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PARALLEL SKEW-SYMMETRIC TENSORS ON 4-DIMENSIONAL METRIC LIE ALGEBRAS

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Abstract. We give a complete classification, up to isometric isomorphism and scaling, of 4-dimensional metric Lie algebras $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ that admit a non-zero parallel skew-symmetric endomorphism. In particular, we distinguish those metric Lie algebras that admit such an endomorphism which is not a multiple of a complex structure, and for each of them we obtain the de Rham decomposition of the associated simply connected Lie group with the corresponding left invariant metric. On the other hand, we find that the associated simply connected Lie group is irreducible as a Riemannian manifold for those metric Lie algebras where each parallel skew-symmetric endomorphism is a multiple of a complex structure.

1. Introduction

Let (M, g) be a Riemannian manifold. A skew-symmetric $(1,1)$-tensor $H : TM \to TM$ is said to be *parallel* if $(\nabla_X H)Y = 0$ for all vector fields $X, Y \in \mathfrak{X}(M)$, where ∇ denotes the Levi-Civita connection associated to g. If in addition $H^2 = -I$ (where I denotes the identity map), then H is a complex structure and (M, g, H) is called a *Kähler manifold*, an object widely studied in the literature. Here, we are interested in (connected) manifolds that admit parallel tensors H that are not a multiple of a complex structure, that is, $H^2 \neq -\lambda^2 I$ for any $\lambda \in \mathbb{R}$.

We focus particularly on pairs (G, g), where G is a 4-dimensional non-abelian Lie group and g is a left invariant metric on G, and we search for left invariant parallel skew-symmetric $(1,1)$-tensors on G. As usual, we work at the Lie algebra level. Namely, we consider non-abelian 4-dimensional metric Lie algebras $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, and look for non-zero skew-symmetric endomorphisms $H : \mathfrak{g} \to \mathfrak{g}$ that are parallel (see Section 2). If we add the condition $H^2 = -I$, then this problem was completely settled by Ovando in [6], where 4-dimensional pseudo-Kähler Lie algebras were classified up to equivalence. In the present paper, we classify up to isometric isomorphism and scaling non-abelian 4-dimensional metric Lie algebras $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ admitting a parallel endomorphism that is not a multiple of a complex structure.

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In addition, given such a pair $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, we classify all possible parallel endomorphisms. These results are included in Section 3; more precisely, in Theorem 3.3 and Proposition 3.8.

Finally, in Section 4, for each non-abelian 4-dimensional metric Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ admitting a non-zero parallel skew-symmetric endomorphism, we study the de Rham decomposition of the associated simply connected Riemannian Lie group (G, g). We find that (G, g) is irreducible (as a Riemannian manifold) if and only if the only parallel endomorphisms on $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ are multiple of complex structures. Note that the 'only if' part is a consequence of the following well-known result: *Let (M, g) be a complete simply connected irreducible Riemannian manifold. Then every parallel skew-symmetric (1,1)-tensor is a multiple of a complex structure* [7, Theorem 10.3.2].

2. Preliminaries

In this section we recall the general definition of parallel skew-symmetric tensor on a Riemannian manifold and then we adapt it to the case of a left invariant parallel skew-symmetric tensor on a Lie group with a left invariant metric. This enables us to work at the Lie algebra level. We also define a notion of equivalence on parallel tensors on a metric Lie algebra.

2.1. **Parallel tensors.** Let (M, g) be a Riemannian manifold and let ∇ be the associated Levi-Civita connection. A skew-symmetric tensor $H : TM \to TM$ is said to be *parallel* if $(\nabla_X H) Y = 0$ for all $X, Y \in \mathfrak{X}(M)$, where ∇H denotes the covariant derivative of H. Recall that

$$(\nabla_X H) Y = \nabla_X (HY) - H (\nabla_X Y).$$

If in addition $H^2 = -I$, then H is a complex structure and (M, g, H) is a Kähler manifold, with the Kähler form given by $\omega(X, Y) = g(HX, Y)$.

2.2. **Left invariant parallel tensors on Lie groups.** Let G be a Lie group, and let $\mathfrak{g}$ be the Lie algebra of left invariant vector fields on G. We assume that g is a left invariant metric, i.e., the left translation $L_p : G \to G$ is an isometry for any $p \in G$. Every left invariant metric g on G determines an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$: $\langle x, y\rangle := g_e(x_e, y_e)$ for $x, y \in \mathfrak{g}$; and conversely, any inner product on $\mathfrak{g}$ determines uniquely a left invariant metric on G.

We fix a left invariant metric g on G and denote as $\langle\cdot,\cdot\rangle$ its induced inner product on $\mathfrak{g}$. Let ∇ be the Levi-Civita connection associated to g. It is a fact that for $x, y \in \mathfrak{g}$, $\nabla_x y \in \mathfrak{g}$, and it is given by the Koszul formula

$$2\langle \nabla_x y, z\rangle = \langle [x, y], z\rangle - \langle [y, z], x\rangle + \langle [z, x], y\rangle. \tag{2.1}$$

It is easy to see that $\nabla_x : \mathfrak{g} \to \mathfrak{g}$ is a skew-symmetric endomorphism with respect to $\langle\cdot,\cdot\rangle$ for any $x \in \mathfrak{g}$.

We consider skew-symmetric $(1, 1)$-tensors $H : TG \to TG$ that are left invariant, i.e., $(\mathrm{d}L_p)_q H_q = H_{pq}(\mathrm{d}L_p)_q$ for all $p, q \in G$. Every such tensor induces a skew-symmetric endomorphism $H : \mathfrak{g} \to \mathfrak{g}$ (again denoted by H) and conversely any skew-symmetric endomorphism on $\mathfrak{g}$ extends uniquely to a skew-symmetric left invariant tensor on G.

It is easy to see that $H : TG \to TG$ as above is parallel if and only if the associated endomorphism $H : \mathfrak{g} \to \mathfrak{g}$ is parallel, in the sense that H commutes with $\nabla_x : \mathfrak{g} \to \mathfrak{g}$ for all $x \in \mathfrak{g}$.

2.3. **Parallel endomorphisms on metric Lie algebras.** The previous discussion enables us to work algebraically. Namely, we fix an abstract real Lie algebra $\mathfrak{g}$ endowed with an inner product $\langle\cdot,\cdot\rangle$, and we search for skew-symmetric linear endomorphisms $H : \mathfrak{g} \to \mathfrak{g}$ satisfying the condition $\nabla_x(Hy) = H\nabla_x y$ for all $x, y \in \mathfrak{g}$, where $\nabla_x y$ is defined via the Koszul formula (2.1).

The pair $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is called a *metric Lie algebra*, and an endomorphism H as above is called a *parallel tensor*. If in addition $H^2 = -I$, then $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is called a *Kähler Lie algebra*.

We intend to classify triples $(\mathfrak{g}, \langle\cdot,\cdot\rangle, H)$, where H is a non-zero parallel skew-symmetric endomorphism. Note that if H is a parallel skew-symmetric endomorphism on $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, then the same holds on $(\mathfrak{g}, \lambda^2\langle\cdot,\cdot\rangle)$ for any $\lambda > 0$. This implies that we may consider the same parallel skew-symmetric endomorphism H on homothetic metrics. In addition, cH will be also parallel on $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ for any $c \in \mathbb{R}$.

A natural notion of equivalence on the set of parallel skew-symmetric endomorphisms on $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is given by the following definition:

Definition 2.1. H_1 and H_2 are said to be *equivalent* if there exists an isometric isomorphism of Lie algebras $\phi : \mathfrak{g} \to \mathfrak{g}$ such that $\phi H_1 = H_2\phi$.

We can now restate the classification problem as follows:

Problem. Classify 4-dimensional metric Lie algebras $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ (up to isometry and homothety) that admit a non-zero parallel skew-symmetric endomorphism H, and classify H up to equivalence.

3. Main results

In this section we determine all triples $(\mathfrak{g}, \langle\cdot,\cdot\rangle, H)$, where $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is a non-abelian 4-dimensional metric Lie algebra and $H : \mathfrak{g} \to \mathfrak{g}$ is a non-zero parallel skew-symmetric endomorphism. The case where $H^2 = -I$ was done in [6] and we start by recalling this.

