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AFFINITY KERNELS ON MEASURE SPACES AND MAXIMAL OPERATORS

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Abstract. In this note we consider maximal operators defined in terms of families of kernels and families of their level sets. We prove a general estimate that extends some classical Euclidean cases and, under some mild transitivity property, we show their basic boundedness properties on Lebesgue spaces. The motivation of these problems has its roots in the analysis associated to affinity kernels on large data sets.

1. Introduction

The history of the relation between harmonic analysis and Euclidean geometry goes back to the origin of potential theory in physics and mathematics. Newtonian and electrostatic Coulombian potentials have the general form

$$K_N(x,y) = \frac{1}{d(x,y)},$$

with d the Euclidean distance in the space. For nuclear forces in the atom, the Yukawa potential takes the form

$$K_Y(x,y) = \frac{e^{-d(x,y)}}{d(x,y)},$$

which is of the order of $K_N(x,y)$ for x and y close together but is much smaller than $K_N(x,y)$ when x and y are far away from each other.

In [2] it is shown that, under some mild transitivity condition on an abstract kernel $K(x,y)$ defined on the abstract set X, $K(x,y) = \varphi(d(x,y))$ for some quasi-metric d on X and some decreasing positive profile function φ.

The kernels K_N, K_Y and K above share a basic property, to wit, the sections of their level sets are metric balls. In some problems related to affinity kernels on data sets, the metric structure underlying is not at all apparent. Let us start from a basic finite situation posed on an undirected graph with $\mathcal{V} = \{1,\dots,n\}$ as its set of vertices and $\mathcal{E}$ the set of all edges. We shall assume that each vertex in $\mathcal{V}$

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has a positive probabilistic measure a_i, which at first glance can be considered to be equal to $\frac{1}{n}$. Set $\overline{a}$ to denote the vector $(a_1, \dots, a_n)$. Let us illustrate the role of the family of kernels on the basic graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \overline{a})$ with a simple example. Assume that the vertices in $\mathcal{V}$ are the vineyards of some fixed region or country. The quality of a single-varietal wine in any of the vineyards depends on several different features: rainfall, temperature, composition of the terroir, et cetera. Each one of these features defines affinities between different vineyards. Each affinity can be considered as a symmetric nonnegative matrix providing the weight of each edge in $\mathcal{G}$. Since we are considering at once several features that are relevant to the quality of the wine, we may think of this situation as a vector-weighted graph. Each edge $\{i, j\} \in \mathcal{E}$ is weighted with a vector weight that takes into account the diversity of features, i.e. $w^k_{i,j}$, where $k = 1, \dots, m$ is the parameter describing the different features. Formally, we have a sequence $\mathcal{G}^k = (\mathcal{V}, \mathcal{E}, \overline{a}, \overline{\overline{w}}^k)$ of weighted graphs, $k = 1, 2, \dots, m$, with $\mathcal{V}, \mathcal{E}$ and $\overline{a}$ fixed. Each $w^k_{i,j}$ is nonnegative with $w^k_{i,j} = w^k_{j,i}$. In other words, we have a basic set $\mathcal{V}$ and a basic probability $\overline{a}$ on $\mathcal{V}$ with a family of nonnegative symmetric kernels given by $\overline{\overline{w}}^k$. In the example above it becomes important to determine a notion of distance between two single-varietals produced by different vineyards. Our result in Theorem 2.1 applies to our example without a priori knowledge of the existence of a metric. The basic shapes are the level sets of the given family of kernels. Of course, the important covering properties of metric balls (Wiener or Vitali type) are generally not satisfied by level sets of kernels. Nevertheless, as shown in [2], under some mild conditions on the affinity matrix we have a natural metric structure on $\mathcal{V}$. On the other hand, the basic boundedness properties of the main operators can be rephrased in terms of the family of kernels, when they satisfy some transitivity property, without any reference to the underlying metric structure. In this note we show that, without any assumption on the metric structure of the kernel, the shape of the level sets always provides an upper control in terms of Hardy–Littlewood-type maximal operators for the integral operators determined by kernels. Some classical results can be obtained as corollaries of this general result. Then we consider the boundedness properties of the maximal operator under transitivity of some kernel of a family of kernels sharing their level sets. We also deal briefly with the associated Muckenhoupt weights, the corresponding BMO and Lipschitz spaces. Connected with the above, we consider in Section 5 some conditions on a radial kernel that can replace the indicator functions of balls in order to obtain the standard Muckenhoupt weights in $\mathbb{R}^n$ or in any α-regular Ahlfors spaces and some related analytical problems.

2. Maximal operators associated to families of kernels and the basic inequality

Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space. We shall consider families of integral operators defined on the nonnegative functions on X. For f measurable and nonnegative, the operator $T_K f$ is defined as $T_K f(x) = \int_{y \in X} K(x, y) f(y)\, d\mu(y)$ for $K : X \times X \to \mathbb{R}^+ \cup \{0\}$ measurable. Of course, with no additional conditions on K

and f, $T_K f$ could be infinite for every x. Nevertheless the operators T_K are well defined.

Let $\mathscr{K}$ be a class of nonnegative measurable kernels K. Let $\mathscr{L}$ be the family of all the level sets of the type $L(\lambda, K) = \{(x,y) \in X \times X : K(x,y) > \lambda\}$ for $K \in \mathscr{K}$ and $\lambda > 0$. For each $x \in X$, set $L_\lambda^K(x)$ to denote the section at x of $L(\lambda, K) \in \mathscr{L}$; in other words,

$$L_\lambda^K(x) = \{y \in X : (x,y) \in L(\lambda, K)\} = \{y \in X : K(x,y) > \lambda\}$$

for $K \in \mathscr{K}$ and $\lambda > 0$. These sections are measurable for almost every $x \in X$. Set $\mathscr{L}_x = \{L_\lambda^K(x) : \lambda > 0,\, K \in \mathscr{K}\}$.

Let us now introduce the two maximal operators that we shall consider. For a given kernel family $\mathscr{K}$ and a given nonnegative measurable function f, set

$$\mathcal{K}^* f(x) = \sup_{K \in \mathscr{K}} T_K f(x).$$

Notice that $\mathcal{K}^* f$ could be identically equal to $+\infty$ or even non-measurable since we have no assumption on the cardinality of $\mathscr{K}$. On the other hand, for the same function f and the same kernel family $\mathscr{K}$, we have a well-defined maximal operator by the family $\mathscr{L}_x$ of sections of $\mathscr{L}$:

$$M_{\mathscr{K}} f(x) = \sup_{L \in \mathscr{L}_x} \frac{1}{\mu(L)} \int_L f(y)\, d\mu(y),$$

which we shall consider to be equal to $+\infty$ when for some L we have that $L \notin \mathscr{F}$ or $\mu(L) = 0$. Again, the measurability of $M_{\mathscr{K}}$ is not guaranteed but nor is it necessary for the pointwise estimate in our result, which we proceed to state and prove.

