivalent to $k = \deg(\gamma) > N - 1 - |N+1-j-i|$. This proves that γ is zero unless $|i-j| \leq k \leq N-1-|N+1-j-i|$, in which case we have

$$\gamma = z_iz_{i+1} \cdots z_{r-1}z_{r-1}^*z_{r-2}^* \cdots z_j^*,$$

where $k = 2r - i - j$. In particular, $k - |i-j|$ must be even. Consequently, the subspace of Γ spanned by such paths has graded dimension at most equal to $t^{|i-j|} + t^{|i-j|+2} + \cdots + t^{N-1-|N+1-j-i|} = Q_{i,j}(t)$.

Assume now that γ is any path of length k between S_i and S_j in Q. Using the relations one can write γ as $\gamma = y_i^a \gamma_1 \gamma_2$, where γ_2 is a cycle around S_j containing only y_l's (therefore a power of $y_j y_{N-j}$), γ_1 is a product of z_l's, and $a \in \{0,1\}$. Note that $\deg(\gamma_2)$ is a multiple of $2N$ and γ_1 is either a path from S_i to S_j if $a=0$ or a path from S_{N+1-i} to S_j if $a=1$. From the previous discussion and Proposition 2.2 we conclude that γ is zero if $\dim_{\mathbb{F}} \operatorname{Ext}_A^k(S_i, S_j) = 0$ or unique modulo I otherwise. By (2.2) and §3.1 this shows that the projection of Γ to the Ext-algebra of A must be an isomorphism. ☐

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