3.1. **4-dimensional Kähler Lie algebras.** We first list the non-abelian real solvable Lie algebras of dimension ≤ 3 with the notation used in [1]. These are:

$$\begin{aligned} \mathfrak{aff}(\mathbb{R}) &: [e_1, e_2] = e_2, \\ \mathfrak{h}_3 &: [e_1, e_2] = e_3, \\ \mathfrak{r}_{3,\lambda} &: [e_1, e_2] = e_2,\ [e_1, e_3] = \lambda e_3, \quad \lambda \in \mathbb{R}, \\ \mathfrak{r}'_{3,\lambda} &: [e_1, e_2] = \lambda e_2 - e_3,\ [e_1, e_3] = e_2 + \lambda e_3,\ \lambda \in \mathbb{R}, \end{aligned} \tag{3.1}$$

where $\{e_1, e_2\}$ is a basis of $\mathfrak{aff}(\mathbb{R})$ and $\{e_1, e_2, e_3\}$ is a basis of $\mathfrak{h}_3$, $\mathfrak{r}_{3,\lambda}$ and $\mathfrak{r}'_{3,\lambda}$. They are all pairwise non-isomorphic, except for $\mathfrak{r}_{3,\lambda} \cong \mathfrak{r}_{3,1/\lambda}$ if $\lambda \neq 0$, and $\mathfrak{r}'_{3,\lambda} \cong \mathfrak{r}'_{3,-\lambda}$.

As a side note, $\mathfrak{r}_{3,-1}$ is the Lie algebra of the group of rigid motions on the Minkowski plane, and it is usually denoted by $\mathfrak{e}(1,1)$. Note also that $\mathfrak{r}_{3,0} = \mathbb{R} \times \mathfrak{aff}(\mathbb{R})$, and that $\mathfrak{r}'_{3,0}$ is the Lie algebra of the group of rigid motions on the 2-dimensional euclidean plane and is usually denoted by $\mathfrak{e}(2)$.

We now list three families of 4-dimensional solvable Lie algebras expressed in the basis $\{e_1, e_2, e_3, e_4\}$ (we follow the notation given in [1]):

$$\begin{aligned} \mathfrak{r}'_{4,\lambda,0} &: [e_4,e_1] = \lambda e_1,\ [e_4,e_2] = -e_3,\ [e_4,e_3] = e_2,\ \lambda > 0, \\ \mathfrak{d}_{4,\lambda} &: [e_4,e_1] = \lambda e_1,\ [e_4,e_2] = (1-\lambda)e_2,\ [e_4,e_3] = e_3,\ [e_1,e_2] = e_3,\ \lambda \geq \frac{1}{2}, \\ \mathfrak{d}'_{4,\lambda} &: [e_4,e_1] = \lambda e_1 - e_2,\ [e_4,e_2] = e_1 + \lambda e_2,\ [e_4,e_3] = 2\lambda e_3,\ [e_1,e_2] = e_3,\ \lambda \geq 0. \end{aligned} \tag{3.2}$$

These are all pairwise non-isomorphic, according to [1, Theorem 1.5].

The content of the next theorem is included in [6, Proposition 3.3], where 4-dimensional pseudo-Kähler Lie algebras were classified. Here we only consider positive definite Kähler Lie algebras.

Theorem 3.1 ([6, Proposition 3.3])**.** *Let $(\mathfrak{g}, \langle\cdot,\cdot\rangle, J)$ be a 4-dimensional Kähler Lie algebra. Then there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ where the Lie brackets and J are as given in Table 1.*

Lie algebra	Lie bracket in an orthonormal basis	Complex structure
$\mathbb{R}^2 \times \mathfrak{aff}(\mathbb{R})$	$[e_1,e_2] = te_2$, $t > 0$	$Je_1 = e_2$, $Je_3 = e_4$
$\mathbb{R} \times \mathfrak{e}(2)$	$[e_1,e_2] = -te_3$, $[e_1,e_3] = te_2$, $t > 0$	$Je_1 = e_4$, $Je_2 = e_3$
$\mathfrak{r}'_{4,\lambda,0}$	$[e_4,e_1] = te_1$, $[e_4,e_2] = -\frac{t}{\lambda}e_3$,	$J_1e_1 = -e_4$, $J_1e_2 = e_3$
$\lambda > 0$	$[e_4,e_3] = \frac{t}{\lambda}e_2$, $t > 0$	$J_2e_1 = -e_4$, $J_2e_2 = -e_3$
$\mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$	$[e_1,e_2] = te_2$, $[e_3,e_4] = se_4$, $t, s > 0$	$Je_1 = e_2$, $Je_3 = e_4$
$\mathfrak{d}_{4,\frac{1}{2}}$	$[e_1,e_2] = te_3$, $[e_4,e_3] = te_3$,	$Je_1 = e_2$, $Je_4 = e_3$
	$[e_4,e_1] = \frac{t}{2}e_1$, $[e_4,e_2] = \frac{t}{2}e_2$, $t > 0$	
$\mathfrak{d}_{4,2}$	$[e_1,e_2] = te_3$, $[e_4,e_3] = \frac{t}{2}e_3$,	$Je_4 = -e_1$, $Je_2 = e_3$
	$[e_4,e_1] = te_1$, $[e_4,e_2] = -\frac{t}{2}e_2$, $t > 0$	
$\mathfrak{d}'_{4,\frac{\delta}{2}}$	$[e_1,e_2] = te_3$, $[e_4,e_1] = \frac{t}{2}e_1 - \frac{t}{\delta}e_2$,	$J_1e_1 = e_2$, $J_1e_4 = e_3$
$\delta > 0$	$[e_4,e_3] = te_3$, $[e_4,e_2] = \frac{t}{\delta}e_1 + \frac{t}{2}e_2$, $t > 0$	$J_2e_1 = -e_2$, $J_2e_4 = -e_3$

TABLE 1. 4-dimensional Kähler Lie algebras

3.2. **Parallel endomorphisms that are not multiple of a complex structure.** We now describe non-abelian 4-dimensional metric Lie algebras that admit a parallel skew-symmetric endomorphism H which is not multiple of a complex structure. We may assume that H is always non-zero in this subsection.

We will use the following lemma whose proof is straightforward.

Lemma 3.2. *Let $A, B \in \mathbb{R}^{4\times 4}$ be two skew-symmetric matrices such that*

$$B = \begin{bmatrix} 0 & -s & 0 & 0 \\ s & 0 & 0 & 0 \\ 0 & 0 & 0 & -t \\ 0 & 0 & t & 0 \end{bmatrix} \qquad \text{with } |s| \neq |t|.$$

If $AB = BA$, then A has also the form of B, that is, there exist $s', t' \in \mathbb{R}$ such that

$$A = \begin{bmatrix} 0 & -s' & 0 & 0 \\ s' & 0 & 0 & 0 \\ 0 & 0 & 0 & -t' \\ 0 & 0 & t' & 0 \end{bmatrix}.$$

Theorem 3.3. *Let $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ be a non-abelian 4-dimensional metric Lie algebra, and let $H : \mathfrak{g} \to \mathfrak{g}$ be a parallel skew-symmetric endomorphism which is not a multiple of a complex structure. Then there exists an orthogonal basis $\{e_1, f_1, e_2, f_2\}$ under which the Lie brackets, $\langle\cdot,\cdot\rangle$ and H are as given in Table 2 with $|a_1| \neq |a_2|$. Moreover, any endormophism appearing in Table 2 is parallel for the corresponding metric Lie algebra.*

Proof. Let $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ be a non-abelian 4-dimensional metric Lie algebra, and let H be any skew-symmetric endomorphism. Fix an orthonormal basis $\{e_1, f_1, e_2, f_2\}$ such that $H(e_i) = a_i f_i$, $H(f_i) = -a_i e_i$ for $i = 1, 2$. Assume now that H is parallel, which means that $\nabla_x H = H\nabla_x$ for all $x \in \mathfrak{g}$, and assume also that H is not a multiple of a complex structure, which implies that $|a_1| \neq |a_2|$. Using Lemma 3.2, ∇_x has the following form (with respect to the fixed basis):

$$\nabla_x = \begin{bmatrix} 0 & -\alpha(x) & 0 & 0 \\ \alpha(x) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta(x) \\ 0 & 0 & \beta(x) & 0 \end{bmatrix}$$

for some linear forms α and β on $\mathfrak{g}$ that are not zero simultaneously. The Lie brackets can be expressed in terms of α and β using that the Levi-Civita connection is torsion-free:

$$\begin{aligned} [e_1, f_1] &= -\alpha(e_1)e_1 - \alpha(f_1)f_1, & [e_1, e_2] &= \beta(e_1)f_2 - \alpha(e_2)f_1, \\ [e_1, f_2] &= -\beta(e_1)e_2 - \alpha(f_2)f_1, & [f_1, e_2] &= \beta(f_1)f_2 + \alpha(e_2)e_1, \\ [f_1, f_2] &= -\beta(f_1)e_2 + \alpha(f_2)e_1, & [e_2, f_2] &= -\beta(e_2)e_2 - \beta(f_2)f_2. \end{aligned} \tag{3.3}$$