Theorem 2.1. *Let $\mathscr{K}$, $\mathcal{K}^*$, and $M_{\mathscr{K}}$ be as before. Then, for $f \geq 0$ measurable, we have*

$$\mathcal{K}^* f(x) \leq \left(\sup_{K \in \mathscr{K}} \int K(x,y)\, d\mu(y) \right) M_{\mathscr{K}} f(x)$$

for almost every $x \in X$.

Proof. Notice first that there is nothing to prove when $M_{\mathscr{K}} f(x) = +\infty$. Hence we need only take into account those $x \in X$ for which $L \in \mathscr{F}$ and $\mu(L) > 0$ whenever $L \in \mathscr{L}_x$. Since we are in a σ-finite measure space, from the Fubini–Tonelli theorem we have that

$$\begin{aligned} T_K f(x) &= \int_{y \in X} K(x,y) f(y)\, d\mu(y) \\ &= \int_{y \in X} f(y) \int_0^{K(x,y)} d\lambda\, d\mu(y) \\ &= \int_0^\infty \int_{\{y : K(x,y) > \lambda\}} f(y)\, d\mu(y)\, d\lambda \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \mu(\{y : K(x,y) > \lambda\}) \left(\frac{1}{\mu(\{y : K(x,y) > \lambda\})} \int_{\{y:K(x,y)>\lambda\}} f(y)\, d\mu(y) \right) d\lambda \\
&\leq \left(\int_0^\infty \mu(\{y : K(x,y) > \lambda\})\, d\lambda \right) M_{\mathscr{K}} f(x) \\
&= \left(\int_{y\in X} K(x,y)\, d\mu(y) \right) M_{\mathscr{K}} f(x).
\end{aligned}
$$

Hence

$$
\mathcal{K}^* f(x) = \sup_{K\in\mathscr{K}} T_K f(x) \leq \left(\sup_{K\in\mathscr{K}} \int_{y\in X} K(x,y)\, d\mu(y) \right) M_{\mathscr{K}} f(x). \qquad \square
$$

In the last section of this note, we derive from the above result some known results in Euclidean and Ahlfors spaces.

3. Transitive affinity structures

Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space. Let $\mathscr{K}$ be a given family of symmetric measurable kernels $K : X \times X \to \mathbb{R}^+ \cup \{0\}$. We say that $(X, \mathscr{F}, \mu, \mathscr{K})$ is an *affinity structure* on X. Given an affinity structure $(X, \mathscr{F}, \mu, \mathscr{K})$ on X, as in the general setting proposed in Section 2, we have at least two maximal operators $\mathcal{K}^*$ and $M_{\mathscr{K}}$. They satisfy the basic estimate provided by Theorem 2.1. Regarding the shapes of the level sets of kernels $K \in \mathscr{K}$, sometimes, under somehow mild transitivity conditions on the kernels, we have that each K has a Newtonian structure (see [2]). That is, $K(x,y) \cong \varphi_K(d_K(x,y))$ with φ_K a decreasing profile and d_K a quasi-metric on X. Here the symbol $\cong$ means that the quotient between the two quantities is bounded above and below by positive constants. In the notation we emphasize the dependence on K of the profile and the metric. Since the level sets of K are essentially those of $\varphi_K \circ d_K$ and φ_K decreasing, the level sets of K are essentially d_K-balls.

Following [2] we say that a symmetric and positive kernel K on $X \times X$ is of Newtonian type if there exist a metric d on X and a (one-to-one) decreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, with $\varphi(0) = \infty$ and $\varphi(\infty) = 0$, such that $\varphi(d(x,y)) = K(x,y)$. In other words, $K = \varphi \circ d$. Actually, this idea of extension of the classical potential $\frac{1}{|x-y|}$ in the three-dimensional Euclidean space can still be generalized by allowing quasi-metrics instead of metrics and profiles φ which are not strictly decreasing. In this section we consider a given affinity structure $(X, \mathscr{F}, \mu, \mathscr{K})$ with some additional transitivity and doubling properties for the sections of the kernels, and the analytical consequences related to the boundedness properties of $\mathcal{K}^*$ and $M_{\mathscr{K}}$.

Before stating the main result of this section let us state the basic metrization lemma given and proved in [2].

Lemma 3.1. *Let X be a set. Let $\mathcal{V}:\mathbb{R}^+\to\mathcal{P}(X\times X)$ be a one-parameter family of subsets of $X\times X$ that satisfies the following properties:*

(S1) *Each $\mathcal{V}(r)$ is symmetric.*
(S2) $\Delta=\{(x,x):x\in X\}\subseteq\mathcal{V}(r)$ *for every* $r>0$.
(S3) $\mathcal{V}(r_1)\subseteq\mathcal{V}(r_2)$ *for* $0<r_1\leq r_2$.
(S4) $\bigcup_{r>0}\mathcal{V}(r)=X\times X$.
(S5) $\bigcap_{r>0}\mathcal{V}(r)\subseteq\Delta$.
(S6) *There exists $T>1$ such that $\mathcal{V}(r)\circ\mathcal{V}(r)\subseteq\mathcal{V}(Tr)$ for every $r>0$, where $U\circ V=\{(x,z)\in X\times X:$ there exists $y\in X$ such that $(x,y)\in V,(y,z)\in U\}$ is the composition of the subsets U and V of $X\times X$.*

Then the function $\delta(x,y)=\inf\{r>0:(x,y)\in\mathcal{V}(r)\}$ is a quasi-metric on X with triangle constant less than or equal to T. Moreover, for every $r>0$, we have

$$\{(x,y)\in X\times X:\delta(x,y)<r\}\subseteq\mathcal{V}(r)\subseteq\{(x,y)\in X\times X:\delta(x,y)<2r\}.$$

Theorem 3.2. *Let $(X,\mathscr{F},\mu,\mathscr{K})$ be an affinity structure on X. Assume that $\mathscr{K}$ satisfies the following properties:*

(i) *The level sets of each $K\in\mathscr{K}$ are the same, precisely, if $K,K'\in\mathscr{K}$ and $\lambda>0$, there exists $\theta>0$ such that $\{K>\lambda\}=\{K'>\theta\}$.*
(ii) *There exists $K_0\in\mathscr{K}$ such that*
  (ii.a) $K_0(x,x)=+\infty$ *for every* $x\in X$;
  (ii.b) *there exists $0<\nu<1$ such that $K_0(x,y)>\lambda$ and $K_0(y,z)>\lambda$ implies $K_0(x,z)>\nu\lambda$;*
  (ii.c) *there exist $M>1$ and $A\geq 1$ such that $\mu(\{y:K_0(x,y)>\frac{\lambda}{M}\})\leq A\mu(\{y:K_0(x,y)>\lambda\})$.*

If $\sup_{K\in\mathscr{K}}\int K(x,y)\,d\mu(y)$ is uniformly bounded, then:

(1) *$\mathcal{K}^*$ and $M_{\mathscr{K}}$ are of weak type $(1,1)$;*
(2) *$M_{\mathscr{K}}$ is bounded in $L^p(X,wd\mu)$, $1<p<\infty$, if and only if w belongs to $\mathcal{A}^p(X,\mathscr{F},\mu,\mathscr{K})$, that is,*

$$\sup_{\substack{K\in\mathscr{K}\\ x\in X,\,\lambda>0}}\left(\fint_{\{y:K(x,y)>\lambda\}}w(y)\,d\mu(y)\right)\left(\fint_{\{y:K(x,y)>\lambda\}}w^{-\frac{1}{p-1}}(y)\,d\mu(y)\right)^{p-1}<\infty$$

(here, as usual, $\fint_E f(x)\,d\mu(x)=\frac{1}{\mu(E)}\int_E f(x)\,d\mu(x)$);
(3) *if $w\in\mathcal{A}^p(X,\mathscr{F},\mu,\mathscr{K})$, then $\mathcal{K}^*$ is bounded in $L^p(X,wd\mu)$, $1<p<\infty$.*

Proof. In the proof of the structure of K_0 we shall follow the lines in [2] for the simple case of the linear transitivity provided by hypothesis (ii.b). Let $\alpha=\frac{\log 2}{\log\nu}$, where ν is the constant in (ii.b). Notice that $\alpha<0$. Now, for $r>0$, define

$$V(r)=\{(x,y)\in X\times X:K_0(x,y)>r^{1/\alpha}\}.$$

Let us check that this family satisfies (S1) to (S6) in Lemma 3.1. The symmetry of each $V(r)$ follows from the symmetry of K_0. Since $K_0(x,x)=+\infty$, from (ii.a) we see that the diagonal Δ of $X\times X$ is contained in each $V(r)$. If $0<r_1\leq r_2$, then $0<r_2^{1/\alpha}\leq r_1^{1/\alpha}$ and $\{K_0>r_1^{1/\alpha}\}\subseteq\{K_0>r_2^{1/\alpha}\}$ or $V(r_1)\subseteq V(r_2)$. Since $K_0>0$, we have that $X \times X = \bigcup_{r>0} V(r)$. Since $K_0(x,y)$ is finite for every $x \neq y$, we also have that $\Delta = \bigcap_{r>0} V(r)$. Let us now consider the composition of $V(r)$ with itself. That is, $V(r) \circ V(r) = \{(x,z) \in X \times X :$ there exists $y \in X$ such that $(x,y) \in V(r)$ and $(y,z) \in V(r)\}$. Take a point $(x,z) \in V(r) \circ V(r)$; then, for $y \in X$ such that $(x,y) \in V(r)$ and $(y,z) \in V(r)$, we have that $K_0(x,y) > r^{1/\alpha}$ and $K_0(y,z) > r^{1/\alpha}$. Hence, from (ii.b) we also have that $K_0(x,z) > \nu r^{1/\alpha} = \nu r^{\frac{\log \nu}{\log 2}} = (2r)^{\frac{\log \nu}{\log 2}}$. Therefore $(x,z) \in V(2r)$ and $V(r) \circ V(r) \subseteq V(2r)$. So we apply Lemma 3.1 to obtain that $\delta(x,y) = \inf\{r > 0 : (x,y) \in V(r)\}$ is a quasi-metric on X with triangular constant $\tau \leq 2$ such that

$$\{\delta(x,y) < r\} \subseteq V(r) \subseteq \{\delta(x,y) < 2r\}$$

for every $r > 0$. The last inclusion can be written in terms of K_0 as follows:

$$\{\delta(x,y) < r\} \subseteq \left\{K_0(x,y) > r^{\frac{1}{\alpha}}\right\} \subseteq \{\delta(x,y) < 2r\}$$

for every $r > 0$. Changing the variable according to $s = r^{1/\alpha}$ we have

$$\left\{\delta^{\frac{1}{\alpha}}(x,y) > s\right\} \subseteq \{K_0(x,y) > s\} \subseteq \left\{\delta^{\frac{1}{\alpha}}(x,y) > 2^{\frac{1}{\alpha}} s\right\}$$

for every $s > 0$. Set $A(s) = \{\delta^{1/\alpha}(x,y) > s\}$, $B(s) = \{K_0(x,y) > s\}$, and $C(s) = \{\delta^{1/\alpha}(x,y) > 2^{1/\alpha} s\}$. So $A(s) \subseteq B(s) \subseteq C(s)$ for every $s > 0$. Hence, taking $K_0(x,y) = s$ we have $(x,y) \notin B(s) = B(K_0(x,y))$, so $(x,y) \notin A(K_0(x,y))$ or, in other words, $\delta^{1/\alpha}(x,y) \leq K_0(x,y)$. On the other hand, since (x,y) does not belong to $C\left(\frac{\delta^{1/\alpha}(x,y)}{2^{1/\alpha}}\right)$, (x,y) does not belong to $B\left(\frac{\delta^{1/\alpha}(x,y)}{2^{1/\alpha}}\right)$ and this fact means that $K_0(x,y) \leq \frac{\delta^{1/\alpha}(x,y)}{2^{1/\alpha}}$. Hence

$$\delta^{\frac{1}{\alpha}}(x,y) \leq K_0(x,y) \leq 2^{\frac{1}{|\alpha|}} \delta^{\frac{1}{\alpha}}(x,y),$$

as desired. In order to finish the proof of (1) we have to show the doubling property for the δ-balls. The last estimate for K_0 becomes