Lie algebra	Metric	Parallel tensor
$\mathbb{R}\times\mathfrak{e}(2)$: $[e_1,e_2]=-f_2$, $[e_1,f_2]=e_2$	$\langle\cdot,\cdot\rangle_t=\begin{bmatrix} t &&& \\ & t && \\ && t & \\ &&& t\end{bmatrix}$, $t>0$	$\begin{bmatrix} 0 & -a_1 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_2 \\ 0 & 0 & a_2 & 0\end{bmatrix}$
$\mathbb{R}^2\times\mathfrak{aff}(\mathbb{R})$: $[e_2,f_2]=f_2$	$\langle\cdot,\cdot\rangle_t=\begin{bmatrix} t &&& \\ & t && \\ && t & \\ &&& t\end{bmatrix}$, $t>0$	$\begin{bmatrix} 0 & -a_1 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_2 \\ 0 & 0 & a_2 & 0\end{bmatrix}$
$\mathfrak{r}'_{4,\lambda,0},\ \lambda>0$: $[e_1,f_1]=\lambda f_1$, $[e_1,f_2]=e_2$, $[e_1,e_2]=-f_2$	$\langle\cdot,\cdot\rangle_t=\begin{bmatrix} t &&& \\ & t && \\ && t & \\ &&& t\end{bmatrix}$, $t>0$	$\begin{bmatrix} 0 & -a_1 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_2 \\ 0 & 0 & a_2 & 0\end{bmatrix}$
$\mathfrak{aff}(\mathbb{R})\times\mathfrak{aff}(\mathbb{R})$: $[e_1,f_1]=f_1$, $[e_2,f_2]=f_2$	$\langle\cdot,\cdot\rangle_{t,s}=\begin{bmatrix} t &&& \\ & t && \\ && ts & \\ &&& ts\end{bmatrix}$, $s,t>0$, $s\leq 1$	$\begin{bmatrix} 0 & -a_1 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_2 \\ 0 & 0 & a_2 & 0\end{bmatrix}$

TABLE 2. 4-dimensional metric Lie algebras that admit a parallel tensor not multiple of a complex structure

Applying the Jacobi identity several times we obtain the following relations:

$$\alpha(e_1)\beta(e_1)+\alpha(f_1)\beta(f_1)=0 \tag{3.4}$$

$$\alpha(e_1)\alpha(e_2)+\alpha(f_2)\beta(f_1)=0 \tag{3.5}$$

$$-\alpha(e_2)\alpha(f_1)+\alpha(f_2)\beta(e_1)=0 \tag{3.6}$$

$$\alpha(e_1)\alpha(f_2)-\alpha(e_2)\beta(f_1)=0 \tag{3.7}$$

$$\alpha(f_1)\alpha(f_2)+\alpha(e_2)\beta(e_1)=0 \tag{3.8}$$

$$\beta(e_2)\alpha(e_2)+\beta(f_2)\alpha(f_2)=0 \tag{3.9}$$

$$\beta(e_2)\beta(e_1)+\beta(f_1)\alpha(f_2)=0 \tag{3.10}$$

$$-\beta(e_1)\beta(f_2)+\beta(f_1)\alpha(e_2)=0 \tag{3.11}$$

$$\beta(e_2)\beta(f_1)-\beta(e_1)\alpha(f_2)=0 \tag{3.12}$$

$$\beta(f_2)\beta(f_1)+\beta(e_1)\alpha(e_2)=0. \tag{3.13}$$

We first rewrite (3.4)–(3.13) as matrix products and computation of determinants. Conditions (3.4) and (3.9) can be written as

$$\det\underbrace{\begin{pmatrix}\alpha(e_1) & -\beta(f_1)\\ \alpha(f_1) & \beta(e_1)\end{pmatrix}}_{=:U}=0,\quad \det\underbrace{\begin{pmatrix}\beta(e_2) & -\alpha(f_2)\\ \beta(f_2) & \alpha(e_2)\end{pmatrix}}_{=:V}=0. \tag{3.14}$$

Conditions (3.5)–(3.8) can be written as

$$\underbrace{\begin{pmatrix}\alpha(e_2) & \alpha(f_2)\\ \alpha(f_2) & -\alpha(e_2)\end{pmatrix}}_{=:A}\underbrace{\begin{pmatrix}\alpha(e_1) & \beta(e_1)\\ \beta(f_1) & \alpha(f_1)\end{pmatrix}}_{=:B} = 0_{2\times 2}, \tag{3.15}$$

Similarly, (3.10)–(3.12) can be written as

$$\underbrace{\begin{pmatrix}\beta(e_1) & \beta(f_1)\\ \beta(f_1) & -\beta(e_1)\end{pmatrix}}_{=:C}\underbrace{\begin{pmatrix}\beta(e_2) & \alpha(e_2)\\ \alpha(f_2) & \beta(f_2)\end{pmatrix}}_{=:D} = 0_{2\times 2}. \tag{3.16}$$

Note that $\det A = -(\alpha(e_2)^2 + \alpha(f_2)^2)$ and $\det C = -(\beta(e_1)^2 + \beta(f_1)^2)$, from which it follows that A is either zero or invertible, and similarly with C. For the rest of the discussion we consider the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}^*$, where $\{e^1, f^1, e^2, f^2\}$, the dual basis of $\{e_1, f_1, e_2, f_2\}$, is orthonormal.

If $C \neq 0$, then it is invertible and (3.16) implies that $D = 0$, that is, $\alpha|_{\mathrm{Span}\{e_2,f_2\}} = \beta|_{\mathrm{Span}\{e_2,f_2\}} = 0$. From $\det U = 0$ we obtain that $(\alpha(e_1), \alpha(f_1)) = \mu(-\beta(f_1), \beta(e_1))$ for some μ. We consider a new orthonormal basis $\{e_1', f_1', e_2, f_2\}$ by setting $e_1' := \frac{\beta(e_1)e_1+\beta(f_1)f_1}{\|\beta\|}$ and $f_1' := \frac{-\beta(f_1)e_1+\beta(e_1)f_1}{\|\beta\|}$. One readily checks that the Lie brackets are given by $[e_1', f_1'] = -\mu\|\beta\| f_1'$, $[e_1', f_2] = -\|\beta\| e_2$, $[e_1', e_2] = \|\beta\| f_2$, and that $He_1' = a_1 f_1'$ and $Hf_1' = -a_1 e_1'$. We now analyze the following two cases:

- (i) Case $\mu = 0$: We redefine $\{e_1, f_1, e_2, f_2\}$ as $\left\{-\frac{e_1'}{\|\beta\|}, -\frac{f_1'}{\|\beta\|}, \frac{e_2}{\|\beta\|}, \frac{f_2}{\|\beta\|}\right\}$. The Lie brackets are now given by $[e_1, e_2] = -f_2$ and $[e_1, f_2] = e_2$, and $He_i = a_i f_i$, $Hf_i = -a_i e_i$ for $i = 1, 2$. Note that this metric Lie algebra is listed in the first row of Table 2 with $t = \frac{1}{\|\beta\|^2}$.
- (ii) Case $\mu \neq 0$: We set $\lambda = |\mu|$ and $\epsilon = -\operatorname{sign}(\mu)$, and redefine $\{e_1, f_1, e_2, f_2\}$ as $\left\{\epsilon\frac{e_1'}{\|\beta\|}, \epsilon\frac{f_1'}{\|\beta\|}, \epsilon\frac{e_2}{\|\beta\|}, \epsilon\frac{f_2}{\|\beta\|}\right\}$. The Lie brackets are $[e_1, f_1] = \lambda f_1$, $[e_1, f_2] = e_2$, $[e_1, e_2] = -f_2$, and $He_i = a_i f_i$, $Hf_i = -a_i e_i$ for $i = 1, 2$. Note that this is the metric Lie algebra of the third row of Table 2 with $t = \frac{1}{\|\beta\|^2}$.

If $A \neq 0$, we interchange $(e_1, f_1, e_2, f_2, \alpha, \beta, a_1, a_2) \leftrightarrow (e_2, f_2, e_1, f_1, \beta, \alpha, a_2, a_1)$ and the conclusion will be the same as in the case $C \neq 0$.

Assume finally that $A = C = 0$, that is, $\alpha|_{\mathrm{Span}\{e_2,f_2\}} = \beta|_{\mathrm{Span}\{e_1,f_1\}} = 0$. According to (3.3), the Lie brackets are given by $[e_1, f_1] = -\alpha(e_1)e_1 - \alpha(f_1)f_1$ and $[e_2, f_2] = -\beta(e_2)e_2 - \beta(f_2)f_2$. We may assume that $\beta \neq 0$, otherwise we interchange $(e_1, f_1, e_2, f_2, \alpha, \beta, a_1, a_2) \leftrightarrow (e_2, f_2, e_1, f_1, \beta, \alpha, a_2, a_1)$ and arrive at this situation. We consider two cases:

- (i) Case $\alpha = 0$: We consider a new orthonormal basis $\{e_1, f_1, e_2', f_2'\}$ by setting $e_2' := \frac{-\beta(f_2)e_2+\beta(e_2)f_2}{\|\beta\|}$ and $f_2' := \frac{-\beta(e_2)e_2-\beta(f_2)f_2}{\|\beta\|}$. The Lie brackets are given by $[e_2', f_2'] = \|\beta\| f_2'$, and $He_2' = a_2 f_2'$, $Hf_2' = -a_2 e_2'$. We redefine $\{e_1, f_1, e_2, f_2\}$ as $\left\{-\frac{e_1}{\|\beta\|}, -\frac{f_1}{\|\beta\|}, \frac{e_2}{\|\beta\|}, \frac{f_2}{\|\beta\|}\right\}$. The Lie brackets are now given by $[e_2, f_2] = f_2$. This is the metric Lie algebra listed in the second row of Table 2 with $t = \frac{1}{\|\beta\|^2}$.

(ii) Case $\alpha \neq 0$: We consider a new orthonormal basis $\{e_1', f_1', e_2', f_2'\}$ by setting $e_1' = \frac{\alpha(f_1)e_1 - \alpha(e_1)f_1}{\|\alpha\|}$, $f_1' = \frac{\alpha(e_1)e_1 + \alpha(f_1)f_1}{\|\alpha\|}$, $e_2' = \frac{\beta(f_2)e_2 - \beta(e_2)f_2}{\|\beta\|}$, $f_2' = \frac{-\beta(e_2)e_2 - \beta(f_2)f_2}{\|\beta\|}$. The Lie brackets are given by $[e_1', f_1'] = \|\alpha\| f_1'$, $[e_2', f_2'] = \|\beta\| f_2'$, and $He_i' = a_i f_i'$ and $Hf_i' = -a_i e_i'$, $i = 1, 2$. We may assume that $\|\alpha\| \leq \|\beta\|$, otherwise we interchange $(e_1, f_1, e_2, f_2, \alpha, \beta) \leftrightarrow (e_2, f_2, e_1, f_1, \beta, \alpha)$. We now redefine $\{e_1, f_1, e_2, f_2\}$ as $\left\{\frac{e_1'}{\|\alpha\|}, \frac{f_1'}{\|\alpha\|}, \frac{e_2'}{\|\beta\|}, \frac{f_2'}{\|\beta\|}\right\}$. The Lie brackets are now given by $[e_1, f_1] = f_1$, $[e_2, f_2] = f_2$. We obtain the metric Lie algebra listed in the fourth row of Table 2, with $t = \frac{1}{\|\alpha\|^2}$ and $s = \frac{\|\alpha\|^2}{\|\beta\|^2}$. Note that $He_i = a_i f_i$ and $Hf_i = -a_i e_i$.

We have completed all the rows of Table 2.

Finally, fix a metric Lie algebra of Table 2. Suppose T is any skew-symmetric endomorphism given as in the third column of Table 2 for the respective metric Lie algebra. Then computing $\nabla_{e_i}, \nabla_{f_i}$ it is straightforward to check that T commutes with $\nabla_{e_i}, \nabla_{f_i}$. Thus T is parallel. □

Remark 3.4. When $|a_1| = |a_2|$ in each metric Lie algebra of Table 2, H is a multiple of a complex structure. This case was studied in [6] as we said previously. Furthermore, the metric Lie algebras of Table 2 are the same metric Lie algebras of the first four rows of Table 1.

On a fixed metric Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, the set of parallel skew-symmetric endomorphisms form a vector space. As a consequence of Theorem 3.3 and Remark 3.4 we obtain the following corollary:

Corollary 3.5. *The vector space of parallel skew-symmetric endomorphisms on each metric Lie algebra of Table 2 has dimension* 2*.*

From Theorem 3.3 and comparing Table 2 with Table 1, we have:

Corollary 3.6. *Let $\mathfrak{g}$ be one of the Lie algebras $\mathfrak{d}_{4,\frac{1}{2}}, \mathfrak{d}_{4,2}, \mathfrak{d}'_{4,\frac{\delta}{2}}$ with $\delta > 0$, and let $\langle\cdot,\cdot\rangle$ be one of the metrics of Table 1. Then the only parallel skew-symmetric endomorphisms on $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ are multiple of complex structures. Thus the vector space of parallel skew-symmetric endomorphisms on these metric Lie algebras is* 1*-dimensional.*

We present the information of the above corollary in Table 3. We choose a presentation so that the outputs of Table 2 and Table 3 look similar. Namely, we fix the structure coefficients of the Lie algebras and vary the metric, and we also write all the non-zero parallel tensors, not just the complex structures.

We now address the problem of distinguishing the metric Lie algebras and the parallel tensors presented in Table 2 and Table 3. Two metric Lie algebras $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ and $(\mathfrak{g}', \langle\cdot,\cdot\rangle')$ are said to be equivalent if there exists an isometric Lie algebra isomorphism $\mathfrak{g} \cong \mathfrak{g}'$. In notation, $(\mathfrak{g}, \langle\cdot,\cdot\rangle) \sim (\mathfrak{g}', \langle\cdot,\cdot\rangle')$. Proposition 3.8 below shows that the metric Lie algebras of Table 2 and Table 3 are all pairwise non-equivalent. In addition, for a fixed metric Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, it describes the moduli space of the parallel endomorphisms according to Definition 2.1.

Lie algebra	Metric	Parallel tensor
$\mathfrak{d}_{4,\frac{1}{2}}$: $[e_1,e_2]=e_3$, $[e_4,e_1]=\frac{1}{2}e_1$, $[e_4,e_2]=\frac{1}{2}e_2$, $[e_4,e_3]=e_3$	$\langle\cdot,\cdot\rangle_t=\begin{bmatrix} t&0&0&0\\ 0&t&0&0\\ 0&0&t&0\\ 0&0&0&t\end{bmatrix}, t>0$	$c\begin{bmatrix} 0&-1&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&-1&0\end{bmatrix}, c\in\mathbb{R}^*$
$\mathfrak{d}_{4,2}$: $[e_1,e_2]=e_3$, $[e_4,e_1]=e_1$ $[e_4,e_2]=-\frac{1}{2}e_2$, $[e_4,e_3]=\frac{1}{2}e_3$,	$\langle\cdot,\cdot\rangle_t=\begin{bmatrix} t&0&0&0\\ 0&t&0&0\\ 0&0&t&0\\ 0&0&0&t\end{bmatrix}, t>0$	$c\begin{bmatrix} 0&0&0&-1\\ 0&0&-1&0\\ 0&1&0&0\\ 1&0&0&0\end{bmatrix}, c\in\mathbb{R}^*$
$\mathfrak{d}'_{4,\frac{\delta}{2}}$, $\delta>0$: $[e_1,e_2]=e_3$, $[e_4,e_1]=\frac{1}{2}e_1-\frac{1}{\delta}e_2$, $[e_4,e_2]=\frac{1}{\delta}e_1+\frac{1}{2}e_2$, $[e_4,e_3]=e_3$	$\langle\cdot,\cdot\rangle_t=\begin{bmatrix} t&0&0&0\\ 0&t&0&0\\ 0&0&t&0\\ 0&0&0&t\end{bmatrix}, t>0$	$c\begin{bmatrix} 0&-1&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&-1&0\end{bmatrix}, c\in\mathbb{R}^*$