$$\delta^{\beta}(x,y) \leq K_0(x,y) \leq 2^{|\beta|} \delta^{\beta}(x,y)$$

for every $(x,y) \in X \times X$, with $\beta < 0$. Hence the δ-balls in X are equivalent to the sections of the level sets of K_0. More precisely, for $x \in X$ and $r > 0$, we have from (ii.c) that

$$\begin{aligned}
\mu(B_\delta(x,2r)) &\leq \mu\Big(\Big\{y : K_0^{\frac{1}{\beta}}(x,y) < 2r\Big\}\Big) \\
&= \mu\big(\{y : K_0(x,y) > 2^{\beta} r^{\beta}\}\big) \\
&\leq \mu\Big(\Big\{y : K_0(x,y) > \frac{1}{M^m}\Big(\frac{r}{2}\Big)^{\beta}\Big\}\Big) \\
&\leq A^m \mu\Big(\Big\{y : K_0(x,y) > \Big(\frac{r}{2}\Big)^{\beta}\Big\}\Big) \\
&= A^m \mu\big(\{y : K_0(x,y) > 2^{|\beta|} r^{\beta}\}\big) \\
&\leq A^m \mu(B_\delta(x,r)),
\end{aligned}$$

where m is the first positive integer such that $4^{\beta} \geq M^{-m}$.

Property (2) follows from (1), (i), and the results in [3]. Now the boundedness properties of $\mathcal{K}^*$ and $M_{\mathscr{K}}$ follow from Theorem 2.1. □

4. BMO and Lipschitz spaces induced by affinity kernels

In the spirit of the result of the last section regarding the operator $M_{\mathscr{K}}$ for a given affinity structure on X satisfying the hypothesis in Theorem 3.2, we may also consider sharp maximal functions $M^{\#}_{\mathscr{K}}$ induced by the level sets of $\mathscr{K}$ and the corresponding bounded mean oscillation and integral Lipschitz spaces. The main results follow from the construction of the quasi-metric δ given in Theorem 3.2 and the results in [7]. See also [1] for the general versions of the John–Nirenberg theorem.

Let us start with the basic definitions. Throughout the section we shall assume that $(X, \mathscr{F}, \mu, \mathscr{K})$ is an affinity structure on X satisfying (i), (ii.a), (ii.b), and (ii.c) in Theorem 3.2 with K_0 as in (ii). Let $f : X \to \mathbb{R}$ be a function which is integrable on every section of every level set of every $K \in \mathscr{K}$. Briefly, we say that f is *$\mathscr{K}$-locally integrable.*

A $\mathscr{K}$-locally integrable function f is said to belong to $\mathrm{BMO}(\mathscr{K})$ if there exists a constant A such that, for every $K \in \mathscr{K}$, every $x \in X$, and every $\lambda > 0$, there exists $C_{x,\lambda,K}$ with

$$\frac{1}{\mu(\{y : K(x,y) > \lambda\})} \int_{\{y:K(x,y)>\lambda\}} |f(y) - C_{x,\lambda,K}|\, d\mu(y) \le A.$$

For $\alpha > 0$, we say that f belongs to $\mathrm{Lip}(\mathscr{K}, \alpha)$ if

$$\int_{\{y:K(x,y)>\lambda\}} \left| f(y) - \fint_{\{y:K(x,z)>\lambda\}} f(z)\, d\mu(z) \right| d\mu(y) \le A\, \mu(\{y : K(x,y) > \lambda\})^{1+\alpha}$$

for some constant A, every $K \in \mathscr{K}$, every $\lambda > 0$, and every $x \in X$. Here, as before, $\fint_E f\, d\mu = \frac{1}{\mu(E)} \int_E f\, d\mu$. For $\gamma > 0$, we say that f belongs to $\Lambda(\mathscr{K}, \gamma)$ if there exists a constant A such that

$$K_0^{\gamma}(x,y)|f(x) - f(y)| \le A$$

for every x and y in X.

Theorem 4.1. *Let $(X, \mathscr{F}, \mu, \mathscr{K})$ be an affinity structure on X satisfying (i) and (ii) in Theorem 3.2.*

(a) *If $f \in \mathrm{BMO}(\mathscr{K})$, then there exist positive constants c_1 and c_2 such that, for every $K \in \mathscr{K}$, every $\lambda > 0$, every $x \in X$, and every $t > 0$, we have the inequality*

$$\mu\Big(\Big\{y : K(x,y) > \lambda \text{ and } \Big|f(y) - \fint_{\{K>\lambda\}} f\, d\mu\Big| > t\Big\}\Big) \le c_1 e^{-\frac{c_2}{A} t} \mu(\{y : K(x,y) > \lambda\}).$$

(b) *If w belongs to the Muckenhoupt class $\mathcal{A}^p(X, \mathscr{F}, \mu, \mathscr{K})$ with respect to the sections of the level sets of kernels in $\mathscr{K}$ for some $p > 1$, then*

$$f = \log(w) \in \mathrm{BMO}(\mathscr{K}).$$

(c) *There exists a constant* $\beta > 0$ *depending only on* $(X, \mathscr{F}, \mu, \mathscr{K})$ *such that*

$$\operatorname{Lip}(\mathscr{K}, \alpha) \subseteq \Lambda(\mathscr{K}, \alpha\beta).$$

Proof. With the arguments in the proof of Theorem 3.2, and with the same notation used there, we have that all the sections of all the level sets of all the kernels in $\mathscr{K}$ are δ-balls. We also have that (X, δ, μ) is a space of homogeneous type with $K_0 \equiv \delta^{-\frac{1}{\beta}}$ for some $\beta > 0$. Then applying the results from [7] and [1] to this setting we obtain the result. Notice that the results in [7] show that if $f \in \operatorname{Lip}(\mathscr{K}, \alpha)$, then $|f(x) - f(y)| \leq C\delta^{\alpha}(x, y) \leq \tilde{C}K_0^{-\alpha\beta}(x, y)$, as desired. □

5. Euclidean and almost Euclidean disquisitions

Let us start from the application of Theorem 2.1 to provide a direct proof of the classical result of convolution kernels with an integrable, radial, and nonincreasing majorant. See [9] for one of the classical proofs.