TABLE 3. Parallel tensors on $(\mathfrak{d}_{4,\frac{1}{2}},\langle\cdot,\cdot\rangle_t)$, $(\mathfrak{d}_{4,2},\langle\cdot,\cdot\rangle_t)$, $(\mathfrak{d}'_{4,\frac{\delta}{2}},\langle\cdot,\cdot\rangle_t)$

Remark 3.7. During the proof of the following proposition we will use that if $\phi:(\mathfrak{g},\langle\cdot,\cdot\rangle)\to(\mathfrak{g},\langle\cdot,\cdot\rangle')$ is an isometric Lie algebra automorphism, then the Ricci operators satisfy the relation $\mathrm{Ric}_{\langle\cdot,\cdot\rangle'}\,\phi=\phi\,\mathrm{Ric}_{\langle\cdot,\cdot\rangle}$; in particular:

(a) $\mathrm{Ric}_{\langle\cdot,\cdot\rangle}$ and $\mathrm{Ric}_{\langle\cdot,\cdot\rangle'}$ have the same characteristic polynomial.
(b) If in addition $\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle'$, then ϕ preserves the eigenspaces of $\mathrm{Ric}_{\langle\cdot,\cdot\rangle}$.

We will also use that if H_1 and H_2 are equivalent parallel endomorphisms on $(\mathfrak{g},\langle\cdot,\cdot\rangle)$, so that there exists an isometric Lie algebra automorphism $\phi:\mathfrak{g}\to\mathfrak{g}$ such that $H_2\phi=\phi H_1$, then

(c) H_1 and H_2 have the same characteristic polynomial.
(d) H_1^2 and H_2^2 have the same eigenvalues and ϕ preserves the corresponding eigenspaces.
(e) Moreover, if H_1 preserves $[\mathfrak{g},\mathfrak{g}]$ then the same holds for H_2, and the restrictions of H_1 and H_2 to $[\mathfrak{g},\mathfrak{g}]$ have the same characteristic polynomial. An analogous conclusion holds if H_1 preserves $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]$ or $[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]]$, or $\mathfrak{z}(\mathfrak{g})$, etc.

Proposition 3.8. *The metric Lie algebras of Table 2 and Table 3 are pairwise non-equivalent. Given one of these metric Lie algebras, any parallel skew-symmetric endomorphism is equivalent to exactly one of the endomorphisms given in Table 4.*

Metric Lie algebra	Parallel endomorphism
$(\mathbb{R}\times\mathfrak{e}(2),\langle\cdot,\cdot\rangle_t)$	$\begin{bmatrix}0&-a_1&0&0\\a_1&0&0&0\\0&0&0&-a_2\\0&0&a_2&0\end{bmatrix}$, $a_1,a_2\geq 0$
$(\mathbb{R}^2\times\mathfrak{aff}(\mathbb{R}),\langle\cdot,\cdot\rangle_t)$	$\begin{bmatrix}0&-a_1&0&0\\a_1&0&0&0\\0&0&0&-a_2\\0&0&a_2&0\end{bmatrix}$, $a_1,a_2\geq 0$
$(\mathfrak{r}'_{4,\lambda,0},\langle\cdot,\cdot\rangle_t)$, $\lambda>0$	$\begin{bmatrix}0&-a_1&0&0\\a_1&0&0&0\\0&0&0&-a_2\\0&0&a_2&0\end{bmatrix}$, $a_1\geq 0$
$(\mathfrak{aff}(\mathbb{R})\times\mathfrak{aff}(\mathbb{R}),\langle\cdot,\cdot\rangle_{t,s})$, $s,t>0$, $s\leq 1$	$\begin{bmatrix}0&-a_1&0&0\\a_1&0&0&0\\0&0&0&-a_2\\0&0&a_2&0\end{bmatrix}$, $a_1,a_2\geq 0$ if $s<1$, or $a_1\geq a_2\geq 0$ if $s=1$
$(\mathfrak{d}_{4,\frac{1}{2}},\langle\cdot,\cdot\rangle_t)$	$c\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}$, $c>0$
$(\mathfrak{d}_{4,2},\langle\cdot,\cdot\rangle_t)$	$c\begin{bmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}$, $c>0$
$(\mathfrak{d}'_{4,\frac{\delta}{2}},\langle\cdot,\cdot\rangle_t),\delta>0$	$c\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}$, $c\in\mathbb{R}^*$

Table 4. Non-equivalent parallel tensors on non-equivalent 4-dimensional metric Lie algebras

Proof. The Lie algebras of the first columns of Table 2 and Table 3 are all pairwise non-isomorphic. We now proceed case by case.

Case $\mathfrak{g}=\mathbb{R}\times\mathfrak{e}(2)$. Let $\phi:(\mathfrak{g},\langle\cdot,\cdot\rangle_t)\to(\mathfrak{g},\langle\cdot,\cdot\rangle_{t'})$ be an isometric Lie algebra isomorphism. Since $[\mathfrak{g},\mathfrak{g}]=\operatorname{Span}\{e_2,f_2\}$ and $\mathfrak{z}(\mathfrak{g})=\operatorname{Span}\{f_1\}$, ϕ has the form

$$\phi=\begin{bmatrix}x_1&0&0&0\\0&x_4&0&0\\0&0&x_9&-x_{10}\\0&0&x_{10}&x_9\end{bmatrix},$$

with $x_1^2t'=x_4^2t'=t$ and $(x_9^2+x_{10}^2)t'=t$. One readily checks that $\phi([e_1,e_2])=[\phi(e_1),\phi(e_2)]$, so that $x_1=1$ and hence $t=t'$.

Now fix t and suppose that H_{a_1,a_2} is a skew-symmetric endomorphism as in Table 2. It is possible to multiply the basis elements by ± 1's so that the Lie brackets in the new basis are the same as before and the endomorphism has $a_1, a_2 \geq 0$. For instance, if $a_1 \geq 0$ and $a_2 < 0$, then we do the change $(e_1, f_1, e_2, f_2) \mapsto (-e_1, -f_1, -e_2, f_2)$. It is only left to show that if H_{a_1,a_2} and $H_{a'_1,a'_2}$ are equivalent, with $a_i, a'_i \geq 0$, then $(a_1, a_2) = (a'_1, a'_2)$. By Remark 3.7 (c) we have $(X^2+a_1^2)(X^2+a_2^2) = (X^2 + a_1'^2)(X^2 + a_2'^2)$; in particular $a_1^2 + a_2^2 = a_1'^2 + a_2'^2$. Moreover, since both H_{a_1,a_2} and $H_{a'_1,a'_2}$ preserve $[\mathfrak{g}, \mathfrak{g}] = \operatorname{Span}\{e_2, f_2\}$, Remark 3.7 (e) tells us that $X^2 + a_2^2 = X^2 + a_2'^2$. It follows that $a_2 = a'_2$, and hence $a_1 = a'_1$.

Case $\mathfrak{g} = \mathbb{R}^2 \times \mathfrak{aff}(\mathbb{R})$. One easily checks that the Ricci operator of $(\mathfrak{g}, \langle\cdot,\cdot\rangle_t)$ with respect to the orthonormal basis $\left\{\frac{e_1}{\sqrt{t}}, \frac{f_1}{\sqrt{t}}, \frac{e_2}{\sqrt{t}}, \frac{f_2}{\sqrt{t}}\right\}$ is given by

$$\operatorname{Ric}_t = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{t} & 0 \\ 0 & 0 & 0 & -\frac{1}{t} \end{bmatrix}.$$

It follows from Remark 3.7 (a) that $(\mathfrak{g}, \langle\cdot,\cdot\rangle_t) \sim (\mathfrak{g}, \langle\cdot,\cdot\rangle_{t'})$ if and only if $t = t'$.

We now fix $t > 0$ and suppose that H_{a_1,a_2} is a skew-symmetric endomorphism as in Table 2. By multiplying f_1 and/or f_2 by ± 1, which does not affect the Lie brackets, we can take $a_1, a_2 \geq 0$. Now suppose that H_{a_1,a_2} and $H_{a'_1,a'_2}$ are equivalent, with $a_i, a'_i \geq 0$. By Remark 3.7 (c) $(X^2 + a_1^2)(X_2^2 + a_2^2) = (X^2 + a_1'^2)(X^2 + a_2'^2)$; in particular $a_1^2 + a_2^2 = a_1'^2 + a_2'^2$. Moreover, since both H_{a_1,a_2} and $H_{a'_1,a'_2}$ preserve $\mathfrak{z}(\mathfrak{g}) = \operatorname{Span}\{e_1, f_1\}$, Remark 3.7 (e) tells us that $X^2+a_1^2 = X^2+a_1'^2$. It follows that $a_1 = a'_1$, and hence $a_2 = a'_2$.

Case $\mathfrak{g} = \mathfrak{r}'_{4,\lambda,0}$, $\lambda > 0$. The Ricci operator of $(\mathfrak{g}, \langle\cdot,\cdot\rangle_t)$ with respect to the orthonormal basis $\left\{\frac{e_1}{\sqrt{t}}, \frac{f_1}{\sqrt{t}}, \frac{e_2}{\sqrt{t}}, \frac{f_2}{\sqrt{t}}\right\}$ is given by

$$\operatorname{Ric}_t = \begin{bmatrix} -\frac{\lambda^2}{t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

By Remark 3.7 (a) we obtain that $(\mathfrak{g}, \langle\cdot,\cdot\rangle_t) \sim (\mathfrak{g}, \langle\cdot,\cdot\rangle_{t'})$ if and only if $t = t'$.

We now fix $t > 0$. Let H_{a_1,a_2} be a skew-symmetric endomorphism as in Table 2. After changing f_1 by $-f_1$ if necessary, which does not affect the Lie brackets, we can take $a_1 \geq 0$. We now assume that H_{a_1,a_2} and $H_{a'_1,a'_2}$ are equivalent, with $a_1, a'_1 \geq 0$ and $a_2, a'_2 \in \mathbb{R}$. By Remark 3.7 (c) $(X^2 + a_1^2)(X^2 + a_2^2) = (X^2 + a_1'^2)(X^2 + a_2'^2)$, in particular $a_1^2 + a_2^2 = a_1'^2 + a_2'^2$. Now let $\phi : \mathfrak{g} \to \mathfrak{g}$ be an isometric isomorphism such that $\phi H_{a_1,a_2} = H_{a'_1,a'_2}\phi$. Then ϕ preserves $[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = \operatorname{Span}\{e_2, f_2\}$, hence $\phi(e_2) = ue_2 + vf_2$ and $\phi(f_2) = -ve_2 + uf_2$ with $u^2 + v^2 = 1$. Now we have $\phi H_{a_1,a_2}(e_2) = a_2\phi(f_2) = -a_2ve_2 + a_2uf_2$ and $H_{a'_1,a'_2}\phi(e_2) = H_{a'_1,a'_2}(ue_2 + vf_2) = a'_2uf_2 - a'_2ve_2$. From this we see that $a_2 = a'_2$, and hence $a_1 = a'_1$.

Case $\mathfrak{g} = \mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$. The Ricci operator with respect to the orthonormal basis $\left\{\frac{e_1}{\sqrt{t}}, \frac{f_1}{\sqrt{t}}, \frac{e_2}{\sqrt{ts}}, \frac{f_2}{\sqrt{ts}}\right\}$ is given by

$$\mathrm{Ric}_{s,t} = \begin{bmatrix} -\frac{1}{t} & 0 & 0 & 0 \\ 0 & -\frac{1}{t} & 0 & 0 \\ 0 & 0 & -\frac{1}{st} & 0 \\ 0 & 0 & 0 & -\frac{1}{st} \end{bmatrix}.$$

It follows from Remark 3.7 (a) that $(\mathfrak{g}, \langle \cdot,\cdot \rangle_{s,t}) \sim (\mathfrak{g}, \langle \cdot,\cdot \rangle_{s',t'})$ if and only if $t = t'$ $s = s'$ (here we are using the constraint $0 < s, s' \leq 1$).

We now fix $t > 0$ and $0 < s \leq 1$. Let H_{a_1,a_2} be a skew-symmetric endomorphism as in Table 2. We can multiply f_1 or f_2 by -1 if necessary, without changing the Lie brackets, and take $a_1, a_2 \geq 0$. If $s = 1$, we can interchange $\{e_1, f_1\}$ and $\{e_2, f_2\}$ if necessary and assume that $a_1 \geq a_2$.

Now suppose that there exists an isometric isomorphism $\phi : \mathfrak{g} \to \mathfrak{g}$ such that $H_{a_1,a_2}\phi = H_{a'_1,a'_2}\phi$, where $a_i, a'_i \geq 0$ and also $a_1 \geq a_2$ and $a'_1 \geq a'_2$ in the case $s = 1$. We have to show that $a_1 = a'_1$ and $a_2 = a'_2$. We consider two cases:

(i) $s < 1$: Since $\mathrm{Span}\{e_1, f_1\}$ and $\mathrm{Span}\{e_2, f_2\}$ are the eigenspaces of the Ricci operator, they are preserved by ϕ according to Remark 3.7 (b). This clearly implies that $a_1 = a'_1$ and $a_2 = a'_2$.

(ii) $s = 1$: In this case we have to assume that $a_1 \geq a_2 \geq 0$, $a'_1 \geq a'_2 \geq 0$. By Remark 3.7 (c), $(X^2 + a_1^2)(X^2 + a_2^2) = (X^2 + a_1'^2)(X^2 + a_2'^2)$. From this we deduce that $a_1 = a'_1$ and $a_2 = a'_2$.

Case $\mathfrak{g} = \mathfrak{d}_{4,\frac{1}{2}}$. In the orthonormal basis $\left\{\frac{e_1}{\sqrt{t}}, \frac{e_2}{\sqrt{t}}, \frac{e_3}{\sqrt{t}}, \frac{e_4}{\sqrt{t}}\right\}$, the Ricci operator is

$$\mathrm{Ric}_{t} = \begin{bmatrix} -\frac{3}{2t} & 0 & 0 & 0 \\ 0 & -\frac{3}{2t} & 0 & 0 \\ 0 & 0 & -\frac{3}{2t} & 0 \\ 0 & 0 & 0 & -\frac{3}{2t} \end{bmatrix}.$$

It follows from Remark 3.7 (a) that $(\mathfrak{g}, \langle \cdot,\cdot \rangle_{t}) \sim (\mathfrak{g}, \langle \cdot,\cdot \rangle_{t'})$ if and only if $t = t'$.

We now fix t. Given a skew-symmetric endomorphism H_c as in Table 3, we can change (e_1, e_3) by $(-e_1, -e_3)$ if necessary, without changing the Lie brackets, and take $c > 0$. Using Remark 3.7 (d), we now easily see that if H_c and $H_{c'}$ are equivalent, with $c, c' > 0$, then $c = c'$.

Case $\mathfrak{g} = \mathfrak{d}_{4,2}$. The Ricci operator in the orthonormal basis $\left\{\frac{e_1}{\sqrt{t}}, \frac{e_2}{\sqrt{t}}, \frac{e_3}{\sqrt{t}}, \frac{e_4}{\sqrt{t}}\right\}$ is

$$\mathrm{Ric}_{t} = \begin{bmatrix} -\frac{3}{2t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{2t} \end{bmatrix}.$$

It follows from Remark 3.7 (a) that if $(\mathfrak{g}, \langle \cdot,\cdot \rangle_{t}) \sim (\mathfrak{g}, \langle \cdot,\cdot \rangle_{t'})$, then $t = t'$.

We now fix t and let H_c be an endomorphism as in Table 3. By changing (e_1, e_2) by $(-e_1, -e_2)$ if necessary, we can take $c > 0$. Finally, if H_c and $H_{c'}$ are equivalent, then by Remark 3.7 (d), we get $c = c'$.