Proposition 5.1. *Let* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$ *be nonincreasing with* $\int_0^\infty \rho^{n-1}\varphi(\rho)\, d\rho < \infty$. *Let* $\mathscr{K}$ *be the family of kernels in* $\mathbb{R}^n$ *given by*

$$\mathscr{K} = \left\{ K_\varepsilon(x, y) = \frac{1}{\varepsilon^n}\varphi\left(\frac{|x - y|}{\varepsilon}\right) : \varepsilon > 0 \right\}.$$

Then

$$\sup_{\varepsilon > 0} \int_{y \in \mathbb{R}^n} \frac{1}{\varepsilon^n}\varphi\left(\frac{|x - y|}{\varepsilon}\right) |f(y)|\, dy \leq CMf(x),$$

with M *the Hardy–Littlewood maximal operator on the Euclidean balls of* $\mathbb{R}^n$.

Proof. Since the sections of the level sets of each K_ε are Euclidean balls, we have that $M_{\mathscr{K}}f \leq Mf$. On the other hand,

$$\int_{y \in \mathbb{R}^n} K_\varepsilon(x, y)\, d\mu(y) = \int_{y \in \mathbb{R}^n} \frac{1}{\varepsilon^n}\varphi\left(\frac{|x - y|}{\varepsilon}\right) dy = \omega_n \int_0^\infty \varphi(\rho)\rho^{n-1}\, d\rho$$

and we are done. □

The situation of the above proposition extends to every metric measure space with a particular relation between the measure of balls and their radii. Then natural settings for these extensions are the α-regular Ahlfors or normal spaces. A metric space (X, d), or more generally a quasi-metric space (see [7]), is said to be *α-Ahlfors regular with respect to the Borel measure* μ if there exist constants $0 < c_1 \leq c_2 < \infty$ such that $c_1 r^\alpha \leq \mu(B(x, r)) \leq c_2 r^\alpha$ for every $x \in X$ and every $r > 0$. Here $B(x, r) = \{x \in X : d(x, y) < r\}$ is the d-ball of radius $r > 0$ at $x \in X$. It is worth noticing that the above situation contains the Euclidean one but also all the classical self-similar fractals with α not necessarily integer and the parabolic metrics in $\mathbb{R}^n$, to mention just a few.

Proposition 5.2. *Let $\alpha > 0$ and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$ be nonincreasing with $\int_0^\infty \rho^{\alpha-1}\varphi(\rho)\,d\rho < \infty$. Let (X, d, μ) be an α-regular Ahlfors space. Let $\mathscr{K}$ be the family of kernels defined in $X \times X$ by*

$$\mathscr{K} = \left\{ K_\varepsilon(x,y) = \frac{1}{\varepsilon^\alpha}\varphi\left(\frac{d(x,y)}{\varepsilon}\right) : \varepsilon > 0 \right\}.$$

Then

$$\sup_{\varepsilon>0} \int_{y\in X} \frac{1}{\varepsilon^\alpha}\varphi\left(\frac{d(x,y)}{\varepsilon}\right) |f(y)|\, d\mu(y) \leq CMf(x),$$

with M the Hardy–Littlewood maximal operator on the d-balls in X.

The proof is the same as that of Proposition 5.1 noticing that, since the space is Ahlfors α-regular and $\int_0^\infty \rho^{\alpha-1}\varphi(\rho)\,d\rho < \infty$, we have that $\sup_{\varepsilon>0} \int_X K_\varepsilon(x,y)\,d\mu(y) < \infty$.

Let us now consider an alternative definition of Muckenhoupt classes, given in (2) of Theorem 3.2 above, in a general affinity structure $(X, \mathscr{F}, \mu, \mathscr{K})$. The family of Euclidean balls in $\mathbb{R}^n$ can also be seen as the family of sections of the level sets of the kernels

$$\mathscr{K} = \left\{ \frac{1}{\omega_n \varepsilon^n} \mathcal{X}\left(\frac{|x-y|}{\varepsilon}\right) : \varepsilon > 0 \right\},$$

where $\mathcal{X}$ is the indicator function of $[0,1)$ and ω_n is the volume of the unit n-dimensional ball. The standard Muckenhoupt condition for $1 < p < \infty$ can be rewritten in terms of the family $\mathscr{K}$ as

$$\left(\int_{\mathbb{R}^n} K(x,y)w(y)\,dy\right)\left(\int_{\mathbb{R}^n} K(x,y)w^{-\frac{1}{p-1}}(y)\,dy\right)^{p-1} \leq C \tag{5.1}$$

for every $K \in \mathscr{K}$.

The point of view presented above suggests another way of defining a class of type $A^p_{\mathscr{K}}(X,\mu)$. The next result shows that condition (5.1) can be considered as a reverse Hölder condition even in the general setting.

Proposition 5.3. *Let $\mathscr{K}$ be a family of nonnegative, symmetric, and measurable kernels K on $X \times X$ with (X,μ) a σ-finite measure space. Assume that there exists $\alpha > 0$ such that $\int_{y\in X} K(x,y)\,d\mu(y) \geq \alpha$ for every $x \in X$. Then, for $1 < p < \infty$, we have*

$$\alpha^p \leq \left(\int_{y\in X} K(x,y)w(y)\,d\mu(y)\right)\left(\int_{y\in X} K(x,y)w^{-\frac{1}{p-1}}(y)\,d\mu(y)\right)^{p-1}$$

for every $K \in \mathscr{K}$.

Proof. From Hölder's inequality we have, with $p + q = pq$,

$$\begin{aligned}\alpha &\leq \int_{y\in X} K(x,y)\,d\mu(y)\\ &= \int_{y\in X} K(x,y)w^{\frac{1}{p}}(y)w^{-\frac{1}{p}}(y)\,d\mu(y)\end{aligned}$$

$$= \int_{y\in X} (K(x,y)w(y))^{\frac{1}{p}} \left(K(x,y)w^{-\frac{1}{p-1}}(y)\right)^{\frac{p-1}{p}} d\mu(y)$$
$$\leq \left(\int_{y\in X} K(x,y)w(y)\,d\mu(y)\right)^{1/p} \left(\int_{y\in X} K(x,y)w^{-\frac{1}{p-1}}(y)\,d\mu(y)\right)^{\frac{p-1}{p}}.$$

□

The next result provides some basic analysis of the relation between the classical $A^p(\mathbb{R}^n)$ and $A^p_{\mathscr{K}}(\mathbb{R}^n)$ when $\mathscr{K} = \left\{\frac{1}{\varepsilon^n}\varphi\left(\frac{|x-y|}{\varepsilon}\right) : \varepsilon > 0\right\}$ with $\varphi : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$, $\omega_n \int_0^\infty \varphi(\rho)\rho^{n-1}\,d\rho = 1$.