Case $\mathfrak{g} = \mathfrak{d}'_{4,\frac{\delta}{2}}$. We consider the orthonormal basis $\left\{\frac{e_1}{\sqrt{t}}, \frac{e_2}{\sqrt{t}}, \frac{e_3}{\sqrt{t}}, \frac{e_4}{\sqrt{t}}\right\}$. The Ricci operator is given by

$$\mathrm{Ric} = \begin{bmatrix} -\frac{3}{2t} & 0 & 0 & 0 \\ 0 & -\frac{3}{2t} & 0 & 0 \\ 0 & 0 & -\frac{3}{2t} & 0 \\ 0 & 0 & 0 & -\frac{3}{2t} \end{bmatrix}.$$

It follows from Remark 3.7 (a) that if $(\mathfrak{g}, \langle\cdot,\cdot\rangle_t) \sim (\mathfrak{g}, \langle\cdot,\cdot\rangle_{t'})$, then $t = t'$.

Fix $t > 0$. Given two skew-symmetric endomorphisms H_c and $H_{c'}$ as in Table 3, with $c, c' \in \mathbb{R}^*$, by Remark 3.7 (d), we have $|c| = |c'|$. We claim that H_c and H_{-c} are non-equivalent. To show this we can assume that $c = 1$. Suppose that there is an isometric isomorphism ϕ such that $H_1\phi = \phi H_{-1}$. Observe that $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}] = \mathrm{Span}\{e_1, e_2, e_3\}$, and that $\mathfrak{g}'' := [\mathfrak{g}', \mathfrak{g}'] = \mathrm{Span}\{e_3\}$. Since ϕ preserves $\mathfrak{g}'$ and is an isometry, it also preserves the orthogonal complement, that is, $\phi(e_4) = ke_4$ with $k = \pm 1$. Since it also preserves $\mathfrak{g}''$, $\phi(e_3) = le_3$ with $l = \pm 1$. Now $le_3 = \phi(e_3) = \phi[e_4, e_3] = [\phi(e_4), \phi(e_3)] = kle_3$, from which $k = 1$. So ϕ can be written in the basis $\{e_1, e_2, e_3, e_4\}$ as

$$\phi = \begin{bmatrix} x_1 & -x_2 & 0 & 0 \\ x_2 & x_1 & 0 & 0 \\ 0 & 0 & l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If we impose the condition that $H_1\phi = \phi H_{-1}$, we get that $x_1 = x_2 = 0$, which contradicts the fact that ϕ is invertible. Hence H_1 and H_{-1} are not equivalent. □

4. The de Rham decomposition of the associated simply connected Riemannian Lie groups

Given a Riemannian manifold (M, g), the holonomy group at a point $p \in M$, denoted by $\mathrm{Hol}_p(M, g)$, is the group formed by parallel translations P_γ around loops $\gamma : [a, b] \to M$ at p, with the usual product of endomorphisms. This is a Lie subgroup of the orthogonal group $O(T_pM)$, and if M is orientable, $\mathrm{Hol}_p(M, g) \subseteq SO(T_pM)$. The restricted holonomy group $\mathrm{Hol}^0_p(M, g)$ is the connected normal subgroup that results from using only contractible loops. This is a closed subgroup of $O(T_pM)$ and hence its action on T_pM is completely reducible. If M is connected, then $\mathrm{Hol}_p(M, g)$ (resp., $\mathrm{Hol}^0_p(M, g)$) is conjugate to $\mathrm{Hol}_q(M, g)$ (resp., $\mathrm{Hol}^0_q(M, g)$) for all $p, q \in M$; therefore the holonomy group and the restricted holonomy group do not depend on the base point, and they are denoted by $\mathrm{Hol}(M, g)$ and $\mathrm{Hol}^0(M, g)$, respectively. We say that (M, g) is irreducible if the action of the restricted holonomy group on T_pM is irreducible. If M is simply connected, then one has $\mathrm{Hol}(M, g) = \mathrm{Hol}^0(M, g)$. If in addition M is complete, then according to the de Rham theorem ([7, Theorem 10.3.1]), a decomposition of T_pM into irreducible subspaces under the action of $\mathrm{Hol}(M, g)$ corresponds to a decomposition of M as a product of irreducible Riemannian manifolds. We refer to this as the de Rham decomposition of (M, g).

Given a metric Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, we denote by (G, g) the associated simply connected Lie group G with the corresponding left invariant metric g. The pair (G, g) is a complete Riemannian manifold.

We will see next that if $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ belongs to Table 3, then (G, g) is irreducible as a Riemannian manifold. One way to show this is by identifying (G, g) with an already known irreducible Riemannian manifold (see Remark 4.2). Another possibility is by showing that $\mathfrak{g}$ has no proper subspaces invariant under the action of $\mathrm{Hol}(G, g)$. Since $\mathrm{Hol}(G, g) = \mathrm{Hol}^0(G, g)$ is connected; this is equivalent to showing that $\mathfrak{g}$ has no proper subspaces invariant by the elements of $\mathfrak{hol}(G, g)$, the Lie algebra of $\mathrm{Hol}(G, g)$. In order to know what $\mathfrak{hol}(G, g)$ is, we use the Ambrose–Singer holonomy theorem, which states that this algebra is the subalgebra of $\mathfrak{so}(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ spanned by the curvature operators and all their covariant derivatives. We begin by following this approach.

Proposition 4.1. *Let $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ be one of the metric Lie algebras of Table 3, and let (G, g) be the associated simply connected Lie group with the corresponding left invariant metric. Then (G, g) is irreducible as Riemannian manifold.*

Proof. We do the proof only for $(\mathfrak{g}, \langle\cdot,\cdot\rangle) = (\mathfrak{d}_{4,2}, \langle\cdot,\cdot\rangle_t)$, the other cases being similar. Since all the metrics $\langle\cdot,\cdot\rangle_t$ are homothetic, we can assume $t = 1$ and we simply denote $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_1$. We can identify $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ with the Euclidean space $\mathbb{R}^4$ so that $\{e_1, e_2, e_3, e_4\}$ is identified with the canonical basis.

As remarked previously, we first have to find $\mathfrak{hol}(G, g)$, which is the Lie subalgebra of $\mathfrak{so}(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ spanned as vector space by the curvature tensors and their covariant derivatives. Then we have to show that $\mathfrak{g}$ has no proper subrepresentations of $\mathfrak{hol}(G, g)$.

First, one can check that $\nabla_{e_4} = 0$ and that

$$\nabla_{e_1} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \nabla_{e_2} = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \end{bmatrix}, \quad \nabla_{e_3} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

Recall that $R(x, y) = \nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]}$ is the curvature tensor for all $x, y \in \mathfrak{g}$; we now compute

$$R(e_1, e_2) = -\frac{1}{2}\nabla_{e_3}, \qquad R(e_1, e_3) = -\frac{1}{2}\nabla_{e_2},$$

$$R(e_1, e_4) = \nabla_{e_1}, \qquad R(e_2, e_3) = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix},$$

$$R(e_2, e_4) = -\frac{1}{2}\nabla_{e_2}, \qquad R(e_3, e_4) = \frac{1}{2}\nabla_{e_3}.$$

In order to facilitate the computation of the covariant derivatives of these tensors, we will make use of the natural identification $\mathfrak{so}(\mathfrak{g}, \langle\cdot,\cdot\rangle) \cong \mathfrak{g}^* \wedge \mathfrak{g}^*$. The tensor $R(e_i, e_j)$ is identified with the 2-form R^{ij} defined by

$$R^{ij}(x, y) = \langle R(e_i, e_j)x, y\rangle, \quad x, y \in \mathfrak{g}.$$

It follows that

$$R^{12}=\frac{1}{4}\left(e^{12}-e^{34}\right),\qquad R^{13}=\frac{1}{4}(e^{13}+e^{24}),\qquad R^{14}=e^{14}+\frac{1}{2}e^{23},$$
$$R^{23}=\frac{1}{2}\left(e^{14}-e^{23}\right),\qquad R^{24}=\frac{1}{4}\left(e^{13}+e^{24}\right),\qquad R^{34}=-\frac{1}{4}(e^{12}-e^{34}),$$

where $\{e^1,e^2,e^3,e^4\}$ is the dual basis of $\{e_1,e_2,e_3,e_4\}$ and e^{ij} denotes $e^i\wedge e^j$.