Proposition 5.4. *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$ be a nonincreasing function with $1 = \omega_n \int_0^\infty \varphi(\rho)\rho^{n-1}\,d\rho$. Let $\mathscr{K} = \left\{\frac{1}{\varepsilon^n}\varphi\left(\frac{|x-y|}{\varepsilon}\right) : \varepsilon > 0\right\}$. Then, for $1 < p < \infty$, the following conditions hold:*

(a) *$A^p_{\mathscr{K}}(\mathbb{R}^n) \subseteq A^p(\mathbb{R}^n)$.*
(b) *Let $s = \sup\{p, q\}$ with $pq = p+q$ and let $\varphi \in L^\infty(\mathbb{R}^+)$ with $\int_1^\infty \varphi(\rho)^{sn-1}\,d\rho < \infty$; then $A^p(\mathbb{R}^n) = A^p_{\mathscr{K}}(\mathbb{R}^n)$.*
(c) *Let $w \in A^p(\mathbb{R}^n)$; then there exists $\delta > 0$ such that, with φ compactly supported and $\int_0^1 \varphi(\rho)^{\delta n-1}\,d\rho < \infty$, we have $w \in A^p_{\mathscr{K}}(\mathbb{R}^n)$.*
(d) *$w \in A^p(\mathbb{R}^n)$ implies that $\mathcal{K}^*$ is bounded in $L^p(w)$. Hence $w \in A^p_{\mathscr{K}}(\mathbb{R}^n)$ implies the boundedness of $\mathcal{K}^*$ in $L^p(w)$.*

Proof. To prove (a) take $w \in A^p_{\mathscr{K}}(\mathbb{R}^n)$; then, since φ is a non-vanishing nonincreasing function, there exist positive numbers a and b such that $\varphi(\rho) \geq b > 0$ for every $\rho \in [0, a]$. Hence

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} w(y)\,dy\right)\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} w^{-\frac{1}{p-1}}(y)\,dy\right)^{p-1}$$
$$\leq \left(\frac{1}{w_n r^n} I_w\right)\left(\frac{1}{w_n r^n} I_{w^{\frac{-1}{p-1}}}\right)^{p-1},$$

where

$$I_v = \int_{y\in\mathbb{R}^n} \frac{1}{b}\varphi\left(\frac{|x-y|a}{r}\right) v(y)\,dy$$

when v is a positive measurable function. Thus, with $\varphi_\varepsilon(|x-y|) = \frac{1}{\varepsilon^n}\varphi\left(\frac{|x-y|}{\varepsilon}\right)$, we have

$$\left(\frac{1}{w_n r^n} I_w\right)\left(\frac{1}{w_n r^n} I_{w^{\frac{-1}{p-1}}}\right)^{p-1}$$
$$\leq \frac{a^{np}}{(bw_n)^p}\left(\int_{y\in\mathbb{R}^n} \varphi_{\frac{r}{a}}(|x-y|)w(y)\,dy\right)\left(\int_{y\in\mathbb{R}^n} \varphi_{\frac{r}{a}}(|x-y|)w^{\frac{-1}{p-1}}(y)\,dy\right)^{p-1}$$
$$\leq \left(\frac{a^n}{b^p}\frac{1}{w_n^p}\right)^p A^p_{\mathscr{K}}(w),$$

with $A^p_{\mathscr{K}}(w)$ the optimal constant in (5.1) for the family $\mathscr{K} = \{\varphi_\varepsilon(|x-y|) : \varepsilon > 0\}$.

Let us now prove (b). Take $w \in A^p(\mathbb{R}^n)$. Let $B_p(w)$ be a constant such that $\|Mf\|_{L^p(w)} \leq B_p^{1/p}(w)\|f\|_{L^p(w)}$, where M is the standard Hardy–Littlewood maximal operator. Then, for every ball B and every measurable subset E of B, we have that

$$\frac{|E|}{|B|} \leq B_p(w) \left(\frac{w(E)}{w(B)}\right)^{1/p}, \tag{5.2}$$

with $w(E) = \int_E w$ (see [4]). The following estimates follow readily from the properties of φ and from (5.2):

$$\begin{aligned}
&\int_{y\in\mathbb{R}^n} \varphi\left(\frac{|x-y|}{\varepsilon}\right) w(y)\,dy \\
&\qquad \leq \|\varphi\|_\infty w(B(x,\varepsilon)) + \sum_{k\geq 1} \int_{\varepsilon 2^{k-1}\leq|x-y|<\varepsilon 2^k} \varphi\left(\frac{|x-y|}{\varepsilon}\right) w(y)\,dy \\
&\qquad \leq \|\varphi\|_\infty w(B(x,\varepsilon)) + \sum_{k\geq 1} \varphi(2^{k-1}) w(B(x,\varepsilon 2^k)) \\
&\qquad \leq \|\varphi\|_\infty w(B(x,\varepsilon)) + \sum_{k\geq 1} \varphi(2^{k-1}) B_p^p(w) 2^{nkp} w(B(x,\varepsilon)) \\
&\qquad \leq \left(\|\varphi\|_\infty + C\int_1^\infty \varphi(\rho)\rho^{pn-1}\,d\rho\right) w(B(x,\varepsilon)).
\end{aligned}$$

On the other hand, since $\sigma = w^{-\frac{1}{p-1}}$ belongs to $A^q(\mathbb{R}^n)$, the estimates above also show that

$$\int_{y\in\mathbb{R}^n} \varphi\left(\frac{|x-y|}{\varepsilon}\right) \sigma(y)\,dy \leq \left(\|\varphi\|_\infty + C\int_1^\infty \varphi(\rho)\rho^{qn-1}\,d\rho\right) \sigma(B(x,\varepsilon)).$$

Hence

$$\begin{aligned}
\left(\int_{y\in\mathbb{R}^n} \varphi\left(\frac{|x-y|}{\varepsilon}\right) w(y)\,dy\right) &\left(\int_{y\in\mathbb{R}^n} \varphi\left(\frac{|x-y|}{\varepsilon}\right) w^{-\frac{1}{p-1}}(y)\,dy\right)^{p-1} \\
&\leq C w(B(x,\varepsilon))\sigma(B(x,\varepsilon))^{p-1} \\
&\leq \tilde{C}|B(x,\varepsilon)|^p.
\end{aligned}$$

Therefore $w \in A^p_{\mathscr{K}}(\mathbb{R}^n)$.

For the case of a singularity of φ at the origin, we have to use the so-called $A^\infty(\mathbb{R}^n)$ condition (see [4]). If $w \in A^p(\mathbb{R}^n)$, there exist positive constants β_1 and δ_1 such that the inequality

$$\frac{w(E)}{w(B)} \leq \beta_1 \left(\frac{|E|}{|B|}\right)^{\delta_1}$$

holds for every ball B and for every measurable subset E of B. Since $\sigma = w^{\frac{1}{p-1}} \in A^q(\mathbb{R}^n)$, there also exist β_2 and δ_2 such that

$$\frac{\sigma(E)}{\sigma(B)} \leq \beta_2 \left(\frac{|E|}{|B|}\right)^{\delta_2}$$

for $E \subseteq B$.

Again, a dyadic decomposition of the integral gives

$$\begin{aligned}\int_{y\in\mathbb{R}^n} \varphi\left(\frac{|x-y|}{\varepsilon}\right) w(y)\,dy &= \sum_{k\geq 1} \int_{\varepsilon 2^{-k}\leq|x-y|<\varepsilon 2^{-k+1}} \varphi\left(\frac{|x-y|}{\varepsilon}\right) w(y)\,dy \\ &\leq \sum_{k\geq 1} \varphi(2^{-k}) w(B(x,\varepsilon 2^{-k+1})) \\ &\leq \sum_{k\geq 1} \varphi(2^{-k})\beta_1 2^{-nk\delta_1} w(B(x,\varepsilon)) \\ &= \beta_1 \left(\sum_{k\geq 1} \varphi(2^{-k}) 2^{-nk\delta_1}\right) w(B(x,\varepsilon)).\end{aligned}$$

Also,

$$\int_{y\in\mathbb{R}^n} \varphi\left(\frac{|x-y|}{\varepsilon}\right) \sigma(y)\,dy \leq \beta_2 \left(\sum_{k\geq 1} \varphi(2^{-k}) 2^{-nk\delta_2}\right) \sigma(B(x,\varepsilon))$$

and

$$\begin{aligned}&\left(\int_{y\in\mathbb{R}^n} \varphi\left(\frac{|x-y|}{\varepsilon}\right) w(y)\,dy\right)\left(\int_{y\in\mathbb{R}^n} \varphi\left(\frac{|x-y|}{\varepsilon}\right) \sigma(y)\,dy\right)^{p-1} \\ &\qquad\leq C\left(\int_0^1 \varphi(\rho)\rho^{n\delta-1}\,d\rho\right)^p |B(x,\varepsilon)|^p\end{aligned}$$

with $\delta = \min\{\delta_1, \delta_2\}$. In order to prove (d) notice that from Proposition 5.1 we have that $\mathcal{K}^*$ is bounded above by the Hardy–Littlewood maximal operator. If $w \in A^p(\mathbb{R}^n)$, we have that M is bounded in $L^p(w)$ and so is $\mathcal{K}^*$. The last assertion follows from the above and (a). □

When the conditions on φ contained in (b) and (c) of Proposition 5.4 are satisfied, no matter what the values of p, q, and δ are, the two classes of weights coincide.

Proposition 5.5. *Let* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$ *be a nonincreasing function such that* $\int_0^1 \frac{\varphi(\rho)}{\rho}\,d\rho < \infty$ *and* $\int_1^\infty \varphi(\rho)\rho^m\,d\rho < \infty$ *for every* $m \in \mathbb{N}$*. Then*

(α) $A^p(\mathbb{R}^n) = A^p_{\mathscr{K}}(\mathbb{R}^n)$ *for* $1 < p < \infty$*;*
(β) $w \in A^p_{\mathscr{K}}(\mathbb{R}^n)$ *if and only if* $\mathcal{K}^*$ *is bounded in* $L^p(w)$*.*

Proof. To prove (α) take $w \in A^p(\mathbb{R}^n)$, the standard Muckenhoupt class. Then

$$\begin{aligned}
&\int_{y\in\mathbb{R}^n} \varphi\left(\frac{|x-y|}{\varepsilon}\right) w(y)\,dy \\
&\qquad = \int_{|x-y|<\varepsilon} \varphi\left(\frac{|x-y|}{\varepsilon}\right) w(y)\,dy + \int_{|x-y|>\varepsilon} \varphi\left(\frac{|x-y|}{\varepsilon}\right) w(y)\,dy \\
&\qquad \le \sum_{k\ge 1} \varphi(2^{-k}) w(B(x,\varepsilon 2^{-k+1})) + \sum_{k\ge 1} \varphi(2^{k-1}) w(B(x,\varepsilon 2^{k})) \\
&\qquad \le \left[\beta_1 \sum_{k\ge 1} \varphi(2^{-k}) 2^{-nk\delta_1} + B_p^p(w) \sum_{k\ge 1} \varphi(2^{k-1}) 2^{nkp}\right] w(B(x,\varepsilon)) \\
&\qquad \le C\left(\int_0^1 \varphi(\rho)\frac{d\rho}{\rho} + \int_1^\infty \varphi(\rho)\rho^{pn-1}\,d\rho\right) w(B(x,\varepsilon)).
\end{aligned}$$

Also,

$$\int_{y\in\mathbb{R}^n} \varphi\left(\frac{|x-y|}{\varepsilon}\right) \sigma(y)\,dy \le C\left(\int_0^1 \varphi(\rho)\frac{d\rho}{\rho} + \int_1^\infty \varphi(\rho)\rho^{pn-1}\,d\rho\right) \sigma(B(x,\varepsilon)).$$

Hence $w \in A^p_{\mathscr{K}}(\mathbb{R}^n)$, as desired. The statement in (β) follows from (a) in Proposition 5.4, Proposition 5.1, and (α), since $Mf(x) \le C\mathcal{K}^* f(x)$. □

A particular case of the above proposition is provided by a local behavior of the type $\frac{1}{\left(\log\frac{1}{\rho}\right)^{1+\varepsilon}}$, $\varepsilon > 0$, close to the origin, and an exponential behavior of the type $e^{-\alpha\rho}$ ($\alpha > 0$) for ρ large. On the other hand, for heavy-tailed profiles φ, the conditions of the above results are not uniformly satisfied and the boundedness of $\mathcal{K}^*$ in $L^p(w)$ for $1 < p < \infty$ does not imply that $w \in A^p_{\mathscr{K}}(\mathbb{R}^n)$.