Using that $\nabla_x(\theta_1\wedge\theta_2)=\nabla_x\theta_1\wedge\theta_2+\theta_1\wedge\nabla_x\theta_2$ for θ_1,θ_2 1-forms, and that $\nabla_x y^*(z)=\langle\nabla_x y,z\rangle$, where y^* is the linear functional such that $y^*(x)=\langle x,y\rangle$, we get the following relations:

$$\begin{aligned}
\nabla_{e_1}R^{12}&=-\frac{1}{8}(e^{13}+e^{24}), & \nabla_{e_2}R^{12}&=\frac{1}{4}(e^{23}-e^{14}),\\
\nabla_{e_3}R^{12}&=0, & \nabla_{e_1}R^{13}&=-\frac{1}{8}(e^{12}-e^{34}),\\
\nabla_{e_2}R^{13}&=0, & \nabla_{e_2}R^{13}&=\frac{1}{4}(e^{14}-e^{23}),\\
\nabla_{e_1}R^{14}&=0, & \nabla_{e_2}R^{14}&=\frac{1}{4}(e^{12}-e^{34}),\\
\nabla_{e_3}R^{14}&=-\frac{1}{4}(e^{24}+e^{13}), & \nabla_{e_1}R^{23}&=0,\\
\nabla_{e_2}R^{23}&=\frac{1}{2}(e^{12}-e^{34}), & \nabla_{e_3}R^{23}&=-\frac{1}{2}(e^{13}+e^{24}).
\end{aligned}$$

We see that the linear space spanned by R^{ij} and $\nabla_{e_k}R^{ij}$ for $i,j,k=1,2,3,4$ is $\mathrm{Span}\{e^{14}+\frac{1}{2}e^{23},e^{13}+e^{24},e^{12}-e^{34},e^{14}-e^{23}\}=\mathrm{Span}\{e^{14},e^{23},e^{13}+e^{24},e^{12}-e^{34}\}$. We conclude that $\mathfrak{hol}(G,g)$ is

$$\mathrm{Span}\left\{\underbrace{\begin{bmatrix}0&0&0&-1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix}}_{=:A},\underbrace{\begin{bmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{bmatrix}}_{=:B},\underbrace{\begin{bmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{bmatrix}}_{=:C},\underbrace{\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}}_{=:D}\right\}.$$

We now have to show that there are no proper subspaces of $\mathfrak{g}=\mathbb{R}^{4\times 1}$ invariant by A, B, C and D. Suppose W is a non-zero invariant subspace of $\mathfrak{g}$, and let $x=(a_1,a_2,a_3,a_4)^T$ be a non-zero element of W. After multiplying by C, if necessary, we can assume that $(a_1,a_4)\neq(0,0)$. Then $Ax=(-a_4,0,0,a_1)^T\in W$, $CAx=(0,-a_1,-a_4,0)^T\in W$ and $DCAx=(a_1,0,0,a_4)\in W$. Note that Ax and $DCAx$ are linearly independent, so we obtain that $e_1,e_4\in W$. Similarly, we get that $e_2,e_3\in W$, and therefore $W=\mathfrak{g}$. The proof is now complete. ☐

Remark 4.2. Proposition 4.1 can also be shown by identifying (G,g) with known Riemannian manifolds. One can easily see that $(\mathfrak{d}_{4,\frac{1}{2}},\langle\cdot,\cdot\rangle_t)$ is equivalent up to scaling to the metric Lie algebra with parameter zero in the second family listed in the main theorem of [5]. Analogously, $(\mathfrak{d}'_{4,\frac{\delta}{2}},\langle\cdot,\cdot\rangle_t)$ is equivalent up to scaling to the metric Lie algebra with parameter $\frac{2}{\delta}$ of the same family. In both cases (G,g) is isometric up to scaling to the Hermitian hyperbolic space $H^2(\mathbb{C}) = SU(2,1)/S(U(2)\times U(1))$, according to [5, Proposition 2]. On the other hand, it is well known that $H^2(\mathbb{C})$ is an irreducible symmetric space (see [2, p. 315]).

In the case of $(\mathfrak{d}_{4,2}, \langle\cdot,\cdot\rangle_t)$, it is not hard to see that $(\mathfrak{d}_{4,2}, \langle\cdot,\cdot\rangle_t)$ is isometric up to scaling to the metric Lie algebra described in $\mathfrak{g}_{4,9}(\frac{1}{2})$, see [3, Theorem 2.2]. It can be shown that (G,g) is isometric up to scaling to the unique irreducible proper 3-symmetric space of dimension four, according to [3, Corollary 3.1]. This space is also viewed as the irreducible Kähler surface corresponding to F_4-geometry, see [9] and [8].

On the other hand, we will analyze the case when $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ belongs to Table 2. First we recall the following theorem.

Theorem 4.3 ([7, Theorem 10.3.2])**.** *Let (M,g) be an irreducible Riemannian manifold, and let H be a parallel skew-symmetric $(1,1)$-tensor which is nowhere zero. Then $H=\lambda J$ for a complex parallel structure J and $\lambda\in\mathbb{R}$.*

Proposition 4.4. *Let $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ be one of the metric Lie algebras of Table 2, and let (G,g) be the associated simply connected Lie group with the corresponding left invariant metric. Then (G,g) is reducible as a Riemannian manifold, and after a suitable scaling (G,g) is isometric to one of the following: $\mathbb{R}^4$, $\mathbb{R}^2\times\mathbb{H}^2$, or $\mathbb{H}^2\times\mathbb{H}^2$ (here the scaling is in each factor).*

Proof. Clearly if $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ belongs to Table 2, then (G,g) is reducible. In the case $(\mathbb{R}\times\mathfrak{e}(2), \langle\cdot,\cdot\rangle_t)$, one can easily check that the curvature tensor R is zero. This implies that $\mathfrak{hol}(G,g)=0$, and hence any 1-dimensional subspace of $\mathfrak{g}=T_eG$ is invariant under $\mathrm{Hol}(G,g)=\{e\}$, where $\{e\}$ is the trivial group. Consequently, any decomposition of T_eG as a direct sum of 1-dimensional subspaces leads to an identification of G with $\mathbb{R}^4$ as Riemannian manifolds.

Before proceeding with the other cases, recall that the hyperbolic plane $\mathbb{H}^2 = \{(x,y): y>0\}$ is a simply connected Lie group with multiplication $(x,y)\cdot(x',y') = (x+yx', yy')$, and that the usual metric $(ds)^2 = \frac{(dx)^2+(dy)^2}{y^2}$ is left invariant. As a Riemannian manifold, it is irreducible with negative curvature. The associated metric Lie algebra is $\mathfrak{aff}(\mathbb{R}) = \mathrm{Span}\{e,f\}$, where $\{e,f\}$ is an orthonormal basis on which the Lie brackets are given by $[e,f]=f$.

With the previous observation, it is straightforward to prove that the cases $(\mathfrak{aff}(\mathbb{R})\times\mathbb{R}^2, \langle\cdot,\cdot\rangle_t)$ and $(\mathfrak{aff}(\mathbb{R})\times\mathfrak{aff}(\mathbb{R}), \langle\cdot,\cdot\rangle_{t,s})$ correspond to Riemannian manifolds $\mathbb{H}^2\times\mathbb{R}^2$ and $\mathbb{H}^2\times\mathbb{H}^2$, respectively.

Finally, we consider the case $(\mathfrak{r}'_{4,\lambda,0}, \langle\cdot,\cdot\rangle_t)$. We can take the parameter $t=1$ in Table 2 since all the metrics are homothetic. The commutator $[\mathfrak{g},\mathfrak{g}] = \mathrm{Span}\{e_2,f_2\}$ is abelian, the subspace $\mathrm{Span}\{e_1,f_1\}$ with the induced metric is a metric Lie algebra isomorphic to $\mathfrak{aff}(\mathbb{R})$ with the usual metric, and $\mathfrak{r}'_{4,\lambda,0}$ has the description $\mathfrak{aff}(\mathbb{R})\ltimes\mathbb{R}^2$. Since the factors are orthogonal, we obtain a decomposition $G=\mathbb{H}^2\ltimes\mathbb{R}^2$, where $\mathbb{H}^2$ has the hyperbolic metric, $\mathbb{R}^2$ has the Euclidean metric and the factors are orthogonal. In other words, the de Rham decomposition of the pair (G,g) associated with $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is, up to scaling, $\mathbb{H}^2\times\mathbb{R}^2$. □

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