The above results extend naturally to Ahlfors regular metric spaces. Now the family of kernels is given by

$$\mathscr{K} = \left\{\frac{1}{\varepsilon^\alpha}\varphi\left(\frac{d(x,y)}{\varepsilon}\right) : \varepsilon > 0\right\},$$

where d is a metric on an abstract set X and α is the dimension of (X, d).

Let us finally briefly illustrate in some simple analytical setting the problems arising when the given family of sections of level sets of kernels involve balls corresponding to different metrics. Let $(X, \mathscr{F}, \mu, \mathscr{K})$ be an affinity structure on X. Assume that the sections of the level sets of each $K \in \mathscr{K}$ are equivalent to balls in X, with respect to some metric d that belongs to some family $\mathcal{D}$ of metrics on X. Hence the corresponding maximal operator on the sections of kernels of $\mathscr{K}$ is

$$M_{\mathcal{D}} f(x) = \sup_{d\in\mathcal{D}} M_d f(x) = \sup_{d\in\mathcal{D}} \sup_{r>0} \frac{1}{\mu(B_d(x,r))} \int_{B_d(x,r)} |f(y)|\,d\mu(y).$$

Here B_d denotes any d-ball in X. This maximal operator coincides with the Hardy–Littlewood maximal operator when $\mathcal{D}$ has only one element and, under a doubling condition on μ, when all the metrics in $\mathcal{D}$ are equivalent with uniform equivalence constants, in the sense that there exist two positive constants c_1 and c_2 such that

$c_1d(x,y) \leq d'(x,y) \leq c_2d(x,y)$ for every choice of d and d' in $\mathcal{D}$ and every x and y in X. However, even in classical harmonic analysis, there are some families of metrics which are not equivalent to the Euclidean one; such is the case of parabolic metrics, which we proceed to briefly describe. It is worth noticing that these metrics still provide the spaces $\mathbb{R}^n$ with their standard topology.

Let us consider the following simple but illustrative situation in $X = \mathbb{R}^n$. Let $\overline{\gamma} = (\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n$ with $\gamma_i \geq 1$. Define $\rho_{\overline{\gamma}}$ on $\mathbb{R}^n$ by $\rho_{\overline{\gamma}}(x) = \sup\{|x_i|^{\frac{1}{\gamma_i}} : i = 1, \dots, n\}$ and $d_{\overline{\gamma}}(x,y) = \rho_{\overline{\gamma}}(x-y)$. Then $\rho_{\overline{\gamma}}$ is a metric (parabolic distance) in $\mathbb{R}^n$ for every such a $\overline{\gamma}$. Notice that each $d_{\overline{\gamma}}$ produces on X a structure of regular Ahlfors space. In fact, $B_{d_{\overline{\gamma}}}(x,r) = \{y \in \mathbb{R}^n : |x_i - y_i| < r^{\gamma_i}\}$ and $|B_{d_{\overline{\gamma}}}(x,r)| = 2^n r^{|\gamma|}$, with $|\gamma| = \sum_{i=1}^n \gamma_i \geq n$. Hence, with $|\cdot|$ the Lebesgue measure, the space $(\mathbb{R}^n, d_{\overline{\gamma}}, |\cdot|)$ is a $|\gamma|$-regular Ahlfors space. Hence all the results of the previous sections hold for families of kernels sharing the $d_{\overline{\gamma}}$ balls as sections of their level sets for every $\overline{\gamma}$ with $\gamma_i \geq 1$. Let us also consider a family of metrics $\widehat{\mathcal{D}}$ on $\mathbb{R}^n$ larger than $\mathcal{D} = \{d_{\overline{\gamma}} : \gamma_i \geq 1\}$. Let $\overline{\gamma} \in \mathbb{R}^n$ with $\gamma_i \geq 1$, $i = 1, \dots, n$, be given. Let A be any symmetric $n \times n$ matrix with eigenvalues $\gamma_1, \gamma_2, \dots, \gamma_n$. The dilation operator defined by the matrix A on $\mathbb{R}^n$ is given by $T^A_\lambda x = e^{A\log(\lambda)}x$, $\lambda > 0$. For a given $x \neq 0$ in $\mathbb{R}^n$, the equation $\|T^A_\lambda x\|_\infty = 1$ has only one solution. Here $\|y\|_\infty = \sup_{1=1,\dots,n} |y_i|$. Clearly, λ depends on x and is positive. Following the lines of Lemma 11.3.1 in [5, Chapter 11], we see that $d_A(x,y) = \frac{1}{\lambda(x-y)}$ defines a metric on $\mathbb{R}^n$ and the d_A balls are orthogonal transformations of the $d_{\overline{\gamma}}$ balls with $\overline{\gamma} = (\gamma_1, \dots, \gamma_n)$ the eigenvalues of A. Hence $M_{\widehat{\mathcal{D}}}$ is the strong maximal function over the family of all rectangles where $\widehat{\mathcal{D}} = \{d_A : \text{eigenvalues}(A) = \{\gamma_1, \dots, \gamma_n\}, \gamma_i \geq 1\}$. On the other hand, $M_{\mathcal{D}}$ is the maximal function defined on the family of the so-called intervals, i.e. parallelepipeds with sides parallel to the coordinate axes.

The classical results regarding the "intervals" and the "rectangles" as differentiation bases in $\mathbb{R}^n$ (see [5, Chapters 7 and 8]) readily give the following proposition.

Proposition 5.6. *Let $\mathcal{D}$, $\widehat{\mathcal{D}}$ be as before and let $\widehat{\mathcal{D}}_F$ be a finite subfamily of $\widehat{\mathcal{D}}$. Then*

(a) $M_{\widehat{\mathcal{D}}_F}$ *is of weak type* $(1,1)$.
(b) *There exists $C > 0$ such that*

$$|\{M_{\mathcal{D}} f > \lambda\}| \leq C \int \frac{|f(x)|}{\lambda}\left(1 + \log^+ \frac{|f(x)|}{\lambda}\right) dx$$

for every $\lambda > 0$ and every f. But $M_{\mathcal{D}}$ is not of weak type $(1,1)$.
(c) $M_{\widehat{\mathcal{D}}}$ *is only bounded on* $L^\infty(\mathbb{R}^n)$.

The estimate in (a) is due to Jessen, Marcinkiewicz, and Zygmund [6]. The result of (c) was proved by Nikodym [8].

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